Classes of similar Markov processes and corresponding exit and entrance spaces

Dedicated to Professor Kôsaku Yosida on his 60th birthday

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1. Introduction

1.1. Fundamentally, our task is the following: a Markov process is given; it is required to describe all similar Markov processes, i.e. processes with the same conditional distributions on any time interval $[s, t]\subset(0, \infty)$ given the positions x_s and x_t of the moving particle at s and t.

To solve this problem, it is sufficient to be able to find for any Markov process:

a) all processes with the same conditional distributions on any interval $[t, \infty)$ given x_t (we call them right similar processes);

b) all processes having the same conditional distributions on any interval $(0, s)$ given x_s (the left similar processes).

Various right similar processes differs from each other only by their behaviour as $t\downarrow 0$ or, figuratively speaking, "by the way of their entrance into the state space E". They will be characterized by measures on the so-called entrance space U. Left similar processes differ from each other by their behaviour as $t\uparrow\zeta$ (the terminal time of the process). They will be characterized by measures on the so-called exit space $\hat{U}$.

To avoid technical difficulties connected with conditional distributions, we shall introduce formally more restricted definitions of similar, right and left similar processes in subsections 1.2 and 1.3.

1.2. A Markov process on a measurable space $(E, \mathscr{B})$ is a pair (x_t, P), where $x_t(\omega)$ is defined for $\omega\in\Omega$, $0<t<\zeta(\omega)$ and takes values in E, P is a σ-finite measure on the space Ω, and for all $s, t>0$, $\Gamma\in\mathscr{B}$

$$P(x_{t+s}\in\Gamma|x_u, 0\leq u\leq s)=p(t, x_s, \Gamma) \qquad (\text{a.s. } P, \Omega_s)^{1)}, \tag{1.1}$$

where $p(t, x, \Gamma)$ is a transition function[2]. We shall assume that the sample space

[1] We set $\Omega_s=\{\omega:\zeta(\omega)>s\}$. The expression (a.s. P, A) means "for P-almost all $\omega\in A$".

[2] The function $p=p(t, x, \Gamma)$ is called a transition function if: a) for fixed t and x, p is

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Ω is the set of all (terminating and non-terminating) paths in E and that $x_t(\omega)=\omega(t)$. Hence the various processes differ from each other only by measures P. All these measures are defined on the σ-field $\mathcal{N}$ generated by the sets $\{\omega: x_u(\omega)\in\Gamma\}$ $(u>0,\ \Gamma\in\mathscr{B})$.

We say that processes P and $\tilde{P}$ are *right similar* if they have the same transition functions. We call them *similar* if their transition functions are connected by relation

$$\tilde{p}(t, x, dy)=\frac{a(t)}{h(x)}p(t, x, dy)h(y)\ , \tag{1.2}$$

where $a(t)$ and $h(x)$ are finite and strictly positive. (The definition of left similarity demands some preparations and will be given in subsection 1.3.)

If P and $\tilde{P}$ are right similar, then

$$P(A|x_t)=\tilde{P}(A|x_t) \qquad (\text{a.s. } P+\tilde{P},\ \Omega_t) \tag{1.3}$$

for any $t>0$, $A\in\mathcal{N}[t, \infty)$[3]. If P and $\tilde{P}$ are similar, then

$$P(A|x_s, x_t)=\tilde{P}(A|x_s, x_t) \qquad (\text{a.s. } P+\tilde{P},\ \Omega_t) \tag{1.4}$$

for any $0<s<t$, $A\in\mathcal{N}[s, t]$[4].

1.3. In the following we shall always accept that a transition function p is fixed. It is easy to prove that the function $\tilde{p}$ defined by (1.2) is a transition function if and only if $a(t)=e^{-\alpha t}$ and

$$\begin{aligned} &e^{-\alpha t}P_t h(x)\le h(x) \ \text{ for all } t>0,\ x\in E,\\ &e^{-\alpha t}P_t h(x)\to h(x) \quad \text{as} \quad t\downarrow 0^{5)}\ . \end{aligned} \tag{1.5}$$

a measure on the σ-field $\mathscr{B}$; $p(t, x, E)\le 1$ and $p(t, x, E)\to 1$ as $t\downarrow 0$; b) for fixed Γ, p is a $\mathscr{B}(0, \infty)\times\mathscr{B}$-measurable function (we denote by $\mathscr{B}(\mathscr{X})$ the class of all Borel sets of a topological space $\mathscr{X}$); c) for all $s, t>0$, $\Gamma\in\mathscr{B}$

$$\int_E p(s, x, dy)p(t, y, \Gamma)=p(s+t, x, \Gamma)\ .$$

3) We denote by $\mathcal{N}(\Delta)$ the σ-field generated by the sets

$$\{\omega: x_u(\omega)\in\Gamma\} \qquad (u\in\Delta, \Gamma\in\mathscr{B})\ .$$

4) The relation (1.3) does not imply right similarity and (1.4) does not imply similarity. However (1.4) and (1.2) are equivalent, for example, if E is denumerable, $p(t, x, \Gamma)$ and $\tilde{p}(t, x, \Gamma)$ vanish for the same triples t, x, Γ and there exists a point x_0 such that $p(t, x_0, y)>0$ for all t, y.

