Products of fractional powers of operators

Dedicated to Professor Kôsaku Yosida on his 60th birthday

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§1. In the study of fractional powers of operators H. Komatsu [3] [4] has investigated precisely the space $D_p^\sigma(A)$, $1\leqq p\leqq\infty$ or $p=\infty-$ for a closed linear operator A in a Banach space X such that

(1) $$\|\lambda(\lambda+A)^{-1}\|\leqq M \quad \text{for} \quad \lambda>0\,,$$

where M is a constant independent of λ. In this paper we consider a family $A=(A_1,\cdots,A_n)$ of linear operators of this kind which satisfy the condition that

(2) $$(\lambda_j+A_j)^{-1}(\lambda_k+A_k)^{-1}=(\lambda_k+A_k)^{-1}(\lambda_j+A_j)^{-1} \quad \text{for} \quad 0<\lambda_j,\ \lambda_k;\ j,\ k=1,\cdots,n\,,$$

and study the properties of the space $D_p^{(\sigma_1,\cdots,\sigma_n)}(A)=\bigcap_{j=1}^n D_p^{\sigma_j}(A_j)$ whose norm is

(3) $$\|x\|_{D_p^{(\sigma_1,\cdots,\sigma_n)}(A)}=\sum_{j=1}^n\|x\|_{D_p^{\sigma_j}(A_j)}\,.$$

To describe our results precisely we shall use the following notations. R_+ denotes the set of positive numbers. For X-valued strongly measurable function $f(t)$ of $t=(t_1,\cdots,t_n)$, we introduce the norm

(4) $$\|f\|_{L_*^p(R_+^n;X)}=\left(\int_{R_+^n}\|f(t)\|^p\frac{dt}{|t|^n}\right)^{1/p} \quad \text{if} \quad 1\leqq p<\infty\,,$$
$$=\sup_{R_+^n}\|f(t)\| \quad \text{if} \quad p=\infty\,,$$

where $|t|$ denotes the Euclidean norm and $dt=dt_1\cdots dt_n$. The space $L_*^p(R_+^n;X)$ is the set of all functions whose norms defined in (3) are finite. $L_*^{\infty-}(R_+^n;X)$ is the subspace of $L_*^\infty(R_+^n;X)$ whose members $f(t)$ tend to zero as $|t|\to\infty$.

For an operator A possessing the above properties, $D_{p,m}^\sigma(A)$, $1\leqq p\leqq\infty$ or $p=\infty-$, $0<\sigma<m$, is the space of all x in X such that $\lambda^\sigma(A(\lambda+A)^{-1})^m x\in L_*^p(R_+;X)$ with the norm

(5) $$\|x\|_{D_{p,m}^\sigma(A)}=\|x\|+\|\lambda^\sigma(A(\lambda+A)^{-1})^m x\|_{L_*^p(R_+;X)}\,.$$

It is known that the space $D_{p,m}^\sigma(A)$ does not depend on m ([4] Proposition 1.2), so that we omit the subscript m on it.

Now let us state our results.

THEOREM 1. *Let $A=(A_1, \cdots, A_n)$ be a family of densely defined closed linear operators in a Banach space X with properties* (1) *and* (2). *Assume that* $0<\mathrm{Re}\,\alpha_j<\sigma_j$ *or* $\alpha_j=0$, $\rho=1-\sum \mathrm{Re}\,\alpha_j/\sigma_j>0$. *Then the space $D_p^{(\sigma_1,\cdots,\sigma_n)}(A)$ contains the domain $D(A^\alpha)$ of the operator $A^\alpha=A_1^{\alpha_1}\cdots A_n^{\alpha_n}$ and the inequality*

$$(6)\qquad \|A^\alpha x\|_{D_p^{(\rho\sigma_1,\cdots,\rho\sigma_n)}(A)} \leqq C\|x\|_{D_p^{(\sigma_1,\cdots,\sigma_n)}(A)}$$

holds for all x in $D_p^{(\sigma_1,\cdots,\sigma_n)}(A)$, where C is a constant independent of x.

