Ideal boundaries of Neumann type associated with elliptic differential operators of second order—II

—Imbedding of the smooth boundary into the ideal boundary—

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Introduction. In the previous paper: "Ideal boundaries of Neumann type associated with elliptic differential operators of second order" in this Volume, pp. 167—186, the author constructed a theory of ideal boundaries of Neumann type associated with the adjoint operator $A^*: A^*u=\operatorname{div}(\nabla u-\boldsymbol{b}u)$, of the elliptic differential operator A of the form:

$$Au(x)=\operatorname{div}[\nabla u(x)]+(\boldsymbol{b}(x)\cdot\nabla u(x))$$
$$=\sum_{i,j}\frac{1}{\sqrt{a(x)}}\frac{\partial}{\partial x^i}\left\{\sqrt{a(x)}a^{ij}(x)\frac{\partial u(x)}{\partial x^j}\right\}+\sum_i b^i(x)\frac{\partial u(x)}{\partial x^i}$$

in a non-compact orientable C^∞-manifold R. In the present paper, as a continuation of the previous paper, we shall establish a theorem for imbedding of the smooth boundary of R into the ideal boundary.

The previous paper mentioned above will be quoted as [IB]. Throughout the present paper, numbers in brackets [ ] refer to those in References at the end of [IB], and sections and theorems will be numbered as continuations of those in [IB].[1)]

§7. Statement of the imbedding theorem. Let R be a subdomain of an orientable C^∞-manifold $\boldsymbol{M}$ of dimension $m\geqq 2$, and assume that a part S of the boundary ∂R of the domain considered in $\boldsymbol{M}$ is an $(m-1)$-dimensional simple hypersurface of class C^3 and that each point $z\in S$ has its neighborhood U_z such that $U_z\cap\partial R=U_z\cap S$. Assume further that $\|a^{ij}\|$ and $\|b^i\|$ are of class C^2 on $R+S$ and that Assumption (A) in [IB] is satisfied for $\boldsymbol{b}=\|b^i\|$ restricted to R. Under these conditions, we shall prove the following two theorems.

THEOREM 7.1. *The hypersurface S is homeomorphically imbedded into the essential part $\hat{S}_1$ of the ideal boundary $\hat{S}$; more precisely, for any point $z\in S$, there corresponds a point $\xi_z\in\hat{S}_1$ in one-to-one way, and the mapping Φ defined by*

(7.1) $\qquad \Phi(x)=x \quad$ *for* $\quad x\in R \quad$ *and* $\quad \Phi(z)=\xi_z \quad$ *for* $\quad z\in S$

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1) Notations also refer to [IB] unless otherwise mentioned.

gives a homeomorphism of $R+S$ *as a subspace of the original manifold* $\boldsymbol{M}$ *onto* $R+\{\xi_z; z\in S\}$ *as a subspace of the compact metric space* $\hat{R}$.

THEOREM 7.2. *The kernel function* $N(x, y)$ *is extended to a continuous function on* $(R+S)\times(R+S)-\{(z, z); z\in R+S\}$, *and*

i) $N(z, y)$ *is an extremal* FH_0 *function of* y *for any* $z\in S$;

ii) $\dfrac{\partial N(z, y)}{\partial \boldsymbol{n}_R(z)}=0$ *for any* $y\in R+S$ *and* $z\in S$ *whenever* $y\neq z$;

iii) $\dfrac{\partial N(x, z)}{\partial \boldsymbol{n}_R(z)}-N(x, z)\beta_R(z)=0$ *for any* $x\in R+\hat{S}$ *and* $z\in S$ *whenever* $x\neq z$.

We shall extend the function $N(x, y)$ continuously up to the boundary points on S in §8, and prove the above theorems in §9.

