On random ellipsoid

Dedicated to Professor Kôsaku Yosida on his 60th birthday

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Let E be the space of $n\times n$ positive definite symmetric matrices of determinant 1. E is an irreducible symmetric Riemanian space. Let L be the Laplace-Beltrami operator:

$$L=\sum_{i,j}\frac{1}{\sqrt{g}}\frac{\partial}{\partial\xi_i}\left(\sqrt{g}\,g^{ij}\frac{\partial}{\partial\xi_j}\right).$$

We consider the diffusion process $\boldsymbol{X}=(x_t,\infty,\mathcal{M}_t,\boldsymbol{P}_x)$ on E with generator L (random ellipsoid). A remarkable property of $\boldsymbol{X}$ consists in its asymptotic behavior, which is formulated as follows.

Let $e^{\lambda_1}\geqq e^{\lambda_2}\geqq\cdots\geqq e^{\lambda_n}$ $(\lambda_1+\cdots+\lambda_n=0)$ be the eigenvalues of $x\in E$. We put

$$x_k=\lambda_k-\lambda_{k+1}\quad(\geqq 0)\,,\quad k=1,\cdots,n-1\,,$$

$$\tilde{x}=(x_1,\cdots,x_{n-1})\,.$$

Then we have

$$\text{(1)}\qquad \boldsymbol{P}_x\left(\lim_{t\to\infty}\frac{\tilde{x}_t}{t}=\rho\right)=1\qquad\text{for any}\quad x\in E\,,$$

where $\rho=(1,\cdots,1)$ and $\tilde{x}_t=\widetilde{x_t}$.

The problem of this type was first discussed by E. B. Dynkin [1]. In [2], he proved, using the theory of Martin boundary, the following

$$\text{(2)}\qquad \boldsymbol{P}_x\left(\lim_{t\to\infty}\frac{\tilde{x}_t}{\|\tilde{x}_t\|}=\frac{\rho}{\|\rho\|},\ x_{t,k}\to\infty,\ 1\leqq k\leqq n-1\right)=1\,.$$

((2) is a consequence of (1)). In this paper we shall prove (1) (for arbitrary symmetric Riemannian space with negative sectional curvature) by means of stochastic integral equation.

1. Let E be a symmetric Riemannian space (of semi-simple type) with negative sectional curvature. We denote by G the connected component of the group of all isometries of E onto E and by K the isotropy subgroup of G at $x_0\in E$. Then, G is a connected semi-simple Lie group (without center), such that all simple components of G are non-compact and K is a maximal compact subgroup of G. E can be identified with $K\backslash G$.

Let $\mathfrak{g}$, $\mathfrak{k}$ be the Lie algebras of G and K, respectively. Then we have

$$\mathfrak{g}=\mathfrak{k}+\mathfrak{p}\ , \tag{3}$$

where $\mathfrak{p}$ is the orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}$ with respect to the Killing form. $\mathfrak{p}$ can be identified with the tangent space of E at x_0. Since the Killing form, restricted on $\mathfrak{p}$, is an Ad K-invariant inner product (X, Y), there exists a G-invariant tensor field $g(X, Y)$ of type $(0, 2)$, which coincides with (X, Y) at x_0. We denote by L the corresponding Laplace-Beltrami operator. L is a G-invariant elliptic differential operator of the second order. In order to construct the diffusion process which has L as generating differential operator, we need the following lemma (Dynkin [3], Theorem 3.13).

LEMMA. *Let E_n $(n=1, 2, \cdots)$ be a sequence of subsets in a measurable space $(E, \mathfrak{B})$, such that $E_n \in \mathfrak{B}$, $E_n \uparrow E$. Further, we suppose that to each n, there corresponds a Markov process in E_n $\boldsymbol{X}^{(n)}=(x_t^{(n)}, \zeta_n, \mathcal{M}_t^{(n)}, \boldsymbol{P}_x^{(n)})$, where $\boldsymbol{X}^{(n)}$ is the part process of $\boldsymbol{X}^{(n+1)}$ on E_n. Then there exists a Markov process $\boldsymbol{X}=(x_t, \zeta, \mathcal{M}_t, \boldsymbol{P}_x)$ on E, such that $\zeta=\lim_{n\to\infty} \zeta_n$ and $\boldsymbol{X}^{(n)}$ is the part process of $\boldsymbol{X}$ on E_n.*

We take as E_n a bounded domain with sufficiently smooth boundary such that $E_n \uparrow E$ and we denote by $P_n(t, x, y)$ the fundamental solution of the boundary value problem:

$$\frac{\partial}{\partial t} v = Lv \qquad \text{in} \quad E_n\ ,$$

$$v|_{\partial E_n}=0\ .$$

It is known that $P_n(t, x, y)$ is the density of the transition function of some Markov process $\boldsymbol{X}_n$ $(n=1, 2, \cdots)$, satisfying the condition of Lemma. Therefore we obtain a Markov process $\boldsymbol{X}=(x_t, \zeta, \mathcal{M}_t, \boldsymbol{P}_x)$, which is, as is easily seen, the required diffusion process.