The inequality (6) is a generalization of the inequality concerning mixed derivatives of functions of several variables belonging to the Besov space $B_{q,p}^{(\sigma_1,\cdots,\sigma_n)}(R^n)$ (cf. [1]):

$$(6')\qquad \|D^\alpha\varphi\|_{B_{q,p}^{(\rho\sigma_1,\cdots,\rho\sigma_n)}} \leqq C\|\varphi\|_{B_{q,p}^{(\sigma_1,\cdots,\sigma_n)}} \quad \text{for} \quad \varphi\in B_{q,p}^{(\sigma_1,\cdots,\sigma_n)}$$

where $D^\alpha=\partial^{|\alpha|}/(\partial x_1)^{\alpha_1}\cdots(\partial x_n)^{\alpha_n}$, $|\alpha|=\alpha_1+\cdots+\alpha_n$. Our inequality (6) is more general than (6)′ in two directions: firstly $A_1, \cdots, A_n$ are abstract operators, secondly $\alpha_1, \cdots, \alpha_n$ are not necessarily integers.

The space $D_p^{(\sigma,\cdots,\sigma)}(A)$ is characterized as the mean interpolation space in the sense of Lions-Peetre [5];

THEOREM 2. *Let $A=(A_1, \cdots, A_n)$ be as in Theorem* 1. *Then the space $D_p^{(\theta m,\cdots,\theta m)}(A)$ coincides with $S(p, \theta, X; p, \theta-1, D^m)$, where $D^m=\cap D(A^\alpha)$, $A^\alpha=A_1^{\alpha_1}\cdots A_n^{\alpha_n}$, α ranges over all n-vectors of non-negative integers with $|\alpha|=\alpha_1+\cdots+\alpha_n=m$, and $0<\theta<1$.*

When $-A_1, \cdots, -A_n$ are infinitesimal generators of bounded strongly continuous semi-groups $G_1, \cdots, G_n$ of operators, respectively, with properties that

$$(7)\qquad \|G_j(t_j)\|\leqq M \quad \text{for} \quad t_j>0,\ j=1,\cdots,n\,,$$

and

$$(8)\qquad G_j(t_j)G_k(t_k)=G_k(t_k)G_j(t_j) \quad \text{for} \quad t_j, t_k>0,\ j,k=1,\cdots,n\,,$$

the space $D_p^{(\sigma,\cdots,\sigma)}(A)$ is characterized as follows:

THEOREM 3. *Let $A_1, \cdots, A_n$ be as above. Then the space $D_p^{(\sigma,\cdots,\sigma)}(A)$ is identical with the space of points x such that*

$$(9)\qquad \|x\|+\||t|^{-\sigma}(I-G(t))^m x\|_{L_*^p(R_+^n;X)}<\infty\,,$$

where $G(t)=G_1(t_1)\cdots G_n(t_n)$. Moreover, the norm of $D_p^{(\sigma,\cdots,\sigma)}(A)$ is equivalent to the norm (9).

§ **2.** For future reference we note here some facts concerning linear operators in Banach spaces.

LEMMA 2.1. *Let A, B be two linear operators and assume that $(\lambda+A)^{-1}$ and $(\mu+B)^{-1}$ exist and are bounded. Then the following conditions are equivalent*:

(i) $(\lambda+A)^{-1}(\mu+B)^{-1}=(\mu+B)^{-1}(\lambda+A)^{-1}$,

(ii) $(\lambda+A)^{-1}x\in D(B)$ *and* $(\lambda+A)^{-1}Bx=B(\lambda+A)^{-1}x$ *for* $x\in D(B)$,

(iii) $(\lambda+B)^{-1}x\in D(A)$ *and* $(\mu+B)^{-1}Ax=A(\mu+B)^{-1}x$ *for* $x\in D(A)$,

where $D(A)$ denotes the domain of A.