5) We set

$$P_t f(x)=\int_E p(t, x, dy)f(y)\ ,\ \ (\nu P_t)(\Gamma)=\int_E \nu(dx)p(t, x, \Gamma)\ .$$

Two operators can be associated in analogous way to any kernel. We denote kernels by small letters and corresponding operators by the same capital letters.

Functions h satisfying (1.5) are called α-excessive. The totality of all such functions will be denoted by $\mathscr{T}^{\alpha}$.

For any Markov process P with the transition function p, (1.1) implies that the measures

$$\nu_t(\Gamma)=P(x_t\in\Gamma) \qquad (t>0)$$

are connected by the relation

$$\nu_t P_s=\nu_{t+s} \qquad (s, t>0)\ . \tag{1.6}$$

Each family of σ-finite measures ν_t $(t>0)$ satisfying (1.6) will be called an entrance law. The set of all entrance laws will be denoted by $\mathscr{S}$.

Let $\mathfrak{M}$ be the class of all similar Markov processes which is determined by the transition function p. An arbitrary process $P\in\mathfrak{M}$ has a transition function of the form

$$\frac{e^{-\alpha t}}{h(x)}p(t, x, dy)h(y) \qquad (h\in\mathscr{T}^{\alpha})\ .$$

The formula

$$\nu_t(dx)=\frac{e^{\alpha t}}{h(x)}P(x_t\in dx) \qquad (t>0)$$

defines an entrance law. Let us write $P=P_{\nu}^{\alpha h}$. It follows from (1.1) that for any $0<t_1<t_2<\cdots<t_n$

$$\begin{aligned}&P_{\nu}^{\alpha h}(x_{t_1}\in dy_1, x_{t_2}\in dy_2, \cdots, x_{t_n}\in dy_n)\\&\quad=\nu_{t_1}(dy_1)p(t_2-t_1, y_1, dy_2)\cdots p(t_n-t_{n-1}, y_{n-1}, dy_n)h(y_n)e^{-\alpha t_n}\ .\end{aligned} \tag{1.7}$$

The formula (1.7) allows to set the one to one correspondence between $\mathfrak{M}$ and the set of all triples ν, α, h where $\nu\in\mathscr{S}$ and $h\in\mathscr{T}^{\alpha}$[6].

Measures $P_{\nu}^{\alpha h}$ with a fixed (α, h) form a class of right similar processes. On the other hand,

$$P_{\nu}^{\alpha h}(A|x_s)=P_{\nu}^{\tilde\alpha\tilde h}(A|x_s) \qquad (\text{a.s. } P_{\nu}^{\alpha h}+P_{\nu}^{\tilde\alpha\tilde h},\ \Omega_s)$$

for all $s>0$, $A\in\mathscr{N}(0, s)$. Therefore it is natural to assume as a definition that any processes $P_{\nu}^{\alpha h}$ and $P_{\nu}^{\tilde\alpha\tilde h}$ are *left similar.* Otherwise the decomposition of the class of similar processes into classes of left similar processes can be obtained by fixing the index ν in (1.7).

[6] For any ν, α, h it is possible to construct the unique measure satisfying (1.7) with the help of a Kolmogorov theorem. To use this theorem we have to impose some restrictions on the state space $(E, \mathscr{B})$. For example, it is sufficient to suppose that E is a Borel set in a separable locally compact metric space and $\mathscr{B}$ is the class of all Borel subsets of E.

1.4. The task of describing all similar, right similar and left similar processes is now reduced to two problems:

PROBLEM A. To describe the set $\mathscr{S}$ of all entrance laws.

PROBLEM B. To describe the set $\mathscr{T}$ of all excessive (i.e. 0-excessive) functions.

Strictly speaking, we have to describe sets $\mathscr{T}^{\alpha}$ for all α but $\mathscr{T}^{\alpha}$ coincides with the set of all excessive functions relative $e^{-\alpha t}p$. (As $e^{-\alpha t}p(t, x, E)$ may be >1 if $\alpha<0$, the reduction requires a slight extension of the notion of a transition function.)

To investigate problems A and B we need certain finiteness conditions.

