On some homogeneous boundary value problems bounded below[*)]

Dedicated to Professor Kôsaku Yosida on his 60th birthday

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§ 1. Introduction.

Let Ω be a compact oriented Riemannian n-space with boundary Γ of class C^{∞}. Let A be a linear partial differential operator on Ω of order $2m$. We assume A is strongly elliptic, that is, there is a constant $c>0$ such that, for any $x\in\bar{\Omega}=\Omega\cup\Gamma$ and for any non zero vector ξ cotangent to Ω at x, we have

$$c^{-1}|\xi|^{2m}\leq \mathrm{Re}\,\sigma_{2m}(A)(x,\xi)\leq c|\xi|^{2m},$$

where $\sigma_{2m}(A)$ is the principal symbol of A. We consider normal systems $\{B_r\}_{r\in R}$, $R=(r_0, r_1, \cdots, r_{m-1})$, of m boundary operators B_{r_j}, r_j being the order of B_{r_j}. We assume $r_j<2m$ for any $j=0, \cdots, m-1$. The coefficients of A and B_{r_j} are assumed to be of class C^{∞}. The problem to be considered is

Problem I. For a given constant $\varepsilon(0\leq\varepsilon\leq m)$ characterize those couples $\{A, \{B_r\}_{r\in R}\}$ which give, with some constants β, $c>0$, the estimate

$$\mathrm{Re}\,((A+\beta)u, u)_{L^2(\Omega)}\geq c\|u\|^2_{H^{m-\varepsilon}(\Omega)} \tag{1}$$

for all u in $H^{2m}_B(\Omega)=\{u\in H^{2m}(\Omega); B_r u|_\Gamma=0$ for any $r\in R\}$. Here $H^s(\Omega)$ stands for the Sobolev space on Ω of order s, $\|\ \|_{H^s(\Omega)}$ is its norm and $(\ ,\ )_{L^2(\Omega)}$ is the inner product in $L^2(\Omega)=H^0(\Omega)$.

If the couple $\{A, \{B_r\}_{r\in R}\}$ admits the estimate (1) with $\varepsilon=m$, then the operator $A+\beta-c$ with definition domain $H^{2m}_B(\Omega)$ is accretive in $L^2(\Omega)$ in the sense of Kato [13]. Grubb [8] has obtained a necessary condition for (1) with $\varepsilon=m$ to hold in the case that A is formally self-adjoint (cf. also Shimakura [17] and Fujiwara and Shimakura [6]). If $\varepsilon=0$ the estimate (1) means the classical estimate which was treated by many authors: cf. for examples, Gårding [7], Agmon [1], Grubb [8].

*) This work was done during the author's stay in Paris. The results of this paper were announced in [4]. The author wishes to express his gratitude for the constant encouragement received from Professor J. L. Lions during his stay in Paris. He also wishes to express his thanks to Professor A. P. Calderón who kindly discussed with him.

In this paper we consider Problem I in the case $\varepsilon=1/2$. So our problem is

Problem II. Characterize those couples $\{A, \{B_r\}_{r\in R}\}$ which give, with some constants β, $c>0$, the estimate

$$\text{Re}\,((A+\beta)u, u)_{L^2(\Omega)} \geq c\|u\|^2_{H^{m-1/2}(\Omega)} \tag{2}$$

for all $u \in H^{2m}_B(\Omega)$. We treat this problem II under the following additional condition (H),

(H) The set $R=\{r_0, \cdots, r_{m-1}\}$ coincides with one of the R_j's defined by

$$R_0=\{0, 1, 2, \cdots, m-1\}\ ,$$

$$R_j=\{0, 1, 2, \cdots, m-j-1, m, m+1, \cdots, m+j-1\}\ , \qquad 1\leq j\leq m-1\ ,$$

$$R_m=\{m, m+1, \cdots, 2m-1\}\ .$$

This condition (H) is necessary if the estimate (1) with $\varepsilon=m$ holds for all couples $\{A+K, \{B_r\}_{r\in R}\}$, K being an arbitrary perturbation "of order $2m-1$". (See Shimakura [17] and Fujiwara-Shimakura [6].)

We give a necessary and sufficient condition for the estimate to hold under assumption (H). This is done through the following steps:

1) The estimate (2) is proved equivalent to an pseudo-differential inequality on Γ between Cauchy data of the function u which satisfies $(A+\beta)u=0$ in Ω.

2) The pseudo-differential inequality on Γ obtained in 1) is reduced to a little weaker pseudo-differential inequality on $\Gamma\times \boldsymbol{T}$, $\boldsymbol{T}$ being the unit circle $\boldsymbol{R}/2\pi\boldsymbol{Z}$.

3) Our necessary and sufficient condition for (2) to hold is the localization theorem of pseudo-differential inequality. This corresponds to "localization theorem of subelliptic inequality" of Hörmander [11] and [12]. The discussion here is a straightforward generalization of his in [11].

The subject treated in 3) is closely related to the recent work of Calderón [3]. One will be able to obtain sufficient conditions for Problem I with general $m>\varepsilon>0$ if one applies the results contained in it to our problem. Our condition which is necessary and sufficient for the estimate (2) to hold involves the first and the second symbols of the pseudo-differential operator on $\Gamma\times \boldsymbol{T}$ mentioned in 2). Sufficient conditions for Problem I with $\varepsilon>1/2$ will imply more terms. But it is very complicated to calculate lower symbols of this operator from symbols of operators A and B_r (and of the shape of the space Ω). This is the main reason why the author presents this note here in its original form.

§ 2. Results.

Now we repeat some of the notations introduced in § 1. Ω is a compact oriented Riemannian n-space with the boundary Γ of C^∞ class. $\bar{\Omega}$ is the union $\Omega \cup \Gamma$. ν denotes the interior unit normal to Γ and D_ν the normal derivative $-i\frac{\partial}{\partial\nu}$ multiplied by $-i=-\sqrt{-1}$. A is a uniformly strongly elliptic linear partial differential operator with $C^\infty(\bar{\Omega})$ coefficients. R_j, $0 \le j \le m$, is the one of the following series of m integers defined by

$$R_0=(0, 1, \cdots, m-1)\ ,$$

$$R_j=(0, 1, \cdots, m-j-1, m, m+1, \cdots, m+j-1)\ , \qquad 1 \le j \le m-1\ ,$$

$$R_m=(m, m+1, \cdots, 2m-1)\ .$$

$\{B_r\}_{r \in R_j}$ is a system of m boundary operators defined by

$$B_r u|_\Gamma = D_\nu^r - \sum_{\substack{\rho \in S_j \\ \rho < r}} B_{r-\rho}^r D_\nu^\rho \tag{3}$$

where S_j is the complementary set of R_j in the set $\{0, 1, 2, \cdots, 2m-1\}$ and $B_{r-\rho}^r$ is a pseudo-differential operator on Γ of order $\le r-\rho$. $H^s(\Omega)$, $s \in \boldsymbol{R}$, is the Sobolev space of order s on Ω. $\|u\|_{H^s(\Omega)}$ is the norm of a function u in $H^s(\Omega)$ and $(u, v)_{L^2(\Omega)}$ is the inner product of two functions u and v in $L^2(\Omega)$.

Once $B=\{B_r\}_{r \in R_j}$ is fixed, $H_B^{2m}(\Omega)$ stands for the space of functions $u \in H^{2m}(\Omega)$ satisfying

$$B_r u|_\Gamma = 0\ , \quad \forall r \in R_j\ .$$

$H_B^s(\Omega)$ is its closure in $H^s(\Omega)$.

Let $\Lambda=(1-\Delta')^{1/2}$ where Δ' is the Laplace-Beltrami operator associated with the induced metric on Γ. Then Λ^k is an isomorphism from $H^s(\Gamma)$ to $H^{s-k}(\Gamma)$.

Let A^* denote the formal adjoint of A. Let $\beta>0$ be so large that for any $\phi_k \in C^\infty(\Gamma)$, $k=0, 1, 2, \cdots, m-1$, we can find the unique function $v \in C^\infty(\bar{\Omega})$ satisfying

$$(A+A^*+2\beta)v=0 \quad \text{on} \quad \Omega\ ,$$

$$D_\nu^k v|_\Gamma = \Lambda^k \phi_k\ , \quad 0 \le k \le m-1\ .$$

Then for any $s \in \boldsymbol{R}$ there is a constant $C>0$ such that for any $\phi_k \in C^\infty(\Gamma)$, $k=0, \cdots, m-1$,

$$C^{-1}\sum_{k=0}^{m-1} \|\phi_k\|^2_{H^{s-1/2}(\Gamma)} \le \|v\|^2_{H^s(\Omega)} \le C\sum_{k=0}^{m-1} \|\phi_k\|^2_{H^{s-1/2}(\Gamma)}\ . \tag{4}$$

(As to the estimate (4) see Lions-Magenes [15].)