PROOF. Assume (i). Then for $x\in D(B)$ we get

$$\begin{aligned} B(\lambda+A)^{-1}x &= B(\lambda+A)^{-1}(\mu+B)^{-1}(\mu+B)x \\ &= B(\mu+B)^{-1}(\lambda+A)^{-1}(\mu+B)x \\ &= \{1-\mu(\mu+B)^{-1}\}(\mu+A)^{-1}(\mu+B)x \\ &= (\lambda+A)^{-1}\{1-\mu(\mu+B)^{-1}\}(\mu+B)x=(\lambda+A)^{-1}Bx . \end{aligned}$$

Conversely, assume (ii). Then we get

$$\begin{aligned} (\mu+B)^{-1}(\lambda+A)^{-1} &= (\mu+B)^{-1}(\lambda+A)^{-1}(\mu+B)(\mu+B)^{-1} \\ &= (\mu+B)^{-1}(\mu+B)(\lambda+A)^{-1}(\mu+B)^{-1}=(\lambda+A)^{-1}(\mu+B)^{-1} . \end{aligned}$$

The equivalence between (i) and (iii) is proved in the same manner.

A closed linear operator A is the negative of the infinitesimal generators of a bounded strongly continuous semi-group $G(t)$ if and only if the domain $D(A)$ is dense and

$$\|\lambda^m(\lambda+A)^{-m}\|\leqq M, \quad \text{for} \quad \lambda>0, \ m=1, 2, \cdots . \tag{10}$$

LEMMA 2.2. *Let $G(t)$ and A be as above. Then*

$$\|A^\alpha I(t)^m\|\leqq C_{\alpha,m}t^{m-\mathrm{Re}\,\alpha} , \tag{11}$$

where $0<\mathrm{Re}\,\alpha<m$, *and* $I(t)=\int_0^t G(\tau)d\tau$.

PROOF. Evidently it is sufficient to prove the inequality for $0<\mathrm{Re}\,\alpha<1=m$. Since $I(t)x\subset D^1_\infty(A)$, we have

$$A^\alpha I(t)x=c_\alpha\int_0^\infty \lambda^{\alpha-1}A(\lambda+A)^{-1}I(t)xd\lambda ,$$

for any x in X ([4]). Combining this with

$$\|\lambda A(\lambda+A)^{-1}I(t)\|=\|\lambda(\lambda+A)^{-1}AI(t)\|=\|\lambda(\lambda+A)^{-1}(1-G(t))\|\leqq 2M^2 ,$$

and $\|I(t)\|\leqq tM$, we obtain

$$\|A^\alpha I(t)x\| \leqq cM^2 t\int_0^{1/t} \lambda^{\mathrm{Re}\,\alpha-1}\|x\|d\lambda + 2cM^2\int_{1/t}^{\infty} \lambda^{\mathrm{Re}\,\alpha-2}\|x\|d\lambda$$
$$= cM^2\left(\frac{1}{\mathrm{Re}\,\alpha}+\frac{2}{1-\mathrm{Re}\,\alpha}\right)t^{1-\mathrm{Re}\,\alpha}\|x\| .$$

LEMMA 2.3. *Let* $-A$ *and* $-B$ *be the infinitesimal generators of bounded strongly continuous semi-groups* $G(t)$ *and* $H(t)$, *respectively. Then* $(\lambda+A)^{-1}$ *and* $(\mu+B)^{-1}$ *commute for all* $\lambda, \mu>0$ *if and only if* $G(t)$ *and* $H(s)$ *commute for all* $t, s>0$.

PROOF. This lemma is an immediate consequence of the representations

$$(\lambda+A)^{-1}x=\int_0^\infty e^{-\lambda t}G(t)xdt , \tag{12}$$

and

$$G(t)x=\lim_{n\to\infty} \exp(-tA(1+n^{-1}A)^{-1})x . \tag{13}$$

(See Hille-Phillips [2], K. Yosida [7].)

We shall frequently use the following well-known fact.

LEMMA 2.4. *Let* (M_1, m_1) and (M_2, m_2) *be* σ*-finite measure spaces. Then the integral operator*

$$Kf(x)=\int_{M_2} K(x, y)f(y)m_2(dy)$$

is a bounded linear operator from $L^p(M_2, m_2)$ *into* $L^p(M_1, m_1)$, $1\leqq p\leqq\infty$, *if*

$$\int|K(x, y)|m_1(dx) , \quad \int|K(x, y)|m_2(dy)\leqq C ,$$

where C *is a constant independent of* x *and* y.