§8. Extension of $N(x, y)$ up to the boundary points on S. For any regular domain Ω containing the compact set K_0 and relatively compact in $\boldsymbol{M}$, we set $\Omega'=\Omega-K_0$ as in [IB], and denote by $N^\Omega(x, y)$ the kernel function of the boundary value problem:

$$(8.1)\qquad Au=-f \ \text{ in } \ \Omega', \quad u\Big|_{\partial K_0}=\varphi_0, \quad \frac{\partial u}{\partial \boldsymbol{n}_\Omega}\Big|_{\partial\Omega}=\varphi_1,$$

which is also the the kernel function of the adjoint problem

$$(8.1^*)\qquad A^*v=-f \ \text{ in } \ \Omega', \quad v\Big|_{\partial K_0}=\varphi_0, \quad \left(\frac{\partial v}{\partial \boldsymbol{n}_\Omega}-\beta_\Omega v\right)\Big|_{\partial\Omega}=\varphi_1.$$

We denote by $G^\Omega(x, y)$, as in [7], the Green function of the elliptic boundary value problem:

$$(8.2)\qquad Au=-f \ \text{ in } \ \Omega', \quad u|_{\partial K_0}=\varphi_0, \quad u|_{\partial\Omega}=\varphi_1,$$

which is also the kernel function of the adjoint problem:

$$(8.2^*)\qquad A^*v=-f \ \text{ in } \ \Omega', \quad v|_{\partial K_0}=\varphi_0, \quad v|_{\partial\Omega}=\varphi_1.$$

As is shown in [7; Theorem 5.1], there exists a sequence $\{D_n\}$ of relatively compact regular subdomains of R such that

$$(8.3)\qquad K_0\subset D_0\subset D_1\subset\cdots\subset D_n\subset\cdots, \quad \bigcup_{n=1}^{\infty} D_n=R$$

and that

$$(8.4)\qquad \lim_{n\to\infty} N^{D_n}(x, y)=N(x, y)$$

uniformly on every compact subset of

$$(8.5)\qquad (R'+\partial K_0)\times(R'+\partial K_0)-\{(z, z); z\in R'+\partial K_0\} \quad (R'=R-K_0).$$

Hereafter we fix such a sequence $\{D_n\}$.

We define $N(x, y)=0$ if $x \neq y$ and at least one of x and y belongs to K_0. Then $N(x, y)$ is continuous on

$$R \times R-\{(z, z) ; z \in R\} . \tag{8.6}$$

Let S_1 be a relatively open subset of S such that $\bar{S}_1$ is compact,[2] and Ω be a regular domain in $\boldsymbol{M}$ satisfying that

$$\begin{cases} \bar{D}_0 \subset \Omega \subset R, \ \bar{\Omega} \text{ is compact and} \\ \partial\Omega \cap S \text{ contains } S_1 \text{ as its relatively compact subset.} \end{cases} \tag{8.7}$$

Define a function $\alpha(x)$ on $\partial\Omega'$ as follows:

$$\alpha(x)=\begin{cases} 1 & \text{on} \quad (\partial\Omega-S)+\partial K_0 \\ 0 & \text{on} \quad \partial\Omega-\overline{(\partial\Omega-S)} . \end{cases} \tag{8.8}$$

We first construct a function $\tilde{G}(x, y)$ which is a Green function of elliptic boundary value problem:

$$\begin{cases} Au=-f & \text{in} \quad \Omega' \\ \alpha u+(1-\alpha) \dfrac{\partial u}{\partial \boldsymbol{n}_\Omega}=\varphi & \text{on} \quad \partial\Omega' , \end{cases} \tag{8.9}$$

and also that of the adjoint problem:

$$\begin{cases} A^* v=-f & \text{in} \quad \Omega' \\ \alpha v+(1-\alpha)\left(\dfrac{\partial v}{\partial \boldsymbol{n}_\Omega}-\beta_\Omega v\right)=\varphi & \text{on} \quad \partial\Omega' . \end{cases} \tag{8.9*}$$

Since $\alpha(x)$ is not continuous on $\partial\Omega$, the existence of such function $\tilde{G}(x, y)$ is not contained in the result of [4], but $\tilde{G}(x, y)$ is constructed as follows.