Now we fix $B=\{B_r\}_{r\in K_j}$. Using the above fact, we decompose any $u\in H_B^{2m}(\Omega)$ into the sum of two functions v and w:

$$u=v+w\,, \tag{5}$$

where

$$(A+A^*+2\beta)v=0 \quad \text{on} \quad \Omega\,, \tag{6}$$

$$D_\nu^k v|_\Gamma=D_\nu^k u|_\Gamma\,, \quad 0\leq k\leq m-1\,. \tag{7}$$

This implies

$$D_\nu^k w|_\Gamma=0 \quad 0\leq k\leq m-1\,, \tag{8}$$

$$(A+A^*+2\beta)w=(A+A^*+2\beta)u \quad \text{on} \quad \Omega\,, \tag{9}$$

$$D_\nu^k v|_\Gamma=0\,, \quad 0\leq k\leq m-j-1\,. \tag{10}$$

We set

$$D_\nu^k u|_\Gamma=D_\nu^k v|_\Gamma=\Lambda^k\varphi_k\,, \quad m-j\leq k\leq m-1\,. \tag{11}$$

$H_B^s(\Omega)$ stands for the closure of $H_B^{2m}(\Omega)$ in $H^s(\Omega)$. Clearly $H_B^m(\Omega)=\{u\in H^m(\Omega);$ $D_\nu^k u|_\Gamma=0,\ 0\leq k\leq m-j-1\}$.

PROPOSITION 1. *The above mapping* $u\to(w, \varphi_{m-j}, \cdots, \varphi_{m-1})$ *can be extended to be an isomorphism from* $H_B^m(\Omega)$ *to* $H_0^m(\Omega)\times H^{m-1/2}(\Gamma)\times\cdots\times H^{m-1/2}(\Gamma)$ *where* $H_0^m(\Omega)=\{u\in H^m(\Omega); D_\nu^k u|_\Gamma=0,\ 0\leq k\leq m-1\}$.

PROOF. Given any $u\in H_B^m(\Omega)$, the φ_k's determined by

$$D_\nu^k u|_\Gamma=\Lambda^k\varphi_k\,, \quad m-j\leq k\leq m-1\,, \tag{12}$$

belong to the space $H^{m-1/2}(\Gamma)$. For any functions $\varphi_{m-j}, \cdots, \varphi_{m-1}$ in $H^{m-1/2}(\Gamma)$, there is uniquely a function v satisfying (6), (10) and (11). The mapping $u\to v$ is continuous because of the estimate (4). Since $w=u-v$, the mapping $u\to(w, \varphi_{m-j}, \cdots, \varphi_{m-1})$ is continuous. Conversely once $\varphi_{m-j}, \cdots, \varphi_{m-1}\in H^{m-1/2}(\Gamma)$ and $w\in H_0^m(\Omega)$ are given, we can find a function v satisfying (6), (10) and (11). So the mapping

$$(w, \varphi_{m-j}, \cdots, \varphi_{m-1})\to w+v=u$$

is continuous. Thus the proposition is proved.

LEMMA 2. *There are pseudo-differential operators* $H_{p,q}(\beta)$, $m-j\leq p,\ q\leq m-1$, *on* Γ *of order* $2m-1$, *such that, for any* $u\in H_B^{2m}(\Omega)$,

$$\operatorname{Re}((A+\beta)u, u)_{L^2(\Omega)}=\operatorname{Re}((A+\beta)w, w)_{L^2(\Omega)}+\sum_{p,q=m-j}^{m-1}(H_{p,q}(\beta)\varphi_q, \varphi_p)_{L^2(\Gamma)}. \tag{13}$$

PROOF. Since the system $\{B_r\}_{r\in R_j}$ of boundary operators is normal, we can find normal systems $\{C_s\}_{s\in S_j}$, $\{M_r\}_{r\in R_j}$, $\{N_s\}_{s\in S_j}$ of boundary operators with the following properties:

a) the coefficients of C_s, M_r, N_s are C^∞,

b) the order of $C_s=\operatorname{ord} N_s=s\in S_j$, S_j being the complementary set of R_j in $(0, 1, 2, \cdots, 2m-1)$,

c) $\operatorname{ord} M_r=r\in R_j$

such that we have Green's formula

$$\int_\Omega Au\cdot\bar{v}dx-\int_\Omega u\cdot\overline{A^*v}dx = \sum_{r\in R_j}\int_\Gamma N_{2m-1-r}u\overline{M_r v}d\sigma-\sum_{r\in R_j}\int_\Gamma B_r u\overline{C_{2m-1-r}v}d\sigma \tag{14}$$

for any u, $v\in H^{2m}(\Omega)$, where dx is the Riemannian measure on Ω and $d\sigma$ is the Riemannian measure on Γ (cf. Lions-Magenes [15]).

Now set $v=u$ in (14), then we have

$$2\operatorname{Re}((A+\beta)u, u)_{L^2(\Omega)}=2\beta\int_\Omega u\cdot\bar{u}dx+\int_\Omega Au\cdot\bar{u}dx+\int_\Omega A^*u\bar{u}dx + \sum_{r\in R_j}\int_\Gamma M_r u\overline{N_{2m-1-r}u}d\sigma-\sum_{r\in R_j}\int_\Gamma C_{2m-1-r}u\overline{B_r u}d\sigma .$$

Since $B_r u=0$ on Γ,

$$2\operatorname{Re}((A+\beta)u, u)_{L^2(\Omega)}=((A+A^*+2\beta)u, u)_{L^2(\Omega)}+\sum_{r\in R_j}\int_\Gamma M_r u\overline{N_{2m-1-r}u}d\sigma .$$

Using the decomposition (5), we obtain

$$2\operatorname{Re}((A+\beta)u, u)_{L^2(\Omega)}=((A+A^*+2\beta)w, u)_{L^2(\Omega)}+\sum_{r\in R_j}\int_\Gamma M_r u\overline{N_{2m-1-r}u}d\sigma .$$

Since $D_\nu^k w|_\Gamma=0$, $0\leq k\leq m-1$, we have

$$2\operatorname{Re}((A+\beta)u, u)_{L^2(\Omega)}=(w, (A+A^*+2\beta)(v+w))_{L^2(\Omega)} + \sum_{r\in k_j}\int_\Gamma M_r u\overline{N_{2m-1-r}u}d\sigma+\sum_{\substack{k+l\leq 2m-1\\ m\leq k\leq 2m-1}}\int_\Gamma D_\nu^k w\overline{P_l u}d\sigma ,$$

where $\{P_l\}_l$ is a normal system of boundary operators with $\operatorname{ord} P_l=l\leq m-1$. Using (6), we have

(15) $$2\,\mathrm{Re}\,((A+\beta)u,u)_{L^2(\Omega)}=(w,(A+A^*+2\beta)w)_{L^2(\Omega)}$$
$$+\sum_{r\in R_j}\int_\Gamma M_r u\cdot\overline{N_{2m-1-r}u}d\sigma+\sum_{k=m}^{2m-1}\int_\Gamma D_\nu^k w\cdot\overline{P_{2m-k-1}u}d\sigma\ .$$

The system $\{B_r\}_{r\in R_j}$ being normal

$$M_r u|_\Gamma=0\ ,\quad r<m,\ r\in R_j\ ,$$

and we can find pseudo-differential operators $Q_{r,k}$ of order $\leq r-k$ on Γ such that

$$M_r u|_\Gamma=\sum_{k=m-j}^{m-1}Q_{r,k}\Lambda^k\varphi_k\ ,\quad m\leq r\in R_j\ .$$

If $M_r u|_\Gamma\neq 0$, then $r\geq m$. This implies $2m-1-r\leq m-1$. So there is a system of pseudo-differential operators $T_{p,q}$ on Γ of order $2m-1-q-p$ such that

(16) $$\sum_{r\in R_j}\int_\Gamma M_r u\overline{N_{2m-1-r}u}d\sigma=\sum_{p,q=m-j}^{m-1}\int_\Gamma T_{p,q}\Lambda^q\varphi_q\overline{\Lambda^p\varphi_p}d\sigma\ .$$

Since $P_l u|_\Gamma=0$ for $0\leq l\leq m-j-1$, we have

$$\sum_{k=m}^{2m-1}\int_\Gamma D_\nu^k w\overline{P_{2m-k-1}u}d\sigma=\sum_{k=m}^{m+j-1}\int_\Gamma D_\nu^k w\overline{P_{2m-k-1}u}d\sigma\ .$$

Since $w=u-v$ and since $\{B_r\}_{r\in R_j}$ is a normal system, we can find pseudo-differential operators $S_{k,l}$ on Γ of order $k-l$, $m-j\leq l\leq m-1$, such that

(17) $$D_\nu^k w=\sum_{l=m-j}^{m-1}S_{k,l}\Lambda^l\varphi_l-D_\nu^k v\ ,\quad m\leq k\leq m+j-1\ .$$

As v satisfies

$$(A+A^*+2\beta)v=0\quad\text{on}\quad\Omega\ ,$$
$$D_\nu^k v|_\Gamma=0\quad 0\leq k\leq m-j-1\ ,$$
$$D_\nu^k v|_\Gamma=\Lambda^k\varphi_k\quad m-j\leq k\leq m-1\ ,$$

we can find pseudo-differential operators $E_{k,l}(\beta)$ on Γ of order $k-l$ such that

(18) $$D_\nu^k v|_\Gamma=\sum_{l=m-j}^{m-1}E_{k,l}(\beta)\Lambda^l\varphi_l\ .$$ [1]

Of course $E_{k,l}(\beta)$ depends on β.

