On the theory of Fourier hyperfunctions and its applications to partial differential equations with constant coefficients

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In this paper we first construct the theory of Fourier transformation of the hyperfunctions on $\boldsymbol{D}^n$, which is the radial compactification of $\boldsymbol{R}^n$ (see Definition 1.1.1), and, as one of its applications, we develop the theory of partial differential operators with constant coefficients, including the convolution operators $S*$, where S is a hyperfunction with compact support.

The theory of hyperfunctions is developed by Sato [33], [34], [35] and studied by Martineau [30], Harvey [10], Komatsu [23], [24] and others.

Our theory is indicated by Sato [33].

The main results of this paper are the following:

(I) We construct a sheaf $\mathscr{R}$ over $\boldsymbol{D}^n$, which coincides with $\mathscr{B}$ (the sheaf of hyperfunction) on $\boldsymbol{R}^n$ and whose global sections are stable under Fourier transformation.

(II) We treat the following problems:

(i) Problem of ellipticity

(ii) Propagation of regularity

(iii) Problem of hyperbolicity.

The remarkable points of our theory are; (i) we have obtained a sheaf (compare the space $\mathscr{S}'$ (Schwartz [41]). This fact turns out to be very useful in the treatment of real-analyticity (see § 5). (ii) The sheaf $\mathscr{R}$ constitutes a flabby sheaf over $\boldsymbol{D}^n$, so that any hyperfunction μ on $\boldsymbol{R}^n$ can be extended to $\boldsymbol{D}^n$ and we can consider its Fourier transform. This fact is used in the treatment of hyperbolicity. (see § 6) (We also use this fact to treat the problem of existence of the solutions of division problems in our forthcoming paper [18].) (iii) When S is a distribution with compact support (which is a special case of our theory), the theory of convolution operators becomes very transparent. Cf. Ehrenpreis [4], [6], Gårding [7]. This is because we consider in the framework of hyperfunctions not in that of distributions.

In the forthcoming paper [18] we develop the theory of modified Fourier hyperfunctions, which is useful in the treatment of division problems.

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Part I. FOURIER HYPERFUNCTIONS

§ 1. Formulations and main results

In this section we give the definition of Fourier hyperfunctions and mention their main properties, which are proved in later sections.

1.1. Definitions

DEFINITION 1.1.1. We denote by $\boldsymbol{D}^n$ the compactification $\boldsymbol{R}^n \sqcup S_\infty^{n-1}$ of $\boldsymbol{R}^n$, where S_∞^{n-1} in an $(n-1)$-dimensional sphere at infinity. When x is a vector in $\boldsymbol{R}^n-\{0\}$, we denote by $x\infty$ the point on S^{n-1} which is represented by x, where we identify S^{n-1} with $\boldsymbol{R}^n-\{0\}/\boldsymbol{R}^+$. The space $\boldsymbol{D}^n$ is given the natural topology, that is: (i) If a point x of $\boldsymbol{D}^n$ belongs to $\boldsymbol{R}^n$, a fundamental system of neighbourhoods of x is the set of all open balls containing the point x. (ii) If a point x of $\boldsymbol{D}^n$ belongs to S_∞^{n-1}, a fundamental system of neighbourhoods of x $(=y\infty)$ is given by $\{(C+a)\cup C_\infty | C_\infty \ni y\infty\}$ where C is an open cone generated by some open neighbourhood y with its vertex at the origin, a is some vector in $\boldsymbol{R}^n$, namely $C+a$ is a cone with its vertex at a, and C_∞ denotes the points at infinity of that cone.

We mainly consider the space $\boldsymbol{D}^n \times \sqrt{-1}\boldsymbol{R}^n$ in what follows.

Now we give the definitions of sheaves $\tilde{\mathcal{O}}$ and $\underset{\sim}{\mathcal{O}}$ over $\boldsymbol{D}^n \times \sqrt{-1}\boldsymbol{R}^n$.

DEFINITION 1.1.2. (The sheaf of slowly increasing holomorphic functions.) We denote by $\tilde{\mathcal{O}}$ the sheaf whose section modules $\tilde{\mathcal{O}}(\Omega)$ over an open set $\Omega\ (\subset \boldsymbol{D}^n \times \sqrt{-1}\boldsymbol{R}^n)$ is the set of all holomorphic functions $f(z) (\in \mathcal{O}(\Omega \cap \boldsymbol{C}^n))$ such that for any positive ε and any compact set K in Ω, the estimate $\sup_{z\in K\cap \boldsymbol{C}^n} |f(z)e^{-\varepsilon|z|}| < \infty$ holds. It is clear that the presheaf $\{\tilde{\mathcal{O}}(\Omega)\}$ constitutes a sheaf over $\boldsymbol{D}^n \times \sqrt{-1}\boldsymbol{R}^n$.

REMARK. By the above definition $\tilde{\mathcal{O}}_z$ is equal to the germ of the sheaf of the holomorphic functions $\mathcal{O}_z$ if z is a point in $\boldsymbol{C}^n (\cong \boldsymbol{R}^n \times \sqrt{-1}\boldsymbol{R}^n)$.

DEFINITION 1.1.3. (The sheaf of rapidly decreasing holomorphic functions.) We denote by $\underset{\sim}{\mathcal{O}}$ the sheaf whose section modules $\{\underset{\sim}{\mathcal{O}}(\Omega)\}$ over an open set $\Omega (\subset \boldsymbol{D}^n \times \sqrt{-1}\boldsymbol{R}^n)$ is the set of all holomorphic functions $f(z) (\in \mathcal{O}(\Omega \cap \boldsymbol{C}^n))$ such that for any compact set K in Ω there exists some positive constant δ_K and the estimate $\sup_{z\in K\cap C^n} |f(z)e^{\delta_K|z|}|<\infty$ holds.

REMARK: By the above definition $\tilde{\mathcal{O}}_z$ coincides with $\mathcal{O}_z$ if z is a point in $\boldsymbol{C}^n$.

DEFINITION 1.1.4 (Topology of $\tilde{\mathcal{O}}(K)$). If K is a compact set in $\boldsymbol{D}^n\times\sqrt{-1}\boldsymbol{R}^n$, then we give $\tilde{\mathcal{O}}(K)$ the inductive limit topology $\varinjlim_m \mathcal{O}_c^m(U_m)$, where U_m is a fundamental system of neighbourhoods of K satisfying $U_m\Supset U_{m+1}$ and $\mathcal{O}_c^m(U_m)$ is the Banach space of all holomorphic functions $f(z)(\in\mathcal{O}(U_m\cap\boldsymbol{C}^n))$ which are continuous in $\bar{U}_m\cap\boldsymbol{C}^n$ and there exists some A such that $|f(z)|\leqq Ae^{-(1/m)|z|}$ with the norm $\|f\|=\sup_{z\in U_m\cap C^n}|f(z)e^{(1/m)|z|}|$. (We have used the symbol $U_m\Supset U_{m+1}$ to denote U_{m+1} has a compact neighbourhood in U_m with respect to the topology of $\boldsymbol{D}^n\times\sqrt{-1}\boldsymbol{R}^n$.)

The topology of $\tilde{\mathcal{O}}(K)$ is well-defined and it becomes DFS-space. (About the notion of DFS-space, see Komatsu [21].) Especially when $K=\boldsymbol{D}^n$, then we denote $\tilde{\mathcal{O}}(\boldsymbol{D}^n)$ as $\mathcal{P}_*$.

We go on to the definition of the space of Fourier hyperfunctions over Ω which is a subset of $\boldsymbol{D}^n$.

DEFINITION 1.1.5. We choose an open set V in $\boldsymbol{D}^n\times\sqrt{-1}\boldsymbol{R}^n$ which contains Ω as a relatively closed set and define $\mathcal{R}(\Omega)$, the space of Fourier hyperfunctions over Ω, by $H^n(V,\tilde{\mathcal{O}})$. (By the excision theorem the space $\mathcal{R}(\Omega)$ is independent of V.)

1.2. Properties of $\mathcal{R}(\Omega)$

(i) The presheaf $\{\mathcal{R}(\Omega)\}$ constitutes a flabby sheaf over $\boldsymbol{D}^n$, whose restriction to $\boldsymbol{R}^n$ coincides with the sheaf of hyperfunctions.

(ii) When K is a compact set in $\boldsymbol{D}^n H_K^n(V,\tilde{\mathcal{O}})\cong(\tilde{\mathcal{O}}(K))'$, especially $\mathcal{R}(\boldsymbol{D}^n)\cong(\mathcal{P}_*)'$.

(iii) If μ is an element of $\mathcal{R}(\boldsymbol{D}^n)$, then we decompose μ as $\sum_{j=1}^{2^n}\mu_j$, with $\operatorname{supp}\mu_j$ contained in the closure of j-th quadrant, which we denote K_j. After this decomposition we define $\mathcal{F}\mu$, Fourier transform of μ, as the cohomology class which is defined by $\{\langle\mu_j, e^{i\langle z,\zeta\rangle}\rangle\}$, here $\langle\mu_j, e^{i\langle z,\zeta\rangle}\rangle$ is defined by the duality given above, as far as $e^{i\langle z,\zeta\rangle}\in\tilde{\mathcal{O}}(K_j)$. On the other hand $\mathcal{P}_*$ is stable under the classical Fourier transformation. From this fact we can define the Fourier transformation $\mathcal{F}_d$, by $\langle\mathcal{F}_d\mu,\varphi\rangle\underset{\text{Df.}}{=}\langle\mu,\mathcal{F}\varphi\rangle$ $(\mu\in\mathcal{R}(\boldsymbol{D}^n), \varphi\in\mathcal{P}_*)$. As for the two definitions we can assert the relation $\mathcal{F}_d\mu=\mathcal{F}\mu$ holds.

About the notation and the proof of the above statements, see §3 of this chapter.

§ 2. Theorems about $\tilde{\mathcal{O}}$ and $\underset{\sim}{\mathcal{O}}$

In this section we prove a vanishing theorem of cohomology groups whose coefficient sheaf is $\tilde{\mathcal{O}}$ and an approximation theorem for $\underset{\sim}{\mathcal{O}}(K)$ (K; compact set in D^n), which are used in § 3.

The essential tools of this section are Hörmander's L^2-estimate (Hörmander [14]) and Komatsu's theory of DFS*-space (Komatsu [21]). We constantly quotate these two papers so we denote [14] by [H] and [21] by [K].

2.1. A cohomology vanishing theorem

Let Ω be a pseudononnex domain in C^n which satisfies $\sup_{z\in\Omega}|\mathrm{Im}\, z|=M<\infty$, and $\psi(z)$ be a plurisubharmonic function in Ω. We denote by X_j, Y_j and Z_j the spaces

$$L^2_{(p,q-1)}(\Omega; (1/j)\|z\|+4\log(1+|z|^2)+\psi(z)),$$

$$L^2_{(p,q)}(\Omega; (1/j)\|z\|+2\log(1+|z|^2)+\psi(z))$$

and

$$L^2_{(p,q+1)}(\Omega;\ (1/j)\|z\|+\psi(z))$$

respectively.

(Here we follow the notation of [H] and we mean by the symbol $\|z\|$ the modification of $\sum_{j=1}^{n}|z_j|$ near $\{z_j=0$ for some $j\}$ so as to become C^∞ and convex. This modification has no essential significance.)

We also define $X=\varprojlim_j X_j$, $Y=\varprojlim_j Y_j$, $Z=\varprojlim_j Z_j$.

LEMMA 2.1.1. *Let $\bar\partial$ be the Cauchy-Riemann operator defined in the distribution sense, then the sequence $X\xrightarrow{\bar\partial} Y\xrightarrow{\bar\partial} Z$ is exact.*

PROOF. When we represent $(X_j)'$ etc. by $L^2_{(p,q-1)}\{\Omega; -(1/j)\|z\|-4\log(1+|z|^2)-\psi(z)\}$ etc., then the adjoint operator of $\bar\partial$ is represented by ϑ, where $\vartheta f=(-1)^{p-1}\sum'_{I,K}\sum_j \partial f_{I,jK}/\partial z_j$.

Now, as is easily checked, $\sum_{j,k} t_j\bar t_k(\partial^2/\partial z_j\partial\bar z_k)\log(1+|z|^2)\geqq(1+|z|^2)^{-2}|t|^2$ for $t\in C^n$, so by a theorem of Hörmander ([H] p. 105, Th. 2.2.1') $X_j\xrightarrow{\bar\partial}Y_j\xrightarrow{\bar\partial}Z_j$ is exact. Therefore $X'_j\xrightarrow{\vartheta}Y'_j\xrightarrow{\vartheta}Z'_j$ is also exact.

On the other hand X' etc. which is an injective limit of Hilbert space X'_j etc. turns out to be a DFS*-space (See [K] p. 368). Therefore the theorem is trivially true if $q>1$ by the so-called Serre-Komatsu duality theorem. (See [K] p. 381, Th. 19.)

We consider the case $q=1$. By the well-known Krein-Shmulian theorem Fréchet space is fully complete, so we need to prove only $\operatorname{Im}\vartheta\cap V^{\circ}$ is closed. (Here V is a neighbourhood of 0 in X and V° is the polar set of V.) Now by the theory DFS*-space, there exists some j such that $\operatorname{Im}\vartheta\cap V^{\circ}=u_j(B_j)$, where B_j is a bounded set in X_j' and u_j is a weak homeomorphism. (See [K] p. 373, Th. 6.) If we assume $\vartheta u_\nu\to f\in V^{\circ}$, then ϑu_ν converges weakly to f in some X_k'. Here we need the following lemma.

LEMMA 2.1.2. *If $u\in Y_j'$ and $\vartheta u\in X_k'$ $(j>k)$, then there exists some v in Y_k' and $\vartheta u=\vartheta v$ holds.*

PROOF OF LEMMA 2.1.2. If we define $\varphi_k(z)=\exp(-1/n\,\bar{z}^2)$, then $\vartheta(\varphi_n u)=\varphi_n\vartheta u$ by the definition of ϑ.

Since we have assumed $\sup_{z\in\Omega}|\operatorname{Im} z|<\infty$, $\varphi_n u$ belongs to Y_k'.

On the other hand $\varphi_n\vartheta u$ converges to ϑu in X_k' by Lebesgue's theorem because φ_n converges to 1 pointwise in Ω. As is remarked above ϑ is a closed range operator from Y_k' to X_k', so from the two facts just proved follows $\vartheta u\in\vartheta(Y_k')$. This means there exists some v in Y_k' such that $\vartheta v=\vartheta u$. Q.E.D.

We return to the proof of Lemma 2.1.1. By Lemma 2.1.2 we may assume not only $\vartheta u_\nu\in X_k'$ but also $u_\nu\in Y_k'$. Since $\vartheta(Y_k')$ is closed in X_k', $\vartheta(Y_k')$ is weakly closed because $\vartheta(Y_k')$ is convex. Therefore there exists some v such that $f=\vartheta v$.

This proves that $\operatorname{Im}\vartheta\cap V^{\circ}$ is closed, hence $\operatorname{Im}\vartheta$ is closed.

Therefore $\operatorname{Im}\bar{\partial}$ is closed because DFS*-space is reflexive, and we can apply Serre-Komatsu duality theorem again. This completes the proof of the theorem for $q=1$.

DEFINITION 2.1.3. We call an open set Ω in $\boldsymbol{D}^n\times\sqrt{-1}\boldsymbol{R}^n$ to be $\tilde{\mathcal{O}}$-pseudoconvex domain if it satisfies following conditions:

(i) $\sup_{z\in V}|\operatorname{Im} z|$, where $V=\Omega\cap\boldsymbol{C}^n$.

(ii) There exists a plurisubharmonic function $\theta(z)$ on V which satisfies $\{z|\theta(z)<c\}\Subset V$ for any c and $\sup_{L\cap\boldsymbol{C}^n}\theta(z)\leqq M_L$ for any $L\Subset\Omega$.

REMARK 1. We prove there exist sufficiently many $\tilde{\mathcal{O}}$-pseudo convex domains later. The easiest but most important example is $D^n\times\sqrt{-1}I^n$, where $I^n=(-1,1)\times\cdots\times(-1,1)$.

REMARK 2. Considering $\theta(z)+\log(1+|z|^2)$, we find $V=\Omega\cap\boldsymbol{C}^n$ is a pseudoconvex domain.

THEOREM 2.1.4. *For any $\tilde{\mathcal{O}}$-pseudoconvex domain Ω in $\boldsymbol{D}^n\times\sqrt{-1}\boldsymbol{R}^n$ we have $H^s(\Omega,\tilde{\mathcal{O}})=0$ $(s\geqq1)$.*

PROOF. We prove the vanishing of Čech cohomology group. It is sufficient to prove $\varinjlim_{\{\Omega_\nu\}} H^s(\{\Omega_\nu\}, \tilde{\mathcal{O}})=0$, where $\{\Omega_\nu\}$ satisfies

(i) $\Omega=\bigcup_N \Omega_\nu$ (locally finite open covering)

(ii) $\Omega_\nu \cap \boldsymbol{C}^n = V_\nu$ is convex.

Now we prove the following Lemma 2.1.5, the theorem follows from the special case of that lemma for $p=q=0$, because we can use Cauchy's integral formula to change the L^2-norm to the sup-norm for holomorphic functions. Before showing Lemma 2.1.5, we define $C^s(Z^{\text{loc}}_{(p,q)}(\{V_\nu\}$; infraexponential)) to be the set of all cochains $c=\{c_\nu\}$ which satisfy

(i) $\bar\partial c_\nu=0$ in V_ν

(ii) For any ε positive and any finite subset M of N $\sum_{\nu\in M}\int_{V_\nu} |c_\nu|^2 e^{-\varepsilon\|\nu\|} dV<\infty$

where dV is the Lebesgue measure on $\boldsymbol{R}^n$.

LEMMA 2.1.5. *Let c belong to $C^s(Z^{\text{loc}}_{(p,q)})(\{V_\nu\}$; infraexponential)) and satisfy $\delta c=0$, then we can find some c' which satisfies*

(i) $\delta c'=c$.

(ii) $c'\in C^{s-1}(Z^{\text{loc}}_{(p,q)}(\{V_\nu\}$, *infraexponential*)) ($\delta$ *means the coboundary operator*).

REMARK. This lemma is essentially due to Hörmander. See [H] p. 114. Th. 2.4.1.

PROOF OF THE LEMMA. We denote by $\{\chi_\nu\}$ the partition of unity subordinate to $\{V_\nu\}$ and define $b_\alpha=\sum_j \chi_j c_{j,\alpha}$. Since $\delta c=0$, we have $\delta b=c$. So $\delta\bar\partial b=0$ because $\bar\partial c=0$. By Cauchy's inequality $\int_{V_\alpha} |b_\alpha|^2 e^{-\varphi(z)} dV \leqq \sum_j \int_{V_\alpha} \chi_j |c_j\alpha|^2 e^{-\varphi(z)} dV$ for any continuous $\varphi(z)$.

By the assumption of the existence of $\theta(z)$, we can find some plurisubharmonic function $\psi(z)$ on V which satisfies

(i) $\sum |\bar\partial \chi_\nu| \leqq e^{\psi(z)}$.

(ii) $\sup_{K\cap C^n} \psi(z) \leqq c_K$ for $\forall K \Subset \Omega$ (Cf. [H] p. 117. Theorem 2.2.4).

Thus it follows from the conditions on c that $\sum_{\alpha\in M}\int |\bar\partial b_\alpha|^2 e^{-\varepsilon\|z\|} dV<\infty$ ($\forall\varepsilon$, $\forall M$).

We consider the case $s=1$. By the fact $\delta(\bar\partial b)=0$, $\bar\partial b$ defines a global section f in this case. Lemma 2.1.1 and the existence of $\psi(z)$ prove the existence of some u such that $\bar\partial u=f$ and $\int_{K\cap C^n} |u|^2 e^{-\varepsilon\|z\|}(1+|z|^2)^{-2} dV<\infty$ ($\forall\varepsilon$, $\forall K \Subset \Omega$).

If we define $c'_\alpha=b_\alpha-u|_{V_\alpha}$, then $\partial c'_\alpha=0$ and $\delta c'=\delta b=c$. Clearly $c'\in C^{s-1}(Z^{\text{loc}}_{(p,q)}(\{V_\nu\}$, infraexponential)).

We go on to the case $s>1$. In this case we use the induction on s, following Hörmander.

By the induction hypothesis there exists b' such that $\partial b'=\bar{\partial} b$ and belongs to $C^{s-2}(Z^{\mathrm{loc}}_{(p,q)}(\{V_\nu\}, \text{infraexponential}))$. Applying Lemma 2.1.1 as above we can find $\{b_\alpha''\}$ which satisfies $b'_\alpha=\bar{\partial} b''_\alpha$ and $\sum_{\alpha\in M}\int_{V_\alpha}|b''_\alpha|^2 e^{-\varepsilon\|z\|}(1+|z|^2)^{-2}dV<\infty$ $(\forall\varepsilon, \forall M)$. Therefore $c'=b-\partial b''$ satisfies all conditions required.

The following theorem shows any open set in $\boldsymbol{D}^n$ has a fundamental system of neighbourhood of $\tilde{\mathcal{O}}$-pseudoconvex domain.

THEOREM 2.1.6 (Cf. Grauert [9] § 3). *Let S be an open set in $\boldsymbol{D}^n$ and U be an open neighbourhood of S in $\boldsymbol{D}^n\times\sqrt{-1}\boldsymbol{R}^n$, then there exists V which satisfies*

(i) *V satisfies the conditions of the theorem.*

(ii) *$V\subset U$ and $S=V\cap\boldsymbol{D}^n$.*

PROOF. If $S\cap S_\infty^{n-1}=\phi$, then the above results of Grauert prove the theorem, so we assume $S\cap S_\infty^{n-1}\neq\phi$.

Since U is open in $\boldsymbol{D}^n\times\sqrt{-1}\,\boldsymbol{R}^n$, there exists $\gamma(z)\in C^\infty(U\cap\boldsymbol{C}^n)$ which satisfies

(i) $\{z\in U\cap\boldsymbol{C}^n|\gamma(z)\leqq c\}\Subset U$ for any c.

(ii) If K is a compact set in $\boldsymbol{R}^n$ or a closed convex cone, then $\sup_L|\gamma(z)|$, $\sup_L|\nabla^2\gamma(z)|\leqq M_{K,\varepsilon}$ as far as $L=K\times\sqrt{-1}(\prod_{j=1}^n[-\varepsilon_j, \varepsilon'_j])\subset U$. Here ∇^2 means any of $\partial^2/\partial x_j\partial x_k$, $\partial^2/\partial x_j\partial y_k$, or $\partial^2/\partial y_j\partial y_k$.

We choose a suitable $a(x)(\in C^\infty(S\cap R^n))$ which grows sufficiently rapidly as x tends to the boundary of $(\boldsymbol{D}^n-S)$ from the interior of $S\cap\boldsymbol{R}^n$ but $\sup_{K\cap R^n}a(x)\leq M_K'$ for any $K\Subset S$, and define $q(z)=a(\mathrm{Re}\,z)\sum_{j=1}^n(\mathrm{Im}\,z_j)^2$ and $p_1(z)=\gamma(z)+q(z)$.

By the definition of the topology of $\boldsymbol{D}^n\times\sqrt{-1}\,\boldsymbol{R}^n$, suitable choice of $a(x)$ ensures the existence of a neighbourhood W of S in $\boldsymbol{D}^n\times\sqrt{-1}\,\boldsymbol{R}^n$, which satisfies that $p_1(z)$ is plurisubharmonic in $W\cap\boldsymbol{C}^n$.

