An abstract formulation of Sobolev type imbedding theorems and its applications to elliptic boundary value problems

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Introduction

We give here an operator theoretical treatment of Sobolev type imbedding theorems and show that the Hardy-Littlewood-Sobolev type inequality holds for certain classes of elliptic boundary value problems. For second order elliptic operators in L^p-space, $1<p<\infty$, realized through zero-Dirichlet or Neumann boundary conditions, the analogy is complete (Theorem 2.5). For $2m$ order elliptic (self-adjoint positive) operators, a weaker result is obtained (Theorem 2.4). In particular, they turn out to be in the class of operators we treated previously ([16], [17]), yielding an estimate of the following type:

$$\|(r+A)^{-k}u\|_{L^q(\Omega)} \leqq \text{Const.}\, r^{\sigma-k}\|u\|_{L^p(\Omega)} \quad \text{for} \quad r>0\,, \tag{0.1}$$

if $0<\sigma=(1/p-1/q)n/2m$, $1<p<q<\infty$, k being any positive integer $>\sigma$. Here A stands for the elliptic operator in question.

Our method is based on the theory of fractional powers of linear operators, as to which we refer to Komatsu's works ([8]-[11]). We note in an abstract way the equivalence of the validity of the Hardy-Littlewood-Sobolev type inequality and the imbedding relations between the domains of fractional powers of operators (Theorem 1.2). These considerations are done in § 1. In § 2, we derive those results spoken about in the above, using the results of § 1 and the Hardy-Littlewood-Sobolev inequality.

§ 1. Imbedding relations of the domains of fractional powers of operators

Let E and F be two Banach spaces continuously imbedded in a Hausdorff linear topological space X. Thus we can define two Banach spaces $E+F$ and $E\cap F$.[1] We consider a closed linear operator A defined in $E+F$ which is non-negative in the sense of Komatsu ([10]), that is, all positive reals $r>0$ are in

[1] $E\cap F$ is the Banach space of $x\in E\cap F$ with the norm $\|x\|_{E\cap F}=\max(\|x\|_E, \|x\|_F)$. $E+F$ is the Banach space of $x=y+z$, $y\in E$, $z\in F$, with the norm $\|x\|_{E+F}=\inf\{\|y\|_E+\|z\|_F\,;\, x=y+z\}$.

the resolvent set $\rho(-A)$ of $-A$, and

$$\|r(r+A)^{-1}\|_{E+F\to E+F}\leqq M \quad \text{for all} \quad r>0 .$$

Here M is a positive constant independent of r. We denote by A_E the restriction of A in E, that is,

$$A_E x=Ax \quad \text{for} \quad x\in D(A_E)=\{x\in E\cap D(A)\,;\,Ax\in E\} .$$

A_F is defined in a similar manner. A_E and A_F are clearly closed operators in E and in F, respectively.

PROPOSITION 1.1. *Let A_1 and A_2 be non-negative operators densely defined in E and in F, respectively. If $(r+A_1)^{-1}x=(r+A_2)^{-1}x$ for $x\in E\cap F$ for all $r>0$, then the operator A in $E+F$ defined by*

$$(1.1) \qquad Ax=A_1x_1+A_2x_2 \quad \textit{for} \quad x=x_1+x_2\in D(A)=D(A_1)+D(A_2)$$

is well-defined. A is a densely defined non-negative operator in $E+F$, and $A_E=A_1$, $A_F=A_2$.

PROOF. Set $J(r)$ for $r>0$ by

$$J(r)x=(r+A_1)^{-1}x_1+(r+A_2)^{-1}x_2 , \quad x=x_1+x_2\in E+F .$$

Since $(r+A_1)^{-1}x=(r+A_2)^{-1}x$ for $x\in E+F$, $J(r)$ is well-defined. It is easily seen that $J(r)$ are bounded in $E+F$ with $\|J(r)\|_{E+F\to E+F}\leqq M/r$, M being a constant independent of r, and that

$$(1.2) \qquad J(r)-J(r')=(r'-r)J(r)J(r') \quad \text{for} \quad r, r'>0 .$$

Now we show that for any $x\in E+F$

$$(1.3) \qquad rJ(r)x\to x \quad \text{strongly as} \quad r\to\infty .$$

In fact,

$$\|rJ(r)x-x\|_{E+F}\leqq\|r(r+A_1)^{-1}x_1-x_1\|_E+\|r(r+A_2)^{-1}x_2-x_2\|_F\to 0 \quad \text{as} \quad r\to\infty ,$$

because A_1 and A_2 are densely defined.

Next we show that

$$(1.4) \qquad A_1x=A_2x \quad \text{for} \quad x\in D(A_1)\cap D(A_2) .$$

In fact, let $x\in D(A_1)\cap D(A_2)$. Then there are $x_1\in E$ and $x_2\in F$ such that $x=(r+A_1)^{-1}x_1=J(r)x_1$ and $x=(r+A_2)^{-1}x_2=J(r)x_2$ for some $r>0$. Hence,

$$J(r')J(r)x_1=J(r')J(r)x_2 \quad \text{for any} \quad r'>0 .$$

By the resolvent equation (1.2),

$$J(r')x_1-J(r')x_2=J(r)x_1-J(r)x_2=0\ .$$

It follows from (1.3) that

$$x_1=\lim r'J(r')x_1=\lim r'J(r')x_2=x_2\ .$$

Hence, we have (1.4).

Now define an operator A in $E+F$ by (1.1). A is well-defined by (1.4). It is immediately seen that

$$(r+A)J(r)x=x \quad \text{for} \quad x\in E+F\ ,$$

and

$$J(r)(r+A)x=x \quad \text{for} \quad x\in D(A)\ .$$

Hence, $J(r)=(r+A)^{-1}$ and A is non-negative. The rest of the assertion is easy to verify.

Since it is enough for our purpose, we assume throughout in this paper that A_E and A_F be densely defined.