REMARK. We extend $P_n(t, x, y)$ to a function on $[0, \infty)\times E\times E$, setting

$$P_n(t, x, y)=0\ , \qquad \text{if} \quad (x, y) \notin E_n\times E_n\ .$$

Then, we have

$$P_n(t, x, y) \leqq P_{n+1}(t, x, y)\ .$$

We put

$$P(t, x, y)=\lim_{n\to\infty} P_n(t, x, y)\ .$$

$P(t, x, y)$ is the density of the transition function of X (with respect to an invariant measure on E).

PROPOSITION 1. $P_x(\zeta=\infty)=1$, *for* $x\in E$.

PROOF. For a bounded continuous function $f(x)$ on E,

$$v(t, x)=\int_E P(t, x, y)f(y)dy$$

is a solution of

$$\frac{\partial}{\partial t}v(t, x)=Lv(t, x) . \tag{4}$$

If $f(x)\equiv 1$ on E, we have $v(t, x)=P(t, x, E)$. In view of G-invariance of L, $P(t, x, y)$ is also G-invariant, i.e.

$$P(t, x, y)=P(t, xg, yg) \qquad \text{for} \quad g\in G .$$

Therefore we have

$$v(t, x)=P(t, x, E)=P(t, xg, E)=v(t, xg) ,$$

that is, $v(t, x)$ is independent of x. On the other hand, by (4), we have

$$\frac{d}{dt}v=Lv=0 .$$

Hence,

$$v(t)=\lim_{t\downarrow 0}v(t)=\lim_{t\downarrow 0}P(t, x, E)=1 .$$

2. Let the notations $\mathfrak{g}, \mathfrak{k}, \mathfrak{p}, \cdots$ be as in §1. We denote by $\mathfrak{a}$ a maximal abelian subalgebra in $\mathfrak{p}$ and by R the root system relative to $\mathfrak{a}$. We put

$$\mathfrak{g}_\alpha=\{X\in\mathfrak{g}; [H, X]=\alpha(H)X, \text{ for } H\in\mathfrak{a}\} , \qquad \text{for} \quad \alpha\in R .$$

Choose some order in $\mathfrak{a}^*$ (the dual space of $\mathfrak{a}$). Then we have

$$\mathfrak{g}=\sum_{\alpha\in R}\mathfrak{g}_\alpha+\mathfrak{a}+\mathfrak{m}=\mathfrak{k}+\mathfrak{a}+\sum_{\alpha>0}\mathfrak{g}_\alpha ,$$

where $\mathfrak{m}$ is the centralizer of $\mathfrak{a}$ in $\mathfrak{k}$.

Let $\{\alpha_1, \cdots, \alpha_p\}$ be the (restricted) fundamental root system and denote by A the set

$$\{H\in\mathfrak{a};\ \alpha_i(H)\geqq 0,\ 1\leqq i\leqq p\} .$$

A is a fundamental domain of the (restricted) Weyl group $W=M\backslash\tilde{M}$ ($\tilde{M}(M)$ is the normalizer (centralizer) of $\mathfrak{a}$ in $\mathrm{Ad}\,K$). For $X\in\mathfrak{p}$, there exist $k\in K$ and $H\in A$, such that

$$X=\mathrm{Ad}\,kH .$$

Now the exponential mapping $X \to \exp X$ is a one to one regular mapping of $\mathfrak{p}$ into G and we may identify $\exp \mathfrak{p}$ with E. H is uniquely determined by X and continuously depends on X. Thus we obtain a continuous mapping of E onto $\tilde{E}=[0, \infty)\times \cdots \times[0, \infty)$

$$\gamma(x)=\{\alpha_1(H), \cdots, \alpha_p(H)\} ,$$

where $H=H(\log x)$. Obviously, $\gamma x=\gamma x'$ if and only if $x'=xk$ for some $k\in K$.