For any z_j in $(\partial W-\partial S)\cap\boldsymbol{C}^n$ we define plurisubharmonic functions $\theta^j(z)$ as follows;

$$\theta^j(z)=\max\{0, (1/\sum_{k=1}^n(\mathrm{Im}\,z_k^j)^2)\times(2\sum_{k=1}^n(\mathrm{Im}\,z_k)^2-\sum_{k=1}^n(\mathrm{Re}\,(z_k-z_k^j))^2\}\,.$$

Since $\theta^j(z_j)=2$, by a suitable choice of z_j $p_2(z)=\sup_j\theta^j(z)$ becomes a well-defined plurisubharmonic function on $W\cap\boldsymbol{C}^n$. (Remark that the supremum is taken over finite indices locally). And moreover we can assume $\{z\in W\cap\boldsymbol{C}^n|p_2(z)\leqq 1\}\cap\partial W=\phi$. After the above preparation we define V to be the interior of $(\{z\in W\cap\boldsymbol{C}^n|p_2(z)\leqq 1\}\cup(W\cap S_\infty^{n-1}))$ with respect to the topology of $\boldsymbol{D}^n\times\sqrt{-1}\,\boldsymbol{R}^n$, and $p(z)$ by $p_1(z)+\sum_{j\geq 0}p_2(z)^j$. Then all conditions we need are satisfied by V and $p(z)$.

REMARK 1. Up to this point we have assumed that $\sup_{z\in\Omega\cap C^n}|\mathrm{Im}\,z|<\infty$, but this condition can be weakend. It will be used in our forthcoming paper [18].

REMARK 2. A little more precise consideration gives us $H^s(\boldsymbol{D}^n\times\sqrt{-1}\,\boldsymbol{R}^n, \tilde{\mathcal{O}})=0$ $(s\geq 1)$, and analogously we can prove the vanishing of cohomology with bounds with respect to the standard covering of $\boldsymbol{C}^n$. This may be used to prove an analogue of Palamodov's Theorem 2 of Ch. 4 §5 (Palamodov [32], if we adopt the notion of Nöther operator. (About the Nöther operator, see [32]). It will be used in treating the overdetermined system in hyperfunctions. (See for e.g. Kaneko [17].) But such a fact seems to have no novelties since it is essentially the same as Ehrenpreis [5], Hörmander [15] and Palamodov [32] except for some technicalities which are used above, so we omit the details.

2.2. Approximation theorem

Next we go on to the proof of an analogue of Runge's theorem.

THEOREM 2.2.1. *Let K be a compact set in $\boldsymbol{D}^n$, then $\mathscr{P}_*=\tilde{\mathcal{O}}(\boldsymbol{D}^n)$ is dense in $\tilde{\mathcal{O}}(K)$. (See Definition 1.1.4 about $\mathscr{P}_*$ and $\tilde{\mathcal{O}}(K)$.)*

PROOF. We define $U_j=\boldsymbol{D}^n\times\sqrt{-1}\{y|\sum_{k=1}^n|y_k|^2<1/j\}$. Using the condition about K, we will prove the existence of $\{\Omega_j\}$ which have the following properties. (The construction of $\{\Omega_j\}$ is done at the end of the proof of this theorem.)

(a) $U_j\supset\Omega_j\Supset K$ and Ω_j's tend to K decreasingly.

(b) For any j and any $T(\Subset\Omega_j)$ there exist an open set V and $\theta(z)$ which is strictly plurisubharmonic in U_j, and they satisfy the following conditions:

(i) $T\Subset V\Subset\Omega_j$;

(ii) $\theta(z)<0$ on $T\cap\boldsymbol{C}^n$;

(iii) $\theta(z)>0$ near $\partial V\cap\boldsymbol{C}^n$;

(iv) For $\forall L\Subset\Omega_j$, $\sup_{L\cap\boldsymbol{C}^n}\theta(z)\leq\exists M_L<\infty$.

Now we begin the proof of the theorem.

At first we define some spaces which we need.

For positive ε and open set Ω we denote by $\mathscr{A}^{2,-2\varepsilon}_{\text{loc}}(\Omega)$ the set of all holomorphic functions $f(z)$ on $\Omega\cap\boldsymbol{C}^n$ which satisfy $\int_{K\cap\boldsymbol{C}^n}|f|^2e^{2\varepsilon\|z\|}dV<\infty$ $(\forall K\Subset\Omega)$, by $L^{2,-\varepsilon}_{\text{loc}}(\Omega)$ the set of all measurable functions $f(z)$ which satisfy $\int_{K\cap\boldsymbol{C}^n}|f|^2e^{\varepsilon\|z\|}dV<\infty$ $(\forall K\Subset\Omega)$ and by $X^\varepsilon(\Omega)$ the closure of $\mathscr{A}^{2,-2\varepsilon}_{\text{loc}}(\Omega)$ in $L^{2,-\varepsilon}_{\text{loc}}(\Omega)$. If we take δ so as to $\varepsilon<\delta(<2\varepsilon)$, $\mathscr{A}^{2,-\delta}_{\text{loc}}(\Omega)$ is contained in $X^{-\varepsilon}(\Omega)$. To prove this fact it is sufficient to show $\mathscr{A}^{2,-\delta,-2\log(1+|z|^2)}_{\text{loc}}(\Omega)$ is dense in $\mathscr{A}^{2,-\delta',-2\log(1+|z|^2)}_{\text{loc}}(\Omega)$, where $\delta'<\delta$ and $\mathscr{A}^{2,-\delta,-2\log(1+|z|^2)}_{\text{loc}}(\Omega)$ is the set of all holomorphic functions $f(z)$ on $\Omega\cap\boldsymbol{C}^n$ which satisfy $\int_{K\cap\boldsymbol{C}^n}|f|^2e^{\delta\|z\|+2\log(1+|z|^2)}dV<\infty$ $(\forall K\Subset\Omega)$.

We also define $L^{2,-\delta,-2\log(1+|z|^2)}_{\text{loc}}(\Omega)$ to be the set of all measurable functions

$f(z)$ which satisfy $\int_{K\cap C^n}|f|^2e^{\delta\|z\|+2\log(1+|z|^2)}dV<\infty$ $(\forall K\Subset\Omega)$.

Let μ belong to $(L_{\mathrm{loc}}^{2,-\delta',-2\log(1+|z|^2)}(\Omega))'$ and be orthogonal to $\mathscr{A}_{\mathrm{loc}}^{2,-\delta,-2\log(1+|z|^2)}(\Omega)$. We want to prove μ is orthogonal to $\mathscr{A}_{\mathrm{loc}}^{2,-\delta',-2\log(1+|z|^2)}(\Omega)$. We use the Hahn-Banach theorem to find some u whose support is compact in Ω with $\int|u|^2e^{-\delta'\|z\|-2\log(1+|z|^2)}dV<\infty$ and $\langle\mu, v\rangle=\int_{\Omega\cap C^n}v\bar{u}\,dV(\forall v\in L_{\mathrm{loc}}^{2,-\delta',-2\log(1+|z|^2)}(\Omega))$. Using a theorem of Hörmander ([H] p. 109, Proposition 2.3.2) and the existence $\theta(z)$ whose properties are given above, we can find some F whose support is compact in Ω with $\vartheta F=u$ and $\int|F|^2e^{-\delta\|z\|-2\log(1+|z|^2)}dV<\infty$, since $\delta'<\delta$ and $\operatorname{supp}u$ is compact in Ω. If φ belongs to $\mathscr{A}_{\mathrm{loc}}^{2,-\delta',-2\log(1+|z|^2)}(\Omega)$, then $\varphi(z)\exp(-(1/n)z^2)$ belongs to $\mathscr{A}_{\mathrm{loc}}^{2,-\delta,-2\log(1+|z|^2)}(\Omega)$ under the condition $\sup_{z\in\Omega\cap C^n}|\operatorname{Im}z|<\infty$. Therefore we have $0=\langle\mu, \varphi(z)\exp(-(1/n)z^2)\rangle=\int(\vartheta F)\varphi(z)\exp(-(1/n)z^2)dV\to\int(\vartheta F)\varphi dV=\langle\mu, \varphi\rangle$ by Lebesgue's theorem. Thus we have proved $\mathscr{A}_{\mathrm{loc}}^{2,-\delta,-2\log(1+|z|^2)}(\Omega)$ is dense in $\mathscr{A}_{\mathrm{loc}}^{2,-\delta',-2\log(1+|z|^2)}(\Omega)$, so that we have proved $\mathscr{A}_{\mathrm{loc}}^{2,-\delta}(\Omega)$ is contained in $X^{-\varepsilon}(\Omega)$.

Now we use these spaces to prove $\mathscr{P}_*$ is dense in $\underset{\sim}{\mathscr{O}}(K)$. Since $\varinjlim_{\varepsilon,j}X^{-\varepsilon}(\Omega_j)=\underset{\sim}{\mathscr{O}}(K)$ to prove the following statement (*).

(*) If an element μ of $[X^{-\varepsilon}(\Omega_{j_0})]'$ is orthogonal to $B\underset{\mathrm{Df.}}{=}\left\{u\in\mathscr{O}(U_{j_0}\cap C^n)\mid\int_{L\cap C^n}|u^2|e^{\delta\|z\|}dV<\infty,\ \forall L\Subset U_{j_0}\right\}$ then μ is zero. From now on we fix ε and j_0, so we denote by X the space $X^{-\varepsilon}(\Omega_{j_0})$. By the Hahn-Banach theorem there exists some u whose support is compact in Ω, $\int|u|^2e^{-\varepsilon\|z\|}dV<\infty$ and $\langle\mu, v\rangle=\int_{\Omega\cap C^n}v\bar{u}\,dV$ $(\forall v\in X)$.

Take $\operatorname{supp}u$ as T and fix V and $\theta(z)$ which correspond to T. We define $C=\bigcup_{\lambda=1}^{\infty}\{v\in L_{\mathrm{loc}}^2(U: \lambda\theta^+-\delta'\|z\|-2\log(1+|z|^2))|\bar{\partial}v=0\}$ where $\theta^+(z)=\max\{0, \theta(z)\}$ and $2\varepsilon>\delta'>\delta$. Then by the condition (iv) on $\theta(z)$, C is contained in B. Since μ is zero on B and $\operatorname{supp}u\Subset\Omega$, $\langle\mu, v\rangle=\int_{\Omega\cap C^n}v\bar{u}\,dV=\int_{V\cap C^n}v\bar{u}\,dV=0$ for any v in C. Moreover by the condition (ii) on $\theta(z)$, $u(z)$ is zero where $\theta(z)>0$. Defining $g_{\delta''}(z)=\cosh(\delta''z)$ we have $\int v\bar{u}\,dV=\int vg_{\delta''}(z)\overline{(u)/g_{\delta''}(\bar{z}))}\,dV$ from the first we choose ε so small as to secure $g_{\delta''}(\bar{z})\neq0$ in U. (The assumption has no essential significance). Taking $\bar{u}=u/g_{\delta''}(\bar{z})$ we have some F which satisfies the follwing conditions (i)$\sim$(iii), by [H] p. 109, Proposition 2.3.2.

(i) $\bar{u}=\vartheta F$

(ii) $F=0$ near ∂V

(iii) $F\in L^2(U; -(\delta''-\delta')\|z\|+2\log(1+|z|^2))$.

So we may consider $f(z)=F(z)g_{\delta''}(\bar{z})$ satisfies

(a) $\vartheta f=u$

(b) $\operatorname{supp} f\subset V\Subset U$

(c) $f\in L^2(U; \delta'\|z\|+2\log(1+|z|^2))$.

Therefore by an integration by parts we can prove the following equality for any v which belongs to $\mathscr{A}^{2,-2\varepsilon}_{\text{loc}}(\Omega)$, where $2\varepsilon>\delta'''>\delta'$.

$$0=\int_{\Omega\cap C^n}(\bar\partial v)\bar f\,dV=\int_{\Omega\cap C^n}\bar\partial(vg_{\delta'''}(z))\,\overline{(f/g_{\delta'''}(\bar z))}\,dV$$
$$=\int_{\Omega\cap C^n}vg_{\delta'''}(z)\vartheta\overline{(f/g_{\delta'''}(\bar z))}\,dV=\int_{\Omega\cap C^n}v\overline{\vartheta f}\,dV$$
$$=\int_{\Omega\cap C^n}v\bar u\,dV=\langle\mu, v\rangle\,.$$

Thus we have proved μ is zero on a dense subset of X, so we conclude μ is zero.

Now we complete the theorem by constructing $\{\Omega_j\}$. We agree to say Ω is of type (E) if $\Omega=\bigcap_{l=1}^{\infty} V^l$ where

$$V^l=\{z|\exp(-\sum_{j=1}^n(z_j-a_j^l)^2)<c_l,\ \sum_{j=1}^n|\operatorname{Im} z_j|^2<d_l,\ \text{where } a_j^l\in \boldsymbol{R}\}\,.$$

Since K is a compact set in D^n, it is clear that K can be approximated by a decreasing sequence of Ω_j, where Ω_j is of type (E).

We construct V and $\theta(z)$ which have the required properties for any $T\Subset\Omega_j$.

From now on we abbreviate Ω_j to Ω. By the definition of T we can find $C_j=K_j\times\sqrt{-1}\,I_j$ such that $T\subset\bigcup_{j=1}^m C_j$, where K_j is a relatively compact open set in $\boldsymbol{R}^n$ or open convex cone, I_j is a direct product of open intervals of $\boldsymbol{R}$, and $C_j\Subset\Omega$. Then taking a suitable set S which is of type (E), we have $T\Subset S$ and $S\cap\{|\operatorname{Re} z|>1\}\Subset\Omega\cap\{|\operatorname{Re} z|>1\}$. On the other hand, recalling that $\Omega=\cap V^l$ by the definition of Ω we have $T\cap\{|\operatorname{Re} z|<2\}\Subset V_\varepsilon^{l\pm,\cdots,\pm}\cap\{|\operatorname{Re} z|<2\}\Subset V^l$ for sufficiently small ε, where $V_\varepsilon^{l\pm,\cdots,\pm}$ is a translation of V^l parallel to the coordinate axis by $\pm\varepsilon$. Thus taking ε sufficiently small we have $T\Subset V_\varepsilon^{l\pm,\cdots,\pm}$ so we define $V=S\cap(\cap V_\varepsilon^{l\pm,\cdots,\pm})$.

By the above construction V can be represented as $\cap V_l$, where $V_l=\{z||f_l(z)|<1,\ \sum_{j=1}^n|\operatorname{Im} z_j|^2<d_l$ where $f_l(z)=c_l\exp(-\sum_{j=1}^n(z_j-a_j^l)^2),\ a_j^l\in\boldsymbol{R}\}$. By the method of construction of Ω and V, we can assume $d_l=d$ without loss of generality.

Defining $\sigma(z)=\sup_l\log|f_l(z)|$ and $\psi(z)=\sigma(z)*\rho_\varepsilon$ (where ρ_ε is a mollifier in $\boldsymbol{R}^{2n}$), we may consider $\psi(z)<0$ on T if ε is sufficiently small. Next we take suitable strictly plurisubharmonic function $\varphi(z)=\varphi(\operatorname{Im} z)$ and define $\chi(z)=\max(\varphi(z), \psi(z))$ so that $\chi(z)<0$ on T and $\chi(z)>0$ near ∂V. At long last we define $\theta(z)=\chi(z)*\rho_\varepsilon+\varepsilon\varphi(z)$, which has the all properties required if ε is sufficiently small.

Thus the proof of Theorem 2.2.1 is finished.

§ 3. Proofs of main properties of $\mathscr{R}(\Omega)$

In this section we prove the main properties of $\mathscr{R}(\Omega)$, which were announced in § 1.

3.1. Representation of $H^p(V, \tilde{\mathcal{O}})$ by differential forms and some extension of Malgrange's theorem on vanishing of n-th cohomology group.

At first we prepare some spaces of differential forms which we will use in this section.

We use Ω to denote some open set in $\boldsymbol{D}^n\times\sqrt{-1}\,\boldsymbol{R}^n$ from now on.

DEFINITION 3.1.1. We define $\mathscr{X}_j(\Omega)$ to be the set of all $(0, j)$-forms u on $\Omega\cap\boldsymbol{C}^n$ which satisfy the following conditions: for any compact set K in Ω and any positive ε

$$\int_{K\cap C^n}|u|^2e^{-\varepsilon\|z\|}dV<\infty \quad\text{and}\quad \int_{K\cap C^n}|\bar\partial u|^2e^{-\varepsilon\|z\|}dV<\infty \quad\text{hold.}$$

DEFINITION 3.1.2. We define $\mathscr{Y}_j(\Omega)$ to be the set of all $(0, j)$-forms u on $\Omega\cap\boldsymbol{C}^n$ which satisfy the following conditions: for any compact set K in Ω we have some positive δ_K such that

$$\int_{K\cap C^n}|u|^2e^{\delta_K\|z\|}\,dV<\infty \quad\text{and}\quad \int_{K\cap C^n}|\bar\partial u|^2e^{\delta_K\|z\|}dV<\infty \quad\text{hold.}$$

We denote by $\mathscr{X}_j$ and $\mathscr{Y}_j$ the sheaves subordinate to the above presheaves $\{\mathscr{X}_j(\Omega)\}$ and $\{\mathscr{Y}_j(\Omega)\}$ respectively. For any compact set K in Ω we can find $C_j=K_j\times\sqrt{-1}\,I_j$ such that $K\subset\bigcup_{j=1}^{m} C_j\Subset\Omega$, where K_j is a relatively compact open set in $\boldsymbol{R}^n$ or open convex cone, I_j is an relatively compact open set in $\boldsymbol{R}^n$. Hence we can find a C^∞ function $\varphi(z)$ on $\boldsymbol{C}^n$ which is equal to 1 on some neighbourhood of $K\cap\boldsymbol{C}^n$ and vanishes outside Ω with $\sup|\varphi(z)|$, $\sup|\nabla\varphi(z)|\leq M$. Therefore the sheaf $\mathscr{X}_j$ and $\mathscr{Y}_j$ are soft sheaves.

By the definitions of $\tilde{\mathcal{O}}$ and $\underset{\sim}{\mathcal{O}}$ and the existence theorem for $\bar\partial u=f$ with bounds (Lemma 2.1.1), we obtain the following soft resolutions of $\tilde{\mathcal{O}}$ and $\underset{\sim}{\mathcal{O}}$ respectively.

$$0\longrightarrow\tilde{\mathcal{O}}\longrightarrow\mathscr{X}_0\xrightarrow{\bar\partial}\mathscr{X}_1\xrightarrow{\bar\partial}\cdots.$$

$$\cdots \xrightarrow{\bar\partial} \mathscr{X}_{n-1} \xrightarrow{\bar\partial} \mathscr{X}_n \longrightarrow 0 \quad \text{(exact)}$$

$$0 \longrightarrow \underset{\sim}{\mathscr{O}} \longrightarrow \mathscr{Y}_0 \xrightarrow{\bar\partial} \mathscr{Y}_1 \longrightarrow \cdots$$

$$\cdots \xrightarrow{\bar\partial} \mathscr{Y}_{n-1} \xrightarrow{\bar\partial} \mathscr{Y}_n \longrightarrow 0 \quad \text{(exact)}\,.$$

(As for the resolution of $\underset{\sim}{\mathscr{O}}$ we can use [H] p. 105, Theorem 2.2.1′ directly.) Therefore we obtain the following Dolbeault isomorphisms:

$$H^p(\Omega, \widetilde{\mathscr{O}}) \simeq \{u \in \mathscr{X}_p(\Omega) | \bar\partial u = 0\} / \bar\partial \mathscr{X}_{p-1}(\Omega) \tag{3.1.3}$$

$$H^p_{\mathrm{comp}}(\Omega, \underset{\sim}{\mathscr{O}}) \cong \{u \in \mathscr{Y}_p(\Omega))_{\mathrm{comp}} | \bar\partial u = 0\} / \bar\partial((\mathscr{Y}_{p-1}(\Omega))_{\mathrm{comp}}\,. \tag{3.1.4}$$

(By $H^p_{\mathrm{comp}}(\Omega, \underset{\sim}{\mathscr{O}})$ we mean the p-th cohomology group with compact support.) Now we introduce the following auxiliary spaces for the sake of convenience.

DEFINITION 3.1.5. We define $X_j(\Omega)$ to be the set of all $(0, j)$-forms u on $\Omega \cap C^n$ satisfying the following condition: for any compact set K in Ω and any positive ε $\int_{K\cap C^n} |u|^2 e^{-\varepsilon\|z\|} dV < \infty$.

DEFINITION 3.1.6. We define $Y_j(\Omega)$ to be the set of all $(0, j)$-forms u with compact support in Ω satisfying $\int_{C^n} |u|^2 e^{\delta\|z\|} dV < \infty$ for some $\delta > 0$.

It is obvious from the above definitions that $X_j(\Omega)$ can be given the natural FS*-space structure, $Y_j(\Omega)$ can be given the natural DFS*-space structure and $Y_{n-j}(\Omega) \cong [X_j(\Omega)]'$. (For the notion of FS*-space and DFS*-space, see [K].) On the other hand, the p-th cohomology group of the complex

$$\{\longrightarrow X_{p-1}(\Omega) \xrightarrow{\bar\partial} X_p(\Omega) \xrightarrow{\bar\partial} X_{p+1}(\Omega) \longrightarrow \cdots\}$$

is isomorphic to the right side of (3.1.3) by the definition, and the p-th cohomology group obtained from the complex

$$\{\longrightarrow Y_{p-1}(\Omega) \xrightarrow{\bar\partial} Y_p(\Omega) \xrightarrow{\bar\partial} Y_{p+1}(\Omega) \longrightarrow \cdots\}$$

is isomorphic to the right side of (3.1.4). Therefore we obtain the following theorem by the Serre-Komatsu duality for the FS*-spaces. (See [K] p. 381, Theorem 19.)

THEOREM 3.1.7. *If* $H^p(\Omega, \widetilde{\mathscr{O}}) = 0$ $(p \geqq 1)$ *then* $[H^j(\Omega, \widetilde{\mathscr{O}})]' \cong H^{n-j}_{\mathrm{comp}}(\Omega, \underset{\sim}{\mathscr{O}})$. (We remark this theorem is also true under a little weaker conditions dim $H^p(\Omega, \widetilde{\mathscr{O}}) < \infty$ $(p \geqq 1)$ but we need not use this fact).

Moreover we can prove the following theorem which can be considered as a generalization of Malgrange's result (Malgrange [28]).

THEOREM 3.1.8. *Let Ω be any open set in $D^n\times\sqrt{-1}R^n$, then $H^n(\Omega, \widetilde{\mathcal{O}})=0$.*

PROOF. From what have been stated, it is sufficient to prove

$$X_{n-1}(\Omega)\xrightarrow{\bar{\partial}}X_n(\Omega)\longrightarrow 0 \quad \text{(exact)}.$$

In particuler it is sufficient to show

$$X_{n-1}(\Omega)\xrightarrow{\bar{\partial}}X_n(\Omega)\longrightarrow 0\ .$$

To prove this fact we need some auxiliary spaces.

Let K_j be an increasing sequence of compact sets which are contained in Ω and exhaust Ω, and define

$$X_l^j(K_j)=\left\{u\in L^{2,\mathrm{loc}}_{(0,l)}(C^n)\,\middle|\,\int_{K_j\cap C^n}|u|^2e^{-(1/j)\|z\|}dV<\infty\right\}$$

then $\varprojlim_j X_l^j(K_j)=X_l(\Omega)$ and $\varinjlim_j (X_l^j(K_j))'\cong Y_{n-l}(\Omega)$. we represent $(X_l^j(K_j))'$ by

$$\left\{u\in L^2_{(0,l)}(C^n)\,\middle|\,\operatorname{supp}u\subset K_j\cap C^n \text{ and } \int_{C^n}|u|^2e^{(1/j)\|z\|}dV<\infty\right\}$$

and obtain $(\partial)'=\vartheta$. (Here we have used the natural identification of $(0,l)$-form with $(0,n-l)$-form.) We want to prove ϑ is injective and of closed range. Since ϑ becomes elliptic operator from $Y_n(\Omega)$ to $Y_{n-1}(\Omega)$, the injectivity is trivially true by the unique continuation property. We prove f is in the range of ϑ when ϑu_ν converges weakly to f in $(X^j_{n-1}(K_j))'$. Then the proof is finished by the usual DFS*-space argument. (Cf. the proof of Lemma 2.1.1.)