Equalities (15), (16), (17) and (18) prove the lemma.

REMARK. Taking the real parts of both sides of (13), if necessary, we may assume that the mapping

[1] As to this, see Seeley [16], Hörmander [11] or Fujiwara [5]. It may be noteworthy that one can calculate the symbol of $T_{k,l}(\beta)$ up to any order (see [5]).

$$\mathscr{H}(\beta):(\varphi_{m-j},\cdots,\varphi_{m-1})\to\Big(\sum_{q=m-j}^{m-1}H_{m-j,q}(\beta)\varphi_q,\cdots,\sum_{q=m-j}^{m-1}H_{m-1,q}(\beta)\varphi_q\Big)$$

is formally self-adjoint.

Let $\boldsymbol{T}$ be the circle $\boldsymbol{T}=\boldsymbol{R}/2\pi\boldsymbol{Z}$. We consider the elliptic operator $\tilde{A}=A+D_s^{2m}$ on $\Omega\times\boldsymbol{T}$. Here s means the generic point of $\boldsymbol{T}$. The boundary operators B_r can be considered as defined on $\Gamma\times\boldsymbol{T}$. $H_B^s(\Omega\times\boldsymbol{T})$, $s\in\boldsymbol{R}$, denotes the closure in $H^s(\Omega\times\boldsymbol{T})$ of $H_B^{2m}(\Omega\times\boldsymbol{T})=\{f\in H^{2m}(\Omega\times\boldsymbol{T})\,;\,B_rf|_{\Gamma\times T}=0,\ \ r\in R_j\}$. Decomposition corresponding to (5) holds for functions in $H_B^m(\Omega\times\boldsymbol{T})$, that is, for any f in $H_B^m(\Omega\times\boldsymbol{T})$,

$$f=g+h\,, \tag{19}$$

$$(\tilde{A}+\tilde{A}^*+2\beta)g=0\quad\text{on}\quad\Omega\times\boldsymbol{T}\,, \tag{20}$$

$$D_\nu^k g|_{\Gamma\times T}=D_\nu^k f|_{\Gamma\times T}\,,\quad 0\le k\le m-1\,. \tag{21}$$

We set $D_\nu^k f|_{\Gamma\times T}=\tilde{\Lambda}^k\phi_k$, $m-j\le k\le m-1$, where $\tilde{\Lambda}=(1-\Delta'+D_s^2)^{1/2}$. Just as Lemma 2 above we have

LEMMA 3. *There are pseudo-differential operators $\tilde{H}_{p,q}(\beta)$ on $\Gamma\times\boldsymbol{T}$ of order $2m-1$ such that for any f in $H_B^{2m}(\Omega\times\boldsymbol{T})$*

$$\begin{aligned}&2\,\mathrm{Re}\,((\tilde{A}+\beta)f,f)_{L^2(\Omega\times\boldsymbol{T})}\\&\quad=2\,\mathrm{Re}\,((\tilde{A}+\beta)h,h)_{L^2(\Omega\times\boldsymbol{T})}+\sum_{p,q=m-j}^{m-1}(\tilde{H}_{p,q}(\beta)\phi_q,\phi_p)_{L^2(\Gamma\times\boldsymbol{T})}\,.\end{aligned} \tag{22}$$

The system of operators $\tilde{H}_{p,q}(\beta)$ is formally self-adjoint. Before going further, we consider the decomposition (19) for function $f=u\otimes e^{i\gamma s}$, γ being an integer and $u\in H_B^m(\Omega)$. Let v be a function determined by

$$(A+A^*+2(\gamma^{2m}+\beta))v=0\quad\text{on}\quad\Omega\,, \tag{23}$$

$$D_\nu^k v|_\Gamma=D_\nu^k u|_\Gamma\,,\quad 0\le k\le m-1\,. \tag{24}$$

Then $g=v\otimes e^{i\gamma s}$ and $h=w\otimes e^{i\gamma s}$ where $w=u-v\in H^m(\Omega)$ satisfying

$$(A+A^*+2(\gamma^{2m}+\beta))w=(A+A^*+2(\gamma^{2m}+\beta))u\,. \tag{25}$$

Setting $$D_\nu^k u|_\Gamma=\Lambda^k\varphi_k\,,\quad m-j\le k\le m-1\,,$$

$$\begin{aligned}D_\nu^k f|_{\Gamma\times T}&=\Lambda^k\varphi_k\otimes e^{i\gamma s}\\&=\Lambda^k(1+\gamma^2-\Delta')^{-k/2}(1+\gamma^2-\Delta')^{k/2}\varphi_k\otimes e^{i\gamma s}\\&=\Lambda^k(1+\gamma^2-\Delta')^{-k/2}(1-\Delta'+D_s^2)^{k/2}(\varphi_k\otimes e^{i\gamma s})\\&=\tilde{\Lambda}^k(\Lambda^k(1+\gamma^2-\Delta')^{-k/2}\varphi_k\otimes e^{i\gamma s})\,.\end{aligned} \tag{26}$$

Thus

$$\phi_k=\Lambda^k(1+\gamma^2-\Delta')^{-k/2}\varphi_k\otimes e^{i\gamma s}\,. \tag{27}$$

Now we have

$$((\tilde{A}+\beta)f, f)_{L^2(\Omega\times T)}=((A+\gamma^{2m}+\beta)u, u)_{L^2(\Omega)}\ , \tag{28}$$

$$((\tilde{A}+\beta)h, h)_{L^2(\Omega\times T)}=((A+\gamma^{2m}+\beta)w, w)_{L^2(\Omega)}\ , \tag{29}$$

$$\sum_{p,q}(\tilde{H}_{pq}(\beta)\phi_q, \phi_p)_{L^2(\Omega\times T)}=\sum_{p,q}(H_{pq}(\gamma^{2m}+\beta)\varphi_q, \varphi_p)_{L^2(\Gamma)} \tag{30}$$

with

$$\phi_p=\Lambda^p(1+\gamma^2-\Delta')^{-p/2}\varphi_p\otimes e^{i\gamma s}\ . \tag{31}$$

PROPOSITION 4. *Let $t\geq 0$. Then there is a constant $C>0$ such that for any f in $H^t(\Omega\times \boldsymbol{T})$*

$$C^{-1}\|f\|^2_{H^t(\Omega\times T)}\leq\sum_l(\|w_l\|^2_{H^t(\Omega)}+(1+l^2)^t\|w_l\|^2_{L^2(\Omega)})\leq C\|f\|^2_{H^t(\Omega\times T)}\ , \tag{32}$$

where

$$f(x, s)=\sum_{l=-\infty}^{\infty}w_l(x)e^{ils} \tag{33}$$

is the Fourier expansion of $f(x, s)$.