In view of the invariance of the transition function,

$$P(t, x, \gamma^{-1}\Gamma)=P(t, x', \gamma^{-1}\Gamma) \qquad \text{for a Borel set } \Gamma \text{ in } \tilde{E}.$$

Therefore, by a theorem of Dynkin ([4], Theorem 10.13), there exists a Markov process on $\tilde{E}$

$$\tilde{\boldsymbol{X}}=(\tilde{x}_t, \infty, \tilde{\mathcal{M}}_t, \tilde{\boldsymbol{P}}_x) ,$$

such that

$$\tilde{x}_t=\gamma x_t , \quad \tilde{P}(t, \gamma x, \Gamma)=P(t, x, \gamma^{-1}\Gamma) .$$

Further, let A, $\tilde{A}$ be the infinitesimal generator of X and $\tilde{X}$, respectively. Then,

$$f\in \mathfrak{D}_{\tilde{A}} \quad \text{if and only if} \quad \gamma^* f\in \mathfrak{D}_A ,$$

$$\gamma^*\tilde{A}f=A\gamma^* f ,$$

where γ^* is a linear mapping of $B(\tilde{E})$ onto $B(E)$, defined by

$$\gamma^* f(x)=f(\gamma x) .$$

($B(E)$ is the space of all bounded measurable functions on E). We want to determine the explicit form of $\tilde{A}$ in $\tilde{E}'$ (the interior of $\tilde{E}$).

PROPOSITION 2. *Let $f(x)$ be a twice continuously differentiable function in $\tilde{E}'$ with compact support. Then we have*

$$f\in \mathfrak{D}_{\tilde{A}} ,$$

and

$$Af=\sum_{i,j=1}^{p} a^{ij}\frac{\partial^2}{\partial x_i\partial x_j}f+\sum_{k=1}^{p} b_k(x)\frac{\partial}{\partial x_k}f , \tag{5}$$

where

$$a^{ij}=(\alpha_i, \alpha_j) ,$$

$$b_k(x)=\frac{1}{2}\sum_{\alpha>0}(\alpha, \alpha_k)\coth\frac{\alpha(H)}{2}$$

(H *is an element of* $\mathfrak{a}$, *such that* $\alpha_i(H)=x_i$).

PROOF. We put $A'=\{h=\exp H;\ \alpha_i(H)>0, 1\leqq i\leqq p\}$. Then the mapping of $M\backslash K\times A$ into $\exp\mathfrak{p}=E$, defined by

$$x=\varphi(\tilde{u}, h)=u^{-1}hu\ ,$$

is one to one and regular. We have to calculate the image of the invariant tensor field $g(X, Y)$ by $\delta\varphi$. Let π be the projection of $\mathfrak{g}$ onto $\mathfrak{p}$. Since $\mathrm{Ad}\,h$ $(h\in A')$ leaves $\mathfrak{m}$ invariant, $\pi\circ\mathrm{Ad}\,h^{1/2}$ induces a linear mapping of $\mathfrak{m}\backslash\mathfrak{k}$ into $\mathfrak{p}$, which is denoted by η. For a tangent vector X at $\tilde{u}\in M\backslash K$, we have

$$d\varphi X=(\eta(2X(u)))_g \qquad (g=h^{1/2}u)\ , \tag{6}$$

where Y_g is a tangent vector of E at x, obtained as an image of $Y\in\mathfrak{p}$ by $x\to xg$. ($X(u)=X_{u^{-1}}$ is defined similarly.) Let θ be an involution of $\mathfrak{g}$ associated with the decomposition (3). Then we have

$$\theta\mathfrak{g}_\alpha=\mathfrak{g}_{-\alpha}\ .$$

We choose a basis of $\mathfrak{g}_\alpha$: $\{X_\alpha^{(1)}, \cdots, X_\alpha^{(n_\alpha)}\}$. $\mathfrak{m}\backslash\mathfrak{k}$ may be identified with the space spanned by $Y_{i,\alpha}=X_\alpha^{(i)}+\theta X_\alpha^{(i)}$, $(i=1, \cdots, n_\alpha,\ \alpha\in R_+)$. Further, $Z_{i,\alpha}=X_\alpha^{(i)}-\theta X_\alpha^{(i)}$ $(i=1, \cdots, n_\alpha,\ \alpha\in R_+)$, together with H_i $(1\leqq i\leqq p)$, constitute a basis of $\mathfrak{p}$ ($\{H_1, \cdots, H_p\}$ is the dual basis of $\{\alpha_1, \cdots, \alpha_p\}$). Let (u_μ) be a local coordinate system around $\tilde{u}\in M\backslash K$. We put

$$X_\mu=\frac{\partial}{\partial u_\mu}\ ,$$

$$X_\mu(u)=\sum_{i,\alpha}c_{i,\alpha}^\mu(u)Y_{i,\alpha}\ .$$

As is easily seen,

$$\mathrm{Ad}\,hY_{i,\alpha}=\mathrm{ch}\,\alpha(H)Y_{i,\alpha}+\mathrm{sh}\,\alpha(H)Z_{i,\alpha}\ .$$

Hence, by (6),

$$d\varphi X_\mu=\left(2\sum\mathrm{sh}\frac{\alpha(H)}{2}c_{i,\alpha}^\mu Z_{i,\alpha}\right)_g\ .$$