We define $\widehat{K}_{j+1}$ to be the closure in $D^n\times\sqrt{-1}R^n$ of the union of $K_{j+1}\cap C^n$ and the connected component of (C^n-K_{j+1}) which is relatively compact with respect to the topology of R^{2n}, then we can find $v_\nu\in(X^j_n(\widehat{K}_{j+1}))'$ such that $\vartheta u_\nu=\vartheta v_\nu$. In fact it follows from $\vartheta u\in[X^j_{n-1}(K_j)]'$ and $u\in[X^k_n(K_k)]'$ $(j<k)$ that $\operatorname{supp}u\subset\widehat{K}_{k+1}$, because ϑ is elliptic.

Now we remark that $\sup_{z\in\widehat{K_{k+1}\cap C^n}}|\mathrm{Im}\,z|<\infty$ by the definition of the topology of $D^n\times\sqrt{-1}R^n$. Therefore it follows from Lebesgue's theorem that

$$0=\int(\varphi_n u)\bar{\partial}g\,dV=\int\vartheta(\varphi_n u)g\,dV=\int\varphi_n(\vartheta u)g\,dV\longrightarrow\int(\vartheta u)g\,dV\ ,$$

where

$$\varphi_n=\exp(-1/n)\bar{z}^2,\ \Omega\supset L\supset\widehat{K}_{k+1} \quad\text{and}\quad g\in X^j(L)\ ,$$

satisfying $\bar{\partial}g=0$. ($g\in X^j(L)$ means $\int_{L\cap C^n}|g|^2e^{-(1/j)\|z\|}dV<\infty$.) Applying [H] p. 109, Proposition 2.3.2, we are allowed to consider $\vartheta u\in\vartheta[X^j(L)]'$, so using the ellipticity of ϑ again we can find $w\in[X_j(K_j)]'$ such that $\vartheta u=\vartheta w$. Thus we can consider

$$\vartheta u_\nu\in[X^j(K_j)]',\ u_\nu\in[X^j(K_j)]' \text{ and } \vartheta u_\nu \xrightarrow[w]{} f \text{ in } [X^j(K_j)]'$$

from the beginning. Choosing L so that $\widehat{K}_j\Subset L\Subset\Omega$, $0=\int\overline{\vartheta u_\nu}gdV\longrightarrow\int\bar{f}gdV$ for any $g\in X^j(L)$, so $f=\vartheta v$, by [H] Proposition 2.3.2 again. This means that f is in the range of ϑ, and the proof is completed.

3.2. Proof of the pure-codimensionality of D^n with respect to $\widetilde{\mathscr{O}}$.

In this section we prove the vanishing of relative cohomology whose coefficient sheaf is $\widetilde{\mathscr{O}}$. (See theorems below.) This implies $\{\mathscr{R}(\Omega)\}$ is a flabby sheaf over D^n. The method of the proof is only a modification of Martineau's theory. (Martineau [30], see also Harvey [10] and Komatsu [23], [24].)

THEOREM 3.2.1. *Let K be a compact set in D^n and V be an open neighbourhood of K. Then we have $H^p_K(V,\widetilde{\mathscr{O}})=0$ $(p\neq n)$ and $H^n_K(\widetilde{\mathscr{O}})\cong[\underset{\sim}{\mathscr{O}}(K)]'$.*

PROOF. By the excision theorem we can assume $H^p(V,\widetilde{\mathscr{O}})=0$ $(p\geqq 1)$. (The results of §2 assures the existence of such V; for example we can take $D^n\times\sqrt{-1}I^n$ as V). We begin the proof by considering the following exact sequence:

$$\begin{aligned}0&\longrightarrow H^0_K(V,\widetilde{\mathscr{O}})\longrightarrow H^0(V,\widetilde{\mathscr{O}})\longrightarrow H^0(V-K,\widetilde{\mathscr{O}})\longrightarrow\\&\longrightarrow H^1_K(V,\widetilde{\mathscr{O}})\longrightarrow H^1(V,\widetilde{\mathscr{O}})\longrightarrow H^1(V-K,\widetilde{\mathscr{O}})\longrightarrow\cdots\\&\longrightarrow H^n_K(V,\widetilde{\mathscr{O}})\longrightarrow H^n(V,\widetilde{\mathscr{O}})\longrightarrow H^n(V-K,\widetilde{\mathscr{O}})\longrightarrow 0\,.\end{aligned}$$

Here $H_p(V,\widetilde{\mathscr{O}})=0$ $(p\geqq 1)$ by the assumption on V, and $H^0_K(V,\widetilde{\mathscr{O}})=0$ by the unique continuation theorem. Thus we obtain the following isomorphisms;

$$\begin{cases}H^1_K(V,\widetilde{\mathscr{O}})\cong H^0(V-K,\widetilde{\mathscr{O}})/H^0(V,\widetilde{\mathscr{O}})\\ H^p_K(V,\widetilde{\mathscr{O}})\cong H^{p-1}(V-K,\widetilde{\mathscr{O}})\ (p\geqq 2)\end{cases}$$

On the other hand we have the following exact sequence;

$$\begin{aligned}0&\longrightarrow H^0_{\text{comp}}(V-K,\underset{\sim}{\mathscr{O}})\longrightarrow H^0_{\text{comp}}(V,\underset{\sim}{\mathscr{O}})\longrightarrow H^0(K,\underset{\sim}{\mathscr{O}})\longrightarrow\\&\longrightarrow H^1_{\text{comp}}(V-K,\underset{\sim}{\mathscr{O}})\longrightarrow H^1_{\text{comp}}(V,\underset{\sim}{\mathscr{O}})\longrightarrow H^1(K,\underset{\sim}{\mathscr{O}})\cdots\\&\longrightarrow H^p_{\text{comp}}(V-K,\underset{\sim}{\mathscr{O}})\longrightarrow H^p_{\text{comp}}(V,\underset{\sim}{\mathscr{O}})\longrightarrow H^p(K,\underset{\sim}{\mathscr{O}})\longrightarrow\cdots\,.\end{aligned}$$

Here $H^p(K,\underset{\sim}{\mathscr{O}})=0$ $(p\geqq 1)$. In fact K is a compact set in D^n, so K has a fundamental system of neighbourhoods composed of $\widetilde{\mathscr{O}}$-pseudoconvex domains Ω_j

(Theorem 2.1.6). Hence it is sufficient to prove $\varinjlim_j H^p(\Omega_j, \underset{\sim}{\mathcal{O}})=0$ $(p\geqq 1)$. On the other hand given any cocycle $\{c_\nu\}$ in $H^p(\Omega_j, \underset{\sim}{\mathcal{O}})$ we can assume that $d=\{d_\nu=\cosh(\varepsilon z)\times c_\nu|\Omega_{j+1}\}$ defines a cocycle in $H^p(\Omega_{j+1}, \underset{\sim}{\tilde{\mathcal{O}}})$ for some positive ε. Therefore Theorem 2.1.3 asserts that $\{d_\nu\}$ is a coboundary in $H^p(\Omega_{j+1}, \tilde{\mathcal{O}})$, that is $d=\delta d'$. Defining c' by $\{d'_\nu\times\cosh(-\varepsilon z)\}$ we have $c|\Omega_{j+1}=\delta c'$, hence the image of $H^p(\Omega_j, \underset{\sim}{\mathcal{O}})$ in $H^p(\Omega_{j+1}, \underset{\sim}{\mathcal{O}})$ is zero. Therefore $\varinjlim H^p(\Omega_j, \underset{\sim}{\mathcal{O}})=H^p(K, \underset{\sim}{\mathcal{O}})=0$. Thus we have the isomorphisms:

$$\begin{cases} H^0(K, \underset{\sim}{\mathcal{O}})\cong H^1_{\mathrm{comp}}(V-K, \underset{\sim}{\mathcal{O}}) \\ H^p_{\mathrm{comp}}(V-K, \underset{\sim}{\mathcal{O}})\cong H^p_{\mathrm{comp}}(V, \underset{\sim}{\mathcal{O}}) \quad (p\geqq 2) \end{cases}$$

Using Theorem 3.1.7 we also have the following isomorphisms:

$$\begin{cases} H^p_{\mathrm{comp}}(V-K, \underset{\sim}{\mathcal{O}})=0 \quad (p\neq 1, n) \\ H^n_{\mathrm{comp}}(V-K, \underset{\sim}{\mathcal{O}})\cong[\tilde{\mathcal{O}}(V)]'\,. \end{cases}$$

As is done in p. 478. we consider the following dual complexes:

$$\begin{array}{ccccccccccc} 0\longrightarrow X_0(V-K) & \xrightarrow{\bar\partial_0} & X_1(V-K) & \xrightarrow{\bar\partial_1} & \cdots & \xrightarrow{\bar\partial_{n-2}} & X_{n-1}(V-K) & \xrightarrow{\bar\partial_{n-1}} & X_n(V-K)\longrightarrow 0 \\ \updownarrow & & \updownarrow & & & & \updownarrow & & \updownarrow \\ 0\longleftarrow Y_n(V-K) & \xleftarrow{-\bar\partial_{n-1}} & Y_{n-1}(V-K) & \xleftarrow{-\bar\partial_{n-2}} & \cdots & \xleftarrow{-\bar\partial_1} & Y_1(V-K) & \xleftarrow{-\bar\partial_0} & Y_0(V-K)\longleftarrow 0\,. \end{array}$$

Taking into account of the fact $H^p_{\mathrm{comp}}(V-K, \underset{\sim}{\mathcal{O}})=0$, the range of $(-\bar\partial_j)$ is closed except for $j=0, n-1$. Since $\bar\partial_{n-1}$ is of closed range by Theorem 3.1.8, $-\bar\partial_0$ is of closed range by the closed range theorem. (Remark the fact DFS*-space is reflexive. See [K].) To prove the closed rangeness of $(-\bar\partial_{n-1})$, we consider the following commutative diagram:

$$\begin{array}{ccccc} 0\longleftarrow & Y_n(V-K) & \xleftarrow{-\bar\partial^{V-K}_{n-1}} & Y_{n-1}(V-K) \\ & \cap\big\downarrow i & & \cap\big\downarrow \\ 0\longleftarrow & Y_n(V) & \xleftarrow{-\bar\partial^V_m} & Y_{n-1}(V)\,. \end{array}$$

(The map i is the natural injection.)

We conclude that $\bar\partial^V_0$ is of closed range since $H^1(V, \tilde{\mathcal{O}})=0$, thus $(-\bar\partial^V_{n-1})$ is of closed range by the Serre-Komatsu duality theorem. ([K] p. 381, Theorem 19.) Therefore $\mathrm{Im}\,(-\bar\partial^{V-K}_{n-1})=i^{-1}(\mathrm{Im}\bar\partial^V_{n-1})$ is closed by the continuity of the map i. Thus we have proved $(-\bar\partial^{V-K}_j)$ are all of closed range, so we can apply the Serre-Komatsu duality theorem and obtain

$$[H^p(V-K, \tilde{\mathcal{O}})]' \cong H^{n-p}_{\mathrm{comp}}(V-K, \underset{\sim}{\mathcal{O}}) .$$

Therefore

$$[H^0(V-K, \tilde{\mathcal{O}})]' \cong H^n_{\mathrm{comp}}(V-K, \underset{\sim}{\mathcal{O}}) \cong H^n_{\mathrm{comp}}(V, \underset{\sim}{\mathcal{O}}) \cong [H^0(V, \tilde{\mathcal{O}})]'$$

Taking into account of the fact $H^0(V-K, \tilde{\mathcal{O}})$ and $H^0(V, \tilde{\mathcal{O}})$ are FS-spaces (*a posteriori* reflexive), we have $H^0(V, \tilde{\mathcal{O}}) \cong H^0(V-K, \tilde{\mathcal{O}})$. Thus

$$H^1_K(V, \tilde{\mathcal{O}}) \cong H^0(V-K, \tilde{\mathcal{O}})/H^0(V, \tilde{\mathcal{O}}) = 0 .$$

If $p \geqq 2$, $p \neq n$, then

$$0 = H^{n-p+1}_{\mathrm{comp}}(V, \underset{\sim}{\mathcal{O}}) = H^{n-p+1}_{\mathrm{comp}}(V-K, \underset{\sim}{\mathcal{O}}) \cong [H^{p-1}(V-K, \tilde{\mathcal{O}})]' \cong [H^p_K(V, \tilde{\mathcal{O}})]' .$$

Hence $H^p_K(V, \tilde{\mathcal{O}}) = 0$.

At last we consider the case $p = n$.

$$[H^n_K(V, \tilde{\mathcal{O}})]' \cong [H^{n-1}(V-K, \tilde{\mathcal{O}})]' \cong H^1_{\mathrm{comp}}(V-K, \underset{\sim}{\mathcal{O}}) \cong H^0(K, \underset{\sim}{\mathcal{O}}) = \underset{\sim}{\mathcal{O}}(K)$$

and since $\underset{\sim}{\mathcal{O}}(K)$ is a DFS-space, *a posteriori* bornologic, Komatsu's theorem asserts the above isomorphism is the topological isomorphism, and we obtain $H^n_K(V, \tilde{\mathcal{O}}) \cong [\underset{\sim}{\mathcal{O}}(K)]'$. Q.E.D.

As is well-known, this theorem combined with Theorem 3.1.8, concludes the pure-codimensionality of $\tilde{\mathcal{O}}$ with respect to D^n, that is,

THEOREM 3.2.2. *Let Ω be in $\boldsymbol{D}^n$, then $H^p_\Omega(V, \tilde{\mathcal{O}}) = 0$ $(p \neq n)$, where $V = (\boldsymbol{D}^n \times \sqrt{-1}\, I^n - \partial_{D^n}\Omega)$. (The symbol $\partial_{D^n}\Omega$ means the boundary of Ω in $\boldsymbol{D}_n$, we abbreviate it to $\partial\Omega$ in the proof.)*

PROOF. Consider the following exact sequence:

$$\begin{aligned} 0 &\longrightarrow H^0_{\partial\Omega}(V, \tilde{\mathcal{O}}) \longrightarrow H^0_{\Omega^a}(V, \tilde{\mathcal{O}}) \longrightarrow H^0_\Omega(V, \tilde{\mathcal{O}}) \longrightarrow H^1_{\partial\Omega}(V, \tilde{\mathcal{O}}) \\ &\longrightarrow \cdots \longrightarrow H^{n-1}_\Omega(V, \tilde{\mathcal{O}}) \longrightarrow H^n_{\partial\Omega}(V, \tilde{\mathcal{O}}) \longrightarrow H^n_{\Omega^a}(V, \tilde{\mathcal{O}}) \\ &\longrightarrow H^n_\Omega(V, \tilde{\mathcal{O}}) \longrightarrow H^{n+1}_{\partial\Omega}(V, \tilde{\mathcal{O}}) \longrightarrow \cdots \end{aligned}$$

(Ω^a means the closure of Ω). Theorem 3.2.1 concludes $H^p_{\partial\Omega}(V, \tilde{\mathcal{O}}) = 0$, $H^p_{\Omega^a}(V, \tilde{\mathcal{O}}) = 0$ $(p \geqq n+1)$, so $H^p_\Omega(V, \tilde{\mathcal{O}}) = 0$ when $p \geqq n+1$. In just the same way Theorem 3.2.1 also gives us $H^p_\Omega(V, \tilde{\mathcal{O}}) = 0$ $(0 \leqq p \leqq n-2)$. Theorem 2.2.1 combined with Theorem 3.2.1 the injectivity of $j : [\underset{\sim}{\mathcal{O}}(\partial\Omega)]' \longrightarrow [\underset{\sim}{\mathcal{O}}(\Omega^a)]'$.

Since

$$0 \longrightarrow H^{n-1}_\Omega(V, \tilde{\mathcal{O}}) \longrightarrow [\underset{\sim}{\mathcal{O}}(\partial\Omega)]' \xrightarrow{\ j\ } [\underset{\sim}{\mathcal{O}}(\Omega^a)]'$$

is exact, we have $H^{n-1}_\Omega(V, \tilde{\mathcal{O}}) = 0$. Q.E.D.

COROLLARY 3.2.3. *$\{\mathscr{R}(\Omega)\}$ constitutes a flabby sheaf over $\boldsymbol{D}^n$.*

PROOF. Direct consequence of the theorem. (Cf. Harvey [10] or Komatsu [23], [24]).

We end this section by giving the explicit pairing between $\mathscr{P}_*$ and $\mathscr{R}(\boldsymbol{D}^n)$ which is given in Theorem 3.2.1 in an abstract way. In the sequel of the proof we also give the notion of the Fourier transformation of the elements of $\mathscr{R}(\boldsymbol{D}^n)$.

If we define $V_0=\boldsymbol{D}^n\times\sqrt{-1}\,I^n$, $V_j=\boldsymbol{D}^n\times\sqrt{-1}\,\{y\in I^n|y_j\neq 0\}$ (where $I=\{-1<y<1\}$), $\mathscr{V}=\{V_j\}_{j=0,\cdots,n}$ and $\mathscr{V}'=\{V_j\}_{j=1,\cdots,n}$, we obtain the isomorphism

$$H^n_{D^n}(\boldsymbol{D}^n\times\sqrt{-1}\,I^n,\tilde{\mathscr{O}})\simeq H^n(\mathscr{V},\mathscr{V}',\tilde{\mathscr{O}})$$

by Leray's theorem. (See for example Komatsu [24].)

Thus we can represent any element μ of $H^n_{D^n}(\boldsymbol{D}^n\times\sqrt{-1}\,I^n,\tilde{\mathscr{O}})$ by some element in $\tilde{\mathscr{O}}(V_1\cap\cdots\cap V_n)$, which we write by $\{\varphi_1,\cdots,\varphi_{2^n}\}=[\varphi]$.

Using this isomorphism the pairing between $\mathscr{R}(\boldsymbol{D}^n)$ and $\mathscr{P}_*$ is given by

$$\langle[\varphi],f\rangle=\sum_{j=1}^{2^n}(-1)^{\prod_{k=1}^{n}\operatorname{sgn}\varepsilon_k}$$
$$\times\int\cdots\int\varphi_j(x_1+i\varepsilon_1,\cdots,x_n+i\varepsilon_n)f(x_1+i\varepsilon_1,\cdots,x_n+i\varepsilon_n)dx_1\cdots dx_n$$

where $|\varepsilon_j|$ is sufficiently small but not zero and $\operatorname{sgn}\varepsilon_j$ is $\varepsilon_j/|\varepsilon_j|$. In fact it is clear any $[\varphi]$ defines an element of $(\mathscr{P}_*)'$ by the above well-defined integration. We denote the map j.

We want to construct the inverse map k of j. For that purpose we must do some preliminaries. (The definition of k is given in Definition 3.2.8.)

PROPOSITION 3.2.4. *If we define $\mathscr{F}\varphi$ by $\int e^{i\langle x,\xi\rangle}\varphi(x)dx$ for $\varphi\in\mathscr{P}_*$ then $\mathscr{F}$ gives a topological isomorphism from $\mathscr{P}_*$ to $\mathscr{P}_*$.*

PROOF. This is obvious from the definition of $\mathscr{P}_*$ and the closed graph theorem for the DFS-space.

DEFINITION 3.2.5. Let μ be an element of $(\mathscr{P}_*)'$, then we define $\mathscr{F}_d\mu$ by the formula $\langle\mathscr{F}_d\mu,\varphi\rangle=\langle\mu,\mathscr{F}\varphi\rangle$ $(\forall f\in\mathscr{P}_*)$.

We also define $\langle\overline{\mathscr{F}}_d\mu,\varphi\rangle=\langle\mu,\overline{\mathscr{F}}\varphi\rangle$ where $\overline{\mathscr{F}}f=\int e^{-i\langle x,\xi\rangle}f(x)dx$.

Denoting the closure of j-th quadrant in $\boldsymbol{D}^n$ by K_j, we obtain the following theorem.

THEOREM 3.2.6. *Every element $\mu\in(\mathscr{P}_*)'$ can be decomposed as $\mu=\sum_{j=1}^{2^n}\mu_j$ where $\mu_j\in(\tilde{\mathscr{O}}(K_j))'$.*

PROOF. This is a direct consequence of Theorem 3.2.1 and Corollary 3.2.3.

REMARK. Theorem 2.2.1 gives a direct proof of this theorem if we proceed as in Martineau [30].

DEFINITION 3.2.7. Using the above decomposition of μ, we define $\mathscr{F}\mu=\{F_j(\zeta)\}$, which is an element of $H^n(\mathscr{V},\mathscr{V}',\tilde{\mathscr{O}})$. Here $F_j(\zeta)=(-1)^{j+1}\langle\mu_j\, e^{i\langle z,\zeta\rangle}\rangle$ ($\operatorname{Im}\zeta$ belongs to the j-th open quadrant).

This definition makes sense, because the vanishing of $H^1(K_j\cup K_k,\underset{\sim}{\mathscr{O}})$ concludes the ambiguity of the decomposition of μ belongs to $[\underset{\sim}{\mathscr{O}}(K_j\cap K_k)]'$, so it is transformed into the coboundary element under the above mapping.

Now we define the map k.

DEFINITION 3.2.8. Let μ belong to $(\mathscr{P}_*)'$, then we define $k(\mu)=\mathscr{F}(\overline{\mathscr{F}}_d\mu)$.

THEOREM 3.2.9. *The composed map $j\circ k:(\mathscr{P}_*)'\longrightarrow(\mathscr{P}_*)'$ is the identity map and j is injective, so j and k are bijective. This proves the statement given in* *p.* 484.

PROOF. At first we prove $j\circ k=\mathrm{id}$. In fact we have the following identity for any f which belongs to $\mathscr{P}_*$.

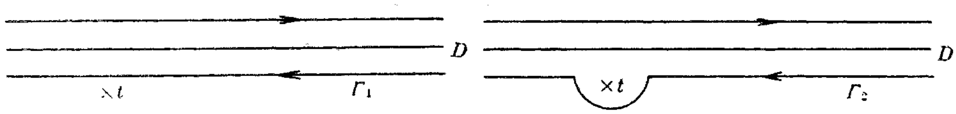
$$\langle j\circ k(\mu),f\rangle=\sum_j\int\cdots\int\langle\nu_j,e^{i\langle z,\zeta\rangle}\rangle f(\zeta)d\zeta=\sum_i\langle\nu_j,\int\cdots\int e^{i\langle z,\zeta\rangle}\rangle f(\zeta)d\zeta$$
$$=\langle\nu,\int\cdots\int e^{i\langle z,\zeta\rangle}\rangle f(\zeta)d\zeta=\langle\nu,\mathscr{F}f\rangle=\langle\overline{\mathscr{F}}_d\mu,\mathscr{F}f\rangle=\langle\mu,f\rangle\,.$$

This proves $j\circ k=\mathrm{id}$.

Next we prove the injectivity of j. For the sake of simplicity we assume $n=1$. (The following arguments succeed in $n\geqq 2$ in the same way.)

Define $f_t(z)=\exp(-(t-z)^2)/2\pi\sqrt{-1}\,(t-z)$ and consider the following path of integration Γ_1 and Γ_2.

