On continuation of regular solutions of partial differential equations to compact convex sets

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§ 1. Introduction.

We consider a simple case of the problem of continuation of solutions of partial differential equations with constant coefficients. A system of partial differential equations with constant coefficients is overdetermined if and only if any solution defined on the complement $U\backslash K$ of a compact convex set K relative to a domain U can be extended to a solution on U (see Malgrange [11] and Palamodov [12] in the case of distribution solutions, and Komatsu [7], [9] in the case of hyperfunctions.). When we consider solutions in some more regular categories, however, the class of differential equations whose solutions never have any compact convex singular set contains more than the overdetermined systems. The first general result in this direction appeared rather recently. Grušin [2] determined the class of single equations which have no infinitely differentiable solutions with unremovable isolated singularities, under the additional condition that one considers only those solutions which can be extended as distributions to the isolated singularity. This result was generalized to the case of systems by Palamodov [12]. On the other hand, Grušin [3] considered the removability of isolated singularities of infinitely differentiable solutions without any additional condition and established a sufficient condition for it. Following the method of [2], Kaneko [5] proved that a single equation $p(D)u=0$ has no real analytic solutions with isolated singularities if and only if no factor of the irreducible decomposition of $p(\zeta)$ is elliptic. We prove here that this class of equations is just the class of those equations which have no real analytic solutions with compact convex singular set. The case of infinitely differentiable solutions is also considered. We employ the method of [3], as well as the theory of hyperfunctions. Especially, the theory of local operators (differential operators of infinite order) plays an important role.

We are concerned with the case of single equations for the sake of simplicity. The case of general systems will be discussed elsewhere. I am grateful to Professor Komatsu for his hearty encouragement and kind advices on corrections and improvements.

§ 2. Hyperfunctions and a regularity theorem

Let $p(\zeta)$ be a polynomial of n variables $\zeta_1, \cdots, \zeta_n$, $p(D)$ be the corresponding partial differential operator, where $D=(D_1, \cdots, D_n)$, $D_1=\sqrt{-1}\frac{\partial}{\partial x_1}$ etc. $\mathscr{B}(U)$ denotes the *hyperfunctions* on the open set $U\subset \boldsymbol{R}^n$, $\mathscr{B}[K]$ denotes the *hyperfunctions with supports in the compact convex set* $K\subset R^n$. We employ similar notations: $\mathscr{A}(U)$ for the *real analytic* functions, $\mathscr{E}^\infty(U)$, $\mathscr{E}^\infty[K]$ for the *infinitely differentiable* functions, and $\mathscr{D}'(U)$, $\mathscr{D}'[K]$ for the *distributions*. $\mathscr{B}_p(U)$ denotes the *hyperfunction solutions* of the equation $p(D)u=0$. $\mathscr{B}_p[K]$, $\mathscr{A}_p(U)$ etc. have similar meanings.

For the general theory of hyperfunctions we refer to [7], [8], [9] etc. We recall here only the following results. $\mathscr{B}[K]$ is considered as the dual of the space $\mathscr{A}(K)$ of *the real analytic functions on* K with the inductive limit topology, namely, $\mathscr{B}[K]$ is considered as the space of *the analytic functionals* with supports contained in K. For $u\in\mathscr{B}[K]$ we define its *Fourier transform* $\tilde{u}(\zeta)$ by the duality just stated: $\tilde{u}(\zeta)=\langle e^{\sqrt{-1}x\cdot\zeta}, u(x)\rangle$, where $x\cdot\zeta=\langle x, \zeta\rangle=x_1\zeta_1+\cdots+x_n\zeta_n$. $\tilde{u}(\zeta)$ is an entire function characterized by the following estimate:

$$|\tilde{u}(\zeta)|\leqq C_\varepsilon \exp(\varepsilon|\zeta|+H_K(\zeta)) \quad \text{for any} \quad \varepsilon>0\,,$$

where $H_K(\zeta)=\sup_{x\in K}\operatorname{Re}\langle x, \sqrt{-1}\zeta\rangle$ is the supporting function of K. Let $u\in\mathscr{B}[K]$ and $v\in\mathscr{B}[L]$ be arbitrary two hyperfunctions of compact support. The *convolution* $u*v$ of u and v is defined to be the hyperfunction with support in $K+L$ which acts as an analytic functional on $\varphi\in\mathscr{A}(K+L)$ by the formula

$$\langle\varphi, u*v\rangle=\langle\langle\varphi(x+y), u(y)\rangle_y, v(x)\rangle_x\,.$$

For the special case $K=\{o\}$, $\mathscr{B}[o]$ is the hyperfunctions with supports at the origin, and its Fourier image is the space of all entire functions which satisfy the following growth condition:

$$|J(\zeta)|\leqq C_\varepsilon \exp(\varepsilon|\zeta|) \quad \text{for any} \quad \varepsilon>0\,.$$

We call these entire functions "*infra-exponential*" after Sato. Infra-exponential functions are also characterized by the growth condition of the coefficients of the Taylor expansion. $J(\zeta)=\sum a_{k_1\cdots k_n}\zeta_1^{k_1}\cdots\zeta_n^{k_n}$ is infra-exponential if and only if $\overline{\lim}\ {}^{k_1+\cdots+k_n}\!\sqrt{|a_{k_1\cdots k_n}|k_1!\cdots k_n!}=0$. In the case of one variable, $J(\zeta)$ is infra-exponential if and only if $J(\zeta)$ has a factorization

$$J(\zeta)=\zeta^m\prod_{k=1}^{\infty}\left(1-\frac{\zeta}{\alpha_k}\right) \quad (\cdots\leqq|\alpha_k|\leqq|\alpha_{k+1}|\leqq\cdots)$$

which converges uniformly on every compact set of $\boldsymbol{C}$, where $\frac{k}{\alpha_k}\to 0$ $(k\to\infty)$ and $\sum\frac{1}{\alpha_k}$ converges to a finite value (Lindelöf [10]).

