An operator theoretical remark on the Hardy-Littlewood-Sobolev inequality

By Atsushi YOSHIKAWA*)

(Communicated by H. Komatsu)

Introduction. As a supplementary remark to our previous papers ([8], [9]), we discuss here the following well-known Hardy-Littlewood-Sobolev inequality:

(0.1) $$\iint_{R^n\times R^n}\frac{f(x)g(y)}{|x-y|^{2\sigma}}\,dx\,dy<\infty$$

for $f\in L^p(R^n)$, $g\in L^q(R^n)$, $1<p,\ q<\infty$, $1/p+1/q>1$, $\sigma=n(2-1/p-1/q)/2>0$. As to this inequality, a very close result is given by the following

THEOREM. *Let (M,dm) be a sigma finite positive measure space. Let $\{T_t\}_{t\geq 0}$ be a family of linear operators mapping functions on M into functions on M. Assume that $\{T_t\}$ forms a semi-group in the sense that $T_tT_s=T_{t+s}$ for t, $s>0$, $T_0=$identity, and that the restriction of $\{T_t\}$ in $L^2(M,dm)$ is a bounded strongly continuous semi-group. Suppose furthermore that the following five conditions hold:*

(i) $\|T_tf\|_p\leqq\|f\|_p$ *if* $f\in L^p(M,dm)$, $1\leqq p\leqq\infty$.

(ii) *Each T_t is self-adjoint in $L^2(M,dm)$.*

(iii) $T_tf\geqq 0$ *if* $f\geqq 0$.

(iv) $T_t1=1$.

(v) *For each $t>0$, $T_tf\in L^\infty(M,dm)$ if $f\in L^1(M,dm)$, and $\|T_tf\|_\infty\leqq Kt^{-\sigma}\|f\|_1$ for some $\sigma>0$.*

Then we have, with the continuous imbedding,

(0.2) $$D(A_p^{\alpha+\tau})\subset D(A_q^{\alpha})^{1)},\ 1<p<q<\infty,\ \mathrm{Re}\,\alpha\geqq 0,\ \tau=\sigma(1/p-1/q)\,.$$

Here A_p is the negative of the infinitesimal generator of T_t in $L^p(M,dm)$: $T_t=\exp(-tA_p)$ in $L^p(M,dm)$.

It is then quite clear in view of our Theorem 1.2 in [9] that (0.1) is close to this Theorem, for we only need to consider the Gauss-Weierstrass transform as T_t in Theorem. In this case, $\sigma=\dfrac{n}{2}$ and (0.1) implies (0.2). The proof of

*) Partly supported by the Sakkokai Foundation.

1) For a closed linear operator T in a Banach space, the definition domain $D(T)$ of T is a Banach space furnished with the graph norm. A fractional power A_p^α of A_p is closed. Details about fractional powers can be found in Komatsu [3], [4], and Yosida [10].

Theorem is, however, not elementary at all. Yet it has, we believe, a certain interest, being entirely in the context of the operator theory. The proof is based on two facts. Firstly, the conditions (i)-(iv) imply a generalized Littlewood-Paley inequality as Stein showed ([6]). It should be noted that this part, though hidden in Stein's monograph, is the core of our proof and the rest is rather straightforward. These conditions thus give a useful information about the inclusion relations between the definition domains of fractional powers of A_p and the real interpolation spaces associated with A_p. Secondly, the condition (v) gives, as we noted previously ([8]), the inclusion relations of real interpolation spaces associated with A_p and with A_q.

Among the conditions in Theorem, (i)-(iv) are rather easily obtained and closely related with second order elliptic differential operators in general. The condition (v) is essentially the Hausdorff-Young inequality.

Proof of Theorem. We first analyze the conditions (i)-(iv) and summarize Stein's results ([6]).