Let $\alpha_n(z)$ be a function of class C^2 on $\partial\Omega'$ satisfying that

$$\alpha_n(z)=\begin{cases} 1 & \text{on} \quad (\partial\Omega \cap \bar{D}_n)+\partial K_0 \\ 0 & \text{on} \quad \partial\Omega-D_{n+1} , \end{cases} \tag{8.10}$$

and $G_n(x, y)$ be the Green function of the boundary value problem:

$$\begin{cases} Au=-f & \text{in} \quad \Omega' \\ \alpha_n u+(1-\alpha_n) \dfrac{\partial u}{\partial \boldsymbol{n}_\Omega}=\varphi & \text{on} \quad \partial\Omega' . \end{cases} \tag{8.11}$$

Then $G_n(x, y)$ is also the Green function of the adjoint problem:

[2] The upper bar $^-$ denotes the closure operation in $\boldsymbol{M}$.

$$(8.11^*)\qquad \begin{cases} A^*v=-f & \text{in } \Omega' \\ \alpha_n v+(1-\alpha_n)\left(\dfrac{\partial v}{\partial \boldsymbol{n}_\Omega}-\beta_\Omega v\right)=\varphi & \text{on } \partial\Omega' \end{cases}$$

as is shown in [4]. Since $\{\alpha_n(z)\,;\,n=1,2,\cdots\}$ is monotone increasing on $\partial\Omega'$ and $\lim_{n\to\infty}\alpha_n(z)=1$ on $\partial\Omega'-S$, we may show that

$$(8.12)\qquad \begin{cases} \{G_n(x,y)\} \text{ converges to a function } \tilde{G}(x,y) \text{ monotone} \\ \text{decreasingly on } \bar{\Omega}'\times\bar{\Omega}' \text{ whenever } x\neq y\,, \end{cases}$$

and

$$(8.13)\qquad \tilde{G}(x,y)=0 \quad \text{if } x\neq y \text{ and at least one of } x \text{ and } y\in(\partial\Omega-S)+\partial K_0\,.$$

By means of Green's formula, we may show that

$$(8.14)\quad G_n(x,y)=G^\Omega(x,y)+\int_{\partial\Omega-D_n}G_n(x,z)\left\{-\frac{\partial G^\Omega(z,y)}{\partial \boldsymbol{n}_\Omega(z)}\right\}dS(z)\quad (x,y\in\overline{\Omega'\cap D_n}\,;\,x\neq y)$$

and that

$$(8.15)\qquad G_n(x,y)=N^\Omega(x,y)-\int_{\partial\Omega-D_n}\left\{G_n(x,z)\beta_\Omega(z)-\frac{\partial G_n(x,z)}{\partial \boldsymbol{n}_\Omega(z)}\right\}N^\Omega(z,y)dS(z)$$
$$(x,y\in\overline{\Omega'\cap D_n}\,;\,x\neq y)\,.$$

Letting $n\to\infty$ in (8.14), we get

$$(8.16)\qquad \tilde{G}(x,y)=G^\Omega(x,y)+\int_{\partial\Omega\cap S}\tilde{G}(x,z)\left\{-\frac{\partial G^\Omega(z,y)}{\partial \boldsymbol{n}_\Omega(z)}\right\}dS(z)\quad (x,y\in\bar{\Omega}'-S\,;\,x\neq y)$$

by virtue of (8.12), and accordingly we may show that

$$(8.17)\qquad \lim_{n\to\infty}\frac{\partial G_n(x,y)}{\partial \boldsymbol{n}_\Omega(y)}=\frac{\partial\tilde{G}(x,y)}{\partial \boldsymbol{n}_\Omega(y)}\leqq 0 \quad \text{for} \quad x\in\Omega' \quad \text{and} \quad y\in\partial\Omega-S$$

and the convergence holds monotone increasingly. Hence, letting $n\to\infty$ in (8.15), we obtain

$$(8.18)\qquad \tilde{G}(x,y)=N^\Omega(x,y)+\int_{\partial\Omega-S}\frac{\partial\tilde{G}(x,z)}{\partial \boldsymbol{n}_\Omega(z)}N^\Omega(z,y)dS(z)\quad (x,y\in\bar{\Omega}'-S\,;\,x\neq y)$$