DEFINITION 1.1. We write $A\in(\sigma, m, E, F)$ for some $\sigma>0$ and for some positive integer m if the following two conditions are satisfied:

(i) A_E and A_F are non-negative in E and in F, respectively;

(ii) for each $r>0$, $(r+A)^{-m}\in\mathscr{L}(E, F)$ with the norm

$$\|(r+A)^{-m}\|_{E\to F}\leqq L_m r^{\sigma-m},\ r>0\ ,$$

L_m being a positive constant independent of r. Here $\mathscr{L}(E, F)$ stands for the set of all bounded linear operators defined on E into F.

PROPOSITION 1.2 *If* $A\in(\sigma, m, E, F)$, *then* $A\in(\sigma, m+1, E, F)$.

In fact, let $x\in E$. Then

$$\|(r+A)^{-m-1}x\|_F\leqq\|(r+A)^{-1}\|_{E\to F}\|(r+A)^{-m}x\|_F\leqq ML_m r^{\sigma-m-1}\|x\|_E\ .$$

PROPOSITION 1.3. *If* $\sigma<m$, *then* $A\in(\sigma, m+1, E, F)$ *implies* $A\in(\sigma, m, E, F)$.

PROOF. We have, for $r>0$,

$$(r+A)^{-m}x=m\int_r^\infty(s+A)^{-m-1}x\,ds \quad \text{for} \ x\in E+F\ .$$

Hence, for $x\in E$,

$$\|(r+A)^{-m}x\|_F\leqq mL_{m+1}\int_r^\infty s^{\sigma-m-1}ds\|x\|_E=m(m-\sigma)^{-1}L_{m+1}r^{\sigma-m}\|x\|_E\ .$$

REMARK 1.1. These two propositions show that $A\in(\sigma, m, E, F)$ does not depend on $m>\sigma$.

Let $G(t)$, $t\geqq 0$, be a bounded continuous semi-group of operators in $E+F$. The negative A of the infinitesimal generator $-A$ of $G(t)$ is non-negative. Here $-A$ is defined in the following way:

$$-Ax=\text{s-}\lim_{t\downarrow 0} t^{-1}(G(t)-I)x \quad \text{for } x\in D(A)\,,$$

where

$$D(A)=\{x\in E+F\,;\ \text{s-}\lim_{t\downarrow 0} t^{-1}(G(t)-I)x \text{ exists}\}\,.$$

We shall also write $G(t)=\exp(-tA)$ in order to show that $-A$ is the infinitesimal generator of $G(t)$ (cf. Yosida [18]).

For every $t\geqq 0$, we denote by $G_E(t)$ the restriction of $G(t)$ in E, that is,

$$G_E(t)x=G(t)x \quad \text{for} \quad x\in D(G_E(t))=\{x\in E\,;\, G(t)x\in E\}\,.$$

$G_F(t)$ is defined similarly. It is clear that, for $t>0$, $G_E(t)$ and $G_F(t)$ are closed in E and in F, respectively.

PROPOSITION 1.4. *Let A_1 and A_2 be non-negative operators in E and in F, respectively, with the relation that $(r+A_1)^{-1}x=(r+A_2)^{-1}x$ for $x\in E\cap F$, $r>0$. If $-A_1$ and $-A_2$ generate bounded continuous semi-groups $G_1(t)$ and $G_2(t)$ in E and in F, respectively, then $-A$ defined as in Proposition 1.1 generates a bounded continuous semi-group $G(t)$ in $E+F$, and $G_E(t)=G_1(t)$, $G_F(t)=G_2(t)$. If, in particular, $G_1(t)$ and $G_2(t)$ are bounded holomorphic, then so is $G(t)$.*

In fact, we only need to verify that $(r(r+A)^{-1})^k$ are uniformly bounded for $k=1, 2, 3,\cdots$, and for $r>0$, but this is obvious from

$$(r(r+A)^{-1})^k x=(r(r+A_1)^{-1})^k x_1+(r(r+A_2)^{-1})^k x_2\,,\ x=x_1+x_2\in E+F\,.$$

If we speak of $\exp(-tA)$ in the sequel, we consider only those cases that $-A_E$ and $-A_F$ generate bounded continuous (or holomorphic) semi-groups, and we simply say that $-A$ generates a bounded (or holomorphic) semi-group instead of saying that $-A$, $-A_E$ and $-A_F$ generate bounded (or holomorphic) semi-groups, respectively.

DEFINITION 1.2. We write $G(t)\in S(\sigma, E, F)$ for some $\sigma>0$ if the following two conditions are satisfied:

(i) $G_E(t)$, $t\geqq 0$, and $G_F(t)$, $t\geqq 0$, are bounded continuous semi-groups in E and in F, respectively;

(ii) for every $t>0$, $G(t)\in \mathscr{L}(E, F)$ with the norm

$$\|G(t)\|_{E\to F}\leqq Kt^{-\sigma}+K'\,,\quad t>0\,,$$

K, K' being constants independent of t, $K>0$, $K'\geqq 0$. For the sake of simplicity we shall only consider the case $K'=0$ in this paper.

PROPOSITION 1.5. *If* $\exp(-tA)\in S(\sigma, E, F)$, *then* $A\in(\sigma, m, E, F)$ *for* $m>\sigma$.

PROOF. Fix an integer $m>\sigma$. Since

$$(r+A)^{-m}x=\Gamma(m)^{-1}\int_0^\infty t^{m-1}e^{-rt}\exp(-tA)dt\ , \quad x\in E+F\ ,$$

we have, for $x\in E$,

$$\|(r+A)^{-m}x\|_F\leqq K\Gamma(m)^{-1}\int_0^\infty t^{m-1-\sigma}e^{-tr}dt\|x\|_E=K\Gamma(m-\sigma)\Gamma(m)^{-1}r^{\sigma-m}\|x\|_E\ .$$

The converse of the above proposition does not hold in general, as can be seen by the translation semi-group in $L^p(R)$, $1\leqq p<\infty$. However, we have

PROPOSITION 1.6. *Suppose that* $-A$ *generates a bounded holomorphic semi-group* $\exp(-tA)$. *Then* $\exp(-tA)\in S(\sigma, E, F)$ *if* $A\in(\sigma, m, E, F)$.