Note that $P_{\nu}^{\alpha h}(\zeta>t)=e^{-\alpha t}\nu_t(h)$ and hence

$$P_{\nu}^{\alpha h}(\Omega)=\lim_{t\downarrow 0}\nu_t(h) . \tag{1.8}$$

We agree to denote this value by $\nu(h)$.[7] Let us say that h is ν-finite and that ν is h-finite if $\nu(h)<\infty$.

We modify the problems A and B in the following way:

PROBLEM A′. To describe the set $\mathscr{S}_q$ of all q-finite entrance laws (for a given α_0-excessive function q).

PROBLEM B′. To describe the set $\mathscr{T}_{\gamma}$ of all γ-finite excessive functions (for a given entrance law γ).

We shall construct two topological spaces: the entrance space $U=U(p, q)$ and the exit space $\hat{U}=\hat{U}(p, \gamma)$. To each point $z\in U$ there corresponds a solution κ^z of the problem A′ and to each point $z\in\hat{U}$ there corresponds a solution k_z of the problem B′. The formula

$$\nu=\int_U \kappa^z\mu^{\nu}(dz)$$

determines the one to one correspondence between $\mathscr{S}_q$ and the set of all finite Borel measures on U. The formula

$$h=\int_{\hat{U}} k_z\mu_h(dz)$$

determines the one to one correspondence between $\mathscr{T}_{\gamma}$ and the set of all finite Borel measures on $\hat{U}$.

1.5. The results formulated in subsection 1.4 are deduced under certain

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7) We denote also by $\nu(h)$ the integral of a function h with respect to a measure ν. If the entrance law ν is defined by the formula $\nu_t=\nu P_t$ where ν is a measure, then both definitions of $\nu(h)$ lead to the same numerical result.

restrictions on the transition function p, entrance law γ and α_0-excessive function q.

For Markov chains these results were, essentially, obtained by Doob [1], T. Watanabe [2] and Hunt [3]. The case of standard (and some other) processes was investigated by Kunita and T. Watanabe [4], [5], [6].

The theory presented in this lecture gives the most general results under minimal assumptions.

2. Support systems, Martin compactums

2.1. We introduce no topology in the state space E. The necessary conditions on the transition function p will be formulated in the terms of support systems.

Let $V=V(E,\mathscr{B})$ be the set of all $\mathscr{B}$-measurable functions on E with the values from the extended half-line $[0,\infty]$. A denumerable set $W\subseteq V$ is called a *support system* if the following conditions are fulfilled:

2.1.A. If $\varphi_1, \varphi_2\in W$ and r_1, r_2 are positive rational numbers, then $r_1\varphi_1+r_2\varphi_2\in W$.

2.1.B. If μ_n $(n=1,2,\cdots)$ are measures on $\mathscr{B}$ and $\mu_n(\varphi)\to l(\varphi)<\infty$ for each $\varphi\in W$, then there exists a measure μ on $\mathscr{B}$ such that $\mu(\varphi)=l(\varphi)$ for all $\varphi\in W$.

2.1.C. Let a set $\tilde{V}\supseteq W$ have the following properties: if $f_1, f_2\in\tilde{V}$, then $f_1+f_2\in\tilde{V}$; if $f_1, f_2\in\tilde{V}$ and $f_1\leq f_2\leq\varphi\in W$, then $f_2-f_1\in\tilde{V}$; if $f_n\uparrow f$ and $f_n\in\tilde{V}$, then $f\in\tilde{V}$. Then $\tilde{V}$ contains V.

If E is a separable locally compact metric space and $\mathscr{B}=\mathscr{B}(E)$, then a support system can be constructed as follows. Consider a sequence of open sets $D_n\uparrow E$ with the compact closures. Choose a denumerable every dense subset $\mathscr{A}_n$ in the set of all positive continuous functions with supports in D_n. Form the sum $\mathscr{A}$ of all $\mathscr{A}_n$ and denote by W the linear span of $\mathscr{A}$ over the set of all positive rational numbers.

2.2. Let a denumerable system $\mathscr{H}$ of bounded functions separate E (i.e. for each $x\neq y\in E$ there exists $f\in\mathscr{H}$ such that $f(x)\neq f(y)$). It is possible to imbed E in a compactum $\mathscr{E}$ in such a way that each $f\in\mathscr{H}$ extends uniquely to $\mathscr{E}$. (We shall denote the extended functions by the same letters.) The compactum $\mathscr{E}$ will be called the compactification of E by means of $\mathscr{H}$.