by virtue of (8.12), (8.13) and (8.17). We may see by means of the continuity of $G^\Omega(x,y)$, $N^\Omega(x,y)$ and $\tilde{G}(x,y)$ that (8.16) and (8.18) hold for any x and $y\in\bar{\Omega}$ whenever $x\neq y$. Similarly we may show that

$$(8.19)\qquad \tilde{G}(x,y)=G^\Omega(x,y)+\int_{\partial\Omega\cap S}\left\{-\frac{\partial G^\Omega(x,z)}{\partial \boldsymbol{n}_\Omega(z)}\right\}\tilde{G}(z,y)dS(z)\quad (x,y\in\bar{\Omega}\,;\,x\neq y)$$

and that

$$(8.20)\qquad \tilde{G}(x,y)=N^\Omega(x,y)+\int_{\partial\Omega-S}N^\Omega(x,z)\frac{\partial\tilde{G}(z,y)}{\partial \boldsymbol{n}_\Omega(z)}dS(z)\quad (x,y\in\bar{\Omega}\,;\,x\neq y)\,.$$

By virtue of the properties of $G^{\Omega}(x, y)$ and $N^{\Omega}(x, y)$, it follows from (8.16), (8.18), (8.19) and (8.20) that $\tilde{G}(x, y)$ is a Green function of the boundary value problem (8.9) and also that of the adjoint problem (8.9*). In particular we note that

$$\frac{\partial \tilde{G}(x, y)}{\partial \boldsymbol{n}_{\Omega}(x)}=0 \quad \text{for} \quad x \in \partial\Omega-\overline{(\partial\Omega-S)} \quad \text{and} \quad y \in \Omega' \tag{8.21}$$

and that

$$\frac{\partial \tilde{G}(x, y)}{\partial \boldsymbol{n}_{\Omega}(y)}-\tilde{G}(x, y)\beta_{\Omega}(y)=0 \quad \text{for} \quad x \in \bar{\Omega} \quad \text{and} \quad y \in \partial\Omega-\overline{(\partial\Omega-S)}\,. \tag{8.21*}$$

LEMMA 8.1. *$N(x, y)$ is extended to a continuous function on $(R+S_1)\times R-\{(z, z)\,;\, z \in R\}$, and we have*

$$N(x, y)=\tilde{G}(x, y)+\int_{\partial\Omega-S}\left\{-\frac{\partial \tilde{G}(x, z)}{\partial \boldsymbol{n}_{\Omega}(z)}\right\}N(z, y)dS(z) \tag{8.22}$$

whenever $x \in \Omega'+S_1+\partial K_0$, $y \in \Omega'+\partial K_0$ and $x \neq y$.

PROOF. Let F be an arbitrary compact subset of $\Omega'+\partial K_0$, Ω_0 be a subdomain of Ω such that $K_0 \cup F \subset \Omega_0 \subset \bar{\Omega}_0 \subset \Omega$ and n_0 be the smallest integer such that $D_{n_0} \supset \bar{\Omega}_0$. Then, by means of Green's formula, we may show that

$$N^{D_n}(x, y)=\tilde{G}(x, y)-\int_{\partial(\Omega\cap D_n)}\left\{\frac{\partial \tilde{G}(x, z)}{\partial \boldsymbol{n}_{\Omega\cap D_n}(z)}-\tilde{G}(x, z)\beta_{\Omega\cap D_n}(z)\right\}N^{D_n}(z, y)dS(z) \tag{8.23}$$

for $x, y \in \Omega \cap D_n'$, and

$$N^{D_n}(x, y)=-\int_{\partial\Omega_0}N^{D_n}(x, z)\frac{\partial G^{\Omega_0}(z, y)}{\partial \boldsymbol{n}_{\Omega_0}(z)}dS(z) \tag{8.24}$$

for $x \in \bar{D}_n-\bar{\Omega}_0$ and $y \in \Omega_0-(K_0)^{\circ}$ whenever $n \geqq n_0$. We have also that (see Lemma A in [7; Appendix])

$$\sup_{n \geqq n_0}\left\{\sup_{x \in \bar{D}_n-\bar{\Omega}_0}\int_{\partial\Omega_0}N^{D_n}(x, z)dS(z)\right\}<\infty\,. \tag{8.25}$$