Figure



If we assume $j([\varphi])=0$, then $\int_{\Gamma_1}\varphi(z)f_t(z)dz=0$. On the other hand we have

$$\int_{\Gamma_2}\varphi(z)f_t(z)dz-\int_{\Gamma_1}\varphi(z)f_t(z)dz=\varphi(t)$$

by Cauchy's formula. Therefore $\varphi(t)=\int_{\Gamma_2}\varphi(z)f_t(z)dz$. But the right side is holomorphic even when $\operatorname{Im}t=0$, so we extend $\varphi(t)$ to the real axis by the right side, and estimate $\varphi(t)$ there. Now we have the estimate $|\varphi(z)|\leqq A_{\varepsilon,\delta,\delta'}e^{\varepsilon|z|}$ on $\{\delta<|\operatorname{Im}z|<\delta'<1\}$ for every ε, δ, δ', by the definition of $H^n(\mathscr{V},\mathscr{V}',\tilde{\mathscr{O}})$. This means

$$|\varphi(t)|=\left|\int \varphi(z)f_t(z)dz\right|\leqq\int_{-\infty}^{\infty}(|\varphi(u+t)|\exp(-u^2)/2\pi u)du\leqq B_\varepsilon e^{\varepsilon|z|}$$

as far as $|\mathrm{Im}\, t|$ is sufficiently small. This proves that $[\varphi]$ is zero as a cohomology class, that is j is injective.

3.3. Fourier transformation and the Paley-Wiener theorem.

In this section we treat the Fourier transformation from the view point of holomorphic functions in tublar domains.

The treatment turns out to be useful in treating the problem of hyperbolicity (§ 6).

THEOREM 3.3.1. *Let Γ be a closed and strictly convex cone in $\boldsymbol{R}^n$ and K be its closure in $\boldsymbol{D}^n$.*

For the sake of simplicity we assume the vertex of the cone Γ be at the origin and $\Gamma\Subset\{x_1\geqq-\varepsilon\}$. (If A and B are cones, then we denote $A\Subset B$ when the closure of A has a compact neighbourhood in the closure of B with respect to the topology of $\boldsymbol{D}^n$.) Then every μ in $[\underset{\sim}{\mathcal{O}}(K)]'$ has the following properties: $\langle\mu, e^{i\langle z,\zeta\rangle}\rangle$ is holomorphic in $\boldsymbol{R}^n\times\sqrt{-1}(\Gamma^0)^i$ and satisfies following estimate (*).
(*) *For every $\Gamma'\Subset\Gamma^0$ and $\varepsilon>0$ we have*

$$\langle\mu, e^{i\langle z,\zeta\rangle}\rangle\leqq C_\varepsilon\exp(\varepsilon|\mathrm{Re}\,\zeta|+\chi_{\Gamma,\varepsilon}(\mathrm{Im}\,\zeta)),\quad \zeta\in\boldsymbol{R}^n\times\sqrt{-1}\,\Gamma',$$

where

$$\chi_{\Gamma,\varepsilon}(\eta)=\sup_{x\in\Gamma-\varepsilon(1,0,\cdots,0)}(-\langle x,\eta\rangle+\varepsilon|x|)\,.$$

(In the above notation Γ^0 means the polar set of Γ, that is $\{\xi|\langle x,\xi\rangle\geqq 0\ \forall x\in\Gamma\}$.)

PROOF. In view of the topology of $\underset{\sim}{\mathcal{O}}(K)$ the proof is immediate.

We go on to the proof of the inverse of the above theorem.

Let $F(\zeta)$ be holomorphic in $\boldsymbol{R}^n\times\sqrt{-1}(\Gamma^0)^i$ for some closed and strictly convex cone Γ and satisfies the growth conditions (*) given in the preceding theorem, then we can consider $F(\zeta)$ to define some cohomology class μ in $H^n(\boldsymbol{D}^n\times\sqrt{-1}\,\Gamma^n, \widetilde{\mathcal{O}})$ (see p. 483.) in a natural way as "boundary value". Then μ can be considered as an element of $(\mathscr{P}_*)'$ and we can find some ν uniquely such that $\mathscr{F}_d\mu=\nu$, by the results of the last parts of 3.2. Then we have the following theorem.

THEOREM 3.3.2. *The element ν can be extended to the linear functional over $\underset{\sim}{\mathcal{O}}(K)$ where K is the closure of Γ in $\boldsymbol{D}^n$, that is ν can be regarded as an element of $[\underset{\sim}{\mathcal{O}}(K)]'$.*

PROOF. The convexity of Γ reduces the situation to the case $n=1$. (The reduction is given at the end of the proof.) At first we give the proof of the theorem when $n=1$.

By the approximation theorem (Theorem. 2.2.1) and the definition of the topology of $\underset{\sim}{\mathcal{O}}(K)$, it is sufficient to prove the following estimate; Let $f(\zeta)$ belong to $\underset{\sim}{\mathcal{O}}^m(D)$, that is $\sup_{|\mathrm{Im}\,\zeta|<1/m} |f(\zeta)e^{(1/m)|\zeta|}|$, then for any positive ε there exists some C_ε such that

$$\left|\int_{-\infty+i\delta}^{\infty+i\delta} F(\zeta)f(\zeta)d\zeta\right| \leqq C_\varepsilon \sup_{z\in\Gamma_\varepsilon} |(\mathcal{F}f)(z)e^{\varepsilon|z|}| \quad (0<\delta\ll 1)$$

where

$$\Gamma_\varepsilon=\{x+iy\,|\,x\geqq-\varepsilon,\ |y|<\varepsilon\} \text{ assuming } \Gamma=\{x\geqq 0\}\,.$$

Then ν belongs to $[\underset{\sim}{\mathcal{O}}(K)]'$. Since $\mathcal{F}:\mathcal{P}_*\longrightarrow\mathcal{P}_*$ is an isomorphism, it is sufficient to prove

$$\left|\int F(\zeta)\int e^{-i\zeta z}g(z)dzd\zeta\right| \leqq C_\varepsilon \sup_{\Gamma_\varepsilon} |g(z)e^{\varepsilon|z|}|$$

where $g(z)=(\mathcal{F}f)(z)$. To prove this inequality we denote the integral in the left side by I. Moreover we define

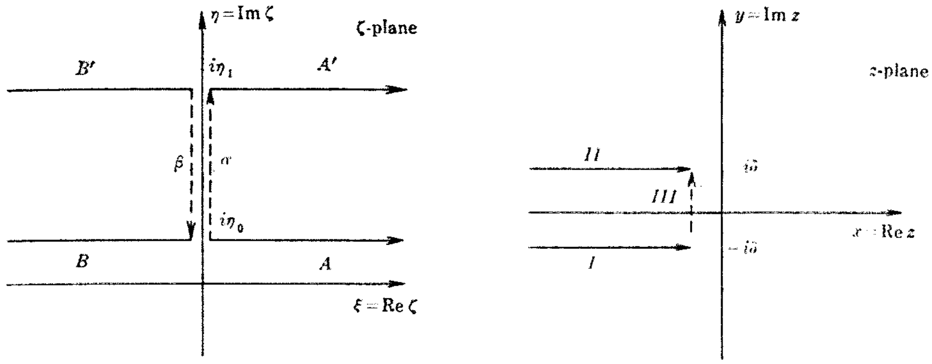
$$\begin{cases} I_+ = \displaystyle\int_{\xi+i\delta,\xi\geqq 0} F(\zeta)\int_{x-i\delta} e^{-i\zeta z}g(z)dzd\zeta \\ I_- = \displaystyle\int_{\substack{\xi+i\delta\\ \xi\leqq 0}} F(\zeta)\int_{x+i\delta} e^{-i\zeta z}g(z)dzd\zeta \end{cases}$$

$$\begin{cases} J_{++} = \displaystyle\int_{\substack{\xi+i\delta\\ \xi\geqq 0}} F(\zeta)\int_{\substack{x-i\delta\\ x\geqq-\delta'}} e^{-i\zeta z}g(z)dzd\zeta \\ J_{+-} = \displaystyle\int_{\substack{\xi+i\delta\\ \xi\geqq 0}} F(\zeta)\int_{\substack{x-i\delta\\ x\leqq-\delta'}} e^{-i\zeta z}g(z)dzd\zeta \\ J_{-+} = \displaystyle\int_{\substack{\xi+i\delta\\ \xi\leqq 0}} F(\zeta)\int_{\substack{x+i\delta\\ x\geqq-\delta'}} e^{-i\zeta z}g(z)dzd\zeta \\ J_{--} = \displaystyle\int_{\substack{\xi+i\delta\\ \xi\leqq 0}} F(\zeta)\int_{\substack{x+i\delta\\ x\leqq-\delta'}} e^{-i\zeta z}g(z)dzd\zeta\,, \quad \text{where} \quad 0<\delta'\ll 1\,. \end{cases}$$

Trivially we have $I=I_++I_-$ and $I_+=J_{++}+J_{+-}$, $I_-=J_{-+}+J_{--}$. Since the values $|J_{++}|$ and $|J_{-+}|$ are smaller than the right side of the required inequality by their definitions, it is enough to prove the following statement to obtain the desired inequality; For every $\theta>0$ and $\delta'>0$, $|J_{+-}+J_{--}|<\theta$ if δ is sufficiently small.

We denote J_{+-} by $J(I, A)$ and J_{--} as $J(II, B)$ respectively, where the paths of integration are as below. Just in the same way we denote by $J(I, A')$ etc. the integral over the path of integration $I\times A'$ etc.

Figure



By the condition (*) we can easily conclude that $J(I, A')$ and $J(II, B')$ tend to zero as η_1 tends to infinity. While we have

$$(J_{+-}+J_{--})-(J(I, A')+J(II, B'))=J(I, \alpha)+J(II, \beta)=J((I-II), \alpha)=-J(III, \alpha)$$

since $J(I+III-II, \alpha)=0$ by the Cauchy's integral theorem. So it is sufficient to prove $|J(III, \alpha)|$ tends to zero as η_1 tends to infinity.

Using the condition (*) again we have the following estimate for every $\delta'>0$:

$$\left|\int_{\eta_0}^{\infty} F(i\tau)\int_{-\delta}^{\delta} \exp(-iz\zeta)g(z)dyd\tau\right|$$
$$\leq C_{\delta'}\int_{\eta_0}^{\infty}\int \exp(\delta'/2\tau)\int_{-\delta}^{\delta} \exp(-\delta'\tau)dyd\tau \leq 2\delta C_{\delta'}\int_{\eta_0}^{\infty} \exp(-\delta'/2\tau)d\tau = K_{\delta'}\cdot\delta\,.$$

Thus we have the desired result.

When the case $n\geqq 2$, we consider as follows. Since Γ is a closed convex cone, we can represent $\Gamma=\bigcap_{\xi} H_\xi$, where $H_\xi=\{x|\langle x, \xi\rangle\geqq 0\}$. Then we can prove the following estimate just as in the case of $n=1$;

$$\left|\int\cdots\int F(\zeta_1, \cdots, \zeta_n)\int\cdots\int e^{-i\langle\zeta, z\rangle}g(z_1, \cdots, z_n)dz_1\cdots dz_n d\zeta_1\cdots d\zeta_n\right|$$

is dominated by $C_\varepsilon \sup_{\substack{x_j\geqq -\varepsilon\\ |y|\leqq\varepsilon}} |g(z)e^{\varepsilon|z|}|$ for every $\varepsilon>0$ when $g(z)$ belongs to $\underset{\sim}{\mathcal{O}}_m(U_m)$, after some affine transformation if necessary.

This concludes that ν can be regarded as an element of $[\underset{\sim}{\mathcal{O}}(H^a_\xi)]'$. On the other hand $[\underset{\sim}{\mathcal{O}}(H^a_\xi)]'$ is isomorphic to $H^n_{H^a_\xi}(D^n\times\sqrt{-1}\,I^n, \tilde{\mathcal{O}})$ by Theorem 3.2.1, thus ν can be considered to belong to

$$\bigcap_{\xi} H^n_{H^a_\xi}(D^n\times\sqrt{-1}\,I^n, \tilde{\mathcal{O}})=H^n_{\bigcap_\xi H^a_\xi}(D^n\times\sqrt{-1}\,I^n, \tilde{\mathcal{O}})$$

since $\{\mathscr{R}(\Omega)\}$ constitutes a sheaf on D^n. This proves that ν belongs to

$$H_K^n(D_n \times \sqrt{-1}I^n, \tilde{\mathscr{O}}) \cong [\underset{\sim}{\mathscr{O}}(K)]',$$

completing the proof.

At last we remark that we can define the Fourier transform of the elements of $\mathscr{R}(D^n)$ via "boundary values" of holomorphic functions. In fact Definition 3.2.7 gives the method. We call this Fourier-Carleman-Leray-Sato-transformation, and denote by $\mathscr{F}$. We have seen this is the same as $\mathscr{F}_d$, defined by the duality. (Cf. Definition 3.2.5.)

Thus the above theorem can be regarded as an analogue of the Paley-Wiener theorem for the Fourier-Carleman-Leray-Sato transformation.

Part II. GENERAL THEORY OF LINEAR PARTIAL DIFFERENTIAL OPERATORS WITH CONSTANT COEFFICIENTS

§ 4. Ellipticity

4.1. Ellipticity and partial ellipticity for local operators

In general when S is a hyperfunction with compact support in $\boldsymbol{R}^n$ and when μ belongs to $\mathscr{R}(\boldsymbol{D}^n)$, we define $S*\mu$ by the following formula $\langle S*\mu, \varphi\rangle = \langle \mu, \check{S}*\varphi\rangle$ $(\varphi \in \mathscr{P}_*)$, where we denote $S(-x)$ by $\check{S}(x)$. It is obvious that the above definition coincides with the definition of convolution given in Martineau [30], Schapira [40] or Sato [35] when μ is a hyperfunction with compact support in $\boldsymbol{R}^n$. Therefore if μ is a hyperfunction on $\boldsymbol{R}^n$ we have $S*\tilde{\mu}|_{\boldsymbol{R}^n} = S*\mu$ where $\tilde{\mu}$ is some extention of μ over $\boldsymbol{D}^n$ which is obtained using the flabbiness of $\mathscr{R}$. In this section we treat the special class of convolution operators $S*$, where S is a hyperfunction with support $\{0\}$. This operator can be considered the most natural generalization of the usual partial differential operator in the theory of hyperfunction, because this operator preserves the support, or equivalently gives a sheaf homomorphism from $\mathscr{B}$ to $\mathscr{B}$. We call such an operator a local operator (with constant coefficients.) (Cf. Sato [36].)

DEFINITION 4.1.1. A local operator $S*$ is said to be *elliptic* if any hyperfunction solution u of $S*u=f$ is real analytic where f is real analytic.

In this section we constantly use the notation $J(\zeta)$ to denote the inverse Fourier transform of S, that is, $\langle S, e^{-i\langle z,\zeta\rangle}\rangle$, and singsupp u to denote the minimal closed set outside of which u is real analytic.

THEOREM 4.1.2. *Let V be the set $\{\zeta \in \boldsymbol{C}^n | J(\zeta)=0\}$ and $\rho(\xi)$ be the distance from ξ to V. Suppose that there exists some positive constant c such that* $\rho(\xi) \geqq c|\xi|$ if ξ belongs to $\boldsymbol{R}^n$ and $|\xi|$ *is sufficiently large. Then $S*$ is elliptic.*

REMARK. There are many elliptic local operators which are not usual elliptic partial differential operators. (Cf. Ehrenpreis [4].) Of course our theorem includes the regularity theorems which are obtained by Bengel [2], Harvey [10] and Komatsu [22], and it seems more constructive than those.

PROOF. It is sufficient to prove the existence of a good parametrix P. Here we mean by a good parametrix P a hyperfunction P which satisfies $S*P=\delta-W$ where W is real analytic and $\operatorname{singsupp} P=\{0\}$. In fact if $S*u\in\mathscr{A}(\Omega)$ then we cut off u outside some compact set K in Ω by the flabbiness of $\mathscr{B}$ and write it $\tilde{u}$. Therefore we have $S*\tilde{u}=g$ where $\operatorname{supp} g\subset K$ and $\operatorname{singsupp} g\subset\partial K$. Then we have $\tilde{u}=u*\delta=\tilde{u}*(S*P-W)=S*\tilde{u}*P-\tilde{u}*W=g*P-\tilde{u}*W$. Therefore $\tilde{u}$ is real analytic in the interior of K if a good parametrix P exists. Thus u is real analytic in Ω since K is arbitrary. Here we have used the relation $\operatorname{singsupp} u*v\subset \operatorname{singsupp} u+\operatorname{singsupp} v$. More refined version of this fact is given in Lemma 4.1.6.

Now we go on to the proof of the existence of a good parametrix for $S*$. (The following proof shows W can be taken entire.)

At first we prove the theorem in case $n=1$. We define $Q(\zeta)$, which is an element of $(\mathscr{P}_*)'$ by the following formula:

$$\langle Q(\zeta), f(\zeta)\rangle=\int_{|\xi|\geqq K} f(\zeta)(1/J(\zeta))d\xi \quad (f(\zeta)\in\mathscr{P}_*)\ .$$

Here K is fixed so large that $J(\zeta)\neq 0$ in $\Omega=\{\zeta\in \boldsymbol{C}\,|\,|\operatorname{Im}\zeta|<c(\operatorname{Re}\zeta-K+1)$ or $|\operatorname{Im}\zeta|<-c(\operatorname{Re}\zeta+K-1)\}$.

Then we can assert that $1/J(\zeta)$ has the following estimate in $\Omega'=\{\zeta\,|\,|\operatorname{Im}\zeta|< c'(\operatorname{Re}\zeta-K)$ or $|\operatorname{Im}\zeta|<-c'(\operatorname{Re}\zeta+K)(c'<c)\}$: for every $\varepsilon>0$ there exists some C_ε such that $|J(\zeta)|\geqq C_\varepsilon e^{-\varepsilon|\zeta|}$. It seems that this statement is essentially well-known but we cannot find any literature to quote, so we give a proof of this proposition in Lemma 4.1.3. This proposition makes the integral on the right side well-defined. Thus $Q(\zeta)\in(\mathscr{P}_*)'$ is well-defined. Next we define $P(x)$ as the Fourier transform of $Q(\zeta)$, and we obtain the representation of $P(x)$ by

$$\left\{\int_{\xi>K} e^{iz\zeta}/J(\zeta)d\zeta, \int_{\xi<-K} e^{iz\zeta}/J(\zeta)d\zeta\right\}\in H^1(\mathscr{V},\mathscr{V}',\tilde{\mathscr{O}})\ ,$$

that is the Čech cohomology of covering. (See the notation of 3.2.)

Now we prove the relation $S*P=\delta-W$ holds on $\boldsymbol{R}^n$. In fact we have

$$\begin{aligned}\langle S*P, f\rangle&=\langle P,\check{S}*f\rangle=\langle\mathscr{F}P,\mathscr{F}(\check{S}*f)\rangle=\langle Q,\mathscr{F}S\mathscr{F}f\rangle\\ &=\int_{|\xi|>K}(1/J(\zeta))\langle S, e^{-iz\zeta}\rangle\mathscr{F}fd\zeta=\int_{|\xi|>K}\mathscr{F}fd\zeta=\int\mathscr{F}fd\zeta-\int_{|\xi|\leqq K}\mathscr{F}fd\zeta\\ &=\langle\delta, f\rangle-\left\langle\int_{|\xi|\leqq K}e^{ix\xi}dx, f\right\rangle\end{aligned}$$

for every $f\in\mathscr{P}_*$. Thus we have $S*P=\delta-\int_{|\xi|\leq K}e^{ix\xi}dx$ on $\boldsymbol{D}$, so restricting this relation to $\boldsymbol{R}$ we have $S*P=\delta-W(x)$ where $W(x)$ is a real analytic function over $\boldsymbol{R}$.

Thus what remains to be proved is singsupp $P=\{0\}$. For that purpose it is sufficient to prove $\int_{\xi>K}e^{iz\zeta}/J(\zeta)d\zeta$, which is defined and holomorphic in $\{\mathrm{Im}\, z>0\}$, is analytically continued across the real axis as far as $\mathrm{Re}\, z\neq 0$. In fact it is easily proved by the change of path of integration. Let us denote $\mathrm{Re}\, z$ by x, $\mathrm{Im}\, z$ by y, $\mathrm{Re}\,\zeta$ by ξ and $\mathrm{Im}\,\zeta$ by η. By the estimate of $|1/J(\zeta)|$ and Cauchy's theorem we have

$$\int_{\xi>K}e^{iz\zeta}/J(\zeta)d\zeta=\int_{\Gamma}e^{iz\zeta}/J(\zeta)d\zeta$$

where $\Gamma=\{\zeta|\eta=k(\xi-K),\ \xi>K\}$, with $0<k<c/2$. On the other hand $\int_{\Gamma}e^{iz\zeta}/J(\zeta)d\zeta$ converges absolutely in $\{z|y>-(k/2)x+(1+k/2)\varepsilon\}$ for every positive ε. This means $\int_{\Gamma}e^{iz\zeta}/J(\zeta)d\zeta$ is real analytic if $x>0$, since k is a fixed constant.

Just in the same way, changing the path of integration to $\Gamma'=\{\zeta|\eta=-k(\xi-K),\ k=c/2\}$ we have the real analyticity of $\int_{\xi>K}e^{iz\zeta}/J(\zeta)d\zeta$ in $\{x<0\}$. This ends the proof when $n=1$.

If $n\geqq 2$, then we define $\{Q_j(\zeta)\}_{j=1,\cdots,2^n}$ by

$$\langle Q_j(\zeta),f(\zeta)\rangle=\int_{\substack{|\xi|\geqq K\\ \xi\in j\text{-th quadrant}}}\cdots\int(1/J(\zeta))f(\zeta)d\xi_1\cdots d\xi_n\quad(f\in\mathscr{P}_*)$$

and $Q(\zeta)=\sum_{j=1}^{2^n}Q_j(\zeta)$. Define $P(x)$ by $\mathscr{F}(Q(\zeta))$. To prove singsupp $P=\{0\}$, we need only to check the regularity of $P(x)$, but the proof given in the case of $n=1$ shows singsupp $P(x)\subset\bigcup_{j=1}^{n}\{x_j=0\}$. Since $P(x)$ is an element of $H^n_{D^n}(\boldsymbol{D}^n\times\sqrt{-1}\,I^n,\tilde{\mathscr{O}})$ we conclude singsupp $P=\{0\}$, because affine transformations do not change singsupp P. The proof of the relation $S*P=\delta-W$ ($W\in\mathscr{A}(\boldsymbol{R}^n)$) is just the same as in the case $n=1$. This ends the proof of the theorem except for the estimate of $|1/J(\zeta)|$, which is given in the next lemma.

LEMMA 4.1.3. *Let Γ be an open cone in $\boldsymbol{C}^n(\cong\boldsymbol{R}^{2n})$ with its vertex at* 0. *Suppose that $J(\zeta)$ is an entire function which is of order* 1 *and of minimum type (that is, $|J(\zeta)|\leqq A_\varepsilon e^{\varepsilon|\zeta|}$, $\forall\varepsilon>0$) and that $J(\zeta)\neq 0$ in Γ, then we have the following estimate: let Γ' be a cone in $\boldsymbol{C}^n\cong\boldsymbol{R}^{2n}$ with $\Gamma'\Subset\Gamma$, then for every $\varepsilon>0$ there exists some C_ε such that $|J(\zeta)|\geqq C_\varepsilon e^{-\varepsilon|\zeta|}$ in Γ'.*

Before giving the proof of this lemma we quote the following two lemmas which are well known in the theory of functions of one complex variable.

LEMMA 4.1.4. *Suppose $f(z)$ is holomorphic and never vanishes in $\{z\,|\,|z|\leqq R\}$. Then we have the following inequality for every r $(r<R)$:*

(1) $\log|f(z)|\geqq -2r/(R-r)\log\sup_{|z|=R}|f(z)|+(R+r)/(R-r)\log|f(0)|$ in $|z|<r$.