PROPOSITION 1. (i) *and* (ii) *imply that* $\{T_t\}$ *in each* $L^p(M, dm)$, $1<p<\infty$, *is a bounded holomorphic semi-group, that is, the mapping* $t \to T_t f$, $f \in L^p(M, dm)$, *is analytically continued in the sector*

$$S_p = \{z \in C;\ |\arg z| < \pi(1-|2/p-1|)/2\}\ .$$

For the proof, see Stein ([6]), pp. 67-71. Roughly speaking, the essential of Stein's proof is as follows. First, (ii) implies that T_t in $L^2(M, dm)$ are self-adjoint non-negative. Thus as functions of t, they are analytically continued in the half-plane $\mathrm{Re}\, t>0$. Second, using the interpolation theorem, a variant of the Riesz-Thorin-Stein theorem, we obtain the result.[2]

Now define the Littlewood-Paley function:

$$g_1(f)(\,.\,) = \left(\int_0^\infty t^2 |\partial T_t f(\,.\,)/\partial t|^2 dt\right)^{1/2}\ . \tag{1.1}$$

PROPOSITION 2. *Let* T_t *be the semi-group in Theorem. If* $f \in L^p(M, dm)$, *then* $g_1(f) \in L^p(M, dm)$, $1<p<\infty$, *and*

$$\|g_1(f)\|_p \leqq M_p \|f\|_p\,,\ 1<p<\infty\ .$$

Conversely,

$$\|f\|_p \leqq M_p(\|g_1(f)\|_p + \|E_0(f)\|_p),\ 1<p<\infty\ .$$

Here E_0 *is defined on* $L^2(M, dm)$ *as the orthogonal projection onto the space of*

[2] Stein's proof only shows that $T_t f$ in L^p, $1<p<\infty$, is weakly right continuous at $t=0$. But this is equivalent to saying that $T_t f$ is strongly continuous (Yosida [10]).

all functions h such that $T_t h = h$ *for* $t>0$ (E_0 turns out to be a bounded operator on each L^p, $1<p<\infty$, and $E_0(f) = \text{s-}\lim_{t\to\infty} T_t f$ in L^p.).

For the proof, see Stein ([6]), Chapter IV. Very roughly speaking, he used the martingale theory, and proved a "martingale maximal theorem", a generalized Paley's theorem, and connecting these in a remarkable theorem (Theorem 8 in [6]). Then, using Rota's "Alternieven de Verfahren", he proved the above.

We shall also need the following generalization of Marcinkiewicz-Mihlin multiplier theorem and its corollary.

PROPOSITION 3. *Write* $T_t = \int_0^\infty e^{-rt} dE(r)$ *in* $L^2(M, dm)$, *using the spectral decomposition. Define an operator*:

$$m(A)f = \int_0^\infty m(r) dE(r) f, \quad f \in L^2(M, dm) ,$$

where $m(r)$ *is a bounded function on* $R_+ = (0, \infty)$. $m(A)$ *is a bounded linear operator on all* $L^p(M, dm)$, $1<p<\infty$, *provided that*

$$m(r) = r \int_0^\infty e^{-rt} M(t) dr , \quad r>0 ,$$

for a bounded measurable function $M(t)$ *on* R_+. *Furthermore,*

$$\|m(A)f\|_p \leqq M_p L \|f\|_p , \quad f \in L^p(M, dm) ,$$

M_p *being a constant depending only on* p *and* $L = \sup|M(t)|$.

PROPOSITION 4. *Let* $-A$ *be the infinitesimal generator of* T_t *in Theorem*: $T_t = \exp(-tA)$. *Then* A_p^{ik} *is a bounded linear operator on all* $L^p(M, dm)$, $1<p<\infty$, *for each real* k. *Furthermore, for* $f \in L^p(M, dm)$,

$$\|A_p^{ik} f\|_p \leqq M_p \exp(\pi|k|) \|f\|_p . \tag{1.2}$$

For the proofs of the above two propositions, see Stein ([6]). (1.2) is a consequence of $r^{ki} = r\int_0^\infty e^{-rt} M_k(t) dt$, $M_k(t) = \Gamma(1-ik)^{-1} t^{-ik}$. Now we are going to study the relations between the definition domains of fractional powers of A_p and the real interpolation spaces associated with A_p.

PROPOSITION 5. *Let* $1<p\leqq 2$. *Then, for any* $\alpha>0$,

$$D(A_p^\alpha) \subset (L^p(M, dm), D(A_p^n))_{\alpha/n, 2}$$

with the continuous imbedding. If $p=2$, *both sides coincide as topological spaces.*

Here n *is an integer* $>\alpha$, *and* $(X, Y)_{\theta, 2}$, $0<\theta<1$, *is the mean space of two Banach spaces* X *and* Y.[3)]

3) See [5], [2], [4].