Combining (8.24) with (8.25), we get

$$\sup_{n \geqq n_0}\left\{\sup_{x \in \bar{D}_n-\bar{\Omega}_0,\, y \in F}N^{D_n}(x, y)\right\}<\infty\,. \tag{8.26}$$

On the other hand, it follows from (8.21*) that

$$\lim_{n\to\infty}\int_{\partial D_n\cap\Omega}\left\{\frac{\partial \tilde{G}(x, z)}{\partial \boldsymbol{n}_{D_n}(z)}-\tilde{G}(x, z)\beta_{\Omega\cap D_n}(z)\right\}dS(z)=0 \tag{8.27}$$

for any $x=\Omega'+\partial K_0$. Letting $n\to\infty$ in (8.23) and using (8.13), (8.26) and (8.27), we obtain (8.22) for any $x, y \in \Omega'+\partial K_0$ $(x \neq y)$ since F in (8.26) is an arbitrary compact subset of $\Omega'+\partial K_0$. Since Ω is an arbitrary regular domain satisfying

(8.7), $N(x, y)$ is continuously extended onto $(R+S_1)\times R-\{(z, z); z\in R\}$.

Similarly we may prove that

LEMMA 8.1*. *$N(x, y)$ is extended to a continuous function on* $R\times(R+S_1)-\{(z, z), z\in R\}$, *and we have*

$$N(x, y)=\tilde{G}(x, y)+\int_{\partial\Omega-S} N(x, z)\left\{-\frac{\partial\tilde{G}(z, y)}{\partial n_\Omega(z)}\right\}dS(z) \tag{8.22*}$$

whenever $x\in\Omega'+\partial K_0$, $y\in\Omega'+S_1+\partial K_0$ *and* $x\neq y$.

Now we fix two domains Ω and Ω_1 such that $\bar{\Omega}_1\cap R\subset\Omega$, that $\partial\Omega\cap S$ contains $\partial\Omega_1\cap S$ as its relatively compact subset and that each of Ω and Ω_1 satisfies (8.7). Let $\tilde{G}(x, y)$ be a Green function of boundary value problems (8.9) and (8.9*) in Ω as mentioned above, and $\tilde{G}_1(x, y)$ be a Green function of boundary value problems of the same type in Ω_1. Then, by means of Green's formula and by (8.21), we may show that

$$N(z, y)=-\int_{\partial\Omega_1-S} N(z, z_1)\frac{\partial\tilde{G}_1(z_1, y)}{\partial n_{\Omega_1}(z_1)}dS(z_1) \tag{8.28}$$

whenever $z\in R-\bar{\Omega}_1$ and $y\in\Omega_1'+S_1+\partial K_0$. Substituting for $N(z, y)$ from (8.28) in (8.22), we get

$$N(x, y)=\tilde{G}(x, y)+\int_{\partial\Omega-S}\int_{\partial\Omega_1-S}\frac{\partial\tilde{G}(x, z)}{\partial n_\Omega(z)}N(z, z_1)\frac{\partial\tilde{G}_1(z_1, y)}{\partial n_{\Omega_1}(z_1)}dS(z_1)dS(z) \tag{8.29}$$

whenever $x, y\in\Omega_1'+S_1+\partial K_0$ and $x\neq y$. Hence $N(x, y)$ is extended to a continuous function on $(\Omega_1'+S_1)\times(\Omega_1'+S_1)-\{(z, z); z\in\Omega_1'+S_1\}$.

Combining this result with Lemmas 8.1 and 8.1*, we have

LEMMA 8.2. *$N(x, y)$ is extended to a continuous function on* $(R+S_1)\times(R+S_1)-\{(z, z); z\in R+S_1\}$.