Let R be a denumerable set. We call the function $k_\alpha(x,\Gamma)\geq 0$ $(\alpha\in R, x\in E, \Gamma\in\mathscr{B})$ a *Martin kernel* (for a given support system W) if $k_\alpha(-,\Gamma)$ is $\mathscr{B}$-measurable for each $\alpha\in R$, $\Gamma\in\mathscr{B}$; $k_\alpha(x,-)$ is a measure for each $\alpha\in R$, $x\in E$

and the functions

$$K_\alpha\varphi \qquad (\alpha\in R,\ \varphi\in W) \tag{2.1}$$

are bounded and separate E. Consider the compactification $\mathscr{E}$ of E by means of system (2.1). Relying on 2.1.B, we can extend $k_\alpha(x, \Gamma)$ for any $\alpha\in R$, $\Gamma\in\mathscr{B}$ to all $z\in\mathscr{E}$ in such a way that for any $z\in\mathscr{E}$, $\alpha\in R$, $k_\alpha(z, -)$ is a measure on $\mathscr{B}$ and for each $\alpha\in R$, $\varphi\in W$, the function $K_\alpha\varphi$ is continuous on $\mathscr{E}$. Let us agree to name $\mathscr{E}$ *the Martin compactum* corresponding to the Martin kernel $k_\alpha(x. \Gamma)$ and the system W.

Starting from the Green kernel

$$g_\alpha(x, \Gamma)=\int_0^\infty e^{-\alpha t}p(t, x, \Gamma)dt$$

we shall construct two Martin kernels and two Martin compactums. The entrance space is a Borel set in one of these compactums and the exit space in the other.

3. Entrance space

3.1. Let us fix an α_0-excessive function q and denote by R the set of all rational numbers $r>\alpha_0$.

Suppose that:

(AI) The functions $g_\alpha(-, \Gamma)$ $(\alpha\in R, \Gamma\in\mathscr{B})$ separate E.

(AII) There exists a support system W satisfying the following conditions: a) $\|\varphi/q\|<\infty$ for any $\varphi\in W$[8]; b) for any $\varphi\in W$, $\alpha>0$ a sequence $f_n\in W$ may be selected such that $f_n\uparrow G_\alpha\varphi$; c) a sequence $\psi_n\in W$ may be selected for which $\psi_n\uparrow q$.

The Martin compactum $\mathscr{E}$ corresponding to the Martin kernel

$$k_\alpha(x, \Gamma)=\frac{g_\alpha(x, \Gamma)}{q(x)}$$

(and the support system W) will be called *the Martin entrance compactum.*

Denote by $\mathscr{E}_1$ the set of all points $z\in\mathscr{E}$ for which

$$K_\beta\varphi(z)+(\beta-\alpha)K_\beta G_\alpha\varphi(z)=K_\alpha\varphi(z) \quad \text{for all} \quad \beta>\alpha\in R,\ \varphi\in W. \tag{3.1}$$

It follows from (3.1) that

$$0\le\frac{(-\alpha)^n}{n!}\frac{d^nK_{\alpha_0+\alpha}\varphi(z)}{d\alpha^n}=\alpha^nK_{\alpha_0+\alpha}G^n_{\alpha_0+\alpha}\varphi(z)\le\frac{1}{\alpha}\left\|\frac{\varphi}{q}\right\|.$$

[8] We set $\|\varphi\|=\sup_x|\varphi(x)|$.

According to a theorem of Bernstein, $K_{\alpha_0+\alpha}\varphi(z)$ is a Laplace transform of a function $F_\varphi^z(t)$ satisfying inequalities $0\le F_\varphi^z(t)\le\|\varphi/q\|$. Using 2.1.B we can choose measures κ_t^z so that $e^{\alpha_0 t}F_\varphi^z(t)=\kappa_t^z(\varphi)$. Then

$$K_\alpha\varphi(z)=\int_0^\infty e^{-\alpha t}\kappa_t^z(\varphi)dt$$

for all $\varphi\in W$ and, hence, for all $\varphi\in V$.

Let us suppose now, that:

(AIII) There exists a denumerable set $\mathscr{H}\subset V$ separating measures[9] and such that for each $\psi\in\mathscr{H}$ $\|\psi/q\|<\infty$ and $P_t\psi(x)$ is right continuous in t.

Then measures κ_t^z determine an entrance law $\kappa^z\in\mathscr{S}_q$. (For $x\in E$ $\kappa_t^x=p(t,x,\Gamma)/q(x)$.)

3.2. Relying on the martingale theory, we construct in the space $\mathscr{E}$ a right continuous in t path $z_t(\omega)$ $(0<t<\zeta(\omega))$ so that for all $\nu\in\mathscr{S}_q$, $h\in\mathscr{T}^{\alpha}$.