For each $J\in\mathscr{B}[o]$, $J*$ acts as a *differential operator of infinite order.* In fact, let $\sum a_k\zeta^k$ be the Fourier transform of J, and let $\varphi\in\mathscr{A}(K)$, where $o\in K$. Then it follows $\langle\varphi, J\rangle=\sum a_k D^k\varphi(o)$, thus $J*\varphi=\langle\varphi(x+y), J(y)\rangle_y=\sum a_k D^k\varphi=J(D)\varphi$, and the last sum converges in the topology of $\mathscr{A}(K)$. The operator $J(D)$ is also applied to the germs of holomorphic functions. These assertions are obvious from the estimate of the Taylor coefficients of $J(\zeta)$. Now, by the duality, the differential operator $J(D)$ is also applied to $\mathscr{B}[K]$. For $u\in\mathscr{B}[K]$, it is easily seen that $J(D)u$ agrees with the convolution $J*u$ defined above. $J(D)$ does not change the support of u. Therefore we can apply $J(D)$ to the germs $\mathscr{B}$ of hyperfunctions, or to $\mathscr{B}(U)$. The operator $J(D)$, defined in this way, is called *the local operator with constant coefficients.* When $\mathscr{B}(U)$ is represented as $\mathscr{B}(U)=H^n_V(U,\mathcal{O})=\mathcal{O}(V\sharp U)/\sum_{j=1}^{n}\mathcal{O}(V_j)$,[1] the operation of $J*$ or $J(D)$ just agrees with one induced by the (sheaf homomorphism of) differential operator of infinite order $J(D)$: $\mathcal{O}\to\mathcal{O}$. Finally we remark that $\mathscr{B}[o]\supsetneqq\mathscr{D}'[o]$, hence all the differential operators with constant coefficients in the ordinary sense are contained in the local operators. For the general theory of local operators we refer to the work of T. Kawai or that of M. Sato for the case of variable coefficients. Their papers will soon appear somewhere (cf. [6], [13]).

THEOREM 1. *Assume that $p\neq 0$ is a differential operator with constant coefficients of finite order, and that K is a compact convex set contained in an open set $U\subset\boldsymbol{R}^n$. Then,*

$$\mathscr{B}_p(U)\cap\mathscr{D}'(U\backslash K)\subset\mathscr{D}'_p(U)\ ,$$
$$\mathscr{B}_p(U)\cap\mathscr{E}^\infty(U\backslash K)\subset\mathscr{E}^\infty_p(U)\ ,$$
$$\mathscr{B}_p(U)\cap\mathscr{A}(U\backslash K)\subset\mathscr{A}_p(U)\ .$$

PROOF. (Cf. Agranovič [1]). $p(D)E=\delta$ has a solution $E\in\mathscr{D}'(\boldsymbol{R}^n)$. Let α be an element of $\mathscr{E}^\infty_0(U)$ such that $\alpha(x)\equiv 1$ on a neighborhood of K. Then, in this neighborhood we have

$$u=\alpha u=\delta*(\alpha u)=E*p(D)\delta*(\alpha u)=E*(p(D)(\alpha u))\ .$$

Because of the choice of α, $p(D)(\alpha u)=0$ on a sufficiently small neighborhood of K. Therefore we find from this equality that if $u\in\mathscr{D}'(U\backslash K)$, then $u\in\mathscr{D}'(U)$, and if $u\in\mathscr{E}^\infty(U\backslash K)$, then $u\in\mathscr{E}^\infty(U)$. In the case of $u\in\mathscr{A}(U\backslash K)$, we know

1) See [7] or [8]. This notation was introduced by R. Harvey.

already that $u\in\mathscr{E}^\infty(U)$. Further, we represent E as $E=\Delta^N F$ in a neighborhood $4V$ of K, where Δ is *the Laplacian*, and F is a summable function on $4V$, and it is assumed that $V\supset K$ is a neighborhood of K and $\operatorname{supp}\alpha\subset 2V$, $\alpha\equiv 1$ on V. (We assumed for simplicity that K contains the origin.) Then, in the neighborhood V of K

$$u=\Delta^N F*p(D)(\alpha u)=F*\Delta^N p(D)(\alpha u)\ ,$$

hence we get the estimate

$$\begin{aligned}\sup_{x\in V}|u(x)|&\leqq C\sup_{x\in U}|\Delta^N p(D)(\alpha u)(x)|\\ &\leqq C\sup\{|\Delta^N p(D)(\alpha u)(x)|,\ x\in\operatorname{supp}\alpha\cap\operatorname{supp}(1-\alpha)\}\ .\end{aligned}$$

This estimate also holds good with the solution $D^k u$ of the equation $p(D)u=0$. Thus

$$\sup_{x\in V}|D^k u(x)|\leqq C\sup\{|\Delta^N p(D)(\alpha D^k u)(x)|,\ x\in\operatorname{supp}\alpha\cap\operatorname{supp}(1-\alpha)\}\ .$$

Let M be the order of the operator $\Delta^N p(D)$. Now we use the Leibniz formula, and notice that u is real analytic on $U\backslash K$, so that on the compact subset $\operatorname{supp}\alpha\cap\operatorname{supp}(1-\alpha)$ of $U\backslash K$, the estimate $\sup|D^k u(x)|\leqq C\cdot B^{|k|}k!$ holds for each k, with some constants C, B independent of k. Thus we can rewrite the above inequality as follows:

$$\begin{aligned}\sup_{x\in V}|D^k u(x)|&\leqq C\sum_{|m|\leqq M}\{|D^{k+m}u(x)|,\ x\in\operatorname{supp}\alpha\cap\operatorname{supp}(1-\alpha)\}\\ &\leqq C\sum_{|m|\leqq M}B^{|k+m|}(k+m)!\ .\end{aligned}$$

Here C denotes the various constants depending on the estimates of the derivatives of α of order up to M, and not depending on k. The last side of the above inequality is estimated by $CM\cdot B^M(B(M+1))^{|k|}k!$ in an elementary way. This shows that u is also real analytic on V. q.e.d.