§3. Proof of Theorem 1 is given by a repeated application of the following

LEMMA 3.1. *Let* A_1, A_2 *satisfy the conditions stated in Theorem 1. If* $0<\mathrm{Re}\,\alpha<\sigma_1$, *then, for* $x\in D_p^{(\sigma_1,\sigma_2)}(A_1, A_2)$,

$$\|A_1^\alpha x\|_{D_p^{(\tau_1,\tau_2)}(A_1,A_2)}\leqq C\|x\|_{D_p^{(\sigma_1,\sigma_2)}(A_1,A_2)} , \tag{14}$$

where $\rho=1-(\mathrm{Re}\,\alpha/\sigma_1)$, $\tau_1=\rho\sigma_1$, *and* $\tau_2=\rho\sigma_2$.

PROOF. Let $x\in D_p^{(\sigma_1,\sigma_2)}(A_1, A_2)$ be given. $x\in D(A_1^\alpha)$ and the inequality

$$\|A_1^\alpha x\|_{D_p^{\tau_1}(A_1)}\leqq C\|x\|_{D_p^{\sigma_1}(A_1)}$$

is already known (Komatsu [4]). Also,

$$A_1^\alpha x=\frac{\Gamma(m)}{\Gamma(\alpha)\Gamma(m-\alpha)}\int_0^\infty \lambda^{\alpha-1}(A_1(\lambda+A_1)^{-1})^m x d\lambda \quad (m \text{ is an integer}>\sigma_1) .$$

Operating $(A_2(\mu+A_2)^{-1})^n$ on both sides and taking norms, we have

(15) $$\|\mu^{\tau_2}(A_2(\mu+A_2)^{-1}A_1^\alpha x\| \leqq f(\mu)+g(\mu) ,$$

where

$$f(\mu)=c_1\int_{\mu^\gamma}^{\infty}\mu^{\tau_2}\lambda^{\mathrm{Re}\,\alpha-1}\|(A_1(\lambda+A_1)^{-1})^m x\|d\lambda ,$$

$$g(\mu)=c_2\int_0^{\mu^\gamma}\mu^{\tau_2}\lambda^{\mathrm{Re}\,\alpha-1}\|(A_2(\mu+A_2)^{-1})^n x\|d\lambda$$

$$=c_2'\mu^{\tau_2+\gamma\mathrm{Re}\,\alpha}\|(A_2(\mu+A_2)^{-1})^n x\| .$$

Now, taking $\gamma=\sigma_2/\sigma_1$, so that $\tau_2+\gamma\,\mathrm{Re}\,\alpha=\sigma_2$, we obtain

(16) $$\|g\|_{L_*^p(R_+;X)} \leqq c_3\|x\|_{D_p^{\sigma_2}(A_2)} .$$

Setting

$$K(\mu,\lambda)=\begin{cases}\mu^{\tau_2}\lambda^{\mathrm{Re}\,\alpha-\sigma_1}, & \text{for } \lambda\geqq\mu^\gamma ,\\ 0, & \text{otherwise,}\end{cases}$$

we have

$$\int_0^\infty K(\mu,\lambda)\frac{d\lambda}{\lambda}=\frac{1}{\sigma_1-\mathrm{Re}\,\alpha} , \qquad \int_0^\infty K(\mu,\lambda)\frac{d\mu}{\mu}=\frac{1}{\tau_2}$$

and

$$f(\mu)=c_1\int_0^\infty K(\mu,\lambda)\lambda^{\sigma_1}\|(A_1(\lambda+A_1)^{-1})^m\|\frac{d\lambda}{\lambda} ,$$

so that

(17) $$\|f\|_{L_*^p(R_+;X)} \leqq c_4\|\lambda^{\sigma_1}(A_1(\lambda+A_1)^{-1})^m x\|_{L_*^p(R_+;X)} \leqq c_4\|x\|_{D_p^{\sigma_1}(A_1)} .$$

Combining (15), (16) and (17), we obtain

$$\|A_1^\alpha x\|_{D_p^{\tau_2}(A_2)} \leqq C\|x\|_{D_p^{(\sigma_1,\sigma_2)}(A_1,A_2)} ,$$

which completes the proof of the lemma.