Since $\mathfrak{g}_\alpha$ $(\alpha\in R)$ is orthogonal to $\mathfrak{a}$ with respect to the Killing form, we have

$$\begin{aligned}g_{i\mu}(x)&=g_x\left(d\varphi\left(\frac{\partial}{\partial x_i}\right),\ d\varphi\left(\frac{\partial}{\partial u_\mu}\right)\right)\\&=\left(H_i, \sum 2\,\mathrm{sh}\frac{\alpha(H)}{2}c_{i,\alpha}^\mu(u)Z_{i,\alpha}\right)=0\ ,\end{aligned}$$

$$g_{ij}(x)=(H_i, H_j)\quad (=a_{ij})\ ,$$

$$\det(g_{\mu\nu}(x))=\left(\prod_{\alpha>0}\mathrm{sh}\frac{\alpha(H)}{2}\right)^2c(u)\ .$$

We put

$$q(x)=\prod_{\alpha>0}\operatorname{sh}\frac{\alpha(H)}{2}$$

(each root in this product is repeated as many times as its multiplicity indicates). From the above result, we see

$$Lf=\sum a^{ij}\frac{\partial^2}{\partial x_i\partial x_j}f+\sum\left(\frac{1}{q}\sum a^{ik}\frac{\partial}{\partial x_k}q\right)\frac{\partial}{\partial x_i}f$$
$$+\text{terms involving derivatives with respect to } u_\mu.$$

Since, for $f\in C_0^2(\tilde{E}')$, $f=\gamma^*f\in C_0^2(E)\subset\mathfrak{D}_A$ and

$$Af=Lf\,,$$

we have

$$\tilde{A}f=\sum_{i,j=1}^{p}a^{ij}\frac{\partial^2}{\partial x_i\partial x_j}f+\sum_{k=1}^{p}b_k(x)\frac{\partial}{\partial x_k}f\,,$$

where

$$\begin{aligned}(a^{ij})&=(a_{ij})^{-1}=((\alpha_i,\alpha_j))\\ b_k(x)&=\frac{1}{q}\sum a^{kj}\frac{\partial}{\partial x_j}q=\sum_{\alpha>0}\frac{1}{2}\sum_{j=1}^{p}(\alpha_j,\alpha_k)\alpha(H_j)\coth\frac{\alpha(H)}{2}\\ &=\frac{1}{2}\sum_{\alpha>0}(\alpha,\alpha_k)\coth\frac{\alpha(H)}{2}\,.\end{aligned}$$

PROPOSITION 3. *Let* $\mathfrak{U}$ *be the collection of all open sets* U *such that* $\bar{U}$ *is a compact set in* $\tilde{E}'$ *and* τ *be the first exit time from* $\mathfrak{U}$ ([4], Chap. 4). *Then we have*

$$\tilde{\boldsymbol{P}}_x(\tau<\infty)=0\qquad \textit{for}\quad x\in\tilde{E}'\,. \tag{7}$$

PROOF. If $\tau<\infty$, $\tilde{x}_\tau$ exists and belongs to $\partial\tilde{E}$. This follows from the continuity of $\tilde{x}_t$ ($t\in[0,\infty)$). Put

$$E_{i_1,\dots,i_s}=\{x\in\tilde{E};\,x_{i_1}=\cdots=x_{i_s}=0,\,x_j>0\quad\text{for}\quad j\neq i_k\}\,.$$

Then,

$$\partial\tilde{E}=\cup E_{i_1,\dots,i_s}\,.$$

For a subset F of ∂E, we put

$$p(F)=\tilde{\boldsymbol{P}}_x(\tau<\infty,\tilde{x}_\tau\in F)\,.$$

We shall show $p(E_{i_1,\dots,i_s})=0$, from which follows (7).

Let $\{\alpha_{j_1}, \cdots, \alpha_{j_r}\}$ be a connected component of $\{\alpha_{i_1}, \cdots, \alpha_{i_s}\}$ and

$$F_{j_1,\dots,j_r}=\{x\in \tilde{E};\, x_{j_k}=0\ (k=1, \cdots, r),\, x_j>0, \text{ if } \alpha_j \text{ is not orthogonal to } \alpha_{j_k}\}\ .$$

Since $E_{i_1\dots,i_s}\subset F_{j_1,\dots,j_r}$, it is sufficient to prove

$$p(F_{j_1,\dots,j_r})=0\ .$$

We may assume that $\alpha_{j_{k+1}}$ is not orthogonal to $\alpha_{j_1}, \cdots, \alpha_{j_k}$ $(k=1, \cdots, r-1)$. Consider the set

$$\Sigma=\{\alpha\in R;\, \alpha=\sum_{k=1}^{r} m'_k\alpha_{j_k},\ m'_k \text{ is a positive integer}\}\ .$$