COROLLARY 2. $\mathscr{B}_p[K]=\mathscr{D}'_p[K]=\mathscr{E}^\infty_p[K]=0$.

PROOF. A similar proof shows that $u=0$ on $U\backslash K$ implies $u=0$ on the whole of U.

§3. Continuation of regular solutions.

Theorem 1 on regularity can be paraphrased as follows:

$$\mathscr{A}_p(U\backslash K)/\mathscr{A}_p(U)\subsetneq\mathscr{E}^\infty_p(U\backslash K)/\mathscr{E}^\infty_p(U)\subsetneq\mathscr{D}'_p(U\backslash K)/\mathscr{D}'_p(U)\subsetneq\mathscr{B}_p(U\backslash K)/\mathscr{B}_p(U)\ ,$$

that is, these are all injections. Hence we can treat all the problem concerning the continuation of solutions in the last space $\mathscr{B}_p(U\backslash K)/\mathscr{B}_p(U)$, where the *flabbiness* of hyperfunctions (the fact that hyperfunctions can always be extended to any larger domain) makes the theory transparent. Our main result is:

THEOREM 3. *Let $p(\zeta)$ be a polynomial which is not a constant, and let $p(D)$ be the corresponding differential operator. Then, in order that the space*[2] $\mathscr{A}_p(U\backslash K)/\mathscr{A}_p(U)=0$, *it is necessary and sufficient that $p(\zeta)$ has no elliptic factor.*

As remarked above, $\mathscr{A}_p(U\backslash K)/\mathscr{A}_p(U)\subsetneq\mathscr{B}_p(U\backslash K)/\mathscr{B}_p(U)$ is injective, so we only have to prove the following: *In order that $\mathscr{A}_p(U\backslash K)\subset\mathscr{B}_p(U)$, it is necessary and sufficient that $p(\zeta)$ has no elliptic factor.*

Proof of the necessity is easy. Suppose that p has an elliptic factor p_1, and $p=p_1^k\cdot q$, $p_1\nmid q$. We suppose for simplicity $o\in K$. Then the fundamental solution E of $p_1^k(D)E=\delta$ belongs to $\mathscr{A}_p(U\backslash K)$, but not to $\mathscr{B}_p(U)$. In fact, if there exists $u\in\mathscr{B}_p(U)$ whose restriction to $U\backslash K$ is equal to E, then it follows

$$p(D)(E-u)=p(D)E=q(D)p_1^k(D)E=q(D)\delta\ .$$

On the other hand, $E-u=0$ on $U\backslash K$, hence we can consider that $E-u\in\mathscr{B}[K]$, and its Fourier transform is an entire function. Thus we apply the Fourier transform on the above equality and get $p(\zeta)\widetilde{(E-u)}=q(\zeta)$. Due to Hilbert's Nullstellensatz, this gives a contradiction if we put $p_1(\zeta)=0$. The necessity is proved.

Before beginning the proof of the sufficiency, we construct an isomorphism of $\mathscr{B}_p(U\backslash K)/\mathscr{B}_p(U)$ to a space of holomorphic functions. Let $p=p_1^{k_1}\cdots p_l^{k_l}$ be the irreducible decomposition. Changing the coordinates, we can suppose, without the loss of generality, that $p_1(\zeta),\cdots,p_n(\zeta)$ are monic polynomials with respect to ζ_n, namely, that the algebraic varieties $N_\lambda=\{p_\lambda(\zeta)=0\}\subset C^n$, $\lambda=1,\cdots,l$ are all normally placed with respect to ζ_n-axis. For the entire function φ on C^n, we define the map $d_\lambda:\varphi\mapsto d_\lambda\varphi={}^t\left(\varphi\Big|_{N_\lambda},\cdots,\dfrac{\partial^{k_\lambda-1}}{\partial\zeta_n^{k_\lambda-1}}\varphi\Big|_{N_\lambda}\right)$ whose image is a column of holomorphic functions on N_λ. The map d is defined to be the direct sum $d=\{d_\lambda\}_{\lambda=1,\cdots,l}$.

Now, choose $u\in\mathscr{B}_p(U\backslash K)$ arbitrarily and let $[u]$ denote an element of $\mathscr{B}(U)$ which is an extention of u. By the assumption for u, we have $p(D)[u]\in\mathscr{B}[K]$. Applying the Fourier transform we obtain an entire function $\widetilde{p(D)[u]}$, and operating d to this, we finally obtain a column of holomorphic functions on $\{N_\lambda\}$ which satisfy the same growth condition as $\widetilde{\mathscr{B}[K]}$, Let $\widetilde{\mathscr{B}[K]}\{N_\lambda\}$ denote the space of columns of holomorphic functions on $\{N_\lambda\}$ which have the growth property just described above. Then we have

[2] We can prove that this quotient space depends only on K, using the existence theorem: $p(D)\mathscr{A}(L)=\mathscr{A}(L)$ for any compact convex set L. See [5].

LEMMA 4. *The operator*

$$\tilde{d}: \mathscr{B}_p(U\setminus K)/\mathscr{B}_p(U) \longrightarrow \widetilde{\mathscr{B}[K]}\{N_\lambda\}$$
$$u \bmod \mathscr{B}_p(U) \mapsto d\cdot \widetilde{p(D)[u]} = \{d_\lambda \widetilde{p(D)[u]}\}$$

is well defined, and injective.

PROOF. Let $[u]$, $[u]_1$ be any two extensions of $u \in \mathscr{B}_p(U\setminus K)$. The difference $[u]-[u]_1$ belongs to $\mathscr{B}[K]$, thus the entire function

$$\widetilde{p(D)[u]} - \widetilde{p(D)[u]_1} = p(\zeta)(\widetilde{[u]-[u]_1})$$

obviously belongs to the kernel of d. Therefore $\tilde{d}\cdot u$ is determined independently of the choice of the extension $[u]$. Clearly, $\mathscr{B}_p(U)$ belongs to the kernel.