LEMMA 4.1.5. *Suppose $f(z)$ is holomorphic in $\{z\,|\,z\leqq 2eR\}$ with $f(0)=1$. $(e=2.7\cdots)$ Let η be an arbitrary positive number not exceeding $3e/2$. Then inside the circle $|z|\leqq R$, but outside a family of excluded circles the sum of whose radii is not greater than $4\eta R$, we have*

(2) $\log|f(z)|>-H(\eta)\log\sup_{|z|=2eR}|f(z)|$ *where* $H(\eta)=2+\log 3e/2\eta$.

See Levin [27] p. 19. Theorem 9 and p. 21. Theorem 11 as for the proofs.

Now we begin to prove Lemma 4.1.3. We can assume $J(0)=1$ without loss of generality. Let S_1 be $\{\zeta\in C^n\,|\,|\zeta|=1\}$. For every $\zeta\in S_1\cap\Gamma$, we find some constant c such that $\{t\zeta\,|\,|\mathrm{Im}\,t|<c\,\mathrm{Re}\,t\}\subset\Gamma$ since Γ is open. Therefore $C_U\underset{\mathrm{Df.}}{=}\{t\zeta\,|\,|\mathrm{Im}\,t|<c\,\mathrm{Re}\,t\}$ is an open cone which satisfies $C_U\cap S_1\Subset\Gamma\cap S_1$, if $U\Subset\Gamma\cap S_1$ and c is sufficiently small.

Since $\Gamma'\Subset\Gamma$, we can cover Γ' by a finite number of such open cones as C_U. According to this fact the proof of the lemma is reduced to the following; $f_\zeta(t)\underset{\mathrm{Df.}}{=}J(t\zeta_1,\cdots,t\zeta_n)$ has the following estimate:

$|f_\zeta(t)|\geqq C_\varepsilon e^{-\varepsilon|t|}$ in $\{t\,|\,|\mathrm{Im}\,t|<(c/2)|\mathrm{Re}\,t|\}$ for every ε, where C_ε depends only on ε but not on ζ. We prove this estimate using the lemmas quoted above. By the assumption of the growth rate of $|J(\zeta)|$ we have $|f_{\zeta^0}(t)|\leqq A_\varepsilon e^{\varepsilon|t|}$, where A_ε does not depend on ζ^0. For any t satisfying $|t|=R$, and $|\mathrm{Im}\,t|<(c/2)\,\mathrm{Re}\,t$ we can find t_0 such that $|t-t_0|<8\eta R$ and $|f_{\zeta^0}(t^0)|\geqq -H(\eta)\log(A_\varepsilon e^{2\varepsilon eR})$ by the above estimate and Lemma 4.1.5. (The constant η will be fixed later.) By the assumption of the destribution of zeros of $J(\zeta)$ we can assume $f_{\zeta^0}(\tau)\neq 0$ in $\{\tau\,|\,|\tau-t^0|<(c/4)(1-8\eta)R\}$, so we have $\log|f_{\zeta^0}(\tau)|\geqq -2\log\sup_{|\tau-t^0|=(c/4)(1-8\eta)R}|f(z)|+3\log|f_{\zeta^0}(t^0)|$ in $\{\tau\,|\,|\tau-t^0|<(c/8)(1-8\eta)R\}$ by Lemma 4.1.4. Taking η so small that the inequality $(1+(c/8)(1-(\eta/8))R>R$ holds, we have $|t-t^0|<(c/8)(1-8\eta)R$. Of course we can assume $\{\tau\,|\,|\tau-t_0|=(c/4)(1-8\eta)R\}\subset\{\tau\,|\,|\tau|<2eR\}$ without loss of generality, we have the following estimate by the above inequalities:

$$\log|f_{\zeta^0}(t)|\geqq -2\log(A_\varepsilon e^{2\varepsilon eR})-3H(\eta)\log(A_\varepsilon e^{2\varepsilon R})\ .$$

Since the constant η depends only on c and A_ε depends only on ε, we have the estimate $|f_\zeta(t)|\geqq C_\varepsilon e^{-2e(2+3H(\eta))\varepsilon|t|}$ in $\{t\,|\,|\mathrm{Im}\,t|<c\,\mathrm{Re}\,t\}$, where C_ε depends only on ε and c.

Because ε can be chosen arbitrarily, we have $|f_\zeta(t)|\geqq C_\varepsilon e^{-\varepsilon|t|}$ in $\{t\,|\,|\mathrm{Im}\,t|<c\,\mathrm{Re}\,t\}$.

Thus we have proved $|J(\zeta)|\geqq C_\varepsilon e^{-\varepsilon|\zeta|}$ in Γ'. Q.E.D.

We can formulate Theorem 4.1.2 in a more precise form using the notion of the sheaf $\mathscr{C}$, which is recently developed by Sato [36], [37], [38].

For the reader's convenience we explain here the notion of the sheaf $\mathscr{C}$ briefly, since Sato's new theory is only available in Japanese. We refer the reader to Sato [37], [38] for the complete theory of the sheaf $\mathscr{C}$. As for the algebraic aspect of the theory Morimoto [31] is also available.

The sheaf $\mathscr{C}$ is constructed on S^*M, the cosphere bundle of a real analytic manifold M. It satisfies $0 \longrightarrow \mathscr{A} \longrightarrow \mathscr{B} \longrightarrow \pi_* \mathscr{C} \longrightarrow 0$ (exact) where π is the projection map from S^*M to M and $\pi_*\mathscr{C}$ means the direct image of the sheaf $\mathscr{C}$ under π. We denote by β the mapping from $\mathscr{B}(U)$ to $\mathscr{C}(\pi^{-1}(U))$. Then we can consider the $\operatorname{supp}\beta(u)$ for any hyperfunction u on M.

Fixing some local coordinate we consider $S^{n-1}\times U$ instead of S^*M where $U\subset M$, and give the rough notion about $\operatorname{supp}\beta(u)$. By $\operatorname{supp}\beta(u)\cap(\{(1, 0 \cdots 0)\}\times U)=\phi$ we mean the hyperfunction u can be locally represented only using the boundary values of holomorphic functions $f_j(\zeta)$, $1\leqq j\leqq l$, on $V\times\sqrt{-1}\Gamma_j$ respectively where Γ_j is an open cone in $\boldsymbol{R}^n$ and $\bigcup_{j=1}^{l}\Gamma_j^0$ does not contain the direction $(1, 0, \cdots, 0)$. See Morimoto [31] about these facts.

Now we have the following Lemma 4.1.6 due to Sato. Before stating it we do some preliminaries. Let M and M' be real analytic manifolds of dimension $m+r$ and m respectively, and φ be a real analytic mapping from M to M' with rank m. For any hyperfunction valued $(m+r)$-form g on M we can consider $\int_{\varphi^{-1}(x')} g(x)$ as far as φ is proper on $\operatorname{supp} g$. This integration is called the integration along fibre and obviously the convolution operation $u*v$, where v is of compact support, is a special type of this integration. (Take $\boldsymbol{R}^{2m}$ and $\boldsymbol{R}^m$ as M and M' respectively, the projection map from $\boldsymbol{R}^{2m}$ to $\boldsymbol{R}^m$ as φ, and $u(x-y)v(y)dxdy$ as g, where we denote the point in $\boldsymbol{R}^{2m}$ by (x, y) and the point in $\boldsymbol{R}^m$ by x.) (See Sato [35] p. 434.) We can also define the integration along fibre of $\mathscr{C}$-valued forms for which the relation

$$\beta\Big(\int_{\varphi^{-1}(x')} g\Big)=\int_{\varphi^{-1}(x')}\beta(g)|_{\varphi^{-1}S^*M'}$$

holds, where $\varphi^{-1}S^*M'=\{(x, \eta)|\varphi(x)=x', \eta=\varphi^*\eta', (x', \eta')\in S^*M\}$. (See Sato [37], [38] for these facts.) Now we state the lemma.

LEMMA 4.1.6 (Sato). *There is the following relation between* $\operatorname{supp}\beta(g)$ *and* $\operatorname{supp}\beta\Big(\int_{\varphi^{-1}(x')} g\Big)$; $\operatorname{supp}\beta\Big(\int_{\varphi^{-1}(x')} g\Big)\subset\varphi(\operatorname{supp}\beta(g)\cap\varphi^{-1}S^*M')$.

PROOF. It is sufficient to prove that $\int_{\varphi^{-1}(x')} g$ is a real analytic function on M' provided that $\operatorname{supp}\beta(g)\cap\varphi^{-1}S^*M'=\phi$, since the sheaf $\mathscr{C}$ is decomposable. (See Sato [38]). But this is immediate by the definition of β.

Now we return to our problems.

DEFINITION 4.1.7. (Characteristic set of the operator $S*$.) We define the characteristic set $V(J)$ to be the intersection of N with S_∞^{n-1} where N is the closure of $\{\zeta\in \boldsymbol{C}^n\cong \boldsymbol{R}^{2n}|J(\zeta)=0\}$ in $\boldsymbol{D}^{2n}$.

It is obvious that if $J(\zeta)$ is a polynomial then $\eta\infty\in N$ is equivalent to $P_m(\eta)=0$ where P_m is the principal symbol of P.

THEOREM 4.1.8. *Suppose that $V(J)$ does not contain $(1, 0, \cdots, 0)$. Then the following statement holds: If $S*u=f$ in Ω and* $\operatorname{supp}\beta(f)\cap(\{(1, 0, \cdots, 0)\}\times\Omega)=\phi$, *then* $\operatorname{supp}\beta(u)\cap(\{(1, 0, \cdots, 0)\}\times\Omega)=\phi$.

PROOF. We define $Q(\zeta)\in(\mathscr{P}_*)'$ by

$$\langle Q(\zeta), f(\zeta)\rangle=\int\cdots\int_{C,\,\xi_1>K} f(\zeta)/J(\zeta)d\xi_1\cdots d\xi_n$$

where C is $\{\xi\in\boldsymbol{R}^n||\xi|<c\xi_1\}$ and K is sufficiently large, and we also define $P(x)$ by $\mathscr{F}(Q(\zeta))$. We denote by I the intersection $C\cap\{|\xi|=1\}$, by $\bar{I}$ its closure, and by ∂I its boundary in $\{|\xi|=1\}$. Then by the condition of $V(J)$ we have $S*P=\delta-W$ where

$$\operatorname{supp}\beta(P(x))\subset(\partial I\times\boldsymbol{R}^n)\cup(\bar{I}\times\{0\}),\ \operatorname{supp}\beta(W(x))\cap(I\times\boldsymbol{R}^n)=\phi.$$

(We regard $P(x)$ as a hyperfunction over $\boldsymbol{R}^n$ by restriction.) From the above relation we have $\tilde{u}=\tilde{u}*\delta=\tilde{u}*(S*P+W)=(S*\tilde{u})*P+\tilde{u}*W$ where $\tilde{u}$ is obtained from u by cutting it off outside some compact set K in Ω by the flabbiness of $\mathscr{B}$. Since $\operatorname{supp}\beta(S*\tilde{u})\cap\{(1, 0, \cdots, 0)\}\times(\text{interior of } K)=\phi$, $\operatorname{supp}\beta(P)\subset(\partial I\times\boldsymbol{R}^n)\cup(\bar{I}\times\{0\})$ and $\operatorname{supp}\beta(W)\cap(I\times\boldsymbol{R}^n)=\phi$, we have $\operatorname{supp}\beta(\tilde{u})\cap\{1, 0, \cdots, 0)\}\times K^i=\phi$ by Lemma 4.1.6. Since K is arbitrary we conclude $\operatorname{supp}\beta(u)\cap\{(1, 0, \cdots, 0)\}\times\Omega=\phi$.

COROLLARY. *If $V(J)$ does not contain $(\pm1, 0, \cdots, 0)$, then u depends real analytically on x_1 as far as $S*u$ does.*

PROOF. This is the immediate consequence of the theorem and the definition of the real analytic dependence. (See Sato [35] Ch. 3, §8 and [38]).

We next prove that the converse of the theorem holds.

THEOREM 4.1.9. *If S does not satisfy the condition of the theorem, then there exists a hyperfunction u which satisfies $S*u=0$ and* $\operatorname{supp}\beta(u)\cap(\{(1, 0, \cdots, 0)\}\times\Omega)\neq\phi$.

(The following method of construction of u was kindly suggested by Prof. Komatsu.)

PROOF. By the assumption on S, we can choose a sequence of $\zeta^m=\xi^m+i\eta^m$ ($\xi^m, \eta^m\in \boldsymbol{R}^n$) which satisfies the following conditions (i) — (iv):

(i) $J(\zeta^m)=0$

(ii) $\xi_1^m \geqq m\xi_1^{m-1}$

(iii) $|\xi_j^m|\leqq \varepsilon_m|\zeta^m|$ $(j=2, \cdots, n)$

(iv) $|\eta_j^m|\leqq \varepsilon_m|\zeta^m|$ $(j=1, \cdots, n)$, where $\varepsilon_m=1/2nm$.

Using this sequence ζ^m, we define $F(z)=\sum_{m=1}^{\infty}\exp(i\langle\zeta^m, z\rangle)$, then it is easily checked that $F(z)$ is holomorphic in $\{\operatorname{Im} z_1>0\}$ and satisfies $S*F(z)=0$ there. We want to prove $\lim_{p\to\infty}\operatorname{Re}F(z_p)=\infty$ for a suitable sequence of z^m with $\operatorname{Im} z_1^p>0$. We take $z^m=(iy_1^m, 0, \cdots, 0)$ where $y_1^m=(\xi_1^m)^{-1}$. Then we have $\cos(y_1^m\eta_1^p)>1/2$ for $m\geqq p$, because we have

$$|y_1^m\eta_1^p|\leqq \varepsilon_p|\zeta^p|(\xi_1^m)^{-1}\leqq(2\xi_1^m)^{-1}\varepsilon_p\xi_1^p$$
$$\leqq(2(p+1)\xi_1^m)^{-1}\varepsilon_p\xi_1^{p+1}\leqq\cdots\leqq(2(p+1)\cdot\cdots\cdot m\xi_1^m)^{-1}\varepsilon_p\xi_1^m$$

by the condition on ζ^m. (Remark that we can assume $|\zeta^p|<(1/2)\xi_1^p$ without loss of generality.) Therefore we have

$$\sum_{m=1}^{p}\exp(-y^p\xi^m)\cos(y^p\eta^m)\geqq(1/2)\sum_{m=1}^{p}\exp(-y_1^m\xi_1^m)\geqq(2e)^{-1}p$$

since $y_1^p=(\xi_1^p)^{-1}\leqq(p\xi_1^{p-1})^{-1}\leqq y_1^{p-1}$. On the other hand we have

$$\Big|\sum_{m\geqq p+1}\exp(-y^p\xi^m)\cos(y^p\eta^m)\Big|\leqq\sum_{m\geqq p+1}\exp(-p^{m-p}/4)\leqq 2\ .$$

Therefore we have $\operatorname{Re}F(z^p)\geqq(2e^{-1})p-2$, that is, $\lim_{p\to\infty}\operatorname{Re}F(z^p)=\infty$. Taking the boundary value of $F(z)$ as u we find the required singular solution of $S*u=0$ by the definition since $S*F=0$ in $\{\operatorname{Im} z_1>0\}$.

REMARK. A weak converse of Theorem 4.1.2 can be easily obtained in the following form.

THEOREM 4.1.9′. *If S does not satisfy the condition of Theorem 4.1.2, then S has no parametrix. (Here we mean by parametrix P a hyperfunction which satisfies $S*P=\delta-W$ where $W\in\mathscr{A}(\boldsymbol{R}^n)$ and whose singular support is compact (not necessarily $\{0\}$.)*

PROOF. First we define a sheaf $\underset{\approx}{\mathscr{O}}$ on $\boldsymbol{D}^{2n}$ giving its section module $\underset{\approx}{\mathscr{O}}(\Omega)$ over Ω as follows. $\underset{\approx}{\mathscr{O}}(\Omega)$ consists of all holomorphic functions $f(z)$ on $\Omega\cap\boldsymbol{C}^n$ which satisfy the following conditions: for any compact set K in Ω there exists some δ_K such that $\sup|f(z)e^{\delta_K|z|}|<\infty$ holds.

We also denote $\underset{\approx}{\mathscr{O}}$ the inverse image of $\underset{\approx}{\mathscr{O}}$ to $\boldsymbol{D}^n$ ($\cong$closure of $\boldsymbol{R}^n\times i\{0\}$ in $\boldsymbol{D}^{2n}$).

We consider the following exact sequence of sheaves on $\boldsymbol{D}^n = \boldsymbol{R}^n \cup S_\infty^{n-1}$:

$$0 \longrightarrow \underset{\approx}{\mathscr{O}} \longrightarrow \mathscr{R} \xrightarrow{\ j\ } \mathscr{R}/\underset{\approx}{\mathscr{O}} \longrightarrow 0$$

We denote by $[\mu]$ the image of μ under the map j.

We have $H^1(\boldsymbol{D}^n, \underset{\approx}{\mathscr{O}})=0$ as in Part I § 3 (cf. 5-41), so we have

$$0 \longrightarrow \underset{\approx}{\mathscr{O}}(\boldsymbol{D}^n) \longrightarrow \mathscr{R}(\boldsymbol{D}^n) \longrightarrow (\mathscr{R}/\underset{\approx}{\mathscr{O}})(\boldsymbol{D}^n) \longrightarrow 0 \cdots (1).$$

On the other hand $\mathscr{R}/\underset{\approx}{\mathscr{O}} = \mathscr{B}/\mathscr{A}$ on $\boldsymbol{R}^n$ by their definitions, so the existence of the parametrix $P(x)$ and the above exact sequence (1) conclude the existence of some $\tilde{P} \in \mathscr{R}(\boldsymbol{D}^n)$ with $\operatorname{supp}[\tilde{P}] = \operatorname{singsupp} P$ such that $S*P=\delta+\varphi$ where φ belongs to $\underset{\approx}{\mathscr{O}}(\boldsymbol{D}^n)$. (Remark that $\operatorname{supp}[P]_{R^n}$ is compact in R^n where $\operatorname{supp}[P]_{R^n}$ is the support of the equivalence class of P in $\mathscr{B}/\mathscr{A}$, that is the $\operatorname{singsupp} P$.) Since $\operatorname{supp}[\tilde{P}]$ is compact in $\boldsymbol{R}^n$, $\mathscr{F}\tilde{P}$ is holomorphic in some neighbourhood of $\boldsymbol{D}^n$ in $\boldsymbol{D}^{2n}$ and $\mathscr{F}\varphi$ tends to zero there as $|\zeta| \longrightarrow \infty$. (We prove these facts in the later section (5.1). The above facts are the most trivial part of that section.)

On the other hand $J(\zeta)$ tends to zero along some ray in that domain by assumption, so $S*P=\delta+\varphi$, that is $J(\zeta)\mathscr{F}P=1+\mathscr{F}\varphi$ gives a contradiction.

Q.E.D.

REMARK. The sheaf $\underset{\approx}{\mathscr{O}}$, which is used above, will play an essential role in our forthcoming paper [18]. Full study of the nature of this sheaf will be given there.

Theorem 4.1.8 gives the most complete results about partial ellipticity. At first we shall give the definition of partial ellipticity and conditional ellipticity as follows after Gårding-Malgrange [8].

DEFINITION 4.1.10. If $\operatorname{supp}\beta(u) \subset \{\xi_1 = \cdots = \xi_m = 0\} \times \Omega$ then we say u depends weakly complex holomorphically on $(x_1, \cdots, x_m)$.

DEFINITION 4.1.11. $S*$ is partially elliptic with respect to $(x_1, \cdots, x_m)$ if any hyperfunction solution of $S*u=0$ depends weakly complex holomorphically on $(x_1, \cdots, x_m)$.

DEFINITION 4.1.12. $S*$ is conditionally elliptic with respect to $(x_1, \cdots, x_m)$ if u is a real analytic function whenever $S*u=0$ and u depends weakly complex holomorphically on $(x_{m+1}, \cdots, x_n)$.

REMARK. If $S*$ is a usual partical differential operator of finite order then those operators which are partially elliptic in $(x_1, \cdots, x_m)$ in the sense of Gårding-Malgrange are partially elliptic in the above sense. Of course this is a corollary of the following theorem, but one can prove it directly using the Gårding-

Malgrange estimate, and Komatsu's method [22], but we omit the details since it seems to have no essential novelties. (In this case u depends complex holomorphically in $(x_1, \cdots, x_m)$: see Sato [35] about the notion of the complex holomorphic dependence. We remark here that complex holomorphic dependence trivially implies the weak complex holomorphic dependence, but the converse is not true in general.

THEOREM 4.1.13. *$S*$ is partially elliptic in $(x_1, \cdots, x_m)$ if and only if $V(J) \subset \{\xi_1 = \cdots = \xi_m = 0\}$.*

THEOREM 4.1.14. *$S*$ is conditionally elliptic in $(x_1, \cdots, x_m)$ if and only if $\{\xi_{m+1} = \cdots = \xi_n = 0\} \cap V(J) = \phi$.*

These are the immediate consequences of Theorem 4.1.8, Theorem 4.1.9 and the definition of weak complex holomorphic dependence.

REMARK 1. The method of the proof of Theorem 4.1.9 implies that we can also prove Theorem 4.1.14 if we define conditional ellipticity using the notion of complex holomorphic dependence instead of the notion of weak complex holomorphic dependence.

REMARK 2. It seems that Theorem 4.1.13 and Theorem 4.1.14 have not known even for usual partial differential operators. A remarkable fact in that case is that partial ellipticity and conditional ellipticity of an operator are determined by its principal part only.

Sato's recent theory has essentially extended this fact even to the variable coefficient case. (Sato [36], [37], [38].)

Remark the fact that Schrödinger operator is not partially elliptic with respect to space variables though the heat equation is in the framework of distributions (Gårding-Malgrange [8].)

We end this section by showing the following theorem which is an extension of Bengel's duality theorem, following the Komatsu method. (Cf. Bengel [1], Komatsu [22], [25].)

THEOREM 4.1.15. *Let K be a compact set in $\boldsymbol{R}^n$ and W be an open neighbourhood of K. If $S*$ is a local operator, we can consider the solution sheaf $\mathscr{B}^{S*}$ and $\mathscr{A}^{S*}$. Assuming $S*$ is an elliptic local operator, we have the following duality theorem*:

$$[\mathscr{A}^{S*}(K)]' \cong H^1_K(W, \mathscr{A}^{\check{S}*}) \cong \mathscr{A}^{\check{S}*}(W-K)/\mathscr{A}^{\check{S}*}(W) .$$

PROOF. We have the regularity theorem (Theorem 4.1.2) and it is essentially well known $S*\mathscr{B}(\Omega) = \mathscr{B}(\Omega)$ for every open set Ω (see for example Schapira [40]; we give the more general existence theorem for convolution operators; see

Theorem 4.2.3.) These two facts combined with the Serre-Komatsu duality theorem ([H] Theorem 19) give the theorem. In fact we have the flabby resolution

$$0 \longrightarrow \mathscr{B}^{\check{S}*} \longrightarrow \mathscr{B} \xrightarrow{\check{S}*} \mathscr{B} \longrightarrow 0$$

and the ellipticity of $\check{S}*$ means $\mathscr{B}^{\check{S}*}=\mathscr{A}^{\check{S}*}$. Just in the same way as above we have

$$0 \longrightarrow \mathscr{A}^{S*} \longrightarrow \mathscr{A} \xrightarrow{\check{S}*} \mathscr{A} \longrightarrow 0 .$$

These two exact sequences give the following dual complexes:

$$\begin{array}{ccccccc} 0 \longrightarrow & \mathscr{B}_K(W) & \xrightarrow{\check{S}*} & \mathscr{B}_K(W) & \longrightarrow 0 \\ & \updownarrow & & \updownarrow & \\ 0 \longleftarrow & \mathscr{A}(K) & \xleftarrow{S*} & \mathscr{A}(K) & \longleftarrow 0 \end{array}$$

Regularity theorem and existence theorem prove $S*$ is of closed range. Thus Serre-Komatsu duality theorem gives us $\mathscr{A}^{S*}(K) \cong [H^1_K(W, \mathscr{A}^{\check{S}*})]'$. (Remark that $\mathscr{A}(K)$ is a DFS-space.) Thus the first part of the isomorphism is proved.