PROOF. First we note, using the holomorphicity of $\exp(-tA)$, that, for $m=1, 2, 3,\cdots$,

$$\exp(-tA)x=(-1)^{m-1}\Gamma(m)^{-1}t^{-m+1}(2\pi i)^{-1}\int_C e^{-zt}(z+A)^{-m}xdz\ , \tag{1.5}$$

by Dunford's integral. Here C is a path in the sector $S_\omega=\{z\in \boldsymbol{C}\,; z\neq 0, |\arg z|<\pi-\omega, 0<\omega<\pi/2\}\subset\rho(-A)\cap\rho(-A_E)\cap\rho(-A_F)$ running from $-\infty\exp(-i(\omega+\varepsilon))$ to $\infty\exp(i(\omega+\varepsilon))$, $0<\omega<\omega+\varepsilon<\pi/2$.

On the other hand, since for $z, z'\in\rho(-A)\cap\rho(-A_E)\cap\rho(-A_F)$

$$(z+A)^{-m}-(z'+A)^{-m}=\sum_{k=0}^{m}\binom{m}{k}(z'^k-z^k)A^{m-k}(z+A)^{-m}(z'+A)^{-m}\ ,$$

we have

$$(z+A)^{-m}\in\mathscr{L}(E, F)\quad\text{for}\quad z\in S_\omega\ ,$$

and

$$\|(z+A)^{-m}\|_{E\to F}\leqq(1+2(2M_\theta+1)^m)L_m|z|^{\sigma-m}\ ,\quad \theta=|\arg z|\ . \tag{1.6}$$

Here M_θ is a constant independent of $|z|$. Since, for every $t>0$, by the holomorphicity of the integrand,

$$\exp(-tA)x=(-1)^{m-1}\Gamma(m)t^{-m}(2\pi i)^{-1}\int_C e^{-z}(t^{-1}z+A)^{-m}xdz\ , \tag{1.5$'$}$$

we have $\exp(-tA)\in S(\sigma, E, F)$ by (1.6).

In the rest of the paper, we assume that $0\in\rho(-A)\cap\rho(-A_E)\cap\rho(-A_F)$. This assumption is not too restrictive. In fact, we can achieve this by considering, if necessary, $1+A$, $1+A_E$ and $1+A_F$ instead of A, A_E and A_F. With this regard, we have the following

PROPOSITION 1.7. *If $F\subset E$ with the continuous imbedding and if $A\in(\sigma, m, E, F)$ for some $\sigma>0$ and for some m, then $0\in\rho(-A)\cap\rho(-A_E)\cap\rho(-A_F)$.*

PROOF. Since $F\subset E$ continuously, there is a constant C such that

$$\|x\|_E\leqq C\|x\|_F \quad \text{for all} \quad x\in F .$$

Using this and $A\in(\sigma, m, E, F)$, we get, for $r>0$,

$$\|(r+A)^{-m}\|_{E\to E}\leqq CL_m r^{\sigma-m} ,$$

and

$$\|(r+A)^{-m}\|_{F\to F}\leqq C^3 L_m r^{\sigma-m} .$$

This implies that A_E and A_F are one-to-one.

On the other hand, since the integral

$$\int_0^\infty r^{-\alpha-1}(r(r+A_E)^{-1})^m x dr , \quad 0<\alpha<\sigma ,$$

converges absolutely for all $x\in E$, it follows that $A_E^{-\alpha}$ is bounded.[2] Hence, A_E^{-1} is bounded. By a similar argument, A_F^{-1} is also bounded.

DEFINITION 1.3. We write $A\in\sum(\sigma, E, F)$ for some $\sigma>0$ if the following two conditions are satisfied:

(i) A_E and A_F are non-negative in E and in F, respectively;

(ii) $A^{-\sigma}\in\mathscr{L}(E, F)$.[2]

PROPOSITION 1.8. *If $A\in\sum(\sigma, E, F)$, then $A\in(\sigma, m\, E, F)$, $m\geqq\sigma$.*

PROOF. Fix an integer m such that $m-1\leqq\sigma\leqq m$. Since

$$(r+A)^{-m}=A^\sigma(r+A)^{-\sigma}(r+A)^{\sigma-m}A^{-\sigma} ,$$

we have, using the fact that A_F is densely defined so that $(A^\alpha)_F=A_F^\alpha$, $0<\operatorname{Re}\alpha$ (cf. Komatsu [8], Theorem 13.1),

$$\begin{aligned}\|(r+A)^{-m}\|_{E\to F}&\leqq\|A^\sigma(r+A)^{-\sigma}\|_{F\to F}\|(r+A)^{\sigma-m}\|_{F\to F}\|A^{-\sigma}\|_{E\to F}\\&\leqq\text{Const.}\ \|(r+A)^{\sigma-m}\|_{F\to F}\\&\leqq\text{Const.}\ r^{\sigma-m} ,\end{aligned}$$

since $0\leqq\sigma-m+1\leqq1$.

REMARK 1.2. The converse statement of the above proposition is in general

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2) Under our assumptions, $A^{-\beta}$, $\beta>0$, in $E+F$ is defined by the formula

$$A^{-\beta}x=c(m, \beta)\int_0^\infty r^{-\beta-1}(r(r+A)^{-1})^m x dr , \quad c(m, \beta)=\Gamma(m)/(\Gamma(m)\Gamma(m+\beta)) ,$$

for $x\in E+F$, m being a positive integer $>\beta>0$. This does not depend on m, and $(A^{-1})^\beta=A^{-\beta}$ (Komatsu [8, 10]).

not true. A counter-example will be given in the Appendix. On the other hand, by Kato's formula for the resolvent of the fractional power (Kato [7], Komatsu [8], Yosida [18]), we immediately see

PROPOSITION 1.9. *If* $A\in(\sigma,1,E,F)$ *for some* σ, $0<\sigma<1$, *then* $A^{\alpha}\in(\sigma\alpha^{-1},1,E,F)$ *for any* α, $\sigma<\alpha<1$. *In particular*, $A\in\sum(\alpha,E,F)$ *for any* α, $\sigma<\alpha<1$.