Next we prove the injectivity. Suppose $\tilde{d}\cdot u = \{d_\lambda \widetilde{p(D)[u]}\} = 0$, where $[u]$ is an extension to $\mathscr{B}(U)$ of an element $u \in \mathscr{B}_p(U\setminus K)$. Then the entire function $\widetilde{p(D)[u]}$ satisfies the condition $\frac{\partial^k}{\partial \zeta_n^k} \widetilde{p(D)[u]}\Big|_{N_\lambda} = 0$, $k=0, 1, \cdots, k_\lambda - 1$ for each λ. Thus $\widetilde{p(D)[u]}$ is divisible by each $p_\lambda^{k_\lambda}$. Applying the theorem of Hartogs we can write $\widetilde{p(D)[u]} = p(\zeta)F(\zeta)$, with an entire function $F(\zeta)$. Here by the well known inequality (see e.g. Hörmander [4] Lemma 3.1.2), again $F(\zeta)$ has the same growth property as $\widetilde{\mathscr{B}[K]}$. Hence there exists $v \in \mathscr{B}[K]$, such that $F(\zeta) = \tilde{v}$, and we have $p(D)[u] = p(D)v$. This shows that u has an extension $[u]-v$ which belongs to $\mathscr{B}_p(U)$. Thus $u=0$ as an element of $\mathscr{B}_p(U\setminus K)/\mathscr{B}_p(U)$. q.e.d.

REMARK. It can be shown that $\tilde{d}$ is an isomorphism onto the subspace of $\widetilde{\mathscr{B}[K]}\{N_\lambda\}$ whose elements belong locally (in the sense of germs) to the image of $d: \mathscr{O} \to \mathscr{O}$. This is proved from the vanishing of cohomology with the growth condition of the type $\widetilde{\mathscr{B}[K]}$.

Regular solutions in $\mathscr{B}_p(U\setminus K)$ are mapped by $\tilde{d}$, constructed above, to holomorphic functions on $\{N_\lambda\}$ satisfying more restricted growth conditions than the type $\widetilde{\mathscr{B}[K]}$. In fact, we have the following:

LEMMA 5. *On $\{N_\lambda\}$ the following estimates hold:*

If $u \in \mathscr{D}'_p(U\setminus K)$, *then* $|\tilde{d}\cdot u| \leqq C_\varepsilon(1+|\zeta|)^k \exp(H_K(\zeta)+\varepsilon|\mathrm{Im}\,\zeta|)$
for $\forall \varepsilon>0$, $\exists k=k_\varepsilon$.

If $u \in \mathscr{E}'^{\infty}_p(U\setminus K)$, *then* $|\tilde{d}\cdot u| \leqq C_{k,\varepsilon}(1+|\zeta|)^k \exp(H_K(\zeta)+\varepsilon|\mathrm{Im}\,\zeta|)$
for $\forall \varepsilon>0$, $\forall k$ *(integer)*.

If $u \in \mathscr{A}_p(U\setminus K)$, *then* $|J(\zeta)\tilde{d}\cdot u| \leqq C_{J,\varepsilon} \exp(H_K(\zeta)+\varepsilon|\mathrm{Im}\,\zeta|)$
for $\forall \varepsilon>0$, $\forall J$ *(entire infra-exponential function)*.

PROOF. First we give another expression of the map $\tilde{d}$. Let $\alpha(x)$ be an element of $\mathscr{C}_0^\infty(U)$ whose value is equal to 1 on a sufficiently small neighborhood of K. Then for $u \in \mathscr{D}'_p(U \backslash K)$, $p(D)(\alpha u)$ is identically zero on some neighborhood of K. Extending this function by zero on K, we obtain an element belonging to $\mathscr{E}'(U \backslash K)$ (distributions with supports contained compactly in $U \backslash K$). Let $[p(D)(\alpha u)]_0$ denote this element. Then we have obviously

$$p(D)(\alpha[u]) = [p(D)(\alpha u)]_0 + p(D)[u] .$$

Here $\alpha[u]$ is really of compact support. Applying the Fourier transform to both sides and operating d, we have

$$0 = d \cdot p(\zeta)\widetilde{\alpha[u]} = d[\widetilde{p(D)(\alpha u)}]_0 + d \cdot \widetilde{p(D)[u]} .$$

Hence

$$\tilde{d} \cdot u = d \cdot \widetilde{p(D)[u]} = -d[\widetilde{p(D)(\alpha u)}]_0 .$$

The last term is the desired expression. (See the remark below.)

By this expression, and by the Paley-Wiener theorem, we immediately obtain the first two statements of the lemma. Concerning the last statement we make another device. Let u be an element of $\mathscr{A}_p(U \backslash K)$. For any entire infra-exponential function $J(\zeta)$, $J(D)u$ again belongs to $\mathscr{A}_p(U \backslash K)$, especially to $\mathscr{C}_p^\infty(U \backslash K)$, (where $J(D)$ is the local operator corresponding to $J(\zeta)$). Thus $\tilde{d} \cdot J(D)u$ satisfies the second estimate of Lemma 5. On the other hand, we have

$$\begin{aligned}
\tilde{d}J(D)u &= \{-d_\lambda\widetilde{[p(D)(\alpha(x)J(D)u]_0}\} \\
&= \{-d_\lambda \widetilde{J(D)[p(D)\alpha u]_0}\} + \{-d_\lambda \widetilde{p(D)[\alpha J(D)u - J(D)\alpha u]_0}\} \\
&= \{-d_\lambda J(\zeta)\widetilde{[p(D)\alpha u]_0}\} + \{-d_\lambda p(\zeta)\widetilde{[\alpha J(D)u - J(D)\alpha u]_0}\} \\
&= \{-d_\lambda J(\zeta)\widetilde{[p(D)\alpha u]_0}\} .
\end{aligned}$$