PROOF. Let Ω' be a copy of Ω and $\hat{\Omega}$ be the double of Ω obtained by sticking Ω to Ω' along Γ. We can extend the metric given on Ω to $\hat{\Omega}$. Let Δ be the Laplace-Beltrami operator associated with it. Consider the eigenvalues $\{\lambda_n^2\}_{n=0}^{\infty}$ and eigenfunctions $\{\chi_n\}$ of $-\Delta$, where χ_n belongs to the eigen value λ_n^2, that is

$$-\Delta\chi_n=\lambda_n^2\chi_n\ . \tag{34}$$

Since the space $H^t(\Omega\times \boldsymbol{T})$ is the restriction of $H^t(\hat{\Omega}\times \boldsymbol{T})$ to $\Omega\times \boldsymbol{T}$, we have only to find a constant $C>0$ such that for any $f=\sum_{l=-\infty}^{\infty}w_l(x)e^{ils}$ in $H^t(\hat{\Omega}\times \boldsymbol{T})$

$$C^{-1}\|f\|^2_{H^t(\hat{\Omega}\times T)}\leq\sum_l(\|w_l\|^2_{H^t(\hat{\Omega})}+(1+l^2)^t\|w_l\|^2_{L^2(\hat{\Omega})})\leq C\|f\|^2_{H^t(\hat{\Omega}\times T)}\ . \tag{35}$$

Let $f(x, s)=\sum_{n,\gamma=-\infty}^{\infty}a_{n\gamma}\chi_n(x)e^{i\gamma s}$ be the representation of f in terms of eigenfunctions of $-\Delta+D_s^2$. Then

$$\begin{aligned}\|f\|^2_{H^t(\hat{\Omega}\times T)}&=\|(1-\Delta'+D_s^2)^{t/2}f\|^2_{L^2(\hat{\Omega}\times T)}\\&=\|\sum_{n,\gamma}a_{n\gamma}(1+\lambda_n^2+\gamma^2)^{t/2}\chi_n\otimes e^{i\gamma s}\|^2_{L^2(\hat{\Omega}\times T)}\\&=\sum_{\gamma,n}\|a_{n\gamma}(1+\lambda_n^2+\gamma^2)^{t/2}\chi_n\|^2_{L^2(\hat{\Omega})}\ .\end{aligned}$$

Since $2^{-t}[(1+\lambda_n^2)^t+(1+\gamma^2)^t]\leq(1+\lambda_n^2+\gamma^2)^t\leq 2^t((1+\lambda_n^2)^t+(1+\gamma^2)^t)$, the square of the norm $\|f\|^2_{H^t(\hat{\Omega}\times T)}$ is equivalent to

$$\sum_{n,\gamma}|a_{n\gamma}|^2\int_{\hat{\Omega}}[(1+\lambda_n^2)^t+(1+\gamma^2)^t]|\chi_n(x)|^2dx \tag{36}$$

$$= \sum_{n,\gamma} |a_{n\gamma}|^2 [\|(1-\Delta)^{t/2}\chi_n\|^2_{L^2(\hat{\Omega})} + (1+\gamma^2)^t \|\chi_n\|^2_{L^2(\hat{\Omega})}]$$
$$= \sum_{\gamma} \{\|(1-\Delta)^{t/2} \sum_n a_{n\gamma}\chi_n\|^2_{L^2(\hat{\Omega})} + (1+\gamma^2)^t \| \sum_n a_{n\gamma}\chi_n\|^2_{L^2(\hat{\Omega})}] .$$

Since $w_\gamma(x) = \sum_n a_{n\gamma}\chi_n(x)$, we turn the last side into

$$\sum_\gamma (\|(1-\Delta)^{t/2} w_\gamma\|^2_{L^2(\hat{\Omega})} + (1+\gamma^2)^t \|w_\gamma\|^2_{L^2(\hat{\Omega})} = \sum_\gamma \{\|w_\gamma\|^2_{H^t(\hat{\Omega})} + (1+\gamma^2)^t \|w_\gamma\|^2_{L^2(\hat{\Omega})}\} .$$

Thus with some $C>0$, we have

$$C^{-1}\|f\|^2_{H^t(\hat{\Omega}\times T)} \leq \sum_\gamma \{\|w_\gamma\|^2_{H^t(\hat{\Omega})} + (1+\gamma^2)^t \|w_\gamma\|^2_{L^2(\hat{\Omega})}\} \leq C\|f\|^2_{H^t(\hat{\Omega}\times T)} . \tag{37}$$

Similarly we can easily prove

COROLLARY 5. *For any ϕ in $L^2(\Gamma\times \boldsymbol{T})$, we have*

$$\|\phi\|^2_{H^{-1/2}(\Gamma\times T)} \leq C \sum_{l=-\infty}^{\infty} (1+l^2)^{-1} \|\varphi_l\|^2_{L^2(\Gamma)} , \tag{38}$$

where

$$\phi = \sum_{l=-\infty}^{\infty} \varphi_l \otimes e^{ils}$$

is the Fourier expansion of ϕ.

Using Agmon's method [2], we have

THEOREM 6. *The following four propositions are equivalent to each other:*

(i) *There are some β_1, $C_1>0$ such that the estimate*

$$\mathrm{Re}\,((A+\beta_1)u, u)_{L^2(\Omega)} \geq C_1\|u\|^2_{H^{m-1/2}(\Omega)} \tag{39}$$

holds for any u in $H^{2m}_B(\Omega)$.

(ii) *There are some constants β_2, $C_2>0$ such that the estimate*

$$\mathrm{Re}\,((\tilde{A}+\beta_2)f, f)_{L^2(\Omega\times T)} \geq C_2\|f\|^2_{H^{m-1/2}(\Omega\times T)} \tag{40}$$

holds for any f in $H^{2m}_B(\Omega\times \boldsymbol{T})$.

(iii) *There are some constants β_3, $C_3>0$ such that the estimate*

$$\sum_{p,q=m-j}^{m-1} (H_{pq}(\beta_3)\varphi_q, \varphi_p)_{L^2(\Gamma)} \geq C_3 \sum_{p=m-j}^{m-1} \|\varphi_p\|^2_{H^{m-1}(\Gamma)} \tag{41}$$

holds for any $\varphi_{m-j}, \cdots, \varphi_{m-1} \in H^{m-1/2}(\Gamma)$.

(iv) *There are some constants γ, β_4, $C_4>0$ such that the estimate*

$$\sum_{p,q=m-j}^{m-1} (\tilde{H}_{p,q}(\beta_4)\phi_q, \phi_p)_{L^2(\Gamma\times T)} + \gamma \sum_{p=m-j}^{m-1} \|\phi_p\|^2_{H^{-1/2}(\Gamma\times T)} \geq C_4 \sum_{p=m-j}^{m-1} \|\phi_p\|^2_{H^{m-1}(\Gamma\times T)} \tag{42}$$

holds for any $\phi_{m-j}, \phi_{m-j+1}, \cdots, \phi_{m-1}$ in $H^{m-1/2}(\Gamma\times \boldsymbol{T})$.

REMARK. A necessary and sufficient condition for the estimate (1) with $0\leq\varepsilon<1/2$ to hold is that there are constants β_5, γ_5 and C_5 such that

$$\sum_{p,q=m-j}^{m-1}(H_{p,q}(\beta_5)\varphi_q,\varphi_p)_{L^2(\Gamma)}+\gamma_5\sum_{p=m-j}^{m-1}\|\varphi_p\|^2_{H^{-1/2}(\Gamma)}\geq C_5\sum_{p=m-j}^{m-1}\|\varphi_p\|^2_{H^{m-1/2-\varepsilon}(\Gamma)} \tag{43}$$

holds for any $\varphi_{m-j},\cdots,\varphi_{m-1}$ in $H^{m-1/2}(\Gamma)$.

PROOF OF THEOREM 6.

(i)⇒(iii): For any $(\varphi_{m-j},\cdots,\varphi_{m-1})$ choose u in Proposition 1 corresponding with $(0,\varphi_{m-j},\cdots,\varphi_{m-1})$. Then by (39) and (13) we have

$$\sum_{p,q=m-j}^{m-1}(H_{p,q}(\beta)\varphi_q,\varphi_p)_{L^2(\Gamma)}\geq C_1\|u\|^2_{H^{m-1/2}(\Omega)}\ . \tag{44}$$

Applying the estimate (4) with $s=m-1/2$ to (44), we have the inequality (41).

Similarly we can prove that (ii) implies (iv).