3.2.A. $P_\nu^{\alpha h}\{z_t\neq x_t\}=0$ except possibly a countable set of t.

3.2.B. For $P_\nu^{\alpha h}$-almost all ω there exists a limit

$$z_0(\omega)=\lim_{t\downarrow 0} z_t(\omega)\ .$$

3.2.C. For any $s>0$, the closure of the path $z_t(\omega)$ $(0<t<s\wedge\zeta(\omega))$ lies in $\mathscr{E}_1$ for $P_\nu^{\alpha h}$-almost all ω.

3.3. We define *the entrance space* U as a set of all $z\in\mathscr{E}_1$ for which

$$P_{\kappa^z}^{\alpha_0 q}\{z_0=z\}=1\ .$$

It is proved that:

3.3.A. For each $\nu\in\mathscr{S}_q$, $h\in\mathscr{T}^{\alpha}$

$$P_\nu^{\alpha h}\{z_0\bar{\in} U\}=0\ .$$

3.3.B. For each positive $\mathscr{N}(0,\infty)$-measurable function ξ, $f\in V(\mathscr{E})$[10], $h\in\mathscr{T}^{\alpha}$

$$M_\nu^{\alpha h}f(z_0)\xi=\int_U f(z)M_{\kappa^z}^{\alpha h}\xi\mu^\nu(dz)\ ,$$

where

$$\mu^\nu(\Gamma)=P_\nu^{\alpha_0 q}\{z_0\in\Gamma\}\ .$$

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[9] i.e. for any two measures $\mu_1\neq\mu_2$ such that $\mu_i(\varphi)<\infty$ for all $\varphi\in W$ there exists $\psi\in\mathscr{H}$ such that $\mu_1(\psi)\neq\mu_2(\psi)$.

[10] We denote by $V(\mathscr{E})$ the set of all positive Borel functions on $\mathscr{E}$.

In particular, for each $\psi \in V(\mathscr{E})$

$$M_\nu^{\alpha h} f(z_0)\psi(z_t) = \int_U e^{-\alpha t} f(z) \kappa_t^z(\psi h) \mu^\nu(dz) . \tag{3.2}$$

Setting $\alpha=0$, $h=1$, $f=1$, $\psi=\chi_\Gamma$, we obtain the following representation for an arbitrary $\nu \in \mathscr{S}_q$

$$\nu_t = \int_U \kappa_t^z \mu^\nu(dz) . \tag{3.3}$$

A measure μ^ν in (3.3) is uniquely determined by ν.

The formula (3.2) also implies that for all $\nu \in \mathscr{S}_q$, $h \in \mathscr{T}^\alpha$

$$P_\nu^{\alpha h}\{z_0 \in dz\} = \kappa^z(h)\mu^\nu(dz) .$$

4. Exit space

4.1. We shall call a function $f \in V$ Λ-continuous if there is a function $F(t, \omega)$ $(\omega \in \Omega,\ 0<t<\zeta(\omega))$, left continuous in t and such that

$$P_\nu\{F(t, \omega) \neq f(x_t)\} = 0 \quad \text{for all} \quad \nu \in \mathscr{S},\ t>0 . \tag{4.1}$$

(The condition (4.1) is fulfilled for all $\nu \in \mathscr{S}$ if it is fulfilled for $\nu_t^x(\Gamma) = p(t, x, \Gamma)$ $(x \in E,\ \Gamma \in \mathscr{B})$).

Let R be the set of all rational numbers $r \geq 0$.

Suppose that the following conditions are satisfied for some measures γ and μ on $\mathscr{B}$ and a support system W:

(BI) The measure γ is finite, the measures m and $\eta = \gamma G$ are finite on all $\varphi \in W$ and η is m-continuous (i.e. $\eta(\Gamma)=0$ if $m(\Gamma)=0$).

(BII) For each $\alpha \in R$, $x \in E$ the measure $g_\alpha(x, -)$ is η-continuous.

(BIII) The density function $k_\alpha(x, y) = \dfrac{g_\alpha(x, dy)}{\eta(dy)}$ can be selected so that the functions

$$\int_E m(dx)\varphi(x)k_\alpha(x, y) \qquad (\varphi \in W,\ \alpha \in R)$$

are Λ-continuous and separate E.

(BIV) There exist constants c_φ^α such that for each α-excessive function h

$$m(\varphi h) \leq c_\varphi^\alpha \gamma(h) \qquad (\varphi \in W,\ \alpha \in R) .\text{[11]}$$

[11] The condition (BIV) is fulfilled, for example, if $\gamma(dx) = \phi(x)m(dx)$ and $\|\varphi/\phi\| < \infty$ for all $\varphi \in W$. It is sufficient also that the measure $\int_E \gamma(dx) g_\alpha(x, -)$ is m-continuous for each $\alpha > 0$ and its density Ψ_α is such that $\|\varphi/\Psi_\alpha\| < \infty$ for all $\varphi \in W$.