Now we prove the second isomorphism. Since

$$\mathscr{B}(W) \xrightarrow{S*} \mathscr{B}(W) \longrightarrow 0 \text{ (exact)}$$

and the real analyticity of $S*u$ implies the real analyticity of u, we have

$$\mathscr{A}(W) \xrightarrow{S*} \mathscr{A}(W) \longrightarrow 0 \text{ (exact)}.$$

Thus we have $H^1(W, \mathscr{A}^{S*})=0$, so

$$0 \longrightarrow \Gamma_K(W, \mathscr{A}^{S*}) \longrightarrow \Gamma(W, \mathscr{A}^{S*}) \longrightarrow \Gamma(W-K, \mathscr{A}^{S*}) \longrightarrow$$
$$\longrightarrow H^1_K(W, \mathscr{A}^{S*}) \longrightarrow H^1(W, \mathscr{A}^{S*})=0 .$$

Therefore we have

$$H^1_K(W, \mathscr{A}^{S*}) \cong \Gamma(W-K, \mathscr{A}^{S*})/\Gamma(W, \mathscr{A}^{S*}) .$$

(Remark that $\Gamma_K(W, \mathscr{A}^{S*})=0$ by the unique continuation property.)

4.2. Ellipticity and partial ellipticity for convolution operators

We treat the problem of ellipticity for convolution operators in this section. The method of the proof is just the same as in the preceding section, but to obtain the result we must assume some additional condition, which we call condition (S), as well as the conditions about the distribution of zeros of $\langle S, e^{-i\langle z,\zeta\rangle}\rangle$.

We also show condition (S) implies $S*\mathscr{B}(\boldsymbol{R}^n)=\mathscr{B}(\boldsymbol{R}^n)$; it will be shown in our forthcoming paper [18] that the condition (S) is not only natural but also best possible one to assure the existence of solutions in a sense.

DEFINITION 4.2.1. Condition (S): For every positive ε there exists some N_ε which satisfies the following condition:

For every $\xi\in\boldsymbol{R}^n$ with $|\xi|>N_\varepsilon$ we can find some $\eta\in\boldsymbol{C}^n$ satisfying $|\xi-\eta|<\varepsilon|\xi|$ and $|J(\eta)|\geqq e^{-\varepsilon|\xi|}$.

REMARK. This notion is indicated by Ehrenpreis [4].

THEOREM 4.2.2. *Let S be a hyperfunction with compact support. Suppose that $J(\zeta)=\langle S, e^{-i\langle z,\zeta\rangle}\rangle$ satisfies the above condition (S) and $V(J)=\phi$. (The definition of $V(J)$ is given in Definition 4.1.10.) Then $u\in\mathscr{A}(\boldsymbol{R}^n)$ whenever $S*u\in\mathscr{A}(\boldsymbol{R}^n)$ and $u\in\mathscr{B}(\boldsymbol{R}^n)$.*

PROOF. If we can prove the estimate (1) below, then the existence of parametrix P can be proved just in the same way as in the proof of Theorem 4.1.2. (Of course we cannot expect $\operatorname{singsupp} P$ is equal to $\{0\}$ but only it is compact in $\boldsymbol{R}^n$, so we have used the terminology "parametrix", not "good parametrix".)

The desired estimate is:

(1) There exists some neighbourhood U of S_∞^{n-1} in $\boldsymbol{D}^{2n}$ for which the following inequality holds; $|J(\zeta)|\geqq C_\varepsilon e^{-\varepsilon|\zeta|}(\forall\varepsilon>0)$ in $U\cap\boldsymbol{C}^n$.

To obtain this estimate we need the following lemma due to Hörmander-Malgrange. (Hörmander [12] p. 154, Lemma 3.1.)

LEMMA 4.2.2. *Suppose that $f(z)$, $g(z)$, $f(z)/g(z)$ are holomorphic in $\{z\in\boldsymbol{C}^n\,|\,|z|<R\}$ with $\sup_{|z|<R}|f(z)|<A$ and $\sup_{|z|<R}|g(z)|<B$, then we have*

$$|f(z)/g(z)|\leqq AB^{2|z|/(R-|z|)}|g(0)|^{-(R+|z|)/(R-|z|)} \quad \textit{in} \quad |z|<R.$$

(Of course we can use the minimum modulus theorem of Ehrenpreis (Ehrenpreis [3]), but here we prefer to use the above lemma because it is much more elementary.)

Now we go on to the proof of estimate (1). Using the condition that $V(J)=\phi$, we can assume $J(\zeta)\neq0$ in $\{\zeta\,|\,3c|\operatorname{Re}\zeta|>|\operatorname{Im}\zeta|, |\operatorname{Re}\zeta|>K\}$ for some constants c and K. We want to estimate $|J(\zeta)|$ from below in $\{\zeta\,|\,c|\operatorname{Re}\zeta|>|\operatorname{Im}\zeta|, |\operatorname{Re}\zeta|>2K\}$. It is sufficient to estimate $|J(\zeta)|$ from below in $\{\zeta\,|\,c|\operatorname{Re}\zeta|>|\operatorname{Im}\zeta|, |\operatorname{Re}\zeta|>N_\varepsilon\}$ for any ε.

Using the condition (S) we can choose some ζ' with $|\mathrm{Re}\,\zeta-\zeta'|<\varepsilon|\mathrm{Re}\,\zeta|$ and $|J(\zeta')|>e^{-\varepsilon|\mathrm{Re}\,\zeta|}$. Consider a ball with redius $2(\varepsilon|\mathrm{Re}\,\zeta|+|\mathrm{Im}\,\zeta|)$ with its center at ζ'. Applying Lemma 4.2.2 we have the following inequality:

$$|1/J(\zeta)|\leqq(A_\varepsilon \exp(\varepsilon(2\varepsilon|\mathrm{Re}\,\zeta|+|\mathrm{Im}\,\zeta|+|\zeta'|)+\alpha(\varepsilon|\mathrm{Re}\,\zeta|+2(\varepsilon|\mathrm{Re}\,\zeta| +|\mathrm{Im}\,\zeta|)))^2\times(\exp(-\varepsilon|\zeta|)^3,$$

since $|J(\zeta)|\leqq A_\varepsilon e^{\varepsilon|\zeta|+\alpha|\mathrm{Im}\,\zeta|}$ holds for some α.

Thus we can conclude $|1/J(\zeta)|<B_\theta \exp(\theta|\zeta|+4\alpha|\mathrm{Im}\,\zeta|)$ for any $\theta>0$. This is the desired estimate and the existence of parametrix follows from this. So the proof is complete.

REMARK 1. It can be easily proved as in Theorem 4.1.9′ that not only the condition $V(J)=\phi$ but also the condition (S) is necessary for the existence of a parametrix.

REMARK 2. It is trivial that Theorem 4.1.8, Theorem 4.1.9, Theorem 4.1.13 and Theorem 4.1.14 hold with some due modifications for the convolution operator $S*$ with the condition (S). We omit the details.

REMARK 3. We can treat more general convolution operator $S*$, that is only assuming singsupp S compact, using the theory of modified Fourier hyperfunctions. The theory will be given in our forthcoming paper [18].

To show the naturality of the condition (S) we exhibit the following existence theorem. More direct proof of this theorem will be given in our forthcoming paper [18]. Further we remark that if S is a distribution with compact support, then the condition (S) is easily seen to hold. Since this fact seems essentially well known, we omit the details. (See for example the argument of Ehrenpreis [4] p. 544. Perhaps we will treat the topics around this problem in some paper which treats the problem of the relation between singular supports and convolutions.)

THEOREM 4.2.4. *Suppose* $\langle S, e^{-i\langle z,\zeta\rangle}\rangle$ *satisfies the condition* (S), *then we have* $S*\mathscr{B}(\boldsymbol{R}^n)=\mathscr{B}(\boldsymbol{R}^n)$.

PROOF. Define $\Omega=\boldsymbol{R}^n\times\sqrt{-1}\{y|y_1>0,\cdots,y_n>0\}$. If we can prove $S*\mathscr{O}(\Omega)=\mathscr{O}(\Omega)$, then the theorem easily follows. The condition (S) combined with Lemma 4.2.3 proves $S*\mathscr{O}(\Omega)=\mathscr{O}(\Omega)$ or equivalently $\check{S}*:[\mathscr{O}(\Omega)]'\longrightarrow[\mathscr{O}(\Omega)]'$ is of closed range by the theorem of Ehrenpreis-Martineau. (About the Ehrenpreis-Martineau theorem, see for example Hörmander [15] Ch. 4, § 4. 5.) For the sake of simplicity of the notation, we assume $n=2$. We want to prove $\check{S}*:[\mathscr{O}(\Omega)]'\longrightarrow[\mathscr{O}(\Omega)]'$ is of closed range. Using the Fourier transform of the analytic functionals, it is sufficient to prove the following statement; if $F(\zeta)/J(\zeta)$ is entire and $|F(\zeta)|\leqq$

$A\exp(\chi_K(\zeta))$ holds for some compact convex set K in Ω then we have $|F(\zeta)/J(\zeta)| \leqq C\exp(\chi_L(\zeta))$ where L is a compact convex set in Ω which depends only $J(\zeta)$ and K. (In the above we denote $\sup_{z\in K}(-\operatorname{Im}\langle z, \zeta\rangle)$ by $\chi_K(\zeta)$.) Let ξ_j be $\operatorname{Re}\zeta_j$ and η_j be $\operatorname{Im}\zeta_j$. In $\{\xi_1 \geqq 0, \xi_2 \geqq 0\}$ the above condition on $|F(\zeta)|$ implies that the estimate $|F(\zeta)| \leqq A\exp(K|\eta| - \delta\xi_1 - \delta\xi_2)$ holds for some positive K and δ. First we consider the case $\xi_1 > \varepsilon_0|\xi|$ and $\xi_2 > \varepsilon_0|\xi|$. Then we have the following estimate on $|F(\zeta)/J(\zeta)|$ there for any positive ε and θ using the condition (S) and Lemma 4.2.3:

$$\begin{aligned}|F(\zeta)/J(\zeta)| &\leqq \{A\exp(K(3\varepsilon|\xi| + |\eta|) - \delta(-3\varepsilon|\xi| - 2|\eta| + |\xi|))\} \\ &\quad \times\{B_\theta \exp(\theta|\zeta| + \alpha(3\varepsilon|\xi| + |\eta|))\}^2 \times \{C_\varepsilon \exp(\varepsilon|\xi|)\}^3 \\ &\leqq C_{\varepsilon,\theta}\exp((2\delta + 3K + 2\theta + 2\alpha)|\eta| + (-\delta + 3\varepsilon(K + \delta + 2\alpha + 1) + 2\theta)|\xi|),\end{aligned}$$

where α depends only on $J(\zeta)$. Thus fixing ε and θ so small that $3\varepsilon(K+\delta+2\alpha+1) + 2\theta < \delta/2$ we have the desired estimate on $|F(\zeta)/J(\zeta)|$ in $\{\xi_1 > \varepsilon_0|\xi|$ and $\xi_2 > \varepsilon_0|\xi|\}$. Secondly we consider the case $\{0 \leqq \xi_1 < \varepsilon_0|\xi|, \xi_2 \geqq 0\}$. Noting that $\xi_2 > c|\xi|$ holds for some positive c in this case and $|F(\zeta)| \leqq A\exp(K|\eta| - K\xi_1 - \delta\xi_2)$ holds in $\{\xi_1 < 0$ and $\xi_2 \geqq 0\}$ we have the following estimate just in the same way as above:

$$\begin{aligned}|F(\zeta)/J(\zeta)| &\leqq \{A\exp(K(3\varepsilon|\xi| + |\eta| - K(-3\varepsilon|\xi| - 2|\eta| + \xi_1) \\ &\quad - \delta(-3\varepsilon|\xi| - 2|\eta| + \xi_2))\} \times \{B_\theta \exp(\theta|\zeta| + \alpha(3\varepsilon|\xi| + |\eta|))\}^2 \times \{C_\varepsilon e^{\varepsilon|\xi|}\}^3 \\ &\leqq C_{\varepsilon,\theta}\exp((2\delta + 5K + 2\theta + 2\alpha)|\eta| + (3\varepsilon(2K + \delta + 2\alpha + 1) + 2\theta)|\xi| \\ &\quad - K\xi_1 - \delta\xi_2) \leqq C_{\varepsilon,\theta}\exp((2\delta + 5K + 2\theta + 2\alpha)|\eta| - K\xi_1 \\ &\quad + (3c\varepsilon(2K + \delta + 2\alpha + 1) + 2c\theta - \delta)\xi_2)\end{aligned}$$

holds for any positive ε and θ. Therefore taking ε and θ so small as to ensure $3c\varepsilon(2K+\delta+2\alpha+1) + 2c\theta < \delta/2$, we have the desired estimate. In other quadrants the situations are essentially the same, so we omit the details. Thus the proof is completed.

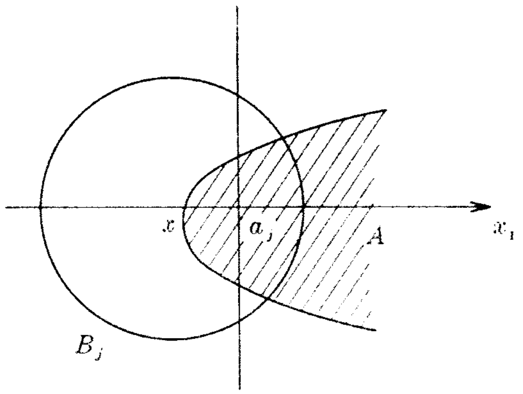
§ 5. Propagation of regularity

5.1. Propagation of regularity for local operators

We treat the problem of propagation of regularity for local operators in this section. The method of the proof seems to be of some interest. (Cf. Malgrange [29].)

THEOREM 5.1.1. *Let S be a hyperfunction with its support at* $\{0\}$. *Suppose A is a convex closed set and x is an extreme point of A. Then there exists a funndamental system of open neighbourhoods $\{U_j\}$ satisfying the following condition: $S*u$ belongs to both $\mathscr{B}(U_j)$ and $\mathscr{A}(U_j - A)$ then u necessarily belongs to $\mathscr{A}(U_j)$. (See the following figure.)*

Figure



$U_j = B_j \cap \{x_1 < a_j\}$

PROOF. Let B_j be a ball with radius j^{-1} and with its center at x. The condition on A and x permits us to conclude U_j $(=B_j \cap \{x_1 < a_j\})$ has the following property $(*)$ after a suitable choice of coordinate system and of a_j.

$(*)$ For every c with $c \leqq a_j$, $A \cap \{x_1 = c\}$ is compact in $U_j \cap \{x_1 = c\}$.

As Sato remarked the sheaf $\mathscr{B}/\mathscr{A}$ is flabby. (This follows easily from the flabbiness of the sheaf $\mathscr{B}$ and the vanishing of $H^1(\Omega, \mathscr{A})$ for every open set Ω.) Therefore we can find some $\tilde{\nu} \in (\mathscr{B}/\mathscr{A})(\boldsymbol{R}^n)$ with $\operatorname{supp} \tilde{\nu} \subset (U_j \cap A)^a$ and $\tilde{\nu} = \nu$ in U_j where $\nu = [u] \in (\mathscr{B}/\mathscr{A})(U_j)$. (The symbol $[u]$ denote the image of u under the map from $\mathscr{B}$ to $\mathscr{B}/\mathscr{A}$). The choice of U_j permits us to conclude $S * \tilde{\nu} = 0$ in U_j and in $x_1 < a_j$. As is remarked in 4.1. $\mathscr{B}/\mathscr{A} \cong \mathscr{R}/\underset{\approx}{\mathscr{O}}$ on $\boldsymbol{R}^n$, we can consider $\tilde{\nu}$ is an element of $(\mathscr{R}/\underset{\approx}{\mathscr{O}})(\boldsymbol{D}^n)$ since $\operatorname{supp} \tilde{\nu}$ is compact in R^n. Thus we can find some ψ, which belongs to both $\mathscr{R}(\boldsymbol{D}^n)$ and $\underset{\approx}{\mathscr{O}}(\boldsymbol{D}^n - \operatorname{supp} \tilde{\nu})$ satisfying $[\psi] = \tilde{\nu}$ by the vanishing of $H^1(\boldsymbol{D}^n, \underset{\approx}{\mathscr{O}})$. Defining $\theta = S * \tilde{\nu}$, we also have some φ which belongs to both $\mathscr{R}(\boldsymbol{D}^n)$ and $\underset{\approx}{\mathscr{O}}(\boldsymbol{D}^n - \operatorname{supp} \theta)$ with $[\varphi] = \theta$. Thus the relation $S * \tilde{\nu} = \theta$ implies $S * \psi - \varphi \in \underset{\approx}{\mathscr{O}}(\boldsymbol{D}^n)$ by the definition. Next we define $F(\zeta) = \int \psi(x) e^{i\langle x, \zeta \rangle} dx$, $G(\zeta) = \int \varphi(x) e^{i\langle x, \zeta \rangle} dx$. (This integration means that $F(\zeta)$ is a sum of two hyperfunctions; one is the integration of hyperfunction with compact support and the other is an integration in the sense of Riemann. Thus $F(\zeta)$ is easily seen to be holomorphic in a conical neighbourhood of $\boldsymbol{R}^n$ in $\boldsymbol{C}^n \cong \boldsymbol{R}^{2n}$. (More precice estimate is given in Lemma 5.1.2 below.) Then we have $J(\zeta) F(\zeta) - G(\zeta) \in \underset{\approx}{\mathscr{O}}(\boldsymbol{D}^n)$ since $\underset{\approx}{\mathscr{O}}(\boldsymbol{D}^n)$ is stable under the Fourier transformation. On the other hand we can decide the convex hull of $\operatorname{supp}[\psi]$ from the growth order of $\int \psi(x) e^{i\langle x, \zeta\rangle} dx$ as is shown in Lemma 5.1.2. Since $\operatorname{supp} S$ is equal to $\{0\}$, we have the following estimate: For any ε and ζ we can find some ζ' with $|J(\zeta')| \geqq C_\varepsilon e^{-\varepsilon|\zeta|}$ and $|\zeta-\zeta'| < \varepsilon|\zeta|$ (Cf. 4.1.). So Lemma 4.2.3 combined with Lemma 5.1.2 concludes that convex hull of $\operatorname{supp}[\psi]$ is equal to that of $[\varphi]$, that is $\operatorname{supp}[\psi] \subset A \cap \{x_1 = a_j\}$. Since $[\psi] = [u]$ in U_j by their definitions, we have $[u] = 0$ in U_j, that is $u \in \mathscr{A}(U_j)$. Thus what remains is to prove the following lemma.

LEMMA 5.1.2. *Let $\mu(x)$ belong to both $\mathscr{B}(D^n)$ and $\underset{\approx}{\mathscr{O}}(D^n - K)$ where K is a compact convex set in R^n. Suppose $F(\zeta)$ is defined by $F(\zeta) = \int \mu(x) e^{i\langle x, \zeta\rangle} dx$, then $F(\zeta)$ satisfies the following condition* (*):

(*) *For any $\varepsilon > 0$ there exist some C_ε and A_ε such that $F(\zeta)$ is holomorphic in $\Gamma_\varepsilon = \{\zeta \mid C_\varepsilon(|\operatorname{Re}\zeta| + 1) > |\operatorname{Im}\zeta|\}$ and $|F(\zeta)| \leqq A_\varepsilon \exp(\varepsilon|\zeta| + \chi_K(\operatorname{Im}\zeta))$ where $\chi_K(\eta) = \sup_{x \in K}(-\langle x, \eta\rangle)$.*

Conversely if $F(\zeta)$ satisfies the above condition (*), *we can obtain a hyperfunction $\mu(x)$ by $\int F(\xi) e^{-i\langle x, \xi\rangle} d\xi$ and this belongs to $\underset{\approx}{\mathscr{O}}(D^n - K)$.*

PROOF. Fix an arbitrary $\varepsilon > 0$. Let K_ε be a convex ε-neighbourhood of K. Taking an rectangle $\prod_{j=1}^{n}(x_j^0, x_j^{0\prime})$ so as to include K we have

$$\int \mu(x) e^{i\langle x, \zeta\rangle} dx = \int_\Gamma \mu(z) e^{i\langle z, \zeta\rangle} dz$$

where $\Gamma = \prod_{j=1}^{n} \Gamma_{(x_j^0, x_j^{0\prime})}$, which is taken to be included in the union of the domain where $\mu(z)$ is holomorphic and the rectangle $\prod_{j=1}^{n}(x_j^0, x_j^{0\prime})$. Here $\Gamma_{(x_j^0, x_j^{0\prime})}$ is the path of integration as below.

Figure

z_j-plane

Then we have $|F(\zeta)| \leqq A \exp(\chi_{\Pi(x_j^0, x_j^{0\prime})}(\operatorname{Im}\zeta))$ in $\{\zeta \mid C_\varepsilon(|\operatorname{Re}\zeta| + 1) > |\operatorname{Im}\zeta|, \operatorname{Re}\zeta_j \geqq 0\}$, where C_ε depends only on the decrease rate of $\mu(z)$ and the angle α formed by $\Gamma_{(x_j^0, x_j^{0\prime})}$ and the real axis. Reflecting $\Gamma_{(x_j^0, x_j^{0\prime})}$ with respect to real axis we obtain the above estimate in $\{\zeta \mid C_\varepsilon(|\operatorname{Re}\zeta| + 1) > |\operatorname{Im}\zeta|\}$. Thus the convexity of K permits us to conclude $|F(\zeta)| \leqq A_\varepsilon \exp(\chi_{K_\varepsilon}(\operatorname{Im}\zeta))$ in some conical neighbourhood of $\boldsymbol{R}^n$ in $\boldsymbol{R}^{2n}$. Therefore we have proved the first part of the theorem.

To prove the second part of the theorem we first remark that $\int F(\xi)e^{-i\langle x,\xi\rangle}d\xi$ can be defined by the inverse Fourier transform of the Fourier hyperfunction $F(\zeta)$, that is defined by $\langle F, f\rangle=\int F(\zeta)f(\zeta)d\zeta\ (\forall f(\zeta)\in \mathscr{P}_*)$. Therefore what we must do is to investigate the singular support of $\mu=\int F(\xi)e^{-i\langle x,\xi\rangle}d\xi$.