(iii)⇒(i) For any $w\in H^{2m}(\Omega)\cap H_0^m(\Omega)$, we have

$$\mathrm{Re}\,((A+\beta_3)w,w)\geq C\|w\|^2_{H^m(\Omega)} \tag{45}$$

(Cf. Gårding [7]). Adding (45) to (41) we have

$$\mathrm{Re}\,((A+\beta_3)w,w)+\sum_{p,q=m-j}^{m-1}(H_{p,q}(\beta_3)\varphi_q,\varphi_p)_{L^2(\Gamma)}\geq C_3\Big(\sum_{p=m-j}^{m-1}\|\varphi_p\|^2_{H^{m-1}(\Gamma)}+\|w\|^2_{H^m(\Omega)}\Big)\ . \tag{46}$$

Let v be the function satisfying

$$(A+A^*+2\beta)v=0\quad\text{on}\quad\Omega\,,$$
$$D_\nu^k v|_\Gamma=\Lambda^k\varphi_k\,,\qquad m-j\leq k\leq m-1\,,$$
$$D_\nu^k v|_\Gamma=0\,,\qquad 0\leq k\leq m-j-1\,.$$

Then (4) gives

$$\|v\|^2_{H^{m-1/2}(\Omega)}\leq C\sum_{k=m-j}^{m-1}\|\varphi_k\|_{H^{m-1}(\Gamma)}\ . \tag{47}$$

This and (13) give

$$\mathrm{Re}\,((A+\beta)u,u)_{L^2(\Omega)}\geq C(\|v\|^2_{H^{m-1/2}(\Omega)}+\|w\|^2_{H^m(\Omega)})\,, \tag{48}$$

where $u=v+w$.

Since the mapping $(v,w)\to u=v+w$ is a continuous mapping $H^{m-1/2}(\Omega)\times H^{m-1/2}(\Omega)$ into $H^{m-1/2}(\Omega)$, we have

(49) $$\|v\|^2_{H^{m-1/2}(\Omega)}+\|w\|^2_{H^{m-1/2}(\Omega)}\geq C\|u\|^2_{H^{m-1/2}(\Omega)}\ .$$

This together with (48) gives

(50) $$\mathrm{Re}\,((A+\beta)u,\,u)\geq C\|u\|^2_{H^{m-1/2}(\Omega)}\ .$$

(i)⇒(ii) Let f be any function in $H^{2m}_{B}(\Omega\times\boldsymbol{T})$. Let

(51) $$f(x,\,s)=\sum_{l=-\infty}^{\infty}w_l(x)\otimes e^{ils}$$

be the Fourier series of $f(x,s)$. Since $w_l(x)$ belongs to $H^{2m}_{B}(\Omega)$, we have

(52) $$\mathrm{Re}\,((A+\beta_1)w_l,\,w_l)_{L^2(\Omega)}\geq C_1\|w_l\|^2_{H^{m-1/2}(\Omega)}\ .$$

So

$$\mathrm{Re}\,((A+\beta_1+l^{2m}+2)w_l,\,w_l)_{L^2(\Omega)}\geq C_1(\|w_l\|^2_{H^{m-1/2}(\Omega)}+(1+l^2)^{m-1/2}\|w_l\|_{L^2(\Omega)})\ .$$

Thus

$$\sum_l \mathrm{Re}\,((\tilde{A}+\beta_1+2)w_l\otimes e^{ils},\,w_l\otimes e^{ils})_{L^2(\Omega\times\boldsymbol{T})}$$
$$\geq C_1\sum_l\{\|w_l\|^2_{H^{m-1/2}(\Omega)}+(1+l^2)^{m-1/2}\|w_l\|_{L^2(\Omega)}\}\ .$$

Since $((\tilde{A}+\beta_1+2)w_l\otimes e^{ils},\,w_k\otimes e^{iks})_{L^2(\Omega\times\boldsymbol{T})}=0$ for $k\neq l$, we have

$$\mathrm{Re}\,((\tilde{A}+\beta_1+2)f,f)_{L^2(\Omega\times\boldsymbol{T})}\geq C_1\sum_l\{\|w_l\|^2_{H^{m-1/2}(\Omega)}+(1+l^2)^{m-1/2}\|w_l\|^2_{L^2(\Omega)}\}\ .$$

This and (32) give

$$\mathrm{Re}\,((\tilde{A}+\beta_1+2)f,f)_{L^2(\Omega\times\boldsymbol{T})}\geq C_2\|f\|^2_{H^{m-1/2}(\Omega\times\boldsymbol{T})}\ .$$

(iv)⇒(iii) Let $\varphi_{m-j},\cdots,\varphi_{m-1}$ be arbitrary functions in $H^{2m-1/2}(\Gamma)$. Apply the inequality (42) to the functions

(53) $$\phi_p=\Lambda^p(1+l^2-\Delta')^{-p/2}\varphi_p\otimes e^{ils}\qquad m-j\leq p\leq m-1\ ,$$

where l is an integer which will be fixed later.

Then, taking (30) and (31) into account, we have

(54) $$\sum_{p,q=m-j}^{m-1}(H_{p,q}(l^2+\beta_4)\varphi_q,\,\varphi_p)_{L^2(\Gamma)}$$
$$+\gamma\sum_{p=m-j}^{m-1}\|\Lambda^p(1+l^2-\Delta')^{-p/2}\varphi_p\otimes e^{ils}\|^2_{H^{-1/2}(\Gamma\times\boldsymbol{T})}$$
$$\geq C_4\sum_{p=m-j}^{m-1}\|\Lambda^p(1+l^2-\Delta')^{-1/2}\varphi_p\otimes e^{ils}\|^2_{H^{m-1}(\Gamma\times\boldsymbol{T})}\ .$$

The corollary to Proposition 4 asserts that there exists a constant $C>0$ such that, for any l,

$$\sum_{p=m-j}^{m-1} \|\Lambda^p(1+l^2-\Delta')^{-p/2}\varphi_p \otimes e^{ils}\|^2_{H^{-1/2}(\Gamma\times T)}$$
$$\leq C(1+l^2)^{-1} \sum_{p=m-j}^{m-1} \|\Lambda^p(1+l^2-\Delta')^{-p/2}\varphi_p \otimes e^{ils}\|_{L^2(\Gamma\times T)} .$$

Therefore if we take l so large that

$$C(1+l^2)^{-1}\gamma < C_4/2 ,$$

then

$$\sum_{p,q=m-j}^{m-1} (H_{p,q}(l^2+\beta)\varphi_q, \varphi_p) \geq \frac{C_4}{2} \sum_{p=m-j}^{m-1} \|\Lambda^p(1+l^2-\Delta')^{-p/2}\varphi_p \otimes e^{ils}\|_{H^{m-1}(L\times T)} .$$

Using (32), from this we have

$$\sum_{p,q=m-j}^{m-1} (H_{p,q}(l^2+\beta_4)\varphi_q, \varphi_p) \geq C \sum_{p=m-j}^{m-1} \|\Lambda^p(1+l^2-\Delta')^{-p/2}\varphi_p\|^2_{H^{m-1}(\Gamma)} .$$

Since the mapping $\Lambda^p(1+l^2-\Delta')^{-p/2}$ is an isomorphism of $H^{m-1}(\Gamma)$ onto itself, we have thus proved

$$\sum_{p,q=m-j}^{m-1} (H_{p,q}(l^2+\beta_4)\varphi_q, \varphi_p) \geq C_3 \sum_{p=m-j}^{m-1} \|\varphi_p\|^2_{H^{m-1}(\Gamma)} . \tag{55}$$

From now on we will consider the estimate (42). The estimate (42) can be localized:

THEOREM 7. *Assume that there exists a family of finite number of real functions* $\{\mu_k(x, s)\}_{k=1}^N$ *in* $\mathscr{D}(\Gamma\times \boldsymbol{T})$ *satisfying*

(i) $$\sum_k \mu_k(x, s)^2 = 1 ,$$

(ii) *for any* $\phi_{m-j}, \phi_{m-j+1}, \cdots, \phi_{m-1} \in \mathscr{D}(\Gamma\times \boldsymbol{T})$ *and for any* k *the following estimate holds*:

$$\sum_{p,q=m-j}^{m-1} (\tilde{H}_{p,q}(\beta_4)\mu_k\phi_q, \mu_k\phi_p)_{L^2(\Gamma\times T)} + \gamma \sum_{p=m-j}^{m-1} \|\mu_k\phi_p\|^2_{H^{-1/2}(\Gamma\times T)}$$
$$\geq C \sum_{p=m-j}^{m-1} \|\mu_k\phi_p\|^2_{H^{m-1}(\Gamma\times T)} . \tag{56}$$

Then for any $\phi_{m-j}, \phi_{m-j+1}, \cdots, \phi_{m-1} \in \mathscr{D}(\Gamma\times \boldsymbol{T})$ *the estimate* (42) *holds with some* β_4, C_4, *and* $\gamma > 0$.