Under these conditions we shall describe the class $\mathscr{T}_\gamma$ of all γ-finite excessive functions (i.e. functions h for which $\gamma(h)<\infty$).

4.2. Let us denote by E_0 the set of all points $x\in E$ for which

$$\int_E m(dy)\varphi(y)k_\alpha(y,x)\leq c^\alpha_\varphi \qquad (\varphi\in W,\ \alpha\in R)\ ,$$

where c^α_φ are defined in (BIV). It is proved that $\gamma(E\setminus E_0)=0$.

The formula

$$\hat{k}(x,dy)=\begin{cases} k_\alpha(y,x)m(dy) & \text{for} \quad x\in E_0\ ,\\ 0 & \text{for} \quad x\in E_0\ ,\end{cases}$$

defines a Martin kernel (with respect to the support system W). The corresponding Martin compactum $\hat{\mathscr{E}}$ will be called *the Martin exit compactum.*

Relying on (BIII) we construct a function $z_t(\omega)\in\hat{\mathscr{E}}$ $(\omega\in\Omega,\ 0<t<\zeta(\omega))$ which is left continuous in t and satisfies the condition $P^h_\nu\{z_t\neq x_t\}=0$ for all $t>0$, $\nu\in\mathscr{S}$, $h\in\mathscr{T}_\gamma$. (Here $P^h_\nu=P^{0h}_\nu$.)

We prove that:

4.2.A. For P^h_ν-almost all ω there exists a limit

$$z_\zeta(\omega)=\lim_{t\uparrow\zeta} z_t(\omega)\ .$$

4.2.B. For each $\varphi\in V$, $f\in V(\hat{\mathscr{E}})$

$$\int_{\hat{\mathscr{E}}} f(z)\hat{K}_\alpha\varphi(z)\mu_h(dz)=M^h_{m^\varphi} e^{-\alpha\zeta}f(z_\zeta)\ , \tag{4.2}$$

where

$$\mu_h(\Gamma)=P^h_\gamma\{z_\zeta\in\Gamma\}\ ,$$

and

$$m^\varphi_t(\Gamma)=\int_E m(dy)\varphi(y)p(t,y,\Gamma)\ .$$

4.3. Using (4.2) we construct a Borel subset $\hat{\mathscr{E}}_1\subseteq\hat{\mathscr{E}}$ with the following properties:

4.3.A. $\mu_h(\hat{\mathscr{E}}\setminus\hat{\mathscr{E}}_1)=0$ for all $h\in\mathscr{T}^\alpha$.

4.3.B. For each $z\in\hat{\mathscr{E}}_1$, $\alpha\in R$

$$\hat{k}_\alpha(z,dy)=k^\alpha_z(y)m(dy)\ ,$$

where k^α_z is an α-excessive function, $\gamma(k^\alpha_z)\leq 1$ and

$$k_z^\alpha = k_z^{\alpha+\lambda} + \lambda G_{\lambda+\alpha} k_z^\alpha \quad \text{for all} \quad \lambda, \alpha \in R\ .^{12)} \tag{4.3}$$

4.3.C. For each $f \in V(\hat{\mathscr{E}})$

$$M_\nu^h e^{-\alpha\zeta} f(z_\zeta) = \int_{\hat{\mathscr{E}}_1} f(z) \nu(k_z^\alpha) \mu_h(dz)\ . \tag{4.4}$$

Setting $\alpha=0$, $f=1$ and taking into account (1.8), we have

$$\nu(h) = \int_{\hat{\mathscr{E}}_1} \nu(k_z^\alpha) \mu_h(dz)\ .$$

Hence for all x

$$h(x) = \int_{\hat{\mathscr{E}}_1} k_z^\alpha(x) \mu_h(dz)\ . \tag{4.5}$$

4.4. We define *the exit space* $\hat{U}$ as the set of all $z \in \hat{\mathscr{E}}_1$ satisfying the condition

$$P_\nu^{k_z}\{z_\nu = z\} = 1 \quad \text{for all} \quad \nu \in \mathscr{S}.$$

It is proved that:

4.4.A. $\mu_h(\hat{\mathscr{E}} \setminus \hat{U}) = 0$ for all $\nu \in \mathscr{S}$; hence the domain of integration in (4.4) and (4.5) can be restricted to $\hat{U}$.

4.4.B. If $h(x) = \int_{\hat{U}} k_z(x) \mu(dz)$ with a finite Borel measure μ, then $\mu = \mu_h$.