Now let L be $\prod_{j=1}^{n} L_j$ where $L_j=[-l_j, l_j]$ and K be contained in L. Define $\Omega_{\delta,\varepsilon}$ to be $\{z\mid \operatorname{Im} z_j>-\delta(\operatorname{Re} z_j-l_j)+\varepsilon\}$. If we assume that z belongs to $\Omega_{\delta,\varepsilon}$ then

$$\int_{\operatorname{Re}\zeta_j\leq 0} F(\zeta)e^{-i\langle z,\zeta\rangle}d\zeta=\int_{\substack{\operatorname{Im}\zeta_j=\delta\operatorname{Re}\zeta_j,\\ \operatorname{Re}\zeta_j\leq 0}} F(\zeta)e^{-i\langle z,\zeta\rangle}d\zeta$$

holds for sufficiently small positive δ. Since $\bigcup_{\substack{\delta>0\\ \varepsilon>0}}\Omega_{\delta,\varepsilon}=\{\operatorname{Im} z_j>0\}$ and the condition (*) proves the above integral defines an element of $\tilde{\mathscr{O}}(\boldsymbol{D}^n-\bigcup_{j=1}^{n}\{x_j<l_j+\varepsilon_0\})$ for any positive ε_0. If we define $\Omega'_{\delta,\varepsilon}=\{z\mid\operatorname{Im} z_j>\delta(\operatorname{Re} z_j+l_j)+\varepsilon\}$ we can change the path of integration $\{\operatorname{Im}\zeta_j=\delta\operatorname{Re}\zeta_j, \operatorname{Re}\zeta_j\leqq 0\}$ to $\{\operatorname{Im}\zeta_j=-\delta\operatorname{Re}\zeta_j, \operatorname{Re}\zeta_j\leqq 0\}$ as long as z belongs to the intersection of $\Omega_{\delta,\varepsilon}$ and $\Omega'_{\delta,\varepsilon}$, thus the above integral defines an element of $\tilde{\mathscr{O}}(\boldsymbol{D}^n-\prod_{j=1}^{n}\{-\varepsilon_0-l_j<x_j<l_j+\varepsilon_0\})$. Since $\mathscr{B}/\mathscr{A}$ is a sheaf and K is convex this completes the proof.

5.2. Propagation of regularity for overdetermined systems

In the preceding section we gave a theorem about propagation of regularity for local operators. In this section we want to generalize the theorem to the overdetermined systems under the very restrictive assumption that they are usual partial differential operators.

THEOREM 5.2.1. *Let $P(D)$ be a partial differential operator form $\mathscr{B}^{r_0}$ to $\mathscr{B}^{r_1}$. (The symbol D denotes $i\partial/\partial x_j$.) Define M by $A^{r_0}/{}^tP(X)A^{r_1}$ where A is the polynomial ring $\boldsymbol{C}[X_1,\cdots,X_n]$.*

Then $\operatorname{Ext}^0(M, A)=0$ *is equivalent to* $\Gamma_*(\boldsymbol{R}^n, (\mathscr{B}/\mathscr{A})^p)=0$ (Here the symbol Γ_* means the section whose support is compact.)

PROOF. If $\operatorname{Ext}^0(M, A)\neq 0$, then we can find some μ which belongs to $\Gamma_*(\boldsymbol{R}^n, \mathscr{B}^p)$ by the definition of $\operatorname{Ext}^0(M, A)$, so we prove the converse. The symbol $[u]$ denote the equivalence class of u in $(\mathscr{B}/\mathscr{A})$ or $(\mathscr{R}/\tilde{\mathscr{O}})$ as usual.

Assume $P(D)u$ belongs to $\mathscr{A}(\boldsymbol{R}^n)^{r_1}$ and $\operatorname{supp}[u]$ is compact in $\boldsymbol{R}^n$. Since $H^1(\boldsymbol{D}^n, \tilde{\mathscr{O}})=0$, we can find some ψ which belongs to $\mathscr{R}(\boldsymbol{D}^n)$ with $\operatorname{supp}[\psi]=\operatorname{supp}[u]$ and $P(D)\psi\in\tilde{\mathscr{O}}^{r_1}(\boldsymbol{D}^n)$. Then we have $P(\zeta)\hat{\psi}(\zeta)=\hat{\theta}(\zeta)$ where $\hat{\psi}(\zeta)$ and $\hat{\theta}(\zeta)$ are the Fourier transforms of ψ and θ respectively, so $\hat{\theta}(\zeta)\in\tilde{\mathscr{O}}^{r_0}(\boldsymbol{D}^n)$ and $\hat{\psi}(\zeta)$ is holomorphic in V, which is the intersection of some open neighbourhood of $\boldsymbol{D}^n$ in $\boldsymbol{D}^{2n}$ and $\boldsymbol{C}^n$. Then the well-known Hörmander-Malgrange inequality for polynomials assures the existence of some $\Psi(\zeta)$ which belong to $\underset{\approx}{\mathcal{O}}^{r_0}(\boldsymbol{D}^n)$. (See Hörmander [15] Proposition 7.6.5.) Since we have assumed $\mathrm{Ext}^0(M, A)=0$, $\Psi(\zeta)$ must coincide with $\hat{\psi}(\zeta)$. Therefore we conclude $\psi(x)$ belongs to $\underset{\approx}{\mathcal{O}}^{r_0}(\boldsymbol{D}^n)$. This means $[\psi]=0$, that is, $[u]=0$. Thus we have proved $\Gamma_*(\boldsymbol{R}^n, (\mathscr{B}/\mathscr{A})^p)=0$ under the condition that $\mathrm{Ext}^0(M, A)=0$.

Theorem 5.2.1 shows that we cannot expect any analogue of the theorem in the preceding section without the assumption that $\mathrm{Ext}^0(M, A)=0$. But the method of the proof of Theorem 5.2.1 applies without any essential modifications using Hörmander-Malgrange's inequality quoted above if we assume the condition that $\mathrm{Ext}^0(M, A)=0$ (See the method of the proof of Theorem 5.2.1.) Thus we have the following theorem.

THEOREM 5.2.2. *Let the system of partial differential equations M satisfy the condition* $\mathrm{Ext}^0(M, A)=0$. *Then the conclusion of Theorem* 5.1.1 *holds.*

REMARK. For the overdetermined system M we have stronger results under the vanishing of higher extension group $\mathrm{Ext}^j(M, A)$, but it requires more algebraic preparations, though they are given in Palamodov [32], so we shall discuss that problem in our separate paper [19]. (As for the analytical tools we have given in this paper almost all needed to treat this problem.)

§ 6. Problem of hyperbolicity

6.1. Hyperbolicity for local operators

We treat the problem of hyperbolicity for local operators from the view point of the nature of the elementary solutions. We begin our theory by giving the definition of hyperbolicity for local operators.

DEFINITION 6.1.1. Let $S*$ be a local operator. (That is, $\mathrm{supp}\, S$ is equal to $\{0\}$.) Then $S*$ is called *hyperbolic in the direction of* $(1, 0, \cdots, 0)$ if and only if there exists some hyperfunction E such that

(i) $S*E=\delta$ in $\boldsymbol{R}^n$.

(ii) $\mathrm{supp}\, E\subset\Gamma$ where Γ is a closed convex cone with its vertex at the origin and $\Gamma\Subset\{x_1>-\delta\}$ $(\delta>0)$. (See 3.3 about the definition of the symbol $\Subset$.)

REMARK 1. When $S*$ is a usual partial differential operator, then the above definition coincides with the usual one except for the fact we admit E to be a hyperfunction. (See for example Hörmander [13] Ch. 5) Further our theory has some interesting results even for the usual partial differential operators. We treat it in 6.3 below.

REMARK 2. In the above definition of hyperbolicity we distinguish the two directions $(1, 0, \cdots, 0)$ and $(-1, 0, \cdots, 0)$. In fact the following theorems show that the hyperbolicity in the direction of $(1, 0, \cdots, 0)$ does not imply that of $(-1, 0, \cdots, 0)$ in general even when $n=1$, except for the case of usual partial differential operators.

THEOREM 6.1.2. *Suppose that a local operator $S*$ is hyperbolic in the direction of $(1, 0, \cdots, 0)$. Then $J(\zeta)$, the Fourier transform of S, satisfies the following condition* (H_1).

(H_1) *There exists some positive c for which the condition* (I) *implies condition* (II).

Condition (I): $J(\zeta)=0$ *and* $|\operatorname{Im}\zeta|<c\operatorname{Im}\zeta_1$.

Condition (II): *For any positive ε we can find some C_ε for which the inequality* $\operatorname{Im}\zeta_1\leqq\varepsilon|\operatorname{Re}\zeta|+C_\varepsilon$ *holds.*

REMARK 1. We use $J(\zeta)=\langle S, e^{i\langle z,\zeta\rangle}\rangle$ for the sake of simplicity, not $\langle S, e^{-i\langle z,\zeta\rangle}\rangle$.

REMARK 2. It is easy to check the uniqueness of the elementary solution E which satisfies the required properties. In fact if we assume $S*E_1=S*E_2=\delta$ and $\operatorname{supp}E_1$, $\operatorname{supp}E_2\subset\Gamma$ for some properly convex cone we have $E_1=E_1*\delta=E_1*(S*E_2)=(S*E_1)*E_2=\delta*E_2=E_2$.

PROOF. By the assumption we can find some E which is an element of $\mathscr{B}(\boldsymbol{R}^n)$ with its support in some convex cone Γ which satisfies the conditions in Definition 6.1.1. Here we use the theory of Fourier hyperfunction. Since we have proved that $\{\mathscr{R}(\Omega)\}$ constitutes a flabby sheaf over $\boldsymbol{D}^n$(Corollary (3.2.3) and by the definition $\mathscr{R}(\Omega)=\mathscr{B}(\Omega)$ if $\Omega\subset\boldsymbol{R}^n$, we find some $\tilde{E}$ which equals to E on $\boldsymbol{R}^n$ and belongs to $\mathscr{R}(\boldsymbol{D}^n)$ with its support in K, which is the closure of Γ in $\boldsymbol{D}^n$. This means $S*\tilde{E}=\delta+\mu$ where $\operatorname{supp}\mu\subset K\cap S_\infty^{n-1}$. We define $F(\zeta)$ by $\langle\tilde{E}, e^{i\langle z,\zeta\rangle}\rangle$ and $M(\zeta)$ by $\langle\mu, e^{i\langle z,\zeta\rangle}\rangle$. Then Theorem 3.3.1 assures $F(\zeta)$ and $M(\zeta)$ are holomorphic in $\{\zeta\,|\,|\operatorname{Im}\zeta|<c'\operatorname{Im}\zeta_1\}$ for some positive c' and $J(\zeta)F(\zeta)=1+M(\zeta)$ there, since $\operatorname{supp}\tilde{E}$ and $\operatorname{supp}\mu\subset K$. Moreover $M(\zeta)$ satisfies following estimate with some $c<c'$. For any positive ε and K there exists some $A_{\varepsilon,K}$ for which the inequality $|M(\zeta)|\leqq A_{\varepsilon,K}e^{\varepsilon|\operatorname{Re}\zeta|-K\operatorname{Im}\zeta_1}$ holds in $\{\zeta\,|\,|\operatorname{Im}\zeta|<c\operatorname{Im}\zeta_1\}$. Fixing K to 1 we denote $A_{\varepsilon,1}$ by A_ε. Suppose ζ satisfy $J(\zeta)=0$ and $|\operatorname{Im}\zeta|<c\operatorname{Im}\zeta_1$ then we have $-1=M(\zeta)$. Therefore we have $0=\log|M(\zeta)|$. From these relations we conclude $\log A_\varepsilon+\varepsilon|\operatorname{Re}\zeta|-\operatorname{Im}\zeta_1\geqq0$. Thus we have $\operatorname{Im}\zeta_1\leqq C_\varepsilon+\varepsilon|\operatorname{Re}\zeta|$ for any ε. Therefore the condition (I) implies the condition (II).

REMARK. We have reduced the situation to "infinity" to obtain the condition (H_1) in the above proof. On the other hand one can also reduce the problem to the "origin" using only the flabbiness of $\mathscr{B}$ (not $\mathscr{R}$) after Gårding. In fact, if $S*$ is hyperbolic in the direction of $(1, 0, \cdots, 0)$, then we can find some hyperfunction F which coincides with E in $\{|x|<1\}$ and whose support is contained in $\Gamma\cap\{|x|\leqq 1\}$. Then we have $S*F=\delta+\nu$ where $\operatorname{supp}\nu\subset\Gamma\cap\{|x|=1\}$. Then the usual Fourier transformation of a hyperfunction with compact support in $\boldsymbol{R}^n$ gives the condition (H_1), as is easily checked. We omit the details.

But we preferred a little more sophisticated proof, which we gave above, because it seems to give the interpretation for the nature of hyperbolicity, that is, the essential part is due to the condition about the support of the elementary solution E and some additional condition must appear because we do not impose any growth conditions on E at infinity.

Now we prove the converse of the above theorem by constructing an elementary solution with the required properties under the condition (H_1).

THEOREM 6.1.3. *Suppose a local operator* $S*$ *satisfies the condition* (H_1). *Then* $S*$ *is hyperbolic in the direction of* $(1, 0, \cdots, 0)$.

PROOF. What we want to do is to construct an elementary solution of $S*$ with the properties required in Definition 6.1.1.

We first treat the case $n=1$, because the essential part of our procedure of constructing the elementary solution is shown most clearly in that case. (In fact, as for the analytical aspect, nothing is needed even in the higher dimensional case except for those used in the case of $n=1$.)

Now the condition (H_1) and Lemma 4.2.3 assure that the estimate $|J(\zeta)|\geqq A_\theta e^{-\theta|\zeta|}$ $(\forall\theta>0)$ holds in $\{\zeta|\operatorname{Im}\zeta\geqq\varepsilon|\operatorname{Re}\zeta|+C_\varepsilon\}$. We can assume C_ε is larger than some fixed C as far as ε is smaller than 1. We define $E_\varepsilon^+(z)$ by $\int_{\Gamma_\varepsilon^+} e^{-iz\zeta}/J(\zeta)d\zeta$ when z belongs $\Omega_{\varepsilon,\delta}$, where $\Omega_{\varepsilon,\delta}$ is $\{z|\operatorname{Im} z<-\varepsilon\operatorname{Re} z-\delta\}$ and Γ_ε^+ is $\{(0+i\eta)|C\leqq\theta\leqq C_\varepsilon\}\cup\{(\xi+i(\varepsilon\xi+C_\varepsilon))|\xi\geqq 0\}$. (See the figures below.)

Figure I

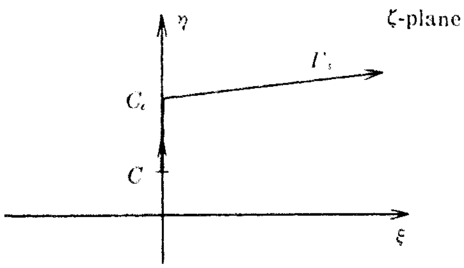
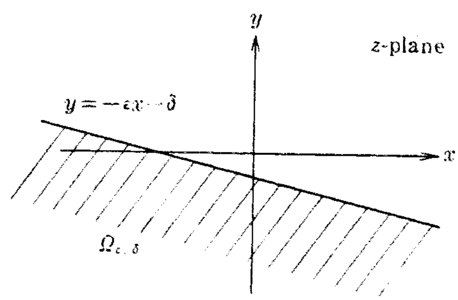


Figure II



On the other hand the Cauchy theorem asserts $E_\varepsilon^+(z)=E_{\varepsilon'}^+(z)$ if z belongs to $\Omega_{\varepsilon,\delta}\cap\Omega_{\varepsilon',\delta'}$. Thus we have obtained $E^+(z)$ which is holomorphic in $\{\mathrm{Im}\,z<0\}$ by the analytic continuation. Just in the same way we can obtain $E^-(z)$ which is holomorphic in $\{\mathrm{Im}\,z>0\}$ using $\int_{\Gamma_\varepsilon^-} e^{-iz\zeta}/J(\zeta)d\zeta$ where Γ_ε^- is $\{(0+i\eta)|C\leqq\eta\leqq C_\varepsilon\}\cup\{(\xi-i(\varepsilon\xi+C_\varepsilon)|\xi\leqq 0\}$.

Now we assert that the hyperfunction $E(x)$, which the pair of holomorphic functions $(-E_-(z), E_+(z))$ define, is the required elementary solution (except for the multiplicative constant (-2π)). (We remark here the isomorphisms $\mathscr{B}(\boldsymbol{R})\cong H_{\boldsymbol{R}}^1(\boldsymbol{C},\mathscr{O})\cong\mathscr{O}(\boldsymbol{C}-\boldsymbol{R})/\mathscr{O}(\boldsymbol{C})$ hold by the definition.) Since $\mathrm{supp}\,S=\{0\}$, $S*$ operates on $\mathscr{O}(\Omega_{\varepsilon,\delta})$ and we have

$$(S*E)(w)=\int_{\Gamma_\varepsilon^+} e^{-iw\zeta}\langle S, e^{iz\zeta}\rangle/J(\zeta)d\zeta$$
$$=\int_{\Gamma_\varepsilon^+} e^{-iw\zeta}d\zeta=i\int_C^{C_\varepsilon} e^{wt}dt+(1+i\varepsilon)\int_0^\infty e^{-iw(1+i\varepsilon)\xi+iC_\varepsilon)}d\xi=e^{Cw}/iw$$

in $\Omega_{\varepsilon,\delta}$. Therefore we have $(S*E^+)(w)=e^{Cw}/iw$ in $\{w|\mathrm{Im}\,w<0\}$. Just in the same way we have $(S*E^-)(w)=-e^{Cw}/iw$ in $\{w|\mathrm{Im}\,w<0\}$. Therefore we have proved $S*E=-2\pi\delta$.

Next we want to prove $\mathrm{supp}\,E\subset\{x\geq 0\}$. To prove it is suffices to prove $E_\varepsilon^+(x)=-E_\varepsilon^-(x)$ in $\{x\leqq a<0\}$ fixing arbitrarily. For that purpose it is sufficient to prove

$$\left|\int_{-\xi_0}^{\xi_0} e^{-ix(\xi+i(C_\varepsilon+\varepsilon\xi_0))}\Big/J(\xi+i(C_\varepsilon+\varepsilon\xi_0))d\xi\right|$$

tends to zero as ξ_0 tends to infinity by Cauchy's integral formula. Now the estimate of $|J(\zeta)|$ from below gives us the following estimates:

$$\left|\int_{-\xi_0}^{\xi_0} e^{-ix(\xi+i(C_\varepsilon+\varepsilon\xi_0))}\Big/(\xi+i(C_\varepsilon+\varepsilon\xi_0))d\xi\right|\leqq\int_{-\xi_0}^{\xi_0} e^{(C_\varepsilon+\varepsilon\xi_0)x}/|J(\xi+i(C_\varepsilon+\varepsilon\xi_0))|d\xi$$
$$\leqq A_\theta e^{(C_\varepsilon+\varepsilon\xi_0)a}\int_{-\xi_0}^{\xi_0} e^{\theta|\xi|}d\xi=2A_\theta e^{(C_\varepsilon+\varepsilon\xi_0)a}\times(e^{\theta\xi_0}-1/\theta)$$

for any $\theta>0$. Then fixing θ so that $\theta<-a\varepsilon$, we can conclude

$$\left|\int_{-\xi_0}^{\xi_0} e^{-ix(\xi+i(C_\varepsilon+\varepsilon\xi_0))}\Big/J(\xi+i(C_\varepsilon+\varepsilon\xi_0))d\xi\right|$$

tends to zero as ξ_0 tends to infinity. This means $\mathrm{supp}\,E\subset\{a\leqq x\}$. Since a in an arbitrary negative number we conclude $\mathrm{supp}\,E$ is contained in $\{0\leqq x\}$. Thus we have proved the theorem in the case $n=1$. Even if $n\geqq 2$, we can go just in the same way since the theorem follows by the convexity of Γ if we only prove it assuming Γ is a direct product of half lines. But we give here a little more sophisticated proof. (Essentially the same.)

The proof is as follows: we want to obtain some element $E(x)$ of $H^n_\Gamma(C^n, \mathcal{O}) \cong H^{n-1}(C^n-\Gamma, \mathcal{O})$, which satisfies $S*E=\delta$. For that purpose we represent $H^{n-1}(C^n-\Gamma, \mathcal{O})$ by some Čech cohomology of covering, that is $H^{n-1}(\mathcal{U}, \mathcal{O})$ where $\mathcal{U}=\{U_\alpha\}$ denotes some family of convex sets determined below for which $\bigcup_\alpha U_\alpha=C^n-\Gamma$ holds. (Of course we can treat the problem representing $H^n_\Gamma(C^n, \mathcal{O})$ by the Čech cohomology of covering. In fact it is just the same thing except for some change of indices.) Now we define U_α as follows. For every τ that belongs to $\{\tau\in C^n\,|\,|\tau|=1$ and $\operatorname{Im}\tau_1>c|\operatorname{Im}\tau|\}$ we choose some κ_τ so that $J(t\tau+\kappa_\tau(\sqrt{-1}, 0, \cdots, 0))\neq 0$ $(t>0)$. We define $U_{\tau,\delta}=\{z\in C^n\,|\operatorname{Im}\langle z, \tau\rangle<-\delta\}$ for positive number δ. We consider (τ, δ) as the index and take $\{U_{\tau,\delta}\}$ as $\mathcal{U}$. By the above condition on τ we can find some Γ for which $\bigcup U_{\tau,\delta}=C^n-\Gamma$ holds. (In fact we can take the dual cone of the base of the tubular domain where $\{t\tau\}$ runs, that is $\{\eta\in R^n\,|\,\eta_1>c|\eta|\}$ in this case. This remark gives the precise information about the convex hull of the support of the elementary solution which we are constructing.) Then for any $(\tau_1, \cdots, \tau_n)$ we assign the integral

$$\int\cdots\int_{K_{\tau_1,\cdots,\tau_n}} e^{-i\langle z,\zeta\rangle}/J(\zeta)d\zeta_1\cdots d\zeta_n$$

where the $K_{\tau_1,\ldots,\tau_n}$ is some (real) n-dimensional cell whose essential part is the (real) n-dimensional cone which is generated by $(\tau_1, \cdots, \tau_n)$ with its vertex at $(\sqrt{-1}\kappa_{\tau_j}, 0, \cdots, 0)$. (We choose j so that $\max_{1\leq k\leq n}\kappa_{\tau_k}=\kappa_{\tau_j}$.) The cell $K_{\tau_1,\ldots,\tau_n}$ is given as follows. For the sake of simplicity we assume $n=2$, and $\kappa_{\tau_1}\geqq\kappa_{\tau_2}$. (We only repeat the procedure in the higher dimensional case.) Then consider the following two cells: K_1 is $\{\zeta\,|\,\zeta=(a\sqrt{-1}, 0)+t\tau_1$, where $\kappa_{\tau_1}\leqq a_2\leqq\kappa_{\tau_2}, t\leqq 0\}$ and K_2 is $\{\zeta\,|\,\zeta=(\kappa_{\tau_2}\sqrt{-1}, 0)+t\tau_1+s\tau_2$, where $t, s\geqq 0\}$. Denoting the above integral by $\varphi_{\tau_1,\ldots,\tau_n}(z)$, which is defined in $\bigcap_{j=1}^n U_{\tau_j,\delta_j}$, a suitable choice of orientation of $K_{\tau_1,\ldots,\tau_n}$ gives a cocycle $\{\varphi_{\tau_1,\ldots,\tau_n}\}$ in $H^{n-1}(\mathcal{U}, \mathcal{O})$ by the estimate of $|J(\zeta)|$ from below, which we have remarked at the beginning of the proof. Therefore if we define $E(x)$ by the element defined by $\varphi_{\tau_1,\ldots,\tau_n}$ we obtain the required elementary solution, since $\operatorname{supp}E(x)$ is contained in Γ by the definition. (The proof of $S*E=(2\pi)^n\delta$ is just the same as in the case of $n=1$.) Thus the proof is complete.

6.2. Hyperbolicity for general convolution operators

We treat the problem of hyperbolicity for the convolution operator S, not assuming $\operatorname{supp} S=\{0\}$. At first we treat the case $\operatorname{supp} S$ is compact. To treat this problem we must modify Definition 6.1.1 a little, that is,

DEFINITION 6.2.1. Let S be a hyperfunction with compact support. Then we say the operator $S*$ is *hyperbolic in the direction of* $(1, 0, \cdots, 0)$ if and only if $S*$ has some elementary solution E whose support is contained in some closed and properly convex cone Γ_a with its vertex at $a(1, 0, \cdots, 0)$ and $\Gamma_a - a(1, 0, \cdots, 0) \Subset \{x_1 \geqq -\delta\}$.