As is easily seen, $S_{\alpha_{j_r}}\cdots S_{\alpha_{j_2}}\alpha_{j_1}\in\Sigma$ where S_α is the reflection of α defined by α. Let $\beta=\sum m_k\alpha_{j_k}$ be the maximal one in Σ. Then we have

$$(\beta, \alpha_{j_k})\geqq 0 \qquad (k=1, \cdots, r)\ .$$

For, if

$$(\beta, \alpha_{j_k})<0\ , \quad \beta'=S_{\alpha_{j_k}}\beta=\beta-\frac{2(\alpha_{j_k}, \beta)}{(\alpha_{j_k}, \alpha_{j_k})}\alpha_{j_k}\in\Sigma\ ,$$

and $\beta'>\beta$, which is impossible. Therefore, for a root $\alpha=\sum_{k=1}^{s}\tilde{m}_k\alpha_{i_k}$ $(\tilde{m}_k\geqq 0)$, we have

$$(\beta, \alpha)\geqq 0\ .$$

Now suppose $p(F_{j_1,\dots,j_r})>0$. Then, in view of the continuity of $\tilde{x}_t$, there exist a compact set $Q\subset F_{j_1,\dots,j_r}$ and a bounded domain $D\subset\tilde{E}$ whose boundary consists of Q and a surface in $\tilde{E}'$ such that

$$f(x)=\tilde{\boldsymbol{P}}_x(\tilde{x}_\sigma\in Q)>0 \qquad \text{for some} \quad x\in D\,,$$

where $\sigma=\sigma_{\partial D}=\inf\{t;\, \tilde{x}_t\in\partial D\}$. $f(x)$ is an L-harmonic function in D, satisfying the boundary condition:

$$\begin{aligned} f(x)&=1 \quad \text{on} \quad Q\,,\\ &=0 \quad \text{on} \quad D-Q\,. \end{aligned}$$

In order to show that $f(x)=0$ in D, we have only to construct a positive superharmonic function in D, $U(x)$, such that $U(x)\to\infty$, uniformly as x tends to Q ([5] Lemma 3.3). Since D is bounded,

$$D\subset\tilde{D}=\{x\in\tilde{E};\ \sum_{k=1}^{r} m_kx_{j_k}\leqq c_1,\quad x_{i_k}\leqq c_1,\quad c_2\leqq x_i\leqq c_1,\quad i\neq i_k\ (k=1, \cdots, s)\}\,,$$

for some $c_1, c_2>0$. We put

$$B(u)=\inf_{x\in\tilde{D},\,\sum m_kx_{j_k}=u}\frac{\sum m_kb_{j_k}(x)}{\sum m_km_ha^{j_kj_h}}\ .$$

By (5), we have

$$\sum m_k m_h a^{j_k j_h} = (\beta, \beta) ,$$

$$\sum m_k b_{j_k}(x) = \frac{1}{2} \sum_{\alpha>0} (\alpha, \beta) \coth \frac{\alpha(H)}{2} .$$

As we have already proved, if $(\alpha, \beta)<0$,

$$\alpha(H_i) = \hat{m}_i > 0 \quad \text{for some index } i \neq i_k \quad (k=1, \cdots, r) .$$

Therefore we have

$$\alpha(H) = \sum_{j=1}^{p} \hat{m}_j \alpha_j(H) \geqq \hat{m}_i \alpha_i(H) = \hat{m}_i x_i \geqq c_2 ,$$

from which follows that

$$\sum_{(\alpha,\beta)<0} (\alpha, \beta) \coth \frac{\alpha(H)}{2} \quad \text{is bounded in } \tilde{D} .$$

Hence we have, in D,

$$\sum m_k b_{j_k}(x) \geqq \frac{1}{2} (\beta, \beta) \coth \frac{\beta(H)}{2} - C_1 ,$$

(8)

$$B(u) \geqq \frac{1}{2} \coth \frac{u}{2} - C_2 \qquad (C_1, C_2 > 0) .$$

We put

$$V(u) = \exp \int_u^* B(v) dv \int_u^* \exp\left(-\int_v^* B(w) dw\right) dv ,$$

$$U(x) = \int_u^* V(v) dv - (u-a)^n + C \Big|_{u=\sum m_k x_{j_k}} .$$

Since $B(u)$ is bounded on $[\varepsilon, c_1]$ $(\varepsilon>0)$,

$$\int_u^* \exp\left(-\int_v^* B(w) dw\right) dv \geqq c' > 0 . \tag{9}$$

From (8) and (9), follows

$$V(u) \geqq \tilde{c} \exp \int_u^* \frac{1}{2} \coth \frac{u}{2} du = c \frac{1}{\operatorname{sh} u/2} .$$

Therefore

$$U(x) \to \infty \quad \text{uniformly as} \quad x \to Q .$$

It is easy to see that

$$L \int_{\sum m_k x_{j_k}}^* V(u) du = -\sum m_k m_h a^{j_k j_h} V'(\sum m_k x_{j_k}) - \sum m_k b_{j_k}(x) V(\sum m_k x_{j_k}) \leqq (\beta, \beta) .$$

Hence, for a suitable choice of $n, a, C, U(x)$ satisfies the required conditions:

$$LU(x)<0, \quad U(x)>0 \quad \text{in} \quad D.$$

(See the proof of Theorem 3.2 [5].)