§4. PROOF OF THEOREM 2. It is obvious that

$$S(p,\theta,X;p,\theta-1,D^m)\subset S(p,\theta,X;p,\theta-1,D(A_j^m))=D_p^{\theta m}(A_j)$$

since D^m is contained in $D(A_j^m)$ (also see [4]).

Conversely, let $x\in D_p^{(\theta m,\cdots,\theta m)}(A)$ be given. From the identity

(18) $$\frac{d}{d\lambda}(F_{m,l}(\lambda))=\lambda^{mn-1}T(\lambda)S(\lambda)^{l-1}R(\lambda)^{m+l} ,$$

where

$$R(\lambda)=(\lambda+A_1)^{-1}\cdots(\lambda+A_n)^{-1} ,$$

$$T(\lambda)=(\sum_j A_j)\lambda^{n-1}+2(\sum_{j\neq k} A_jA_k)\lambda^{n-2}+\cdots+nA_1\cdots A_n ,$$
$$S(\lambda)=(\sum_j A_j)\lambda^{n-1}+(\sum_{j\neq k} A_jA_k)\lambda^{n-2}+\cdots+A_1\cdots A_n ,$$

and

$$F_{m,l}(\lambda)=\sum_{k=0}^{l-1}(-1)^k\binom{l-1}{k}\frac{\lambda^{mn+nk}}{m+k}R(\lambda)^{m+k} ,$$

which will be proved later on, it follows that

$$\int_1^N \lambda^{mn-1}T(\lambda)S(\lambda)^{l-1}R(\lambda)^{m+l}x d\lambda=F_{m,l}(N)x-F_{m,l}(1)x .$$

Consequently, setting

$$u(\lambda)=\begin{cases} C\lambda^{mn}T(\lambda)S(\lambda)^{l-1}R(\lambda)^{m+l}x & \text{for} \quad \lambda\geqq 1 \\ Cm\lambda^m F_{m,l}(1)x & \text{for} \quad \lambda<1 , \end{cases}$$

where

$$C=\left\{\sum_{k=0}^{l-1}(-1)^k\binom{l-1}{k}\frac{1}{m+k}\right\}^{-1}=m\binom{m+l-1}{l-1} ,$$

we find that

$$x=\int_0^\infty u(\lambda)\frac{d\lambda}{\lambda} ,$$

for $[N(N+A_j)^{-1}]^m x$ tends to x as $N\to\infty$. Therefore, it is sufficient to show that

$$\|\lambda^{\theta m-|\beta|}A^\beta u(\lambda)\|_{L^p_*(R_+;X)}\leqq C\|x\|_{D_p^{(\theta m,\cdots,\theta m)}(A)}$$

is valid for all n-vectors $\beta=(\beta_1,\cdots,\beta_n)$ of non-negative integers with $|\beta|\leqq m$. Since the inequality

$$\int_0^1 \|\lambda^{\theta m-|\beta|}A^\beta u(\lambda)\|^p\frac{d\lambda}{\lambda}\leqq \text{const.}\|x\|^p$$

is obvious, we only consider the itegration over $1\leqq\lambda<\infty$. From the definition of $u(\lambda)$ and the facts

$$\|\lambda(\lambda+A_j)^{-1}\| , \quad \|A_j(\lambda+A_j)^{-1}\|\leqq M$$

we have

$$\lambda^{\theta m-|\beta|}\|A^\beta u(\lambda)\|\leqq C_1\lambda^{\theta m}\sum_{|\alpha|=l}\|A^\alpha(\lambda+A_1)^{-\alpha_1}\cdots(\lambda+A_n)^{-\alpha_n}x\| .$$