Under the above definition of (generalized) hyperbolicity we easily conclude the condition (H_1) (see Theorem 6.1.2) is necessary for the hyperbolicity of $S*$. (In fact just the same proof holds in this case.) But only the condition (H_1) cannot assure the existence of elementary solutions with the required properties since $|1/J(\zeta)|$ may become too large at finity. The proof of Theorem 6.1.2 indicates us that we should impose some growth conditions on $|1/J(\zeta)|$ at infinity to obtain an elementary solution E. (As is shown later, the estimate is automatically satisfied for local operators and this makes the situation so simple when $S*$ is a lacal operator.) The growth condition required is as follows.

THEOREM 6.2.2. *Suppose* $S*$ *is hyperbolic in the direction of* $(1, 0, \cdots, 0)$. *Then we have the following condition* (EH_1):

(EH_1) *There exist some constants* c *and* a *for which the following estimate* (E) *holds in* $\{\zeta \,|\, |\operatorname{Im} \zeta| < c \operatorname{Im} \zeta_1\}$: (E) *For any positive* ε *we can find some* C_ε *such that for any positive* θ *the estimate* $|J(\zeta)| \geqq A_\theta e^{-\theta|\operatorname{Re}\zeta| - a \operatorname{Im} \zeta_1}$ *holds in* $\{\operatorname{Im} \zeta_1 > \varepsilon|\operatorname{Re} \zeta| + C_\varepsilon\}$.

PROOF. Using the notation of Theorem 6.1.2 we obtain $1+M(\zeta)=J(\zeta)F(\zeta)$ in $\{\zeta \,|\, |\operatorname{Im} \zeta| < c \operatorname{Im} \zeta_1\}$ from the condition on hyperbolicity. Moreover we have the following estimate in $\{\zeta \,|\, |\operatorname{Im} \zeta| < c' \operatorname{Im} \zeta_1\}$ for any $c' < c$: $|M(\zeta)| \leqq B_\varepsilon e^{\varepsilon|\operatorname{Re}\zeta| - K \operatorname{Im}\zeta_1}$ $(\forall \varepsilon > 0, K > 0)$ and $|F(\zeta)| \leqq B_{\theta'} e^{\theta|\operatorname{Re}\zeta| + a \operatorname{Im}\zeta_1}$ $(\forall \theta > 0)$. Therefore we can assume $|M(\zeta)| < 1/2$ in $\{\zeta \,|\, \operatorname{Im} \zeta_1 > \varepsilon|\operatorname{Re} \zeta| + C_\varepsilon\}$ for suitable C_ε. This means $|J(\zeta)| \geqq (1/2)|F(\zeta)|^{-1} \geqq C_\theta e^{-\theta|\operatorname{Re}\zeta| - a \operatorname{Im}\zeta_1}$ must hold there. Thus we obtain the condition (EH_1).

On the other hand it is easy to construct an elementary solution with the required property. In fact the proof of Theorem 6.1.3 holds without any essential change. (It is sufficient to make only two amendments. The first one is as for the definition of $\Omega_{\varepsilon,\delta}$ (Cf. 6.1): We must take $\{z \,|\, \operatorname{Im} z < \varepsilon(\operatorname{Re} z + a) - \delta\}$ there, and this is the reason why $\operatorname{supp} E$ must extend over some cone with its vertex translated from the origin. (As for $U_{\tau,\delta}$ (Cf. Theorem 6.1.3) the situation is the same.) The second is that $S*$ does not operate on $\mathcal{O}(\Omega_{\varepsilon,\delta})$ but it sends the element of $\mathcal{O}(\Omega_{\varepsilon,\delta})$ to that of $\mathcal{O}(\Omega_{\varepsilon,\delta} + \operatorname{supp} S)$. But it does not give any difficulties since $\operatorname{supp} S$ is compact.) Thus we have the following theorem:

THEOREM 6.2.3. *If the convolution operator $S*$ satisfies the condition* (EH_1), *then $S*$ is hyperbolic in the direction of* $(1, 0, \cdots, 0)$.

But we are not psychologically satisfied completely with the condition (EH_1) though it is complete from the logical view point, so we give another sufficient condition for the hyperbolicity using the very deep result of Ehrenpreis, that is the minimum modulus theorem. (See Ehrenpreis [3] p. 317, Theorem 5.)

THEOREM 6.2.4. *Suppose that the convolution operator $S*$ satisfies the condition* (S) *as well as* (H_1) *then the condition* (EH_1) *holds with some a. (See* 4.2 *about the definition of condition* (S). *See also the remarks given before Theorem* 4.2.4.)

REMARK. Since condition (S) is always satisfied for a distribution with compact support, only the condition (H_1) assures the hyperbolicity of $S*$ if S is a distribution with compact support. This fact seems to deserve to be remarked (Cf. Ehrenpreis [6], Gårding [7].)

PROOF OF THEOREM 6.2.4. We only give the case $n=1$, since the estimate in the higher dimensional case is a routine repetition of that procedure. Using the condition (S) for any positive ε and any ζ with $\operatorname{Im}\zeta>0$ we can find some ζ_1 with $|\zeta_1-\operatorname{Re}\zeta|<\theta|\operatorname{Re}\zeta|$ and $|J(\zeta_1)|\geqq C_\theta e^{-\theta|\operatorname{Re}\zeta|}$. On the other hand the minimum modulus theorem assures the existence of some ζ_2 with $\operatorname{Im}\zeta<\operatorname{Im}\zeta_2<3\theta|\operatorname{Re}\zeta|+2\operatorname{Im}\zeta$ and $|J(\zeta_2)|\leqq K_{\theta'}\exp(-B\theta'(1+\theta)|\operatorname{Re}\zeta|-C\alpha(3\theta|\operatorname{Re}\zeta|+2\operatorname{Im}\zeta)-11\theta|\operatorname{Re}\zeta|)$ for any $\theta'>0$. (Here α is the exponential type of $J(\zeta)$ and B and C are some constants.) Thus applying Lemma 4.2.3 to the circle with radius $3(\operatorname{Im}\zeta+\theta|\operatorname{Re}\zeta|)$ and its center at ζ_2, we obtain the following estimate from the condition (H_1),

$$\begin{aligned}|J(\zeta)|&\geqq C_{\theta'}\{\exp(\theta'(3\operatorname{Im}\zeta+(1+3\theta)|\operatorname{Re}\zeta|+\alpha(5\operatorname{Im}\zeta+6\theta|\operatorname{Re}\zeta|)\}\\&\quad\times\{\exp(-B\theta'(1+\theta)|\operatorname{Re}\zeta|-C\alpha(3\theta\operatorname{Re}\zeta+2\operatorname{Im}\zeta)-11\theta|\operatorname{Re}\zeta|)\}^2\\&\geqq C_{\theta'}\exp((\beta\theta'+\gamma\theta)|\operatorname{Re}\zeta|-\delta\operatorname{Im}\zeta)\,.\end{aligned}$$

Here the constants β, γ and δ depend neither on θ nor on θ', we have the estimate (EH_1). Q.E.D.

Thus we have a complete characterization on the hyperbolicity as far as $\operatorname{supp}S$ is compact. However Prof. Gårding kindly suggested me that I should investigate the case where $\operatorname{supp}S$ is contained in some properly convex cone, as he had investigated using distributions in his unpublished paper [7]. The author expresses his sincere gratitude to Prof. Gårding for his kindness.

PROBLEM (Gårding). Let S be a hyperfunction with its support in some properly convex closed cone Γ for which $\Gamma\Subset\{x_1\geqq -c\}$ holds for some c. Investigate the conditions on S under which there exists some hyperfunction E with its support in another properly convex closed cone which is compactly contained in $\{x_1 \geqq -c'\}$ for some c' with respect to the topology of $\boldsymbol{D}^n$.

The following gives an answer to this problem.

Let $\tilde{S}$ be a Fourier hyperfunction which coincides with S on $\boldsymbol{R}^n$ and with its support in K, where K is the closure of Γ in $\boldsymbol{D}^n$. (We have used here the flabiness of $\mathscr{R}$.) If $S*$ has an elementary solution E indicated above, we have some F which belongs to $\mathscr{R}(\boldsymbol{D}^n)$ with its support in L, where L is a properly convex cone C_1 if restricted to $\boldsymbol{R}^n$, and $S*F=\delta+\nu$ holds, where $\operatorname{supp}\nu \subset L \cap S_\infty^{n-1}$. (Here we assume $L \supset K$ where L is the closure of C_1 without loss of generality.) Thus we have the condition (EH_1) again using Theorem 3.3.1. But we must check whether the above condition remains unchanged if we use another extension of S or not. In this case the answer is affirmative, that is for another choice of $\tilde{\tilde{S}}$ we have also the condition (EH_1). It is obvious from the result of Theorem 3.3.2 and the fact that $\operatorname{supp}(\tilde{S}-\tilde{\tilde{S}})$ is contained in $K \cap S_\infty^{n-1}$ and $\operatorname{supp} E$ is contained in properly convex cone in $\boldsymbol{R}^n$.

Thus we have the necessary condition for the "hyperbolicity" of S. Next we prove the sufficiency of the realization of (EH_1) for some (*a posteriori* for any) extension of S. In fact we can construct $E(x)$ just in the same way as in Theorem 6.2.3; only we remark that S cannot operate on $\mathcal{O}(\Omega_{\varepsilon,\delta})$, since $\operatorname{supp} S$ is not compact.

This fact makes it difficult to prove $S*E=(2\pi)^n\delta$ directly. To avoid this difficulty we use the following trick: We first decompose $\tilde{S}$ into the sum of S_m and T_m where $\operatorname{supp} S_m$ is contained in $\{x \in \boldsymbol{R}^n | x_1 \leqq m\}$, and denote the restriction of T_m to $\boldsymbol{R}^n$ by T_m^R. Obviously $S=S_m+T_m^R$. The conditions on the $\operatorname{supp} S$ and $\operatorname{supp} E$ make T_m*E well-defined. Now we fix some domain $\Omega_c=\{x \in \boldsymbol{R}^n | x_1<c\}$ and prove $S*E=(2\pi)^n\delta$ holds there. Since c is an arbitrary number we can obviously conclude that $S*E=(2\pi)^n\delta$ holds on $\boldsymbol{R}^n$ from this fact. Remark first it is clear that $T_m^R*E=0$ in Ω_c for sufficiently large m. Now let $J_m(\zeta)$ be the Fourier transform of S_m: Then S_m*E can be represented by the cochain defined by

$$\left\{\int_{K_{\tau_1,\cdots,\tau_n}} J_m(\zeta)e^{-i\langle z,\zeta\rangle}/J(\zeta)\, d\zeta_1 \cdots d\zeta_n\right\}.$$

(See the proof of Theorem 6.1.3. We must use the covering translated according to the size of $\operatorname{supp} S_m$ as in the proof of Theorem 6.2.2.) Just in the same way $S_m*E-(2\pi)^n\delta$ can be represented by the cochain

$$\left\{\int_{K_{\tau_1,\cdots,\tau_n}} (J_m(\zeta)-J(\zeta))e^{-i\langle z,\zeta\rangle}/J(\zeta)\, d\zeta_1 \cdots d\zeta_n\right\}.$$

On the other hand $\mathrm{supp}(\tilde{S}-S_m)=\mathrm{supp}\,T_m$ is contained in $\{x|x_1\geqq m\}$. Therefore we have the estimate $|J_m(\zeta)-J(\zeta)|\leqq e^{-(m+a)\mathrm{Im}\,\zeta_1}$ in $\{\zeta|\mathrm{Im}\,\zeta_1\geqq\varepsilon|\mathrm{Re}\,\zeta|+C_\varepsilon$ and $|\mathrm{Im}\,\zeta|<c\,\mathrm{Im}\,\zeta_1\}$, where a is a fixed constant determined by S only. This estimate proves as usual that $\mathrm{supp}(S_m*E-(2\pi)^n\delta)$ is contained in $\{x_1\geqq m+a\}$. Therefore we have $S_m*E=(2\pi)^n\delta$ in Ω_c for sufficiently large m. Thus we have $S*E=(S_m+T_m^R)*E=S_m*E+T_m^R*E=(2\pi)^n\delta$ in Ω_c for sufficiently large m. This ends the proof.

We summarize the above statements as a theorem.

THEOREM 6.2.5. *Let S be a hyperfunction with its support in some properly convex closed cone Γ. Then the operator $S*$ is "hyperbolic" in the sense given in the problem of Gårding, if and only if some (a posteriori any) extension of S to D^n with its support in K satisfies the condition* (EH_1). (*Here K denotes the closure of Γ in D^n.*)

REMARK. We can treat the above problem by reducing it to Theorem 6.2.3. It is the idea of Prof. Gårding [7]. In fact if we find some hyperfunction F with its support in some properly convex closed cone satisfying $S_m*F=\delta$, then $\mathrm{supp}\,T_m^R*F$ does not contain 0 for sufficiently large m, so there exists some G with $(\delta+T_m^R*F)*G=\delta$. Therefore $F*G$ satisfies the required properties. See the paper [7] for details.

6.3. Remarks on hyperbolic polynomials

In this section we analyze the condition (H_1) obtained in 6.1, when $S*$ is a usual partial differential operator.

THEOREM 6.3.1. *$P(D)$ is hyperbolic in the direction of* $(1, 0, \cdots, 0)$ *if and only if $P_m(D)$ is hyperbolic in the sense of Gårding with respect to* $(1, 0, \cdots, 0)$ *where $P_m(D)$ is the principal part of $P(D)$.*

REMARK. Those operators which are hyperbolic in the sense of Gårding (see for example Hörmander [13] Ch. 5.) are hyperbolic in the sense of our definition, so "strong hyperbolicity" is equivalent to "hyperbolicity" in our formulation.

Prof. Schapira informed me that he had recently obtained this result also (Schapira [40bis]).

PROOF. We first prove that the hyperbolicity of $P(D)$ implies that of $P_m(D)$. We begin the proof by showing $P_m(1, 0, \cdots, 0)\neq 0$. We use the technics used in Hörmander [13] Ch. 5 and Larsson [26] Theorem 9, that is the Puiseux expansion. Assume $P_m(1, 0, \cdots, 0)=0$. Of course we can choose real numbers α_j $(j=2, \cdots, n)$ which satisfy $P_m(1, \alpha_2, \cdots, \alpha_n)=0$. Let $Q(\lambda, \mu)$ be $P(\lambda, \lambda\mu\alpha_2, \cdots, \lambda\mu\alpha_n)$ and we rearrange $Q(\lambda, \mu)$ as $\sum_{\nu=0}^{m}\lambda^\nu R_\nu(\mu)$. Since $R_m(\mu)=P_m(1, \mu\alpha_2, \cdots, \mu\alpha_m)\not\equiv 0$, we can write $Q(\lambda, \mu)=R_m(\mu)\prod_{j=1}^{m}(\lambda-\lambda_j(\mu))$ where $\lambda_j(\mu)$ can be expanded as $\sum_{k\geq N_j} a_k^j(\mu^{1/p})^k$ in $0<|\mu|<\delta$ for some δ. (See the appendix of Hörmander [13].) Now condition (H_1) assures the existence of ν_0 for which $R_{\nu_0}(0)\neq 0$. In fact if all $R_\nu(0)$ should reduce to zero, $P(\lambda, 0, \cdots, 0)=Q(\lambda, 0)$ would become zero by the definition. Taking $\lambda=\sqrt{-1}\eta$, where η is positive, we would have $P(\sqrt{-1}\eta, 0, \cdots, 0)\equiv 0$. Obviously it contradicts the condition (H_1). Thus we have some ν_0 for which $R_{\nu_0}(0)\neq 0$ holds. On the other hand the assumption $P_m(1, 0, \cdots, 0)=0$ implies $R_m(0)=0$. Therefore if we choose a sequence $\{\mu_l\}$ which converges to 0 with $R_m(\mu_l)\neq 0$ then we would have $|R_{\nu_0}(\mu_l)/R_m(\mu_l)|$ tends to infinity. Since $R_{\nu_0}(\mu_l)/R_m(\mu_l)$ can be represented by the fundamental symmetric function of $\{\lambda_j(\mu_l)\}$, we should have some j_0 for which $|\lambda_{j_0}(\mu_l)|$ tends to infinity. Considering the Puiseux expansion of $\lambda_{j_0}(\mu)$, we have $\lambda_{j_0}(\mu)\sim a_{N_0}(\mu^{1/p})^{-N_0}$ with N_0 positive. Assume $\mathrm{Im}\,a_{N_0}>0$. Then the condition (H_1) gives $(\mathrm{Im}\,a_{N_0})\mu^{-N_0/p}<\varepsilon\mu^{1-N/p_0}+C_\varepsilon$ for sufficiently small μ. From this we have $\mathrm{Im}\,a_{N_0}<\varepsilon\mu+C_\varepsilon\mu^{N_0/p}$. This is a contradiction.

Even when $\mathrm{Im}\,a_{N_0}=0$ or $\mathrm{Im}\,a_{N_0}<0$, the same method of arguments gives a contradiction. (When $\mathrm{Im}\,a_{N_0}=0$, then we take the element of the type $e^{(N_0/2p)\pi\sqrt{-1}}\mu$ $(\mu>0)$, and when $\mathrm{Im}\,a_{N_0}<0$, we use $e^{(N_0/p)\pi\sqrt{-1}}\mu$ $(\mu>0)$. Thus we conclude $P_m(1, 0, \cdots, 0)\neq 0$.

Next we prove $P_m(\xi+i\tau(1, 0, \cdots, 0))=0$ $(\xi\in \boldsymbol{R}^n)$ implies $\mathrm{Re}\,\tau=0$. By the homogeneity of P_m we can assume $\mathrm{Re}\,\tau\geqq 0$. On the other hand we have $0=P_m(\xi+i\tau(1, 0, \cdots, 0))=\lim_{\sigma\to+\infty}\sigma^{-m}P(\sigma\xi+i\sigma\tau(1, 0, \cdots, 0))$. Since $P_m(1, 0. \cdots, 0)\neq 0$, as we have proved above, the zeros τ of $\sigma^{-m}P(\sigma\xi+i\sigma\tau(1, 0, \cdots, 0))$ depend continuously on σ^{-1} for sufficiently large σ. Therefore the condition (H_1) implies $\mathrm{Re}\,\tau\leqq\varepsilon$ for any positive ε. This means $\mathrm{Re}\,\tau=0$. Thus we have proved that the hyperbolicity of $P(D)$ implies that of $P_m(D)$.

REMARK. To prove that $P_m(1, 0, \cdots, 0)\neq 0$ we can also use the null-solution of Hörmander [13]. In fact if $P_m(1, 0, \cdots, 0)=0$ then there exists some u with its support in $\{x_1\geqq 0\}$ ([13] Theorem 5.3.2.) If $P(D)$ is hyperbolic with respect to $(1, 0, \cdots, 0)$ then $u=(P(D)E)*u=P(D)u*E=0$; this is a contradiction.

Now we prove the converse. Using the results which are obtained in the framework of the generalized distributions (see Larsson [26] and Schapira [39]) we can prove this indirectly, but we prefer to prove directly, since it is much easier. In fact the next lemma gives a proof when combined with Theorem 6.1.3.

LEMMA 6.3.2. *Let an m-th order homogeneous differential operator* $P_m(D)$ *be hyperbolic with respect to* $(1, 0, \cdots, 0)$, *and* $Q(D)$ *be a differential operator of order at most* $(m-1)$. *We set* $P(D)=P_m(D)+Q(D)$. *Then* $P(\xi+i\tau N)=0$ $(\xi\in\boldsymbol{R}^n)$

implies $|\mathrm{Re}\,\tau|\leqq C_N(|\xi|+1)^{1-1/m}$ *where* N *is sufficiently near to* $(1, 0, \cdots, 0)$ *and* C_N *depends continuously on* N.

The proof of this lemma can be obtained just in the same way as in Hörmander [13] p. 148, Lemma 5.7.3, so we omit the details.

Thus we have completed the proof of Theorem 6.3.1.

EXAMPLE. Theorem 6.3.1 states that the operator $\partial^2/\partial x_1^2+\partial/\partial x_2$ is hyperbolic in the direction of $(1, 0)$, though it is not so in the sense of Gårding, namely in the sense of distribution. Here we give an expilcit form of its elementary solution with its support in $\{(x_1, x_2)\,|\,x_1\geqq 0$ and $x_2=0\}$. We represent $H^2_{\{z_1\geqq 0\}\times\{z_2=0\}}(C^2, \mathcal{O})$ by the Čech cohomology of covering $H^2(\mathcal{U}, \mathcal{U}', \mathcal{O})$ where $\mathcal{U}=\{U_j\}_{j=0}^2$, $\mathcal{U}'=\{U_1, U_2\}$ are given by $U_0=C_2$, $U_1=\{(z_1, z_2)\in C^2|z_1\geqq 0\}$ and $U_2=\{(z_1, z_2)\in C^2| \neq 0\}$. We define $G(z_1, z_2)=-1/4\pi^2\sum_{k=0}^{\infty}(k!/(2k+1)!)\ z_1^{2k+1}/z_2^{k+1}\,\mathrm{Log}\,z_1$, where $\mathrm{Log}\,z_1$ is the principal value of $\log z_1$ and uniform and holomorphic in $\{z_1\in C|z_1\geqq 0\}$, then it is easy to check that $G(z_1, z_2)$ is well-defined and holomorphic in $U_1\cap U_2$. We define the cohomology class defined by $G(z_1, z_2)$ as $E(x_1, x_2)$, then we easily prove $(\partial^2/\partial x_1^2+\partial/\partial x_2)E(x_1, x_2)=\delta(x_1, x_2)$. (The proof of these facts needs only calculations, so we omit the details.) We also express the above series $\sum_{k=0}^{\infty}(k!/(2k+1)!)z_1^{2k+1}/z_2^{k+1}=S$ by the following integral (see Hitotumatu et al [11] p. 58, 3°):

$$S=(1/2)z_1+(1/\sqrt{z_2})(1+w^2/2)\exp(w^2/4)\int_0^{w/2}e^{-s^2}ds$$

where $w=z_1/\sqrt{z_2}$. (The right side is uniform and holomorphic in $\{(z_1, z_2)z_2\neq 0\}$.) It seems a little worthwhile to remark that the elementary solution of $(\partial^2/\partial x_1^2+\partial/\partial x_2)$, whose support is just $\{x_1\geqq 0\}\times\{x_2=0\}$ is given by the "boundary values" of a rather simple function given above, so we spend half a page on this example.

REMARK. In the proof of Theorem 6.3.1 we obtained $P_m(1, 0, \cdots, 0)\neq 0$ if $P(D)$ is hyperbolic in the direction of $(1, 0, \cdots, 0)$. In view of Theorem 4.1.8 or its corollary this proves every hyperfunction u which satisfies $P(D)u=0$ depends real analytically on x_1 especially we can consider its specialization to $x_1=x_1^0$. This remark permits us to treat the initial value problem from a very general view point. From this view point we will treat in our forthcoming paper such problems as (i) Cauchy problems for hyperbolic partial differential operators with constant coefficients, (ii) the problem of non-admissible data (Cf. John [16]) (iii) Holmgren's uniqueness theorem, and (iv) generalized Cauchy problems.

In that paper we develop the theory of the Fourier-Borel transformation with parameters to solve (i), we use Theorem 4.1.8 to solve (ii) (without using the technics of regularization (Cf. Hörmander [13]); in fact it is impossible to regularize a hyperfunction in view of Theorem 4.2.3), to solve (iii) we use Sato's fundamental theorem about the regularity of the solutions of a partial differential operator with variable coefficients (Sato [36], [37], [38]), and to solve (iv) we use some duality theorem analogous to that of Bengel [1] using the technics of Komatsu [25]. (See Kawai [20], [20$^{\text{bis}}$] and Komatsu and Kawai [25$^{\text{bis}}$].

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