PROOF. By Kato's formula,

$$(r+A^{\alpha})^{-1}x=\frac{\sin\pi\alpha}{\pi}\int_0^{\infty}\frac{t^{\alpha}}{r^2+2rt^{\alpha}\cos\pi\alpha+t^{2\alpha}}(t+A)^{-1}xdt\ .$$

Hence,

$$\|(r+A^{\alpha})^{-1}x\|_F\leqq\frac{\sin\pi\alpha}{\pi}\int_0^{\infty}\frac{L_1t^{\alpha+\sigma-1}}{r^2+2rt^{\alpha}\cos\pi\alpha+t^{2\alpha}}dt\|x\|_E$$
$$=\text{const. }r^{\sigma/\alpha-1}\|x\|_E\ .$$

The second assertion follows from the resonance theorem.

REMARK 1.3. Using Theorems 1.1 and 1.2 below, we have a more general result. That is,

PROPOSITION 1.9'. *If* $A\in(\sigma,m,E,F)$, *then* $A\in\sum(\alpha,E,F)$ *for any* $\alpha>\sigma$.

In our previous paper (Yoshikawa [16]), we proved

THEOREM 1.1.[3] *Assume either* $A\in(\sigma,m,E,F)$ *or* $\exp(-tA)\in S(\sigma,E,F)$. *Then*

$$(E,D(A_E^k))_{\sigma/k+\theta,p}\subset(F,D(A_F^k))_{\theta,p}\ ,$$

for $0<\theta<\sigma/k+\theta<1$, $1\leqq p\leqq\infty$, *k being a positive integer. The imbedding is continuous. Here* $(Y,Z)_{\theta,p}$, $0<\theta<1$, $1\leqq p\leqq\infty$, *donotes the mean space of two Banach spaces Y and Z* (Lions-Peetre [14], cf. Grisvard [6], Komatsu [9]).

COROLLARY 1.1. *Under the hypothesis of Theorem* 1.1, *we have*

$$D(A_E^{\alpha+\sigma})\subset D(A_F^{\beta})\quad\textit{for any }\alpha,\beta,\ \mathrm{Re}\,\alpha>\mathrm{Re}\,\beta>0\ ,$$

the imbedding being continuous.

The main result in this section is the following

THEOREM 1.2. *The following three conditions are mutually equivalent*:

(i) $A\in\sum(\sigma,E,F)$;

(ii) $D(A_E^{\sigma})\subset F$ *with the continuous imbedding*;

(iii) $D(A_E^{\alpha+\sigma})\subset D(A_F^{\alpha})$ *for any* α, $\mathrm{Re}\,\alpha>0$, *the imbedding being continuous.*

PROOF. First we prove that (iii) implies (i). In fact, let $x\in E$. Then $A^{-\alpha-\sigma}x\in D(A_E^{\alpha+\sigma})\subset D(A_F^{\alpha})$. Hence, $A^{-\sigma}x=A^{\alpha}A^{-\alpha-\sigma}x\in F$, and (i) follows from the closed graph theorem. Now we prove that (i) and (ii) imply (iii). In fact, let

[3] In reality, we assumed $A\in(\sigma,1,E,F)$ in [16]. But it is not difficult to reduce the problem to the case $A\in(\sigma,1,E,F)$, $0<\sigma<1$.

$x\in D(A_E^{\alpha+\sigma})$. Then $A^{-\alpha}x\in D(A_F^{\sigma})\subset F$. On the other hand, by Proposition 1.8 and Corollary 1.1, $x\in F$. Hence, $x\in D(A_F^{\alpha})$. For $(A^{\alpha})_F=A_F^{\alpha}$ since A_F is densely defined (see Theorem 13.1 in Komatsu [8]). The continuity of the imbedding results from the closed graph theorem. Finally, the equivalence of (i) and (ii) follows from Proposition 1.10 below.

PROPOSITION 1.10. *Let B be a closed linear operator in $E+F$ such that $0\in\rho(B)\cap\rho(B_E)\cap\rho(B_F)$. Then $B^{-1}\in\mathscr{L}(E, F)$ if and only if $D_E(B)\subset F$ with the continuous imbedding.*

PROOF. Define $B^{\sharp}$ by

$$B^{\sharp}x=B^{-1}x \quad \text{for} \quad x\in D(B^{\sharp})=\{x\in E\,;\,B^{-1}x\in F\}\,.$$

$B^{\sharp}$ is clearly closed. If $D(B_E)\subset F$, then $D(B^{\sharp})=E$. Hence, $B^{-1}=B^{\sharp}\in\mathscr{L}(E, F)$.

Conversely, let $B^{-1}\in\mathscr{L}(E, F)$. Since $F, D(B_E)\subset E+F$, it follows that the operator J:

$$Jx=x \quad \text{for} \quad x\in D(J)=\{x\in D(B_E)\,;\,x\in F\}$$

is closed from $D(B_E)$ into F. Take any $x\in D(B_E)$ and set $y=B_Ex$. Then $z=B^{-1}y\in F$ and $z=x$ in $E+F$. This shows that $D(J)=D(B_E)$ and $D(B_E)\subset F$.

THEOREM 1.3. *If there exists a Banach space W such that $D(A_E)\subset W$ and that $F=[E, W]_{\theta}$ for some $\theta, 0<\theta<1$, then $[E, D(A_E)]_{\theta}\subset F$. In particular if we have $[E, D(A_E)]_{\theta}=D(A_E^{\theta})$, then $A\in\sum(\theta, E, F)$. Here $[Y, Z]_{\theta}$, $0<\theta<1$, denotes the complex interpolation space of two Banach spaces Y and Z* (Calderón [3], Lions [13]).