Thus the problem B′ of subsection 1.4 is completely solved[13].

4.5. The set $\hat{U}_0$ of all $z \in \hat{U}$ such that $k_z^\alpha = 0$ for all $\alpha > 0$ is called *the passive exit space*; the set $\hat{U}_a = \hat{U} - \hat{U}_0$ is called *the active exit space.* For all $\nu \in \mathscr{S}$, $h \in \mathscr{T}_\gamma$

$$M_\nu^h f(z_\zeta) \chi_{\zeta=\infty} = \int_{\hat{U}_0} f(z) \nu(k_z) \mu_h(dz)\ ,$$

$$M_\nu^h f(z_\zeta) e^{-\alpha\zeta} = \int_{\hat{U}_a} f(z) \nu(k_z^\alpha) \mu_h(dz) \qquad (\alpha > 0)\ .$$

A function $h \in \mathscr{T}_\gamma^-$ can be represented in the form

$$h = \int_{\hat{U}_0} k_z \mu(dz)\ ,$$

12) Fix some $z \in \hat{\mathscr{E}}_1$, $x \in E$. We conclude from (4.3) that $k_z^\alpha(x)$ is uniformly continuous in $\alpha \in R$. Therefore it can be continuously extended to all positive real α. The properties listed in 4.3. B remain valid for the extended functions.

13) Notice that the set of all minimal γ-finite excessive function is given by the expression ck_z where $z \in \hat{U}$ and c is an arbitrary positive constant.

with the finite Borel measure μ if and only if $P_t h = h$ for all $t>0$. It can be represented in the form

$$h = \int_{\hat{U}_a} k_z \mu(dz)$$

with the finite Borel measure μ if and only if $P_t h \to 0$ as $t \to \infty$.

The formula

$$h = \int_{\hat{U}_a} k_z^\alpha \mu(dz) \qquad (\mu(\hat{U}_a) < \infty,\ \alpha > 0)$$

gives a general form of α-excessive functions for which $\gamma(h) < \infty$ and $\gamma(Gh) < \infty$ (moreover,

$$h + \alpha Gh = \int_{\hat{U}_a} k_z \mu(dz)\ ).$$

4.6. To any closed set Γ in the Martin exit compactum $\hat{\mathscr{E}}$ there corresponds an operator P_Γ on the set of all excessive functions. To define it we consider the time $\tau_\Gamma = \inf\{t : t > 0,\ z_t \in \Gamma\}$, construct the right continuous regularization $H(t)$ of the supermartingale $h(x_t)$ and set $P_\Gamma h(x) = M_x H(\tau)$ [14].

Let $\mathscr{A}$ be a system of closed sets of $\hat{\mathscr{E}}$. We call an excessive function h $\mathscr{A}$-harmonic if $P_\Gamma h = h$ for all $\Gamma \in \mathscr{A}$. An $\mathscr{A}$-harmonic function is called A-harmonic if $\mathscr{A}$ is the totality of all closed subsets of an open set A.

We prove that:

4.6.A. For each $z \in \hat{U}$ the class of all $\hat{\mathscr{E}} \backslash \{z\}$-harmonic functions is given by the expression ck_z where c is an arbitrary positive constant. An excessive function h is A-harmonic if and only if $\mu_h(A) = 0$.

4.6.B. In order that a function $k_z (z \in \hat{U})$ be $\mathscr{A}$-harmonic, it is sufficient that z belongs to no set $\Gamma \in \mathscr{A}$, and it is necessary that there is a sequence $z_n \to z$ with a finite number of members in any $\Gamma \in \mathscr{A}$. An excessive function h is $\mathscr{A}$-harmonic if and only if the measure μ_h is concentrated on the set

$$\{z : k_z \text{ is } \mathscr{A}\text{-harmonic}\}\ .$$

5. Excessive measures

5.1. We fix a suport system W and say that a measure η defined on the σ-field $\mathscr{B}$ is excessive if for all $\varphi \in W$ $\eta(P_t \varphi) \leq \eta(\varphi) < \infty$ $(t>0)$ and $\eta(P_t \varphi) \to \eta(\varphi)$ as $t \downarrow 0$. An excessive measure η is P_t-invariant if $\eta P_t = \eta$ for all $t>0$; it is P_t-vanishing if $\eta(P_t \varphi) \to 0$ as $t \to \infty$ for all $\varphi \in W$. Each excessive measure η can be

[14] According to usual notations we set $P_x = P^{\alpha\ h}_{\nu\ x}$ where $\alpha = 0$, $h = 1$ and $\nu_t^x(\Gamma) = p(t, x, \Gamma)$.

written uniquely as the sum of a P_t-invariant and a P_t-vanishing measure.