Now, let l be an integer greater than θmn, then $\alpha_j>\theta m$ for some j. Since

$$\|A^\alpha(\lambda+A_1)^{-\alpha_1}\cdots(\lambda+A_n)^{-\alpha_n}x\|\leqq M^{l-\alpha_j}\|A_j^{\alpha_j}(\lambda+A_j)^{-\alpha_j}x\| ,$$

it follows that

$$\int_1^\infty \{\lambda^{\theta m}\|A^\alpha(\lambda+A_1)^{-\alpha_1}\cdots(\lambda+A_n)^{-\alpha_n}x\|\}^p\frac{d\lambda}{\lambda}\leqq C_2\|x\|_{D_p^{\theta m}(A_j)} .$$

Thus we have

$$\int_1^\infty \|\lambda^{\theta m-|\beta|}A^\beta u(\lambda)\|^p\frac{d\lambda}{\lambda}\leqq C_3\|x\|^p_{D_p^{(\theta m,\cdots,\theta m)}(A)} ,$$

as we wished to show.

PROOF OF (18). Induction on l. For $l=1$, the identity can be shown from a simple calculation as follows:

$$\frac{d}{d\lambda}(\lambda^{mn}R(\lambda)^m)=mn\lambda^{mn-1}R(\lambda)^m-\sum_{j=1}^n m\lambda^{mn}R(\lambda)^m(\lambda+A_j)^{-1}$$
$$=m\lambda^{mn-1}T(\lambda)R(\lambda)^{m+1} .$$

Assuming that the identity is valid for $l-1$, we have

$$\frac{d}{d\lambda}F_{m,l}=\frac{d}{d\lambda}(F_{m,l-1}-F_{m+1,l-1})$$
$$=\lambda^{mn-1}T(\lambda)S(\lambda)^{l-2}R(\lambda)^{m+l-1}-\lambda^{mn+n-1}T(\lambda)S(\lambda)^{l-2}R(\lambda)^{m+l}$$
$$=\lambda^{mn-1}T(\lambda)S(\lambda)^{l-2}\{(\lambda+A_1)\cdots(\lambda+A_n)-\lambda^n\}R(\lambda)^{m+l}$$
$$=\lambda^{mn-1}T(\lambda)S(\lambda)^{l-1}R(\lambda)^{m+l} ,$$

which completes the proof of (18).

REMARK. Our proof also gives

$$D_p^{(\theta m,\cdots,\theta m)}(A)=S(p,\theta,X;p,\theta-1,\bigcap_{j=1}^n D(A_j^m)) .$$

§5. PROOF OF THEOREM 3. First we shall show that

$$\|u^{-\sigma}(1-G_j(u))^m x\|_{L_*^p(R_+;X)}\leqq C\||t|^{-\sigma}(1-G(t))^m x\|_{L_*^p(R_+^n;X)} , \tag{19}$$

for $j=1,\cdots,n$. To this end we employ the following identity

$$\sum_{k=0}^m\binom{m}{k}(-1)^k\{(t^k-1)^m u^{m-k}-(t^k-u)^m\}=0 , \tag{20}$$

which is verified as follows:

$$\sum_{k=0}^m\binom{m}{k}(-1)^k(t^k-1)^m u^{m-k}=\sum_{k=0}^m\binom{m}{k}(-1)^k\sum_{j=0}^m\binom{m}{j}(-1)^{m-j}t^{kj}u^{m-k}$$
$$=\sum_{j=0}^m\binom{m}{j}(-1)^j\sum_{k=0}^m\binom{m}{k}(-1)^{m-k}t^{jk}u^{m-k}$$
$$=\sum_{j=0}^m\binom{m}{j}(-1)^j(t^j-u)^m .$$

Using (20) with u replaced by $G_1(u)$ and t replaced by $G_1(u)G(t)$, we have

$$\{1-G_1(u)\}^m=\sum_{k=1}^{m}(-1)^k\binom{m}{k}\{(G_1(ku)G(kt)-1)^m G_1(mu-ku)$$
$$-(G(kt)G_1(ku-u)-1)^m G_1(mu)\}\ .$$