PROPOSITION 4. *Let* $\tilde{\sigma}=\inf\{t; \tilde{x}_t \in \tilde{E}'\}$. *Then, we have*

$$\tilde{P}_x(\tilde{\sigma}=0)=1 \qquad \textit{for} \quad x \in \partial\tilde{E}.$$

PROOF. Let

$$X=(\mathrm{Ad}\, k)H \quad (H \in A).$$

$\mathrm{ad}\, X$ and $\mathrm{ad}\, H$ have the same characteristic polynomial

$$F(\lambda)=\sum_{k=n_0}^{n} c_k \lambda^k,$$

where $n=\dim \mathfrak{g}$, $n_0=\dim(\mathfrak{a}+\mathfrak{m})$. Obviously, $c_{n_0}=0$ if and only if $\alpha_i(H)=0$ for some i. Since $c_k=c_k(X)$ is a polynomial function on $\mathfrak{p}$, we have

$$\gamma^{-1}(\partial\tilde{E})=\{x \in E; \Phi(x)=0\} \quad (=\Xi)$$

for some $\Phi(x) \in C^\infty(E)$. Therefore, we have, for $x \in \Xi$,

$$P_x(\sigma'=0)=1$$

where $\sigma'=\inf\{t; x_t \notin \Xi\}$ (cf. [4] chap. 13, §2). This proves the proposition.

3. Let D_n be a bounded domain such that $\bar{D}_n \subset D_{n+1}$, $D_n \uparrow \tilde{E}'$ and $b_k^{(n)}(x)$ a bounded continuous function on $\boldsymbol{R}^p$ which satisfies Lipschitz condition and coincides with $b_k(x)$ in D_n. We denote by $\boldsymbol{X}_n$ the Markov process on $(\boldsymbol{R}^p, \mathfrak{B})$ governed by the stochastic integral equation

$$x_{t,k}^{(n)}=x_k+\sum_{j=1}^{p} \tilde{a}^{kj} \int_0^t d\xi_{s,j}+\int_0^t b_k^{(n)}(x_s^{(n)})ds \quad (k=1, \cdots, p)$$

where $\xi_t=(\xi_{t,1}, \cdots, \xi_{t,p})$ is the path of the Brownian motion and $(\tilde{a}^{ij})$ is a positive symmetric matrix such that

$$\frac{1}{2}(\tilde{a}^{ij})^2=(a^{ij})$$

(see [4] Theorem 11.4). Let $\boldsymbol{X}^{(n)}$ be the part process of $\boldsymbol{X}_n$ on D_n. Then, by the uniqueness of the solution of stochastic integral equation ([4] Theorem 11.6, Corollary), $\boldsymbol{X}^{(n)}$ $(n=1, 2, \cdots)$ satisfy the condition of Lemma in §1.

Thus we obtain a Markov process on $\tilde{E}'$, which is equivalent to the part process of $\tilde{\boldsymbol{X}}$ on $\tilde{E}'$ and satisfies the stochastic integral equation

$$x_{t,k}=x_k+\sum \tilde{a}^{kj}\int_0^t d\xi_{s,j}+\int_0^t b_k(x_s)ds \qquad (k=1, \cdots, p) \tag{10}$$

(c.f. Proposition 3). In the same way, a Markov process $\boldsymbol{Y}$ is constructed by the stochastic integral equation

$$y_{t,k}=y_k+\sum \tilde{a}^{kj}\int_0^t d\xi_{s,j}+\int_0^t \tilde{b}_k(y_s)ds \qquad (k=1, \cdots, p) , \tag{11}$$

where

$$\tilde{b}_k(x)=\frac{n_k}{2}(\alpha_k, \alpha_k)\coth\frac{x_k}{2}+n'_k(\alpha_k, \alpha_k)\coth x_k .$$

(n_k, n'_k are the multiplicities of α_k and $2\alpha_k$, respectively. In case $2\alpha_k$ is not a root, $n'_k=0$.) The termination time of $\boldsymbol{Y}$ is its first exit time $\tilde{\tau}$ from $\mathfrak{U}$.