PROOF. Let $\phi_{m-j}, \cdots, \phi_{m-1} \in \mathscr{D}(\Gamma\times \boldsymbol{T})$ be any functions. Then

$$\sum_{p,q=m-j}^{m-1} (\tilde{H}_{p,q}(\beta_4)\phi_q, \phi_p)_{L^2(\Gamma\times T)} = \sum_k \sum_{p,q=m-j}^{m-1} (\mu_k \tilde{H}_{p,q}(\beta_4)\phi_q, \mu_k\phi_p)_{L^2(\Gamma\times T)}$$
$$= \sum_{k,p,q} (\tilde{H}_{p,q}(\beta_4)\mu_k\phi_q, \mu_k\phi_p)_{L^2(\Gamma\times T)} + \sum_{k,p,q} ([\mu_k, \tilde{H}_{p,q}(\beta_4)]\phi_q, \mu_k\phi_p)_{L^2(\Gamma\times T)}$$

where $[\mu_k, \tilde{H}_{p,q}(\beta_4)]$ means the commutator $\mu_k\tilde{H}_{p,q}(\beta_4)-\tilde{H}_{p,q}(\beta_4)\mu_k$. Taking it into account that the principal symbol of $[\mu_k, \tilde{H}_{p,q}(\beta_4)]$ is skew hermitian, we have the estimate

$$\sum_{p,q=m-j}^{m-1}(\tilde{H}_{p,q}(\beta_4)\varphi_q, \varphi_p)_{L^2(\Gamma\times T)}$$
$$\geq C\sum_{k,p,q}\|\mu_k\phi_p\|^2_{H^{m-1}(\Gamma\times T)}-\gamma\sum_{k,p,q}\|\mu_k\phi_p\|^2_{H^{-1/2}(\Gamma\times T)}-C\sum_{p=m-j}^{m-1}\|\phi_p\|^2_{H^{m-3/2}(\Gamma\times T)}\ .$$

This together with the trivial inequality

$$\sum_k\|\mu_k\phi_p\|^2_{H^{m-1}(\Gamma\times T)}\geq\frac{1}{2}\|\phi_p\|^2_{H^{m-1}(\Gamma\times T)}-C\|\phi_p\|^2_{H^{-1/2}(\Gamma\times T)}$$

gives the estimate

$$\sum_{p,q=m-j}^{m-1}(\tilde{H}_{p,q}(\beta)\varphi_q, \varphi_p)\geq C\sum_{p,q=m-j}^{m-1}\|\phi_p\|^2_{H^{m-1}(\Gamma\times T)}-C\sum_{p=m-j}^{m-1}\|\phi_p\|^2_{H^{-1/2}(\Gamma\times T)}\ .$$

According to Theorem 7 we have only to consider the estimate (42) under the following situation:

Let Ω be any open set (not necessarily connected) in $\boldsymbol{R}^{n+1}$. Let $H_{p,q}$, $m-j\leq p,q\leq m-1$, be pseudo-differential operators of order 1 defined in Ω and let $h_{p,q}(x,\xi)\sim\sum_{j=0}h^j_{p,q}(x,\xi)$ be the symbol of $H_{p,q}$ (cf. Kohn-Nirenberg [14] or Hörmander [10]). We assume the matrix of principal symbol $(h^0_{p,q}(x,\xi))_{pq}$ of the operator $H_{p,q}$ is hermitian. Then the following result holds:

THEOREM 8. *The following two properties are equivalent*:

(i) *For any compact set K in Ω, there are constants C_0 and $C_1>0$ such that, for any $\phi_{m-j},\cdots,\phi_{m-1}\in\mathscr{D}(K)$,*

$$\text{(57)}\qquad \mathrm{Re}\sum_{p,q=m-j}^{m-1}(H_{p,q}\phi_q, \phi_p)_{L^2(\Omega)}+C_1\sum_{r=m-j}^{m-1}\|\phi_p\|^2_{H^{-1/2}(\Omega)}\geq C_0\sum_{p=m-j}^{m-1}\|\phi_p\|^2_{L^2(\Omega)}\ .$$

(ii) *For any compact set K_1 in Ω, there exist constant $C>0$, integer $N>0$ and a function $\varepsilon(\xi)$, with $\lim_{|\xi|\to\infty}\varepsilon(\xi)=0$ such that, for any $x\in K_1$, $\psi_{m-j},\cdots,\psi_{m-1}\in C_0^\infty(\boldsymbol{R}^{n+1})$,*

$$\text{(58)}\quad \mathrm{Re}\sum_{p,q=m-j}^{m-1}\sum_{|\alpha|+|\beta|\leq 2}\frac{|\xi|^{(|\beta|-|\alpha|)/2}}{\alpha!\beta!}h^{0(\beta)}_{p,q(\alpha)}(x,\xi)\int_{R^{n+1}}(iD_y)^\beta\psi_s(y)\overline{(-iy)^\alpha\psi_r(y)}dy$$
$$+\mathrm{Re}\sum_{p,q=m-j}^{m-1}h^1_{p,q}(x,\xi)\int_{R^{n+1}}\psi_q(y)\overline{\psi_p(y)}dy$$
$$+\varepsilon(\xi)\sum_{|\alpha|+|\beta|\leq N}\sum_{p=m-j}^{m-1}\int_{R^{n+1}}|D^\alpha_y y^\beta\psi_p(y)|^2dy\geq C\sum_{p=m-j}^{m-1}\int_{R^{n+1}}|\psi_p(y)|^2dy$$

where

$$q^{0(\beta)}_{(\alpha)}(x,\xi)=D^\alpha_x D^\beta_\xi q^0(x,\xi)\ .$$

REMARK. The estimate (43) holds if and only if the matrix defined by the principal symbols $\sigma_{2m-1}(H_{pq}(\beta))(x', \xi')$ is uniformly positive definite on $|\xi'|=1$.

The proof of Theorem 8 will be given in the next section.

§3. Proof of Theorem 8.

In this section we slightly change notations. Let Ω be a bounded open set in $\boldsymbol{R}^n$ (not necessarily connected). Let P be a pseudo-differential operator of order ρ on Ω which is given by

$$(Pu)(x)=(2\pi)^{-n}\int_{R^n} p(x, \xi)\hat{u}(\xi)e^{ix\cdot\xi}d\xi\ , \quad x\in\Omega\ , \tag{59}$$

where $\hat{u}(\xi)$ is Fourier transform of $u\in\mathscr{D}(\Omega)$ and the functions $p(x, \xi)$ admits the asymptotic expansion

$$p(x, \xi)\sim\sum_{j=0}^{\infty} p_j(x, \xi)\ . \tag{60}$$

As to the meaning of this asymptotic expansion, see Hörmander [10]. We assume that the homogeneous degree of $p_j(x, \xi)$ in ξ is $\rho-j$.

THEOREM 9. *Let K be any compact set in Ω and φ be a function in $\mathscr{D}(\Omega)$ which is identically* 1 *in some neighborhood of K. Let σ be a number satisfying $\rho-\sigma<0$, $\sigma>0$. Then for any $N>0$, there is a constant C_K such that for any x in K, any $\xi\in\boldsymbol{R}^n$, $|\xi|>1$ and for any ψ, ϕ in $\mathscr{D}(\boldsymbol{R}^n)$, we have the estimate*

$$\begin{aligned}\Big| |\xi|^{n/2}\int_{\Omega} &(P\varphi v_1)(y)\overline{\varphi(y)v_2(y)}dy \\ &-\sum_{|\alpha|,|\beta|<N}\frac{(\sqrt{|\xi|})^{|\beta|-|\alpha|}}{\alpha!\beta!}D_x^\alpha D_\xi^\beta p(x, \xi)\int_{R^n} D_y^\beta\psi(y)\overline{(-iy)^\alpha\phi(y)}dy\Big| \\ &\leq C_K|\xi|^{\rho+\frac{n-N-3\sigma}{2}}(\|\psi\|_{H^{2N+\frac{n+1}{2}}(R^n)})(\sum_{|\alpha|\leq N}\|y^\alpha\phi\|_{H^\sigma(\boldsymbol{R}^n)})\end{aligned} \tag{61}$$

where

$$v_1(y)=\psi((y-x)\sqrt{|\xi|})e^{iy\cdot\xi}\ ,$$
$$v_2(y)=\phi((y-x)\sqrt{|\xi|})e^{iy\cdot\xi}\ .$$

PROOF. Since

$$v_1(y)=\psi((y-x)\sqrt{|\xi|})e^{iy\cdot\xi}\ ,$$
$$v_2(y)=\phi((y-x)\sqrt{|\xi|})e^{iy\cdot\xi}\ ,$$

we have

$$\hat{v}_1(\eta)=\int_{R^n} e^{-iy\cdot\eta}\varphi((y-x)\sqrt{|\xi|})e^{iy\cdot\xi}dy=e^{-ix\cdot(\eta-\xi)}|\xi|^{-n/2}\hat{\varphi}\left(\frac{\eta-\xi}{\sqrt{|\xi|}}\right),$$

and

$$\hat{v}_2(\eta)=e^{-ix\cdot(\eta-\xi)}|\xi|^{-n/2}\hat{\phi}\left(\frac{\eta-\xi}{\sqrt{|\xi|}}\right).$$