PROOF. It is enough to prove the proposition for an $\alpha, 0<\alpha<1$.[4] Let $f\in D(A_p^\alpha)$. Then we can find an $h\in L^p(M, dm)$ such that $(1+A_p)^{-\alpha}h=f$. We have

$$A^2T_tf=A^2T_t(1+A)^{-\alpha}h=A^\alpha(1+A)^{-\alpha}A^{2-\alpha}T_th\,,\quad t>0\,,$$

since $\{T_t\}$ is holomorphic. Thus

$$\|A^2T_tf\|_p\leqq\|A^\alpha(1+A)^{-\alpha}A^{1-\alpha}T_{t/2}AT_{t/2}h\|_p\leqq\text{const.}\,t^{\alpha-1}\|AT_{t/2}h\|_p\,,$$

for $A^\alpha(1+A)^{-\alpha}$ is a bounded operator in $L^p(M, dm)$[5] and $\|A^{1-\alpha}T_t\|_{L^p\to L^p}\leqq$const. $t^{\alpha-1}$, $t>0$.

Note that an equivalent norm for the space $(L^p(M, dm), D(A_p^n))_{\alpha/n,2}$ is given by[6]

$$\|f\|=\|f\|_p+|f|\,,$$
$$|f|=\left(\int_0^\infty\|t^{2-\alpha}A^2T_tf\|_p^2\,dt/t\right)^{1/2}.$$

Now,

$$\begin{aligned}|f|&\leqq\text{const.}\left(\int_0^\infty t^{4-2\alpha}t^{2\alpha-2}\|AT_{t/2}h\|_p^2\,dt/t\right)^{1/2}\\&=\text{const.}\left(\int_0^\infty t^2\|AT_th\|_p^2\,dt/t\right)^{1/2}\\&=\text{const.}\left(\int_0^\infty t^2\|\partial T_th/\partial t\|_p^2\,dt/t\right)^{1/2}.\end{aligned}$$

By Minkowski's inequality, we have, for $1<p\leqq2$,

$$\left(\int_0^\infty t^2\|\partial T_th/\partial t\|_p^2\,dt/t\right)^{1/2}$$
$$\leqq\left(\int_M\left(\int_0^\infty t^2|\partial T_th(m)/\partial t|^2\,dt/t\right)^{p/2}dm\right)^{1/p}=\|g_1(h)\|_p\,.$$

Hence, by Proposition 2,

$$|f|\leqq\text{const.}\,\|h\|_p\leqq\text{const.}\,\|f\|_{D(A_p^\alpha)}\,.$$

It follows that $f\in(L^p(M, dm), D(A_p^n))_{\alpha/n,2}$, and

$$\|f\|_{(L^p,D(A_p^n))_{\alpha/n,2}}\leqq\text{const.}\,\|f\|_{D(A_p^\alpha)}\,.$$

The assertion for the case $p=2$ is trivial since A_2 is self-adjoint non-negative.

We can now prove Theorem, using Propositions 4 and 5 and the condition (v) in Theorem. However, before that, we are going to prove a dual statement

[4] Komatsu [4], Theorem 2.7.
[5] Komatsu [3], Proposition 6.2.
[6] Komatsu [4], Theorem 5.3.

of Proposition 5 since it has a certain interest in itself.

PROPOSITION 6. *Let* $2\leqq p<\infty$. *Then, for any* $\alpha>0$,

$$(L^p(M, dm), D(A_p^n))_{\alpha/n,2}\subset D(A_p^\alpha)$$

with the continuous imbedding. Here n is an integer $>\alpha$. *For the case* $p=2$ *the same remark as in Proposition* 5 *holds.*

PROOF. It suffices to prove the proposition for an $\alpha, 0<\alpha<1$. Firstly, we note[7] that if $\|T_t f\|_p\to 0$ as $t\to\infty$ for $f\in L^p$, then $E_0(f)=0$ and the converse inequality in Proposition 2 reduces to:

$$\|f\|_p\leqq M_p\|g_1(f)\|_p\,,\ 1<p<\infty\,.$$

Secondly, since $L^p, 1<p<\infty$, is reflexive, it is enough to prove that $A_p^\alpha(r(r+A_p)^{-1})f$ is bounded for $r>0$ if $f\in(L^p, D(A_p^n))_{\alpha/n,2}$.[8] Let $f\in(L^p, D(A_p^n))_{\alpha/n,2}$. Then, by Minkowski's inequality, we have, for $2\leqq p<\infty$,