Moreover, we have

$$d_\lambda J(\zeta)\widetilde{[p(D)\alpha u]_0} = \begin{pmatrix} J(\zeta)\widetilde{[p(D)\alpha u]_0}\Big|_{N_\lambda} \\ \vdots \\ \dfrac{\partial^{k_\lambda - 1}}{\partial \zeta_n^{k_\lambda - 1}} (J(\zeta)\widetilde{[p(D)\alpha u]_0})\Big|_{N_\lambda} \end{pmatrix}$$

$$= \begin{pmatrix} J(\zeta) & & 0 \\ \vdots & \ddots & \\ \dfrac{\partial^{k_\lambda - 1}}{\partial \zeta_n^{k_\lambda - 1}} J(\zeta), & \cdots, & J(\zeta) \end{pmatrix} \times \begin{pmatrix} \widetilde{[p(D)\alpha u]_0}\Big|_{N_\lambda} \\ \vdots \\ \dfrac{\partial^{k_\lambda - 1}}{\partial \zeta_n^{k_\lambda - 1}} \cdot \widetilde{[p(D)\alpha u]_0}\Big|_{N_\lambda} \end{pmatrix} .$$

Now that $\tilde{d}\cdot J(D)u$ satisfies the second estimate of Lemma 6, it follows from the first row of this formula that $\widetilde{[p(D)\alpha u]_0}|_{N_\lambda}$ satisfies the last estimate of the lemma for any infra-exponential function $J(\zeta)$. Since $\frac{\partial}{\partial\zeta_n}J(\zeta)$ is also infra-exponential, we see from the second row of the formula that

$$J(\zeta)\frac{\partial}{\partial\zeta_n}\widetilde{[p(D)\alpha u]_0}\Big|_{N_\lambda}=\frac{\partial}{\partial\zeta_n}\Big(J(\zeta)\widetilde{[p(D)\alpha u]_0}\Big)\Big|_{N_\lambda}-\Big(\frac{\partial}{\partial\zeta_n}J(\zeta)\Big)\widetilde{[p(D)\alpha u]_0}\Big|_{N_\lambda}$$

also satisfies the same estimate. Repeating this, we get the desired estimate.

q.e.d.

REMARK. 1) Grušin [3] and Palamodov [12] employ this expression of $\tilde{d}$ as their definition throughout. 2) If we take for $\alpha(x)$ a characteristic function, this paraphrase of $\tilde{d}$ holds as well for any hyperfunction u, but for the present, this gives nothing more.

In order to reformulate the last condition of the preceding lemma to a more convenient one, we give another lemma.

LEMMA 6. *There exist entire infra-exponential functions* $J_\nu(\zeta)$, $\nu=1,\cdots,N$, *satisfying*

$$C\cdot\exp\Big(A\sum_{i=1}^{n}\frac{|\zeta_i|}{\log(|\zeta_i|+M)}\Big)\leqq|J_1(\zeta)|+\cdots+|J_N(\zeta)|\ .$$

Here A is a positive constant which may be chosen arbitrarily large. C, M are positive constants depending on A.

PROOF. First we consider the case of one variable. We make somewhat general treatment for the sake of simplifying the notations. Let $\varphi(t)$ be a function of t defined for $t\geqq 1$ such that $\varphi(1)=1$, and $\varphi(t)\to\infty$ monotonely when $t\to\infty$. We consider the function $J(z)$ defined by the infinite product

$$J(z)=\prod_{m=1}^{\infty}\Big(1-\frac{z^2}{(m\varphi(m))^2}\Big)\ .$$

This is an entire function because $\sum\frac{1}{(m\varphi(m))^2}\leqq\sum\frac{1}{m^2}<\infty$ by the assumption. This function admits the following estimate from below: Let x and y be respectively the real and the imaginary part of z. Then in the region where $y^2\geqq 3x^2$,

$$\begin{aligned}\Big|\Big(1-\frac{z^2}{(m\varphi(m))^2}\Big)\Big|^2&=\Big(1-\frac{x^2-y^2}{(m\varphi(m))^2}\Big)^2+\Big(\frac{2xy}{(m\varphi(m))^2}\Big)^2\\&=1-\frac{2(x^2-y^2)}{(m\varphi(m))^2}+\frac{(x^2+y^2)^2}{(m\varphi(m))^4}\end{aligned}$$

$$\geqq 1+\frac{x^2+y^2}{(m\varphi(m))^2}+\frac{(x^2+y^2)^2}{4(m\varphi(m))^4}=\left(1+\frac{|z|^2}{2(m\varphi(m))^2}\right)^2 .$$

Thus for any z satisfying $|\mathrm{Im}\, z|\geqq\sqrt{3}\,|\mathrm{Re}\, z|$,

$$|J(z)|\geqq \prod_{m=1}^{\infty}\left(1+\frac{|z|^2}{2(m\varphi(m))^2}\right)\geqq \prod_{m=1}^{M}\left(1+\frac{|z|^2}{2(m\varphi(m))^2}\right).$$

Choosing M so as to satisfy $M\varphi(M)\leqq|z|\leqq(M+1)\varphi(M+1)$, we have $\frac{|z|^2}{(m\varphi(m))^2}\geqq 1$ for $1\leqq m\leqq M$, and $M\leqq M\varphi(M)\leqq|z|$, so that

$$M\geqq\frac{|z|}{\varphi(M+1)}-1\geqq\frac{|z|}{\varphi(|z|+1)}-1 .$$

Here we used the fact that $\varphi(t)$ monotonely increases. Thus we obtain

$$|J(z)|\geqq\left(\frac{3}{2}\right)^M\geqq\frac{2}{3}\left(\frac{3}{2}\right)^{\frac{|z|}{\varphi(|z|+1)}}=\frac{2}{3}\exp\left(\left(\log\frac{3}{2}\right)\cdot\frac{|z|}{\varphi(|z|+1)}\right).$$

The function $J(z)$ is evidently infra-exponential by the criterion of Lindelöf mentioned at §2. Direct verification is also easy. In fact, for any $\varepsilon>0$, we can prove, by an elementary calculation, that

$$|J(z)|\leqq \prod_{m=1}^{\infty}\left(1+\frac{|z|^2}{(m\varphi(m))^2}\right)\leqq C_\varepsilon \prod_{m=1}^{\infty}\left(1+\frac{\varepsilon^2|z|^2}{m^2}\right)\leqq C_\varepsilon e^{\pi\varepsilon|z|} .$$

Here $C_\varepsilon=\varepsilon^{-2\varphi^{-1}(\varepsilon^{-1})}$ is a constant depending only on ε.