Fix a function $l\in V$ for which $q=Gl$ is finite-valued and strictly positive.

Let us assume the conditions (AI)-(AIII) and the following condition

(CI) $$\left\|\frac{G\varphi}{q}\right\|<\infty \quad \text{for any} \quad \varphi\in W .$$

It is proved that the formula

$$\eta=\int_0^\infty \nu_t dt$$

defines the one to one correspondence between q-finite entrance laws ν and l-finite P_t-vanishing measures η. Relying on (1.3) and setting

$$\delta^z=\int_0^\infty \kappa_t^z dt \tag{5.1}$$

we conclude that the formula

$$\eta=\int_U \delta^z \mu(dz)$$

determines the one to one correspondence between finite Borel measures on U and l-finite P_t-vanishing measures η.

5.2. Consider an P_t-invariant measure η and an excessive function h. Let Ω be the set of all paths in E defined on all time intervals $(-\infty, \lambda)$, where λ range through the set $(-\infty, +\infty]$. There exists a unique measure P_ν^h on the σ-field $\mathcal{N}(-\infty, +\infty)$ such that for each $-\infty<t_1<t_2< \cdots <t_n$

$$P_\eta^h\{x_{t_1}\in dy_1, x_{t_2}\in dy_2, \cdots, x_{t_n}\in dy_n\} \\ =\eta(dy_1)P(t_2-t_1, y_1, dy_2) \cdots P(t_n-t_{n-1}, y_{n-1}, dy_n)h(y_n) .$$

(The measure P_η^h determines a stationary Markov process.)

Let W, l and q have the same meaning as in subsection 5.1. Suppose that

(CII) $$\|G\varphi\|<\infty \quad \text{for all} \quad \varphi\in W .$$

Assume also the conditions (CI) and (AI) (but no conditions (AII)-(AIII)). Denote by R the set of all rational numbers $r\geq 0$ and construct the Martin compactum $\mathcal{E}'$ corresponding to the Martin kernel

$$k_\alpha(x, \Gamma)=\frac{g_\alpha(x, \Gamma)}{q(x)} \qquad (\alpha\in R, x\in E, \Gamma\in \mathcal{B}) .$$

It is possible to define a function $z_t(\omega)$ $(\omega\in\Omega, -\infty<t<\zeta(\omega))$ taking values in $\mathcal{E}'$, right continuous in t and such that $P_\eta^h(z_t\neq x_t)=0$ for all $t\in(-\infty, +\infty)$,

l-finite P_t-invariant η and $h\in\mathscr{T}^-$. It is proved that there exists P^h_η-almost surely a limit

$$z_{-\infty}=\lim_{t\downarrow-\infty} z_t\ .$$

The passive entrance space U_0' is defined as a set of all $z\in\mathscr{E}'$ satisfying conditions:

5.2.A. $k_\alpha(z,\Gamma)=0$ for all $\alpha>0$.

5.2.B. $\delta^z(-)=k(z,-)$ is a P_t-invariant measure and $\delta^z(l)=1$.

5.2.C. $P^h_{\delta^z}(z_{-\infty}=z)=1$ for all $h\in\mathscr{T}$.

We prove that for all η, h

$$P^h_\eta\{z_{-\infty}\bar{\in} U_0'\}=0$$

and for any $\mathscr{N}(-\infty,+\infty)$-measurable function $\xi\geqslant 0$ and $f\in V(\mathscr{E}')$

$$M^h_\eta f(z_{-\infty})\xi=\int_{U_0'} f(z)M^h_{\delta^z}\xi\mu^h(dz)$$

where

$$\mu^\eta(\Gamma)=M_\eta\chi_\Gamma(z_{-\infty})l(z_0)\ .$$

Setting $f=1$, $h=1$, $\xi=\chi_A$ we have

$$\eta=\int_{U_0'}\delta^z\mu^\eta(dz)\ . \tag{5.2}$$

The measure μ^η is finite and is uniquely determined by the formula (5.2).

5.3. Assume now the conditions (AI)-(AIII) and (CI)-(CII). The results of subsection 3 remain valid if we replace $\mathscr{E}$ by $\mathscr{E}'$. The condition

$$P^h_{\kappa^z}\{z_0=z\}=1$$

defines a subset U_a' of the set $\mathscr{E}'$ which is called *the active entrance space.* It follows from subsections 5.1 and 5.2 that each l-finite excessive measure η is represented uniquely in the form

$$\eta=\int_{U'}\delta^z\mu(dz)\ ,$$

where $U'=U_0'\cup U_a'$, δ^z are the measures described by (5.1) and 5.2.B and μ is a finite Borel measure.

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