$$\begin{aligned}\|g_1(A_p^\alpha(r(r+A_p)^{-1})f)\|_p &\leqq\left(\int_0^\infty \|t\partial T_t A^\alpha(r(r+A)^{-1})f/\partial t\|_p^2\,dt/t\right)^{1/2}\\ &\leqq\left(\int_0^\infty \|tA^{1+\alpha}T_t(r(r+A)^{-1})f\|_p^2\,dt/t\right)^{1/2}\\ &\leqq M\left(\int_0^\infty \|tA^{1+\alpha}T_t f\|_p^2\,dt/t\right)^{1/2}=M\langle f\rangle\,,\end{aligned}$$

since $r(r+A_p)^{-1}$ are uniformly bounded for $r>0$. Now using that $\langle f\rangle+\|f\|_p$ gives an equivalent norm for the space $(L^p, D(A_p^n))_{\alpha/n,2}$,[9] we have

$$\|g_1(A_p^\alpha(r(r+A_p)^{-1})f)\|_p\leqq\text{const.}\ \|f\|_{(L^p,D(A_p^n))_{\alpha/n,2}}\,.$$

On the other hand,

$$\|T_t A_p^\alpha(r(r+A_p)^{-1})f\|_p\leqq\text{const.}\ t^{-\alpha}\|r(r+A_p)^{-1}f\|_p\to 0$$

as $t\to\infty$, because $\{T_t\}$ is holomorphic. Hence, by the remark at the beginning of the proof,

$$\begin{aligned}\|A_p^\alpha(r(r+A_p)^{-1})f\|_p&\leqq\text{const.}\ \|g_1(A_p^\alpha(r(r+A_p)^{-1})f\|_p\\ &\leqq\text{const.}\ \|f\|_{(L^p,D(A_p^n))_{\alpha/n,2}}\,.\end{aligned}$$

It follows that $f\in D(A_p^\alpha)$ and

$$\|f\|_{D(A_p^\alpha)}\leqq\text{const.}\ \|f\|_{(L^p,D(A_p^n))_{\alpha/n,2}}\,.$$

REMARK. For the case of the Poisson kernel in the Euclidean n-space, a

---

7) See Stein [6], p. 123, and his discussion in pp. 55-56.

8) Komatsu [3], Proposition 4.5.

9) Komatsu [4], Theorem 5.3.

more detailed result was given by Taibleson ([7]). He also gave counter-examples, showing that his result could not be improved. The proofs of our Propositions 5 and 6, especially that of Proposition 5, are an abstract transposition of Taibleson's proof of his Theorem 15. Now we study the implications of the condition (v) in Theorem.

PROPOSITION 7. *If* $f\in L^p(M, dm)$, $1\leqq p<\infty$, *then* $T_t f\in L^q(M, dm)$, $p<q\leqq\infty$, *for any* $t>0$, *and*

$$\|T_t f\|_q\leqq K_{p,q}t^{-\sigma(1/p-1/q)}\|f\|_p .$$

PROOF. We only need to use the interpolation theorem. In fact, the case when $p=1$ and $q=\infty$ is nothing but the condition (v). If $p\neq 1$, or $q\neq\infty$, we can find an r, $1\leqq r\leqq\infty$, such that

$$1/p=1-\theta+\theta/r,\ 1/q=\theta/r,\ 0<\theta<1 ,$$

since $0<1/p-1/q<1$. Interpolating

$$T_t: L^1(M, dm)\to L^\infty(M, dm) ,$$

and

$$T_t: L^r(M, dm)\to L^r(M, dm) ,$$

we have, for $f\in L^p(M, dm)$,

$$\|T_t f\|_q\leqq K_{p,q}(t^{-\sigma})^{1-\theta}\|f\|_p=K_{p,q}t^{-\sigma(1/p-1/q)}\|f\|_p .$$

PROPOSITION 8. *Let* $1<p<q<\infty$. *Then*

$$(L^p(M, dm), D(A_p^n))_{\alpha/n,2}\subset(L^q(M, dm), D(A_q^n))_{\beta/n,2}$$

if $\alpha=\beta+\sigma(1/p-1/q)>0$, $\beta>0$, *and* n *an integer* $>\alpha$. *Here the imbedding is continuous.*

This is an immediate consequence of Proposition 7 and our Theorem 2.2 in [8] (or Theorem 1.1 in [9]).