This is immediate from the definition of the complex interpolation space and Theorem 1.2. Note that $[E, D(A_E)]_{\theta}=D(A_E^{\theta})$ is realized if $A^{ir}\in\mathscr{L}(E, E)$ and $\|A^{ir}\|_{E\to E}=\text{const.}\exp(\text{const.}|r|)$ for real r, $r\neq 0$ (Lions [13], Fujiwara [5]).

Now we consider another situation. Suppose that there be given two Banach spaces E_1, F_1, and two operators L, R; $L\in\mathscr{L}(E_1, E)\cap\mathscr{L}(F_1, F)$, $R\in\mathscr{L}(E, E_1)\cap\mathscr{L}(F, F_1)$. Assume furthermore that $RL=$the identity operator on each of E_1 and F_1. Let $P=LR$. Then it is clear that

$$P\in\mathscr{L}(E, E)\cap\mathscr{L}(F, F) \quad \text{and} \quad PP=P\,.$$

PROPOSITION 1.11. *Assume that $(r+A)^{-1}P=P(r+A)^{-1}$ for every $r>0$. Then $A\in(\sigma, m, E, F)$ and $A\in\sum(\sigma, E, F)$ imply $RAL\in(\sigma, m, E, F)$ and $RAL\in\sum(\sigma, E, F)$, respectively.*

This is immediate, since $(r+RAL)^{-1}=R(r+A)^{-1}L$ on each of E_1 and F_1. In fact, put $J(r)=R(r+A)^{-1}L$ for $r>0$. Then for $x\in E_1$ or F_1,

$$J(r)(r+RAL)x=R(r+A)^{-1}P(r+A)Lx$$
$$=RP(r+A)^{-1}(r+A)Lx=RLx=x\ ,$$

and

$$(r+RAL)J(r)x=R(r+A)P(r+A)^{-1}Lx$$
$$=R(r+A)(r+A)^{-1}PLx=RLx=x\ .$$

2. Applications to the elliptic boundary value problems

We begin with the following well-known rosult:

THEOREM 2.1. *Let* $1<p<\infty$. *Define the operator* A_p *in* $L^p(R^n)$ *by*

$$A_p u=(1-\Delta)u \quad for \quad u\in D(A_p)=W^{2,p}(R^n)\ ,$$

where

$$\Delta=\sum_{j=1}^{n}\partial^2/\partial x_j^2=-\sum_{j=1}^{n}D_j^2\ ,\quad D_j=-i\partial/\partial x_j\ .$$

Then we have

(i) $D(A_p^\alpha)=L_{2\alpha}^p=D(A_p^{\mathrm{Re}\,\alpha})$ *for* $\mathrm{Re}\,\alpha\geqq 0$. *Here* L_α^p *is Calderón's Lebesgue space* ([2]);

(ii) $A\in\sum(n(1/p-1/q)/2,\ L^p(R^n),\ L^q(R^n),\ 1<p<q<\infty$.

PROOF. By Mihlin's theorem, we see that A_p is non-negative in each $L^p(R^n)$, $1<p<\infty$. The first assertion is then clear. In fact, $D(A_p^\alpha)=D(A_p^{\mathrm{Re}\,\alpha})$ follows from the fact that, for real r, $A_p^{ir}\in\mathscr{L}(L^p(R^n), L^p(R^n))$ with the norm not exceeding const. $(1+|r|)^n$. For the proof of (ii), we note that

$$(A^{-\sigma}f)(x)=\int_{R^n}k_\sigma(x-y)f(y)dy,\quad \sigma=n(1/p-1/q)/2\ ,$$

where

$$k_\sigma(x-y)=|x-y|^{2\sigma-n}b_\sigma(x-y)\ ,$$

$$b_\sigma(x-y)=2^{-\sigma}\pi^{-n/2}\Gamma(\sigma)^{-1}\int_0^\infty s^{n/2-\sigma-1}\exp(-s-|x-y|^2/2s)ds\ .$$

Since b_σ is bounded, we obtain (ii) by the Hardy-Littlewood-Sobolev inequality.

Using this theorem and Proposition 1.11, we have the following

THEOREM 2.2. *Let* $1<p<\infty$. *We define the operator* $A_{i,p}$, $i=0$ *or* 1, *in* $L^p(R_+^n)$ *as follows:*

$$A_{i,p}u=(1-\Delta)u \quad for \quad u\in D(A_{i,p})\ ,$$

where

$$D(A_{0,p})=\{u\in W^{2,p}(R_+^n)\ ;\ u=0\ on\ x_n=0\}\ ,$$

or

$$D(A_{1,p})=\{u\in W^{2,p}(R^n_+)\,;\,\partial u/\partial x_n=0 \text{ on } x_n=0\}\,.$$

Then, for each i, i=0 or 1,

$$A_i\in\sum(n(1/p-1/q)/n,\,L^p(R^n_+),\,L^q(R^n_+))\,,\quad 1<p<q<\infty\,.$$

PROOF. We employ Fujiwara's technique ([4]). Define R and L as follows:

$$(L_0g)(x)=\begin{cases} g(x) & \text{for } x_n>0\,,\\ -g(x',-x_n) & \text{for } x_n<0\,,\quad x'=(x_1,\cdots,x_{n-1})\end{cases}$$

for $g\in L^r(R^n_+)$, and

$$(R_0f)(x)=(f(x',x_n)-f(x',-x_n))/2\Big|_{x_n>0}$$

for $f\in L^r(R^n)$, $1<r<\infty$. Then A_0, L_0 and R_0 satisfy the hypothesis of Proposition 1.11 with $E_1=L^p(R^n_+)$, $F_1=L^q(R^n_+)$, $E=L^p(R^n)$ and $F=L^q(R^n)$.