Set $Q=P\varphi$, then P and Q have the same symbol on K. Let $q(x, \xi)$ denote the Fourier integral kernel of Q, that is, for any u in $\mathcal{D}(\Omega)$,

$$(Qu)(y)=(2\pi)^{-n}\int_{R^n} q(y, \eta)\hat{u}(\eta)e^{iy\cdot\eta}d\eta .$$

Set

$$q(y, \eta)=\sum_{|\alpha|\leq N-1}\frac{(i(y-x))^\alpha}{\alpha!}D_x^\alpha p(x, \eta)+\sum_{|\alpha|=N}(y-x)^\alpha R_\alpha(x, y, \eta) . \tag{62}$$

The operator R_α defined by

$$(R_\alpha u)(y)=(2\pi)^{-n}\int R_\alpha(x, y, \eta)\hat{u}(\eta)e^{iy\cdot\eta}d\eta \tag{63}$$

is a pseudo-differential operator of order ρ. Therefore we have

$$\begin{aligned}\left|\int(y-x)^\alpha(R_\alpha v_1)(y)\overline{\varphi(y)v_2(y)}dy\right| &\leq C\|R_\alpha\varphi v_1\|_{H^{-\sigma}(\mathbf{R}^n)}\|(y-x)^\alpha v_2\|_{H^\sigma(\mathbf{R}^n)}\\ &\leq C\|v_1\|_{H^{\rho-\sigma}(\mathbf{R}^n)}\|(y-x)^\alpha v_2\|_{H^\sigma(\mathbf{R}^n)} .\end{aligned} \tag{64}$$

Since $\rho-\sigma<0$, it is trivial that

$$\|v_1\|^2_{H^{\rho-\sigma}(R^n)}\leq C\|v_1\|^2_{L^2(R^n)}=\|\xi\|^{-n/2}\|\varphi\|^2_{L^2(R^n)} . \tag{65}$$

On the other hand the following estimate is valid:

$$\|(y-x)^\alpha v_2\|^2_{H^\sigma(R^n)}\leq C|\xi|^{-(n/2)+4\sigma-N}\|y^\alpha\phi\|^2_{H^\sigma(R^n)} . \tag{66}$$

In fact,

$$\begin{aligned}\|(y-x)^\alpha v_2\|^2_{H^\sigma(R^n)} &=\int_{R^n}|(-D_\eta-x)^{-\alpha}\hat{v}_2(\eta)|^2(1+|\eta|)^{2\sigma}d\eta\\ &=|\xi|^{-n}\int_{R^n}\left|D_\eta^\alpha\hat{\phi}\left(\frac{\eta-\xi}{\sqrt{|\xi|}}\right)\right|^2(1+|\eta|)^{2\sigma}d\eta\\ &=|\xi|^{-n/2-N}\int_{R^n}|(D_\eta^\alpha\hat{\phi})(\eta)|^2(1+|\xi+\sqrt{|\xi|}\eta|)^{2\sigma}d\eta\\ &\leq C|\xi|^{-n/2-N+\sigma}\int_{R^n}|D_\eta^\alpha\hat{\phi}(\eta)|^2(1+|\xi|)^{2\sigma}(1+|\eta|)^{2\sigma}d\eta\\ &\leq C|\xi|^{-n/2-N+3\sigma}\|y^\alpha\phi\|^2_{H^\sigma(R^n)} .\end{aligned}$$

It follows from (64), (65) and (66) that

(67)
$$\left|\int_{R^n}(y-x)^\alpha(R_\alpha v_1)(y)\overline{\varphi(y)v_2(y)}dy\right| \le C|\xi|^{-(n+N+3\sigma)/2}\|\psi\|_{L^2(R^n)}\|y^\alpha\phi\|_{H^\sigma(R^n)}\ .$$

Now we consider the mapping P_α defined by

(68)
$$(P_\alpha u)(y)=(2\pi)^{-n}\int_{R^n}D_x^\alpha p(x,\eta)\hat{u}(\eta)e^{iy\cdot\eta}d\eta\ .$$

By change of variables we turn this into

(69)
$$P_\alpha v_1(y)=(2\pi)^{-n}e^{iy\cdot\xi}\int_{R^n}D_x^\alpha p(x,\xi+\sqrt{|\xi|}\eta)\hat{\phi}(\eta)e^{i\sqrt{|\xi|}\cdot(y-x)\cdot\eta}d\eta\ .$$

Putting

(70)
$$w_\alpha(y)=(2\pi)^{-n}\int_{R^n}D_x^\alpha p(x,\xi+\sqrt{|\xi|}\eta)\hat{\phi}(\eta)e^{iy\cdot\eta}d\eta\ ,$$

we have

(71)
$$P_\alpha v_1(y)=e^{iy\cdot\xi}w_\alpha(\sqrt{|\xi|}\cdot(y-x))\ .$$

By Taylor's theorem the function given by

$$R_N^\alpha=D_x^\alpha p(x,\xi+\sqrt{|\xi|}\eta)-\sum_{|\beta|<N}\frac{D_x^\alpha}{\beta!}D_\xi^\beta p(x,\xi)(i\sqrt{|\xi|}\eta)^\beta$$

is estimated as

$$|R_N^\alpha|\le\begin{cases}C|\sqrt{|\xi|}\eta|^N(1+|\xi|)^{\rho-N} & \text{when}\quad |\eta|<(1/2)\sqrt{|\xi|}\ ,\\ C|\sqrt{|\xi|}\eta|^N & \text{everywhere}\ .\end{cases}$$

And we have

$$\begin{aligned}&\left|\int R_N^\alpha(x,\xi,\eta)\hat{\phi}(\eta)e^{iy\cdot\eta}d\eta\right|\\ &\le C\int_{R^n}(1+|\xi|)^{\rho-N}(\sqrt{|\xi|}\eta)^N|\hat{\phi}(\eta)|d\eta+C\int_{|\eta|\ge(1/2)\sqrt{|\xi|}}|\hat{\phi}(\eta)||\xi|^{N/2}|\eta|^Nd\eta\\ &\le C|\xi|^{\rho-N/2}\|\phi\|_{H^{N+(n+1)/2}(R^n)}+C|\xi|^{(n-N)/2}\|\phi\|_{H^{2N}(R^n)}\\ &\le C|\xi|^{\rho+(n-N)/2}\|\phi\|_{H^{2N+(n+1)/2}}\ .\end{aligned}$$

This implies that

(72)
$$\left|w_\alpha(y)-\sum_{|\beta|<N}\frac{1}{\beta!}D_x^\alpha D_\xi^\beta p(x,\xi)(i\sqrt{|\xi|})^{|\beta|}D_y^\beta\phi(y)\right|\le C|\xi|^{\rho+(n-N)/2}\|\phi\|_{H^{2N}(R^n)}\ .$$

If $|\xi|$ is so large that $\varphi\left(\frac{y}{\sqrt{|\xi|}}+x\right)=1$ for any y in Ω,

$$(73)\qquad \int_{R^n} P_\alpha v_1(y)\overline{\varphi(y)v_2(y)}dy=\int_{R^n} w_\alpha(\sqrt{|\xi|}(y-x))\varphi(y)\overline{\phi((y-x)\sqrt{|\xi|})}dy$$
$$=|\xi|^{-n/2}\int_{R^n} w_\alpha(y)\varphi\Big(\frac{y}{\sqrt{|\xi|}}+x\Big)\overline{\phi(y)}dy$$
$$=|\xi|^{-n/2}\int_{R^n} w_\alpha(y)\overline{\phi(y)}dy\ .$$

Thus we have obtained

$$\Big||\xi|^{n/2}\int P_\alpha v_1(y)\varphi(y)\overline{v_2(y)}dy-\sum_{|\beta|<N}\frac{(i\sqrt{|\xi|})^{|\beta|}}{\beta!}D_x^\alpha D_\xi^\beta p(x,\xi)\int D_y^\beta\varphi(y)\overline{\phi(y)}dy\Big|$$
$$\leq C|\xi|^{\rho+(n-N)/2}\|\varphi\|_{H^{2N}(R^n)}\|\phi\|_{L^2(R^n)}\ .$$

This combined with (62) and (67) gives

$$(74)\qquad |\xi|^{n/2}\int_{R^n}(P\varphi v_1)(y)\overline{\varphi(y)v_2(y)}dy$$
$$-\sum_{|\alpha|+|\beta|<N}\frac{(\sqrt{|\xi|})^{|\beta|-|\alpha|}}{\alpha!\beta!}D_x^\alpha D_\xi^\beta p(x,\xi)\int_{R^n}D_y^\beta\varphi(y)\overline{(-iy)^\alpha\phi(y)}dy$$
$$\leq C|\xi|^{-(n+N+3\sigma)/2}\|\varphi\|^2_{L^2(\Omega)}\||y|^N\phi\|_{H^\sigma(\Omega)}$$
$$+C|\xi|^{\rho+(n-N)/2}\|\varphi\|_{H^{2N+(n+1)/2}(R^n)}\Big(\sum_{|\alpha|<N}\|y^\alpha\phi\|_{L^2(R^n)}\Big)$$
$$\leq C|\xi|^{\rho+(n-N-3)/2}\|\varphi\|_{H^{2N+(n+1)/2}(R^n)}\Big(\sum_{|\alpha|<N}\||y|^\alpha\phi\|_{H^\sigma(R^n)}\Big)\ .$$

Theorem 9 is now proved.