At last, we give

PROOF OF THEOREM. In view of our Theorem 1.2 in [9], it is enough to prove Theorem for an $\alpha>0$. For $1<p\leqq 2\leqq q<\infty$, we have, for any $\alpha>0$,

$$D(A_p^{\alpha+\tau})\subset D(A_q^\alpha),\ \tau=\sigma(1/p-1/q)>0 ,$$

with the continuous imbedding. For, by Propositions 5, 6 and 8, we have

$$D(A_p^{\alpha+\tau})\subset(L^p(M, dm), D(A_p^n))_{\alpha+\tau/n,2}$$
$$\subset(L^q(M, dm), D(A_q^n))_{\alpha/n,2}\subset D(A_q^\alpha) .$$

Here n is an integer $>\alpha+\tau$, and all the imbeddings are continuous. Now we are going to prove Theorem for $1<p<q<2$ and for a positive integer α. The case $2<p<q<\infty$ can be proved similarly. Since $1<p<q<2$, we can find a θ, $0<\theta<1$, such that $1/q=(1-\theta)/p+\theta/2$. Interpolating

$$D(A_p^\alpha)=D(A_p^\alpha) ,$$

and

$$D(A_p^{\alpha+\sigma'})\subset D(A_2^\alpha),\ \sigma'=\sigma(1/p-1/2) ,$$

we have

$$[D(A_p^\alpha), D(A_p^{\alpha+\sigma'})]_\theta\subset[D(A_p^\alpha), D(A_2^\alpha)]_\theta$$

with the continuous imbedding. Here $[X, Y]_\theta$, $0<\theta<1$, denotes the complex interpolation space of two Banach spaces X and Y.[10] By the commutativity result of Grisvard ([2] p. 171), we have

$$[D(A_p^\alpha), D(A_2^\alpha)]_\theta=D(A_q^\alpha)$$

with equivalent norms. On the other hand, by Proposition 4,

$$[D(A_p^\alpha), D(A_p^{\alpha+\sigma'})]_\theta=D(A_p^{\alpha+\sigma'\theta})$$

with equivalent norms. Hence,

$$D(A_p^{\alpha+\tau})\subset D(A_q^\alpha),\ \tau=\sigma'\theta=\sigma(1/p-1/q) ,$$

and the imbedding is continuous.

References

[1] Calderón, A. P., Intermediate spaces and interpolation, the complex method, Studia Math. **24** (1964), 113-190.

[2] Grisvard, P., Commutativité de deux foncteurs d'interpolation et applications, J. Math. Pures Appl. **45** (1966), 143-290.

[3] Komatsu, H., Fractional powers of operators, Pacific. J. Math. **19** (1966), 285-346.

[4] Komatsu, H., Fractional powers of operators, II, Interpolation spaces, Pacific J. Math. **21** (1967), 89-111.

[5] Lions, J. L. and J. Peetre, Sur une classe d'espaces d'interpolation, Pub. Math. I.H.E.S. N° 19 (1964), 5-68.

[6] Stein, E. M., Topics in Harmonic Analysis Related to the Littlewood-Paley Theory, Ann. Math. Studies, No. 63, Princeton Univ. Press and Univ. Tokyo Press, Princeton, 1970.

[7] Taibleson, M. H., On the theory of Lipschitz spaces of distributions on Euclidean n-space, I, Principal properties, J. Math. Mech. **13** (1964), 407-479.

[8] Yoshikawa, A., Remarks on the theory of interpolation spaces, J. Fac. Sci. Univ. Tokyo, Sec. I **15** (1968), 209-251.

10) Calderón [1], Lions-Peetre [5].

[9] Yoshikawa, A., An abstract formulation of sobolev type imbedding theorems and its applications to elliptic boundary value problems, to appear.
[10] Yosida, K., Functional Analysis, Springer, Berlin, 1965.

(Received July 14, 1970)

University of Tokyo
Presently at
Department of Mathematics
Faculty of Science
Hokkaido University
Sapporo
060 Japan

Added in the proof. A complete treatment, including an operator theoretical proof of the Hardy-Littlewood-Sobolev inequality is possible. The improved result will be published shortly.