1° $$\tilde{\boldsymbol{P}}_x(\tilde{\tau}=\infty)=1 \qquad \text{for} \quad x\in\tilde{E}' .$$

For, if $\tilde{\tau}<\infty$, $\lim_{t\uparrow\tau} y_t=y_{\tau}$ exists (note that the first term of the right side of (11) is constinuous and the second term is monotone increasing). Therefore, by the definition of $\tilde{\tau}$,

$$y_{\tau,i}=\infty , \quad \text{or} \quad =0 \qquad \text{for some } i .$$

But, if $y_{\tau,i}=\infty$, $b_i(y_t)$ is bounded on $[0, \tau]$. Hence the right side of (11) is finite, which is a contradiction. In the same way as in Proposition 3, we can show that the latter is also impossible.

2° $$b_k(x)<\tilde{b}_k(x) \quad (k=1, \cdots, p) \qquad \text{in} \quad \tilde{E}' .$$

For a positive root $\alpha\neq\alpha_i$, $2\alpha_i$, $S_{\alpha_i}\alpha$ is also positive. Let $\{\alpha^{(1)}, \cdots, \alpha^{(s)}\}$ be the set of all positive roots such that $(\alpha, \alpha_i)>0$ and put $\beta^{(k)}=S_{\alpha_i}\alpha^{(k)}$ $(k=1, \cdots, s)$. Then we have

$$(\beta^{(k)}, \alpha_i)=-(\alpha^{(k)}, \alpha_i)<0$$

and

$$(\alpha, \alpha_i)=0 \qquad \text{for} \quad \alpha\neq\alpha^{(k)},\ \beta^{(k)} \ (1\leqq k\leqq s) .$$

Therefore

$$\begin{aligned} b_k(x)=&(\alpha_k, \alpha_k)\Big(\frac{n_k}{2}\coth\frac{x_k}{2}+n'_k\coth x_k\Big) \\ &+\sum_{i=1}^{s}(\alpha_k, \alpha^{(i)})n^{(i)}\Big(\coth\frac{\alpha^{(i)}(H)}{2}-\coth\frac{\beta^{(i)}(H)}{2}\Big) \end{aligned} \tag{12}$$

where $n^{(i)}$ is the multiplicity of $\alpha^{(i)}$. Since

$$\beta^{(i)}(H)=\alpha^{(i)}(H)-\frac{2(\alpha_k, \alpha^{(i)})}{(\alpha_k, \alpha_k)}\alpha_k(H)<\alpha^{(i)}(H)\ ,$$

the second term is negative. 2° immediately follows from (12).

3° $\qquad \tilde{P}_x(\tilde{x}_t<y_t,\ \text{for all } t>0)=1 \quad \text{for} \quad x\in\tilde{E}'$.

$$\varphi(t)=\tilde{x}_{t,k}(x)-y_{t,k}(x)=\int_0^t(b_k(\tilde{x}_s)-\tilde{b}_k(y_s))ds$$

is a continuously differentiable function such that

$$\varphi(0)=0 \quad \text{and} \quad \varphi(t_0)=0 \quad \text{implies} \quad \varphi'(t_0)<0\ .$$

Therefore ([6] Chap. 4, 16, Lemma 4), we have

$$\varphi(t)<0\ , \quad \text{for} \quad t>0\ .$$

4° $\qquad \tilde{P}_x\left(\lim_{t\to\infty}\frac{y_{t,k}}{t}=\left(\frac{n_k}{2}+n'_k\right)(\alpha_k, \alpha_k)\right)=1$.

This follows from

$$\coth u>1,\quad \lim_{u\to\infty}\coth u=1 \quad \text{and} \quad \tilde{P}_x\left(\lim_t \frac{\tilde{\xi}_{t,k}}{t}=0\right)=1\ .$$

From 3°, 4°, we have

$$\tilde{P}_x\left(\overline{\lim_{t\to\infty}}\frac{x_{t,k}}{t}\leqq\left(\frac{n_k}{2}+n'_k\right)(\alpha_k, \alpha_k)\right)=1\ . \tag{13}$$

We put

$$\rho=\frac{1}{2}\sum_{\alpha>0}\alpha=\sum_{k=1}^{p}\mu_k\alpha_k\ .$$

(Every root in this sum is repeated as many times as its multiplicity indicates.) Since

$$\frac{2(\rho, \alpha_k)}{(\alpha_k, \alpha_k)}=n_k+2n'_k\ ,$$

we have

$$(\rho, \rho)=\sum_{i=1}^{p}\frac{(\alpha_i, \alpha_i)}{2}(n_i+2n'_i)\mu_i\ . \tag{14}$$

From (10), we have

$$\sum_{i=1}^{p}\mu_i\tilde{x}_{t,i}=\sum\mu_i x_i+\sum_{i,j=1}^{p}\tilde{a}^{ij}\mu_i\int_0^t d\xi_{t,j}+\int_0^t\sum_{i=1}^{p}\mu_i b_i(\tilde{x}_s)ds\ .$$

On the other hand,

$$\sum_{i=1}^{p} \mu_i b_i(x) = \sum_{\alpha>0} \frac{1}{2}(\alpha, \rho) \coth \frac{\alpha(H)}{2} \geqq (\rho, \rho) .$$