From the regularity of $N(x, y)$ which follows from (8.29), we may show that

$$\sup_{x\in R-\bar{\Omega}}\int_{\partial\Omega-S} N(x, z)dS(z)<\infty \tag{8.30}$$

(this can be proved, for instance, by the same argument as the proof of Lemma A in [7; Appendix]). Replacing Ω_1 in (8.28) by Ω, we have

$$N(x, y)=\int_{\partial\Omega-S} N(x, z)\left\{-\frac{\partial\tilde{G}(z, y)}{\partial n_\Omega(z)}\right\}dS(z) \tag{8.31}$$

for any $x\in R-\bar{\Omega}$ and $y\in\Omega'+S_1+\partial K_0$. Combining (8.31) with (8.30), we obtain the following Lemma 8.3 since the domain Ω can be so chosen that $\bar{\Omega}$ contains any pre-assigned compact subset F of $R+S_1$.

LEMMA 8.3. *If F is a compact subset of $R+S_1$ and E is a subset of $R-F$ relatively closed in R, then $N(x, y)$ is bounded on $E\times F$.*

§ **9. Proof of imbedding theorem.** Under the assumption stated in § 7, we denote by $\mathrm{dis}(x, y)$ the Riemannian distance between the points x and y in $R+S$ defined by $\|a_{ij}(x)\|$. Let S_1 be a relatively open subset of S such as mentioned in the preceding §.

LEMMA 9.1. *For any $z\in S_1$, there corresponds one and only one point $\xi_z\in\hat{S}$ such that $\lim_{\nu\to\infty}\rho(x_\nu, \xi_z)=0$ holds for any sequence $\{x_\nu\}\subset R$ satisfying $\lim_{\nu\to\infty}\mathrm{dis}(x_\nu, z)=0$, and we have $N(\xi_z, y)=N(z, y)$ for any $y\in R$.*

PROOF. For any given $z\in S$, we may take a sequence $\{z_n\}\subset R$ such that $\lim_{n\to\infty}\mathrm{dis}(z_n, z)=0$. The sequence $\{z_n\}$ has no accumulating point in R with respect to the metric ρ, while $\hat{R}$ is compact with respect to ρ. Hence there exists a subsequence $\{z_{n_\nu}\}$ of the sequence $\{z_n\}$ and a point $\xi\in\hat{S}$ such that $\lim_{\nu\to\infty}\rho(z_{n_\nu}, \xi)=0$. Let $\{x_\nu\}$ be an arbitrary sequence in R satisfying $\lim_{\nu\to\infty}\mathrm{dis}(x_\nu, z)=0$. Then, by Lemma 8.2, we have

$$\lim_{\nu\to\infty}|N(x_\nu, y)-N(z_{n_\nu}, y)|=0 \quad \text{for any} \quad y\in D_0 .$$

Hence we obtain $\lim_{\nu\to\infty}\rho(x_\nu, z_{n_\nu})=0$ by means of the definition of the metric ρ (see (3.4) in [IB]), and accordingly we get $\lim_{\nu\to\infty}\rho(x_\nu, \xi)=0$. This result implies also that the point $\xi\in\hat{S}$ is uniquely determined by $z\in S$; so we may write $\xi=\xi_z$ and accordingly $N(\xi_z, y)=N(z, y)$ for any $y\in R$ by virtue of Lemma 8.2.

LEMMA 9.2. *If $\{y_n\}\subset R$ and $\lim_{n\to\infty}\mathrm{dis}(z, y_n)=0$, then $\lim_{n\to\infty}N(\xi_z, y_n)=\infty$.*

PROOF. It follows from (8.20) and (8.22) that $\lim_{n\to\infty}\mathrm{dis}(z, y_n)=0$ implies $\lim_{n\to\infty}N(z, y_n)=\infty$. Hence this lemma follows immediately from the preceding lemma.