Now we define

$$J_1^0(z)=J(z),\quad J_2^0(z)=J(e^{\frac{\sqrt{-1}}{3}\pi}z),\cdots,\quad J_6^0(z)=J(e^{\frac{5}{3}\sqrt{-1}\pi}z) .$$

For these entire infra-exponential functions we have, for any $z\in \boldsymbol{C}$,

$$|J_1^0(z)|+|J_2^0(z)|+\cdots+|J_6^0(z)|\geqq\frac{2}{3}\cdot\exp\left(\left(\log\frac{3}{2}\right)\frac{|z|}{\varphi(|z|+1)}\right).$$

In the case of several variables, we take the products $\prod_{i=1}^{n} J_{\nu_i}^0(\zeta_i)$ in all the possible way, where $J_{\nu_i}^0(\cdot)$ is one of $J_1^0,\cdots,J_6^0$ constructed above. Then if $J_1(\zeta),\cdots,J_N(\zeta)$ are the whole of these product functions ordered in an appropriate way, we have

$$\begin{aligned}|J_1(\zeta)|+\cdots+|J_N(\zeta)|&=\prod_{i=1}^{n}(|J_1^0(\zeta_i)|+\cdots+|J_6^0(\zeta_i)|)\\&\geqq\left(\frac{2}{3}\right)^n\exp\left(\log\frac{3}{2}\sum_{i=1}^{n}\frac{|\zeta_i|}{\varphi(|\zeta_i|+1)}\right).\end{aligned}$$

Now we apply the estimates obtained above to the function $\varphi(t)=\dfrac{\log(t+M-1)}{\log M}$. This function satisfies $\varphi(1)=1$, and $\varphi(t)\to\infty$ monotonely when $t\to\infty$. Thus, when $J_1,\cdots,J_N$ are the functions constructed above corresponding to $\varphi(t)=\dfrac{\log(t+M-1)}{\log M}$, we have the estimate from below

$$|J_1(\zeta)|+\cdots+|J_N(\zeta)|\geqq\left(\frac{2}{3}\right)^n \exp A\cdot\sum_{i=1}^{n}\frac{|\zeta_i|}{\log(|\zeta_i|+M)}.$$

Here the constant $A=\log\left(\dfrac{3}{2}\right)\cdot\log M$ is as large as we wish when we choose M large. Thus the lemma is proved.

REMARK. The formulation of this lemma for general φ is possible. It is of some interest, but it is not necessary for further discussions.

The advantage of the function $\dfrac{|\zeta|}{\log|\zeta|}$ is shown by the following lemma.

LEMMA 7. *Let $F(z)$ be a holomorphic function of one variable on* $\operatorname{Re}z\geqq0$, *which satisfies for some constants $A>0$, $M\geqq0$, $C>0$*

$$|F(z)|\leqq C\exp\left(-2A\frac{|z|}{\log(|z|+M)}+A|\operatorname{Re}z|\right).$$

Then $F(z)$ is bounded on $\operatorname{Re}z\geqq0$.

PROOF. Consider the transform $z=\dfrac{w}{\log(w+1)}$ of the variables, where log denotes the branch for which $\log 1=0$. This maps $\operatorname{Re}w\geqq0$ into $\operatorname{Re}z\geqq0$. For, we have

$$\operatorname{Re}z=\operatorname{Re}\frac{w}{\log(w+1)}=\operatorname{Re}\frac{\operatorname{Re}w+\sqrt{-1}\operatorname{Im}w}{\log|w+1|+\sqrt{-1}\arg(w+1)}$$
$$=\frac{\operatorname{Re}w\cdot\log|w+1|+\operatorname{Im}w\arg(w+1)}{(\log|w+1|)^2+(\arg(w+1))^2}.$$

Because of our choice of the branch of log, $\operatorname{Im}w$ and $\arg(w+1)$ have the same sign, so that this value is non-negative when $\operatorname{Re}w\geqq0$. Thus $\operatorname{Re}z\geqq0$ if $\operatorname{Re}w\geqq0$. The composed function $G(w)=F\left(\dfrac{w}{\log(w+1)}\right)$ is holomorphic on $\operatorname{Re}w\geqq0$, and satisfies the estimate

$$|G(w)|\leqq C\exp(A|z|)\leqq C\exp\left(A\frac{|w|}{\log|w+1|}\right).$$

Therefore $|G(w)|\leqq C_\varepsilon\exp(\varepsilon|w|)$ for any $\varepsilon>0$. Moreover $G(w)$ is bounded on $\operatorname{Re}w=0$. In fact, put $w=\sqrt{-1}\eta$, $-\infty<\eta<\infty$, then,

$$z=\frac{w}{\log(w+1)}=\frac{\sqrt{-1}\eta}{\log\sqrt{1+\eta^2}+\sqrt{-1}\arg(1+\sqrt{-1}\eta)},$$

so that $|z|\leqq|\eta|$ when $|\eta|\geqq\sqrt{e^2-1}$. Thus we have, for $|\eta|\geqq\sqrt{e^2-1}$,

$$
\begin{aligned}
&-2A\frac{|z|}{\log(|z|+M)}+A|\mathrm{Re}\,z| \\
&\leqq -2A\frac{|\eta|}{\sqrt{(\log\sqrt{1+\eta^2})^2+(\arg(1+\sqrt{-1}\eta))^2}}\frac{1}{\log(|\eta|+M)} \\
&+A\frac{|\eta|\cdot|\arg(1+\sqrt{-1}\eta)|}{(\log\sqrt{1+\eta^2})^2}\ .
\end{aligned}
$$

Here the two terms have the same growth order $\dfrac{|\eta|}{(\log|\eta|)^2}$ when $|\eta|$ is large. Therefore, noticing that $|\arg(1+i\eta)|\leqq\dfrac{\pi}{2}<2$, we see that the last side becomes negative when $|\eta|$ is large enough. Thus $G(w)$ is bounded when $\mathrm{Re}\,w=0$.