Therefore, as in 4°, we have

$$\tilde{P}_x\left(\underline{\lim}_{t\to\infty} \frac{\sum \mu_i \tilde{x}_{t,i}}{t} \geqq (\rho, \rho) \right) = 1 .$$

Since $$\underline{\lim}_{n\to\infty} (a_n + b_n) \leqq \underline{\lim}_{n\to\infty} a_n + \overline{\lim}_{n\to\infty} b_n ,$$

$$\begin{aligned}(\rho, \rho) &\leqq \underline{\lim}_{t\to\infty} \sum_{i=1}^{p} \frac{\mu_i \tilde{x}_{t,i}}{t} = \underline{\lim}_{t\to\infty} \frac{\mu_k \tilde{x}_{t,k}}{t} + \sum_{i\neq k} \overline{\lim}_{t\to\infty} \frac{\mu_i \tilde{x}_{t,i}}{t} \\ &\leqq \underline{\lim}_{t\to\infty} \frac{\mu_k \tilde{x}_{t,k}}{t} + \sum_{i\neq k} (\alpha_i, \alpha_i) \left(\frac{n_i}{2} + n_i' \right) \mu_i .\end{aligned}$$

Therefore, by (14),

$$\tilde{P}_x\left(\underline{\lim}_{t\to\infty} \frac{\tilde{x}_{t,k}}{t} \geqq (\alpha_k, \alpha_k) \left(\frac{n_k}{2} + n_k' \right) \right) = 1 . \tag{15}$$

From (13) and (15), we have

$$\tilde{P}_x\left(\lim_{t\to\infty} \frac{\tilde{x}_{t,k}}{t} = (\alpha_k, \alpha_k) \left(\frac{n_k}{2} + n_k' \right), k=1, \cdots, p \right) = 1$$

or

$$\tilde{P}_x\left(\lim_{t\to\infty} \frac{\tilde{x}_t}{t} = \rho \right) = 1 , \qquad \text{for} \quad x \in \tilde{E}' ,$$

where we identify $\rho \in \mathfrak{a}^*$ with an element $H_\rho \in \mathfrak{a}$ such that $\rho(H) = (H, H_\rho)$.

Now let $x \in \partial\tilde{E}$. Then, for $t_0 > 0$, we have

$$\tilde{P}_x(\tilde{x}_{t_0} \in \tilde{E}') = 1 . \tag{16}$$

For, by Proposition 4,

$$\tilde{P}_x(\tilde{x}_{r_n} \in \partial\tilde{E}, n = 1, 2, \cdots) = 0 \tag{17}$$

($r_n, n = 1, 2, \cdots$ are rational numbers in $(0, t_0)$). On the other hand, by Proposition 3,

$$\begin{aligned}\tilde{P}_x(\tilde{x}_t \in \tilde{E}', \text{ for all } t \geqq t_0) &= \tilde{P}_x(\tilde{x}_{t_0} \in \tilde{E}', \theta_{t_0}\{\tilde{x}_s \in \tilde{E}', \text{ for all } s \geqq 0\}) \\ &= \tilde{M}_x(\chi_{\{\tilde{x}_{t_0} \in \tilde{E}'\}} \tilde{x}_{t_0} \in \tilde{E}'\} \tilde{P}_{\tilde{x}_{t_0}}(\tilde{x}_s \in \tilde{E}', \text{ for all } s \geqq 0)) \\ &= \tilde{P}_x(\tilde{x}_{t_0} \in \tilde{E}') .\end{aligned}$$

Consequently, by (17),

$$\tilde{P}_x(\tilde{x}_{t_0} \in \tilde{E}') \geqq \tilde{P}_x(\tilde{x}_{\tau_n} \in \tilde{E}', \text{ for some } n) = 1 .$$

Therefore, we have

$$\begin{aligned}\tilde{P}_x\left(\lim_{t\to\infty} \frac{\tilde{x}_t}{t} = \rho\right) &= \tilde{P}_x\left(\lim_{t\to\infty} \frac{\tilde{x}_{t+t_0}}{t} = \rho\right) \\ &= \tilde{P}_x\left(\theta_{t_0}\left\{\lim_{t\to\infty} \frac{\tilde{x}_t}{t} = \rho\right\}\right) \\ &= \tilde{M}_x \tilde{P}_{\tilde{x}_{t_0}}\left(\lim_{t\to\infty} \frac{\tilde{x}_t}{t} = \rho\right) = 1, \text{ by (16)} .\end{aligned}$$

Hence we have obtained the following theorem.

THEOREM.

$$P_x\left(\lim_{t\to\infty} \frac{\tilde{x}_t}{t} = \rho\right), \qquad \text{for all} \quad x \in E .$$

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