LEMMA 9.3. *If $\{x_n\}\subset R$, $z\in S_1$ and $\lim_{n\to\infty}\rho(x_n, \xi_z)=0$, then $\lim_{n\to\infty}\mathrm{dis}(x_n, z)=0$.*

PROOF. Suppose that $\lim_{n\to\infty}\mathrm{dis}(x_n, z)=0$ does not hold. Then there exists a neighborhood $U(z)$ of z and a subsequence $\{x_{n_\nu}\}$ of the sequence $\{x_n\}$ such that $x_{n_\nu}\notin U(z)$ for any ν. Let $\{y_n\}$ be a sequence in $U(z)\cap R$ such that $\lim_{n\to\infty}\mathrm{dis}(y_n, z)=0$. Then, by virtue of Lemma 8.3, there exists a constant C such that

$$N(x_{n_\nu}, y_n)\leqq C \quad \text{for any} \quad \nu \quad \text{and} \quad n . \tag{9.1}$$

Since $\lim_{\nu\to\infty}\rho(x_{n_\nu}, \xi_z)=0$ and $N(x, y)$ is continuous in $x\in\hat{R}$ with respect to the metric ρ for any fixed y, it follows from (9.1) that

$$N(\xi_z, y_n)\leqq C \quad \text{for any} \quad n ,$$

which contradicts Lemma 9.2.

LEMMA 9.4. *For any point $z\in S_1$, there corresponds a point $\xi_z\in\hat{S}$ in one-to-one way, and the mapping Φ_1 defined by*

$$\tag{9.2}\begin{cases}\Phi_1(x)=x & \text{for } x\in R\\ \Phi_1(z)=\xi_z & \text{for } z\in S_1\end{cases}$$

gives a homeomorphism of $R+S_1$ as a subspace of $\boldsymbol{M}$ into $\hat{R}$.

PROOF. For any $z\in S_1$, there corresponds one and only one point $\xi_z\in\hat{S}$ with the property stated in Lemma 9.1. By virtue of Lemma 9.3, $z_1\neq z_2$ $(z_1, z_2\in S)$ implies $\xi_{z_1}\neq\xi_{z_2}$. Hence (9.2) defines a one-to-one mapping of $R+S_1$ into $\hat{R}$ $(=R+\hat{S})$. The bi-continuity of the mapping Φ at any point $x\in R$ is obvious. We shall prove the bi-continuity at any point $z\in S$.

For any sequence $\{x_n\}\subset R$ and any $z\in S$, $\lim_{n\to\infty}\mathrm{dis}(x_n, z)=0$ implies, and is implied by, $\lim_{n\to\infty}\rho(\Phi(x_n), \Phi(z))\equiv\lim_{n\to\infty}\rho(x_n, \xi_z)=0$ by means of Lemmas 9.1 and 9.3. Therefore it is sufficient to prove, under the condition: $\{z, z_1, z_2, \cdots\}\subset S$, that $\lim_{n\to\infty}\rho(\xi_{z_n}, \xi_z)=0$ if and only if $\lim_{n\to\infty}\mathrm{dis}(z_n, z)=0$. For each $z_n\in S$, we may take $x_n\in R$ such that both $\mathrm{dis}(x_n, z_n)<1/n$ and $\rho(x_n, \xi_{z_n})<1/n$ hold (by Lemma 9.1), and consequently

$$\mathrm{dis}(x_n, z)-\frac{1}{n}\leqq\mathrm{dis}(z_n, z)\leqq\mathrm{dis}(x_n, z)+\frac{1}{n}$$

and

$$\rho(x_n, \xi_z)-\frac{1}{n}\leqq\rho(\xi_{z_n}, \xi_z)\leqq\rho(x_n, \xi_z)+\frac{1}{n}\ .$$

Since $\lim_{n\to\infty}\mathrm{dis}(x_n, z)=0$ implies and is implied by $\lim_{n\to\infty}\rho(x_n, \xi_z)=0$ (by Lemmas 9.1 and 9.3), we may see from the above inequalities that $\lim_{n\to\infty}\mathrm{dis}(z_n, z)=0$ is equivalent to $\lim_{n\to\infty}\rho(\xi_{z_n}, \xi_z)=0$.