For A_1, we take R_1 and L_1 as

$$(L_1g)(x)=\begin{cases} g(x) & \text{for } x_n>0\,,\\ g(x',-x_n) & \text{for } x_n<0\,,\quad x'=(x_1,\cdots,x_{n-1})\,,\end{cases}$$

for $g\in L^r(R^n_+)$, and

$$(R_1f)(x)=(f(x',x_n)+f(x',-x_n))/2\Big|_{x_n>0}$$

for $f\in L^r(R^n)$, $1<r<\infty$. Then A_1, L_1 and R_1 satisfy the hypothesis of Proposition 1.11 with the same E_1, F_1, E and F as in the case of A_0.

Another application of Theorem 2.1 gives

PROPOSITION 2.1. *Let Ω be a domain in R^n with a compact smooth boundary $\partial\Omega$. Then we have, for $1<p<\infty$,*

(2.1) $W^{m,p}(\Omega)\subset L^q(\Omega)$ *if* $1/q=1/p-m/n>0$, *m being a positive integer. More generally, for* $1<p<\infty$,

(2.2) $$[L^p(\Omega),\,W^{m,p}(\Omega)]_\theta\subset L^q(\Omega)\quad \text{if}\quad 1/q=1/p-m\theta/n>0,\ 0<\theta<1\,.$$

PROOF. We prove (2.2). The proof of (2.1) is essentially the same. Let $\{\varphi_i\}_{i=0}^N$ be a partition of unity such that φ_i is supported by U_i. U_0 is disjoint from $\partial\Omega$. For each $i=1,\cdots,N$, U_i is diffeomorphic with the half-disc $D=\{x\,;\,x_n\geqq 0,\ |x|^2<1\}$. Here $U_i\cap\partial\Omega$ is mapped into $x_n=0$. Let $f\in[L^p(\Omega),\,W^{m,p}(\Omega)]_\theta$, and set $f_i=\varphi_i f$. Then each $f_i\in[L^p(\Omega),\,W^{m,p}(\Omega)]_\theta$ and $f=\sum f_i$. Since U_0 is open in R^n, it follows that $f_0\in L^p_{m\theta}\subset L^q(\Omega)$, the imbedding being a consequence of Theorem 1.2 and Theorem 2.1. Now consider f_i, $i=1,\cdots,N$. Let ψ_i be a diffeomorphism from the half-disc D onto U_i, mapping $D\cap\{x_n=0\}$ to $U_i\cap\partial\Omega$. Consider the function $g(x)$ which coincides with $f_i(\psi_i(x))$ on D, and zero elsewhere in R_+^n. Then $g \in [L^p(R_+^n), W^{m,p}(R_+^n)]_\theta$. Define the extension mapping L by

$$(Lh)(x)=\begin{cases} h(x) & \text{for } x_n>0 , \\ \sum_{k=1}^{m} a_k f(x', -kx_n) & \text{for } x_n<0 , \ x'=(x_1, \cdots, x_{n-1}) \end{cases}$$

with the constants a_k satisfying

$$\sum_{k=1}^{m} (-k)^j a_k=1 \quad \text{for} \quad j=0, \cdots, m-1 .$$

Here $h(x)$ stands for a function defined in R_+^n. Then L maps $L^p(R_+^n)$ into $L^p(R^n)$ and $W^{m,p}(R_+^n)$ into $W^{m,p}(R^n)$, both continuously. Hence, for the above g, $Lg \in [L^p(R^n), W^{m,p}(R^n)]_\theta$. By Theorem 2.1, noting that $W^{m,p}(R^n)=L_m^p$, it follows that $Lg \in L^q(R^n)$. Since $Lg=g$ on R_+^n, we conclude $g \in L^q(R_+^n)$. Therefore $[L^p(\Omega), W^{m,p}(\Omega)]_\theta \subset L^q(\Omega)$. The continuity of the imbedding mapping follows from the closed graph theorem.

COROLLARY 2.1. *For* $1<p<\infty$, *we have*

$$W^{m,p}(R_+^n) \subset L^q(R_+^n) \quad \textit{if} \quad 1/q=1/p-m/n>0 , \tag{2.3}$$

and

$$[L^p(R_+^n), W^{m,p}(R_+^n)]_\theta \subset L^q(R_+^n) \quad \textit{if} \quad 1/q=1/p-m\theta/n>0, \ 0<\theta<1 . \tag{2.4}$$

From this corollary and Shimakura's result ([15]), we have

THEOREM 2.3. *Let* $A(D)=1-\Delta$, $B(D)=D_n-\sum_{j=1}^{n-1} b_j D_j - b_0$, b_j, b_0 *being complex constants. We define the operator* A_p *in* $L^p(R_+^n)$, $1<p<\infty$, *by*

$$A_p u=A(D)u \quad \textit{for} \quad u \in D(A_p) ,$$

where

$$D(A_p)=\{u \in W^{2,p}(R_+^n) \,;\, B(D)u=0 \ \textit{on} \ x_n=0\} .$$

Then, for $1<p<q<\infty$,

$$A \in \Sigma(n(1/p-1/q)/2, \ L^p(R_+^n), L^q(R_+^n))$$

if the b_j *satisfy*

$$\sum_{j=0}^{n-1} |\mathrm{Im}\, b_j|^2<1 , \tag{2.5}$$

or more generally

$$|i\sqrt{|\xi'|^2+1+r}-\sum_{j=0}^{n-1} b_j \xi_j - b_0|>c\sqrt{|\xi'|^2+1+r} , \tag{2.6}$$

$\xi'=(\xi_1, \cdots, \xi_{n-1}) \in R^{n-1}$, $r \geqq c$, *c being a positive constant.*