Thus, setting $f(t)=\|(1-G(t))^m x\|$, we obtain

$$\|(1-G_1(u))^m x\|\leqq M\sum_{k=1}^{m}\binom{m}{k}\{f(ku+kt_1\,,\,kt')+f(ku-u+kt_1\,,\,kt')\}\ ,$$

where $t'=(t_2\,,\cdots,t_n)$. Integrating this inequality over $0\leqq t_1\,,\cdots,t_n\leqq u$, we have

$$\|(1-G_1(u))^m x\|\leqq\frac{M}{u^n}\sum_{k=1}^{m}\binom{m}{k}\int_0^u\cdots\int_0^u\{f(ku+kt_1\,,\,kt')+f(ku-u+kt_1\,,\,kt')\}dt\ ,$$
$$\leqq\frac{cM}{u^n}\int_0^{2mu}\int_0^{mu}\cdots\int_0^{mu}f(t)dt\ .$$

If we set

$$K(u,t)=u^{-n-\sigma}|t|^{n+\sigma},\quad\text{for}\quad 0\leqq t_1\leqq 2mu\ ,\ 0\leqq t_2\,,\cdots,t_n\leqq mu\ ,$$
$$=0,\qquad\text{otherwise,}$$

then $u^{-\sigma}$ times the right hand side of the above inequality is expressed as

$$cM\int K(u,t)\,|t|^{-\sigma}f(t)\frac{dt}{|t|^n}\ ,$$

so that its $L_*^p(R_+\,;X)$-norm does not exceed

$$cC_2M\||t|^{-\sigma}f(t)\|_{L_*^p(R_+^n;X)}\ ,$$

since

$$\int_{R_+^n}K(u,t)\frac{dt}{|t|^n}\ ,\qquad\int_0^\infty K(u,t)\frac{du}{u}\leqq C_2\ .$$

This completes the proof of (19) for $j=1$. The other cases are proved analogously. If the right hand side of (19) is finite, then $u^{-\sigma}(1-G_j(u))^m x\in L_*^p(R_+\,;X)$, $j=1,\cdots,n$, so that $x\in D_p^{(\sigma,\cdots,\sigma)}(A)$ by [4] Theorem 4.3.

To prove the converse inequality

$$\||t|^{-\sigma}(1-G(t))^m x\|_{L_*^p(R_+^n;X)}\leqq C\|x\|_{D_p^{(\sigma,\cdots,\sigma)}(A)} \tag{21}$$

it is sufficient to verify

$$\||t|^{-\sigma}(1-G_1(t_1))^{k_1}\cdots(1-G_n(t_n))^{k_n}x\|_{L_*^p(R_+^n;X)}\leqq C\|x\|_{D_p^{(\sigma,\cdots,\sigma)}(A)} \tag{22}$$

for any n-vector $k=(k_1\,,\cdots,k_n)$ of non-negative integers with $|k|=k_1+\cdots+k_n=m$. Since $m>\sigma$, there exist real numbers $\alpha_1\,,\cdots,\alpha_n$ such that

$$\sigma=\alpha_1+\cdots+\alpha_n,\ 0\leqq\alpha_j\leqq k_j,\ j=1,\cdots,n,$$

and that atmost one of them, say $0<\alpha_1<k_1$. From Lemma 2.2 and the fact that

$$1-G_j(t_j)=A_jI_j(t_j),\quad I_j(s)=\int_0^s G_j(u)du,$$

it follows that

$$\begin{aligned}\|\Pi(1-G_j(t_j))^{k_j}x\|&=\|\prod_{j=2}^n A_j^{k_j-\alpha_j}I_j(t_j)^{k_j}(1-G_1(t_1))^{k_1}y\|\\&\leqq C_1\prod_{j=2}^n t_j^{\alpha_j}\|(1-G_1(t_1))^{k_1}y\|,\end{aligned}$$

where $y=A_2^{\alpha_2}\cdots A_n^{\alpha_n}x$. Consequently, we obtain

$$\begin{aligned}&\||t|^{-\sigma}\prod_{j=1}^n(1-G_j(t_j))^{k_j}x\|_{L_*^p(R_+^n;X)}\\&\leqq C_1\left\{\int_0^\infty\|(1-G_1(t_1))^{k_1}y\|^p dt_1\int_{R_+^{n-1}}\frac{t_2^{\alpha_2 p}\cdots t_n^{\alpha_n p}}{|t|^{n+p\sigma}}dt'\right\}^{1/p}\\&=C_2\left\{\int_0^\infty\|(1-G_1(t_1))^{k_1}y\|^p t_1^{-p\alpha_1-1}dt_1\right\}^{1/p}\\&\leqq C_3\|y\|_{D_p^{\alpha_1}(A_1)}\ \text{(see [4] Theorem 4.3).}\end{aligned}$$