LEMMA 9.5. *For any $z\in S_1$, $N(\xi_z, y)$ is an extremal FH_0 function of y.*

PROOF. In view point of the preceding lemma, we may consider S_1 as a subset of $\hat{S}$. Since $N(\xi_z, y)$ is an FH_0 function of y (by Theorem 5.1 in [IB]), it is represented by

$$\tag{9.3} N(\xi_z, y)=\int_{\hat{S}_1}N(\xi, y)d\mu(\xi)$$

where μ is a measure on $\hat{S}_1$ such that

$$\tag{9.4}\mu(\hat{S}_1)=\int_{\partial K_0}\frac{\partial N(\xi_z, y)}{\partial\boldsymbol{n}_{K_0}(y)}dS(y)<\infty\ .$$

Let φ be a function of class C^2 on S such that

$$(9.5)\qquad \begin{cases} \varphi(y)=0 & \text{on} \quad (S-S_1)\cup\{z\} \quad \text{and} \\ \varphi(y)>0 & \text{on} \quad S_1-\{z\}\,, \end{cases}$$

and let Ω be a domain satisfying (8.7). Then we may construct a function w of class C^2 on $R+S$ satisfying that

$$(9.6)\qquad w\Big|_{S\cap\bar{\Omega}}=\varphi\,,\quad \frac{\partial w}{\partial n_\Omega}\Big|_{S\cap\bar{\Omega}}=0$$

and

$$(9.7)\qquad w=0 \quad \text{in} \quad R-\Omega\,.$$

Since $N(x, y)$ has the expressions (8.28) and (8.29), we may show that

$$(9.8)\qquad w(x)=-\int_\Omega N(x, y)Aw(y)dy$$

for any $x\in\bar{\Omega}$. Putting $x=z(=\xi_z$ in view point of the preceding lemma) and using (9.3), (9.5) and (9.6), we obtain that

$$(9.9)\qquad \begin{aligned} 0=\varphi(z)&=-\int_\Omega N(\xi_z, y)\cdot Aw(y)dy=-\int_\Omega\int_{\hat{S}_1} N(\xi, y)\cdot Aw(y)d\mu(\xi)dy \\ &=-\int_{(S_1-\{z\})\cap\hat{S}_1}\int_\Omega N(x, y)\cdot Aw(y)dyd\mu(x)=\int_{(S_1-\{z\})\cap\hat{S}_1} w(x)d\mu(x)\,. \end{aligned}$$

Since $w(x)=\varphi(x)>0$ on $S_1-\{z\}$, we get

$$\mu((S_1-\{z\})\cap\hat{S}_1)=0$$

and accordingly

$$(9.10)\qquad (1-c)N(\xi_z, y)=\int_{\hat{S}_1-S_1} N(\xi, y)d\mu(\xi)$$

where c is the μ-measure of the point ξ_z and $0\leqq c\leqq 1$. Let y in the equality (9.10) tend to z. Then $N(\xi_z, y)$ tends to ∞ by Lemma 9.2 while the right-hand side remains bounded by means of (9.4) and Lemma 8.3. Hence c must be equal to one and accordingly $\mu(\hat{S}_1-S_1)=0$. Thus we may see that $N(\xi_z, y)$ is an extremal FH_0 function of y.

PROOF OF THEOREMS 7.1 AND 7.2. By Lemma 9.4, there corresponds $\xi_z\in\hat{S}$ for any $z\in S_1$ in one-to-one way, and the mapping Φ_1 defined by (9.2) gives a homeomorphism of $R+S_1$ as a subspace of $\boldsymbol{M}$ into $\hat{R}=R+\hat{S}$. Furthermore, for any $z\in S_1$, $N(\xi_z, y)$ is an extremal FH_0 function of y by Lemma 9.5, and hence $\xi_z\in\hat{S}_1$ by Theorem 6.4 in [IB]. Since S_1 can be chosen "arbitrarily large" in S, the mapping Φ_1 is uniquely extended to the mapping Φ defined by (7.1) which gives a homeomorphism of $R+S$ onto $R+\{\xi_z; z\in S\}\subset\hat{R}$. Theorem 7.1 is thus proved. Combining this result with Lemmas 8.1, 8.1* and 8.2, we may obtain Theorem 7.2.

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