PROOF. Since $D(A_p) \subset W^{2,p}(R_+^n)$, we have Corollary 2.1, for $1<p<\infty$,

$$[L^p(R^n_+), D(A_p)]_\theta \subset L^q(R^n_+)$$

if $1/q=1/p-2\theta/n>0$, $0<\theta<1$. However, by Shimakura's result ([15]) $A_p^{ir} \in \mathscr{L}(L^p(R^n_+), L^p(R^n_+))$ for real r with norm dominated by constant times exp(const. $|r|$). It follows that $[L^p(R^n_+), D(A_p)]_\theta = D(A_p^\theta)$. Hence, by Theorem 1.2, we have $A \in \sum(n(1/p-1/q)/2, L^p(R^n_+), L^q(R^n_+))$. For more general p, q, $1<p<q<\infty$, insert $p_1, \cdots, p_N$ such that $p=p_0<p_1\cdots<p_N<p_{N+1}=q$ with $1/p_{j+1}=1/p_j-2\theta_j/n>0$, $0<\theta_j<1$, $j=0, \cdots, N$. Then from the above result and using Theorem 1.2 several times, we have

$$A \in \sum(n(1/p-1/q)/2, L^p(R^n_+), L^q(R^n_+)) .$$

REMARK 2.1. Theorem 2.2 can be proved in the same way as the above theorem, since in this case also we have bounded pure imaginary powers for $A_{i,p}$.

THEOREM 2.4. *Let Ω be a bounded domain in R^n with a smooth boundary. Let $A(x, D)$ be an elliptic partial differential operator of order $2m$ defined on Ω, $\{B_j(x, D)\}_{j=1}^m$ the set of normal boundary differential operators on $\partial\Omega$. We define the operator A_p in $L^p(\Omega)$, $1<p<\infty$, by*

$$A_p u = A(x, D)u \quad \textit{for} \quad u \in D(A_p) ,$$

where

$$D(A_p) = \{u \in W^{2m,p}(\Omega)\,;\, B_j(x, D)u=0 \ \textit{on}\ \partial\Omega,\ j=1, \cdots, m\} .$$

Assume that A_2 be self-adjoint positive. Assume furthermore that $D(A_2^l) \subset W^{2ml,2}(\Omega)$ for $l=[n/4m]+1$, $[h]$ being the greatest integer less than h for real h. Then we have

$$A \in \sum(n(1/p-1/q)/2m, L^p(\Omega), L^q(\Omega)) \quad \textit{for} \quad 1<p\leqq 2\leqq q<\infty . \tag{2.7}$$

Furthermore, for an integer $k>n(1/p_1-1/q_1)/2m>0$,

$$A \in (n(1/p_1-1/q_1)/2m, k, L^{p_1}(\Omega), L^{q_1}(\Omega)) , \quad 1<p_1<q_1<\infty . \tag{2.8}$$

PROOF. Since $D(A_2^l) \subset W^{2ml,2}(\Omega)$ for a positive integer with $n\leqq 4ml$, we can find a θ, $0<\theta<1$, for any q, $2<q<\infty$, such that $1/q=1/2-2m\theta l/n>0$. Hence, we have, by Proposition 2.1,

$$[L^2(\Omega), D(A_2^l)]_\theta \subset L^q(\Omega) .$$

Since, under our assumptions, A_2^{ir} are uniformly bounded for real r, we have $[L^2(\Omega), D(A_2^l)]_\theta = D(A_2^{\theta l})$. On the other hand, by Agmon [1], all A_p, $1<p<\infty$, are non-negative. In fact, any direction $\omega=\arg z$, $0\leqq|\omega|<\pi$, is the direction of minimal growth for A_p. Hence, we have

$$A \in \sum(\sigma, L^2(\Omega), L^q(\Omega)), \quad \sigma=n(1/2-1/q)/2m . \tag{2.9}$$

Taking the anti-duals, and noting that in this case $(A^{-\sigma})^*=(A^*)^{-\sigma}=A^{-\sigma}$, we have

(2.10) $$A\in\sum(\sigma, L^{q'}(\Omega), L^2(\Omega)),\ q'=q/(q-1),\ \sigma=n(1/q'-1/2)/2m\ .$$

From (2.9) and (2.10), using Theorem 1.2, we have (2.7). Thus, by Proposition 1.8,

$$A\in(n(1/p-1/q)/2m, k, L^p(\Omega), L^q(\Omega)),\ 1<p\leqq 2\leqq q<\infty$$

for any integer $k\geqq n(1/p-1/q)/2m$. Since

$$\|(s+A_r)^{-k}\|_{L^r(\Omega)\to L^r(\Omega)}\leqq M_r s^{-k}$$

for all $s>0$ and for all r, $1<r<\infty$, we have by the (complex) interpolation

$$(s+A)^{-k}\in\mathscr{L}(L^{p_1}(\Omega), L^{q_1}(\Omega)),\ s>0\ ,$$
$$1/p_1=(1-\theta)/p+\theta/r,\ 1/q_1=(1-\theta)/q+\theta/r,\ 0<\theta<1\ ,$$

and

$$\|(s+A)^{-k}\|_{L^{p_1}(\Omega)\to L^{q_1}(\Omega)}\leqq\text{Const.}\, s^{\sigma-k}$$

with $\sigma=n(1/p_1-1/q_1)/2m$. This shows (2.8).

REMARK 2.2. From the above proof, we can say a little more. First,

$$(z+A)^{-k}\in\mathscr{L}(L^{p_1}(\Omega), L^{q_1}(\Omega)),\ 1<p_1<q_1<\infty\ ,$$

if z does not lie on the negative real axis and if $k>\sigma=n(1/p_1-1/q_1)/2m$. Furthermore, in this case, we have

(2.11) $$\|(z+A)^{-k}\|_{L^{p_1}(\Omega)\to L^{q_1}(\Omega)}\leqq\text{Const.}\,|z|^{\sigma-k}\ .$$

For the proof, we only need to note that every direction $\omega=\arg z$, $0\leqq|\omega|<\pi$, is the direction of minimal growth.

Second, (2.8) holds if A_2 is regularly accretive, since in this case A_2^{ir}, $r\in R$, are bounded operators (Lions [12], Komatsu [9]).