Thus the theorem of Fragmen-Lindelöf confirms the boundedness of $G(w)$ on $\mathrm{Re}\,w\geqq0$. Returning to the $F(z)$, it follows that $F(z)$ is bounded on the positive real axis (where $z=\dfrac{w}{\log(w+1)}$ is certainly solved for z). Hence $F(z)$ is bounded on the whole $\mathrm{Re}\,z\geqq0$, again by the Phragmén-Lindelöf theorem. q.e.d.

End of proof of the sufficiency of Theorem 3. Let u be an element of $\mathscr{A}_p(U\backslash K)$. The image $\tilde{d}\cdot u(\zeta)$ is a column of holomorphic functions on $\{N_\lambda\}$ which satisfy, by Lemma 5, the estimate $|J(\zeta)\tilde{d}\cdot u(\zeta)|\leqq C_{J,\varepsilon}\exp(H_K(\zeta)+\varepsilon|\mathrm{Im}\,\zeta|)$ for any $\varepsilon>0$ and for any entire infra-exponential function $J(\zeta)$. Now choose $J_1(\zeta),\cdots,J_N(\zeta)$ so as to satisfy Lemma 6, apply for each of them the estimate above, and add all these inequalities obtained, then, after dividing the both sides by $|J_1(\zeta)|+\cdots+|J_N(\zeta)|$, we are lead to the following estimate:

$$|\tilde{d}\cdot u(\zeta)|\leqq C_\varepsilon\exp\Big(-A\sum_{i=1}^{n}\frac{|\zeta_i|}{\log(|\zeta_i|+M)}+H_K(\zeta)+\varepsilon|\mathrm{Im}\,\zeta|\Big)\,.$$

Here, as remarked in the Lemma 6, A may be chosen as large as desired. Now suppose that any of the N_λ is not elliptic, then each N_λ contains a real infinite point. Fix an N_λ and choose the coordinates so that one of the real infinite points on N_λ agrees with $(\infty,0,\cdots,0)$. On this choice of the coordinates, we can also assume that N_λ is in the normal place with respect to ζ_n-axis. Expanding the defining equation of N_λ to the Puiseux series with respect to ζ_1 at $(\infty,0,\cdots,0)$, we see that

$$\{(\zeta_1,\zeta_2,\cdots,\zeta_{n-1},f(\zeta_1))\,;\,|\zeta_1|>\frac{1}{\delta}\,,\,|\zeta_2|<1,\cdots,|\zeta_{n-1}|<1\}$$

where $$f(\zeta_1)=\sum_{k=k_0}^{-\infty}a_k\zeta_1^{k/q}$$

is an open subset of N_λ. Here the coefficients a_k, q, k_0 and the radius of convergence $\frac{1}{\delta}$ all depend on $\zeta_2,\cdots,\zeta_{n-1}$. We note that this expansion takes place at $(\infty,0,\cdots,0)$, from which it follows $f(\zeta_1)/\zeta_1\to 0$ when $\zeta_1\to\infty$. This shows that $\frac{k_0}{q}<1$. Now, for fixed $\zeta_2,\cdots,\zeta_{n-1}$, the function $F(\zeta_1)$ of one variable ζ_1:

$$F(\zeta_1)=\tilde{d}\cdot u(\zeta_1,\cdots,\zeta_{n-1},f(\zeta_1))$$

is a column of q-valued holomorphic functions on $|\zeta_1|>\frac{1}{\delta}$, and satisfies

$$|F(\zeta_1)|\leqq C_\varepsilon\exp\left\{-A\left(\frac{|f(\zeta_1)|}{\log(|f(\zeta_1)|+M)}+\sum_{j=1}^{n-1}\frac{|\zeta_j|}{\log(|\zeta_j|+M)}\right)\right.$$
$$\left.+H_K(\zeta_1,\cdots,\zeta_{n-1},f(\zeta_1))+\varepsilon|\mathrm{Im}(\zeta_1,\cdots,\zeta_{n-1},f(\zeta_1))|\right\}.$$

Here, $|f(\zeta_1)|\leqq C|\zeta_1|^{\frac{k_0}{q}}$ and $\frac{k_0}{q}<1$. Therefore the $f(\zeta_1)$ in the symbol $H_K(\cdot)$ and $\varepsilon|\mathrm{Im}\,f(\zeta_1)|$ are both cancelled by the term $-\frac{A}{2}\frac{|\zeta_1|}{\log(|\zeta_1|+M)}$ when we choose A sufficiently large. Thus, omitting unnecessary terms and replacing the terms including only $\zeta_2,\cdots,\zeta_{n-1}$ by constants, we have

$$|F(\zeta_1)|\leqq C'_\varepsilon\exp\left(-\frac{A}{2}\frac{|\zeta_1|}{\log(|\zeta_1|+M)}+a|\mathrm{Im}\,\zeta_1|\right)$$

for sufficiently large $|\zeta_1|$. Here a is a positive constant depending only on the diameter of K. We can assume that $\frac{A}{2}>2a$, so that, applying Lemma 7 to $F(\zeta_1)$ on each of the half spaces $|\mathrm{Im}\,\zeta_1|>\frac{1}{\delta}$, we conclude that all the branches of $F(\zeta_1)$ are bounded. Hence $F(t^q)$ $(t=\zeta_1^{1/q})$ is a bounded, one-valued function of t on $|t|>\delta^{-1/q}$. Thus by Riemann's theorem, it is also holomorphic at $t=\infty$. On the other hand, we have, by the estimate above, $t^kF(t^q)\to 0$ when $t\to+\infty$, for any $k=0,1,2,\cdots$. This implies that all the Taylor coefficients of $F(t^q)$ at $t=\infty$ vanishes. Thus we have $F(\zeta_1)\equiv 0$. We note that this takes place for each fixed $\zeta_2,\cdots,\zeta_{n-1}$ for $|\zeta_2|<1,\cdots,|\zeta_{n-1}|<1$.