But, by virtue of Theorem 1, it follows that

$$\|y\|_{D_p^{\alpha_1}(A_1)}\leqq C_4\|x\|_{D_p^{(\sigma,\cdots,\sigma)}(A)},$$

and Theorem 3 is established.

§ 6. EXAMPLE. Let $X=L^q(R^n)$, $1\leqq q\leqq\infty$, $G_j(t_j)\varphi(x)=\varphi(x_1,\cdots,x_j-t_j,\cdots,x_n)$. Then $G_1,\cdots,G_n$ satisfy the conditions stated in Theorem 3, and their generators are $-\partial/\partial x_1,\cdots,-\partial/\partial x_n$. The space $D_p^\alpha(\partial/\partial x_j)$ is the space of all φ such that

$$\left\{\int_0^\infty t^{-\sigma p-1}\|\Delta^m(te_j)\varphi\|_{L^q(R^n)}^p dt\right\}^{1/p}<\infty, \tag{23}$$

where $\Delta(a)\varphi(x)=\varphi(x)-\varphi(x-a)$, $e_1=(1,0,\cdots,0),\cdots,e_n=(0,0,\cdots,1)$. By virtue of [4] Theorem 2.6 (23) is equivalent with

$$\left\{\int_0^\infty t^{-\alpha p-1}\|\Delta^2(te_j)\left(\frac{\partial}{\partial x_j}\right)^\nu\varphi\|_{L^q(R^n)}^p dt\right\}^{1/p}, \tag{24}$$

where $\alpha+\nu=\sigma$, $0<\alpha\leqq1$, ν is an integer. Thus $D_p^{(\sigma_1,\cdots,\sigma_n)}(\partial/\partial x_1,\cdots,\partial/\partial x_n)$ coincides with the Besov space $B_{q,p}^{(\sigma_1,\cdots,\sigma_n)}(R^n)$ (see Besov [1]), and (6)′ is a special case of (6).

It is obvious that the space D^m in Theorem 2 for the present case is the Sobolev space $W_q^m(R^n)$, so that from Theorem 2 we have

$$B_{q,p}^{(\theta m,\cdots\theta m)}(R^n)=S(p,\theta,L^q;p,\theta-1,W_q^m(R^n)) .$$

Finally, Theorem 3 gives us that the Besov space $B_{q,p}^{(\sigma,\cdots,\sigma)}$ is identical with the Lipschitz space $\Lambda(\sigma;q,p;R^n)$ in the sense of Taibleson [6].

References

[1] Besov, O. V., Investigation of a class of function spaces in connection with imbedding and extension theorems, Trudy Mat. Inst. Steklov. **60** (1961), 42-81; English transl., Amer. Math. Soc. Transl. (2) **40** (1964), 85-126.

[2] Hille, E. and R. S. Phillips, Functional analysis and semi-groups, Amer. Math. Soc. Colloq. Publ. **31**.

[3] Komatsu, H., Fractional powers of operators, Pacific J. Math. **19** (1966), 285-346.

[4] Komatsu, H., Fractional powers of operators, II, Interpolation spaces, Pacific J. Math. **21** (1967), 89-111.

[5] Lions, J. L. et J. Peetre, Sur une classe d'espaces d'interpolation, Publ. Math. I. H. E. S. N°19 (1964), 5-68.

[6] Taibleson, M. H., On the theory of Lipschitz spaces of distributions on Euclidean n-space, I, Principal properties, J. Math. Mech. **13** (1964), 407-479.

[7] Yosida, K., Functional Analysis, Springer-Verlag, 1965.

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