If $m=1$, a better result is obtained (cf. Theorems 2.2 and 2.3 in the above). Namely,

THEOREM 2.5. *Let Ω be a bounded domain in R^n with a smooth boundary. Let $A(x, D)$ be given by*

$$A(x, D)=\sum_{i,j=1}^n a_{ij}(x)D_iD_j+\sum_{i=1}^n a_i(x)D_i+a_0(x)\ ,$$

where $a_{ij}(x)$ are real and at every point $x\in\Omega$, $\sum_{i,j=1}^n a_{ij}\xi_i\xi_j$ is a positive definite quadratic form. Then for $B_1u=u$ or $B_1u=\partial u/\partial\nu$, ν being the outer normal at the boundary, we have

(2.12) $$A\in\sum(n(1/p-1/q)/2, L^p(\Omega), L^q(\Omega))\ ,\quad \textit{for}\quad 1<p<q<\infty\ .$$

Here the operator A_p is defined as in Theorem 2.4. (2.8) *also holds in this case.*

PROOF. Since, in this case, for real r, $A_p^{ir}\in\mathscr{L}(L^p(\Omega), L^p(\Omega))$ with

$$\|A_p^{ir}\|_{L^p(\Omega)\to L^p(\Omega)}\leqq \text{Const.}\, e^{\varepsilon|r|}, \quad \varepsilon>0$$

(Fujiwara [5]), we have $[L^p(\Omega), D(A_p)]_\theta=D(A_p^\theta)$, $0<\theta<1$. Hence, (2.12) follows as in the proof of Theorem 2.3. (2.8) holds in this case by Proposition 1.8.

Appendix.

In § 1, we remarked that $A\in\sum(\sigma, E, F)$ did not necessarily follow from $A\in(\sigma, m, E, F)$. We give here an example of operator A which is in $(\sigma, 1, E, F)$ but not in $\sum(\sigma, E, F)$.

We define a semi-group of operators $G(t)=\exp(-tA)$ in $L^p(R)$, $1\leqq p<\infty$, by

$$(G(t)f)(x)=\exp(-t(1+x^2))f(x)\,, \quad t\geqq 0\,.$$

Then it is clear that $G(t)$ is a holomorphic semi-group in $L^p(R)$ for each p, $1\leqq p<\infty$.

PROPOSITION A.1. *For $1\leqq q<p<\infty$, we have*

$$G(t)=\exp(-tA)\in S((1/q-1/p)/2, L^p(R), L^q(R))\,.$$

In particular,

$$A\in((1/q-1/p)/2, 1, L^p(R), L^q(R))\,,$$

but, for any choice of p, q, $1\leqq q<p<\infty$,

$$A\notin\sum((1/q-1/p)/2, L^p(R), L^q(R))\,.$$

PROOF. The first two assertions are an immediate consequence of Hölder's inequality. For the proof of the last part, we need a

LEMMA. *Let (Ω, μ) be a sigma finite positive measure space. Denote by $L_p(\Omega, \mu)$ the space of complex valued measurable functions f with $|f|^p$ summable over Ω. Then*

$$L_p(\Omega, \mu)\subset L^q(\Omega, \mu) \quad \textit{with the continuous imbedding}$$

for some p, q, $1\leqq q<p\leqq\infty$, if and only if $\mu(\Omega)<\infty$.

COMPLETION OF THE PROOF OF PROPOSITION A.1. $A\in\sum((1/q-1/p)/2, L^p(R), L^q(R))$ means that

$$\text{(A.1)} \qquad (1+x^2)^{-\sigma}f(x)\in L^q(R) \quad \text{for every} \quad f\in L^p(R),\ \sigma=(1/q-1/p)/2\,.$$

Now, denote by E_r, $1\leqq r<\infty$, the space of all functions $f(x)$ defined on R with $\|f\|_{E_r}^r=\int_R |f(x)|^r(1+x^2)^{-1/2}dx<\infty$. E_r is clearly a Banach space with the norm $\|f\|_{E_r}$. On the other hand, define a linear operator M_r by

$$(M_rf)(x)=(1+x^2)^{1/2r}f(x)\,.$$

Then it is clear that M_r is a one-to-one mapping of $L^r(R)$ onto E_r and that M_r and M_r^{-1} are continuous. Now we see that (A.1) is equivalent to

(A.2) $E_p \subset E_q$, the imbedding being continuous.

In fact, we have the following diagram:

$$\begin{array}{ccc} L^p(R) \ni f \longrightarrow & (1+x^2)^{-(1/q-1/p)/2} f(x) \in L^q(R) \\ \Big\downarrow M_p & M_q^{-1} \Big\uparrow \\ E_p \ni M_p f \quad \longrightarrow & M_p f \in E_q \,. \end{array}$$

But (A.2) does not hold by virtue of the above lemma. Hence,

$$A \in \sum((1/q-1/p)/2, L^p(R), L^q(R)).$$

PROOF OF LEMMA. The if part is evident from Hölder's inequality. For the proof of the only if part, we may assume without loss of generality that p be finite. In any case we can reduce to this by the interpolation. Suppose $\mu(\Omega)=\infty$. Then we have $L^r(\Omega, \mu) \not\subset L^1(\Omega, \mu)$ for any r, $1<r<\infty$. In fact, if $L^r(\Omega, \mu) \subset L^1(\Omega, \mu)$, then the mapping

$$L^r(\Omega, \mu) \ni f \longrightarrow \int_\Omega f \, d\mu \,,$$

would be a continuous linear functional on $L^r(\Omega, \mu)$. It would follow that $1 \in L^{r'}(\Omega, \mu)$, $r'=r/(r-1)$, which contradicts our assumption $\mu(\Omega)=\infty$. Hence there exists a function $g \in L^r(\Omega, \mu)$ such that $g \notin L^1(\Omega, \mu)$. Next we show $L^p(\Omega, \mu) \not\subset L^q(\Omega, \mu)$ for any p, q, $1<q<p<\infty$, if $\mu(\Omega)=\infty$. In fact, let $r=p/q>1$, and take the above g. Set $f=|g|^{1/q}$. Then $f \in L^p(\Omega, \mu)$ but $f \notin L^q(\Omega, \mu)$.

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