Returning back, we conclude that $\tilde{d}\cdot u(\zeta)\equiv 0$ on some open subset of N_λ. Since N_λ is irreducible by the assumption, the set of the regular points on N_λ is connected and we conclude that $\tilde{d}\cdot u\equiv 0$ on the whole of N_λ. Making the same arguments for each N_λ, $\lambda=1,\cdots,l$ we finally obtain $\tilde{d}\cdot u\equiv 0$, which means, on account of Lemma 4, the triviality of the image of the map $\mathscr{A}_p(U\backslash K)/\mathscr{A}_p(U)\to\mathscr{B}_p(U\backslash K)/\mathscr{B}_p(U)$. q.e.d.

A similar proof gives the following

PROPOSITION 8. *In order that* $\mathscr{C}_p^\infty(U\backslash O)/\mathscr{C}_p^\infty(U)=0$, *it is sufficient that each*

N_λ has a real infinite point where the defining equation of N_λ has a Puiseux expansion beginning from the constant term. Here $O \in U$ is a point belonging to U.

This can be proved by using the second estimate of Lemma 5 and by the Fragmen-Lindelöf theorem of the usual type.

As a realization of this rather unsightly condition, we have

THEOREM 9. *Let $p(\zeta_1, \cdots, \zeta_n)$ be an irreducible polynomial*[3] *of degree m. Let p_m be it's principal part. If for some $k \geqq 1$*

1) $p_m(1, 0, \cdots, 0) = 0, \cdots, \quad \dfrac{\partial^{k-1}}{\partial \zeta_n^{k-1}} p_m(1, 0, \cdots, 0) = 0, \quad \dfrac{\partial^k}{\partial \zeta_n^k} p_m(1, 0, \cdots, 0) \neq 0$

2) *p is of degree $\leqq m-k$ with respect to ζ_1*

then $C_p^\infty(U \backslash O)/C_p^\infty(U) = 0$.

PROOF. By assumption, $(\infty, 0, \cdots, 0)$ is a real infinite point of $N(p) = \{\zeta; p(\zeta) = 0\}$. Expanding the solution ζ_n of $p(\zeta) = 0$ to the Puiseux series at this point, with respect to ζ_1, we see that this series begins with a constant term. In fact, we can make the following elementary consideration. Rewrite

$$p(\zeta) = p_1(\zeta) + \zeta_n^k p_2(\zeta) + c\zeta_n^k \zeta_1^{m-k} + p_3(\zeta) = 0 ,$$

where $p_1(\zeta)$ is of degree $\geqq k+1$ with respect to ζ_n, $p_2(\zeta)$ is of degree $\leqq m-k-1$ with respect to ζ_1, p_3 is of degree $\leqq k-1$ with respect to ζ_n, and c is a constant. By assumption $c \neq 0$. On the other hand, $p_1(\zeta)$ is of degree $< m-k$ with respect to ζ_1. Noticing that $\dfrac{\zeta_n}{\zeta_1} \to 0$ when $\zeta_1 \to \infty$, we can rewrite the terms $q(\zeta_2, \cdots, \zeta_{n-1})\zeta_1^{k_1}\zeta_n^{k_2}$ in $p_1(\zeta)$ as

$$\left(\frac{\zeta_n}{\zeta_1}\right)^{k_2-k} \cdot q \cdot \zeta_1^{k_1+k_2-k} \zeta_n^k = \zeta_1^{m-k} \cdot \zeta_n^k \cdot o\left(\frac{\zeta_n}{\zeta_1}\right) \cdot q ,$$

because $k_2 - k \geqq 1$ and $k_1 + k_2 - k \leqq m-k$ on account of $k_1 + k_2 \leqq m$. Also we can rewrite the terms in $\zeta_n^k p_2(\zeta)$ as $\zeta_n^k \cdot \zeta_1^{m-k} \cdot o\left(\dfrac{1}{\zeta_1}\right) \cdot q$. Thus dividing the both sides of the equation by $c\zeta_1^{m-k}$, we get the following equation on ζ_n, which is equivalent to the initial one when $|\zeta_1|$ is large:

$$\zeta_n^k \left(1 + o\left(\frac{\zeta_n}{\zeta_1}\right)\right) + (\text{a polynomial of degree} \leqq k-1 \text{ on } \zeta_n) = 0.$$

We see easily that this equation has bounded coefficients, and the coefficient of the highest degree is indeed apart from zero when $|\zeta_1|$ is large enough. Thus

[3] If p is an irreducible polynomial, the normality with respect to ζ_n is not necessary, since the associated Noetherian operator is only the restriction to $N(p)$.

our equation has only bounded solutions when $|\zeta_1|$ is large. Therefore the Puiseux expansion of ζ_n with respect to ζ_1 must begin with a constant term.

q.e.d.

REMARK. 1) The equivalent condition for $C_p^\infty(U\backslash O)/C_p^\infty(U)=0$ is not yet known. We only remark that this is not determined by the principal part alone (see e.g. examples of Grušin [3]). 2) In the case of $k=1$, the second condition is excessive, and this is Grušin's theorem ([3]). 3) It is easily seen that if K has an interior $\mathring{K}\neq\phi$, $C_p^\infty(U\backslash K)/C_p^\infty(U)\neq 0$ always. For example, take a solution $u\in\mathscr{E}^\infty(U)$ of $p(D)u=f$ for $f\in\mathscr{E}_0^\infty(\mathring{K})$. Then, this u is a non-trivial element of $\mathscr{E}_p^\infty(U\backslash K)/\mathscr{E}_p^\infty(U)$. On the other hand, when K is convex and thin, some interesting aspects occur.

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