In the following we shall consider C^l valued functions and $l\times l$ matrix valued pseudo-differential operator P. If a function u has its value in a finite dimensional complex vector space, $|u(x)|$ denotes the norm of the vector $u(x)$. If u and v are C^l valued functions, then $\langle u(x), v(x)\rangle$ is the sesquilinear scalar product of $u(x)$ and $v(x)$. In the following we assume the operator P is given by

$$(75)\qquad Pu(x)=(2\pi)^{-n}\int_{R^n} p(x,\xi)\hat{u}(\xi)e^{ix\cdot\xi}d\xi$$

where $p(x,\xi)$ is an $l\times l$ matrix valued C^∞ function with an asymptotic expansion

$$(76)\qquad p(x,\xi)\sim\sum_{j=0}^\infty p_j(x,\xi)\ .$$

We assume that $p_j(x,\xi)$ is homogeneous in ξ of degree $\rho-j$.

THEOREM 10. *Let P be an $l\times l$ matrix valued pseudo-differential operator of order* 1 *defined by* (75) *and* (76) *with $\rho=1$. If for any compact set K' in Ω there are constants $C_0>0$ and $C_1>0$ such that, for any C^l valued function u in $\mathscr{D}(K')^l$, we have*

$$\mathrm{Re}\,(Pu, u)_{L^2(\Omega)} \geq C_0\|u\|^2_{L^2(\Omega)} - C_1\|u\|^2_{H^{-1/2}(\Omega)}\ , \tag{77}$$

then for any compact set K in Ω, there are a constant $C>0$, an integer $N>0$ and a function $\varepsilon(\xi)$ with $\lim_{|\xi|\to\infty}\varepsilon(\xi)=0$ such that for any $x\in K$ and $\psi\in\mathscr{D}(\mathbf{R}^n)^l$ we have

$$\mathrm{Re}\left[\sum_{|\alpha|+|\beta|\leq 2}\frac{|\xi|^{(|\beta|-|\alpha|)/2}}{\alpha!\beta!}\int_{R^n}\langle p_{0(\alpha)}^{(\beta)}(x,\xi)(iD_y)^\beta\psi(y), (-iy)^\alpha\psi(y)\rangle\, dy + \int_{R^n}\langle p_1(x,\xi)\psi(y),\psi(y)\rangle\, dy + \varepsilon(\xi)\sum_{|\alpha|+|\beta|\leq N}\int_{R^n}|D^\alpha y^\beta\psi(y)|^2\, dy\right] \geq C\int_{R^n}|\psi(y)|^2 dy\ . \tag{78}$$

PROOF. It follows from Theorem 9 that for any compact set K in Ω and φ in $\mathscr{D}(\Omega)$ which is identically 1 in some neighbourhood of K there are a constant $C>0$, an integer $N>0$ and a function $\varepsilon(\xi)$ with $\lim_{|\xi|\to 0}\varepsilon(\xi)=0$ such that, for any ψ in $\mathscr{D}(K)^l$, we have

$$\Big||\xi|^{1/2}\int_\Omega\langle P\varphi v(y), \varphi v(y)\rangle dy - \int_{R^n}\langle p_1(x,\xi)\psi(y),\psi(y)\rangle dy - \sum_{|\alpha|+|\beta|\leq 2}\frac{(|\xi|)^{(|\beta|-|\alpha|)/2}}{\alpha!\beta!}\int_{R^n}\langle p_{0(\alpha)}^{(\beta)}(x,\xi)(iD_y)^\beta\psi(y), (-iy)^\alpha\psi(y)\rangle dy\Big| \leq \varepsilon(\xi)\sum_{|\alpha|+|\beta|<N}\int_{R^n}|D^\alpha y^\beta\psi(y)|^2 dy \tag{79}$$

where $v(y)=\psi((y-x)\sqrt{|\xi|})e^{iy\cdot\xi}$ and

$$p_{0(\alpha)}^{(\beta)}(x,\xi)=D_x^\alpha D_\xi^\beta p_0(x,\xi)\ .$$

On the other hand, Hörmander proved in his paper [11] that

$$\int_{R^n}|\psi(y)|^2 dy \leq |\xi|^{n/2}\int_\Omega|\varphi v(y)|^2 dy + \delta^{-2}|\xi|^{-1}\int_{R^n}|y|^2|\psi(y)|^2 dy\ , \tag{80}$$

and

$$\|\varphi v\|^2_{H^{-1/2}(\Omega)} \leq C|\xi|^{-1-n/2}\sum_{|\alpha|\leq 1}\int_{R^n}|D^\alpha\psi|^2 dy\ , \tag{81}$$

where δ is so determined that $\varphi=1$ at any point within distance δ from K.

Take $K'=\mathrm{supp}\,\varphi$ and apply the inequality (77) to $u=\varphi v$. Then we have

$$\mathrm{Re}\,|\xi|^{n/2}\int_\Omega\langle(P\varphi v)(y),\varphi v(y)\rangle dy \geq C_0|\xi|^{n/2}\int_\Omega|\varphi v(y)|^2 dy - C_1|\xi|^{n/2}\|\varphi v\|^2_{H^{-1/2}(\Omega)}\ . \tag{82}$$

This together with (79), (80) and (81) proves Theorem 10.

To complete the proof of Theorem 8, we make some preparation (See Hörmander [11]).

Let Θ be a nonnegative function in $\mathscr{D}(\boldsymbol{R}^n)$ which is positive in $\{x; |x_j|\leq 1/2,$ for any $j\}$. Label the lattice points in $\boldsymbol{R}^n$ as $g_0=0, g_1, g_2, \cdots$ and let

(83) $$\varphi_k(x)=\Theta(x-g_k)/[\sum_j \Theta(x-g_j)^2]^{1/2} .$$

Then we have

(84) $$\sum_k \varphi_k(x)^2=1 ,$$

(85) $$\sum_k |(D^\alpha\varphi_k)(x)|^2\leq C_\alpha .$$

Note that for any x in some neighbourhood of the origin 0, $\varphi_0(x)=1$ and $\varphi_k(x)=0$, $k\neq 0$. If x, $y\in \operatorname{supp}\varphi_k$, then

(86) $$|x-y|<C \qquad C=3\sqrt{n}/2 .$$

Setting

(87) $$\psi_k(\xi)=\varphi_k(\xi|\xi|^{-1/2}) ,$$

we have

(88) $$\sum_k \psi_k(\xi)^2=1$$

and

(89) $$|\xi|^\alpha \sum_k |D^\alpha\psi_k(\xi)|^2\leq C_\alpha .$$

For ξ, $\eta\in \operatorname{supp}\psi_k$, we have

(90) $$|\xi-\eta|\leq C|\xi|^{1/2} .$$

If $|\xi-\eta|\leq (1/2)|\xi|$

(91) $$(\sum_j |\psi_j(\xi)-\psi_j(\eta)|^2)^{1/2}\leq C|\xi|^{-1/2}|\xi-\eta| .$$

Let Q be an $l\times l$ matrix valued pseudo-differential operator in Ω of order2 s. Let $0\neq\xi^j\in \operatorname{supp}\psi_j$, δ in $(0,1)$ and for any $u\in\mathscr{D}(\boldsymbol{R}^n)^l$ let

$$(Q_j^\delta u)(x)=\sum_{|\beta|\leq 2}\frac{1}{\beta!}q_0^{(\beta)}\left(x, \frac{\xi^j}{\delta}\right)\left(i\left(D-\frac{\xi^j}{\delta}\right)\right)^\beta u(x)+q_1\left(x, \frac{\xi^j}{\delta}\right)u(x) .$$

Here q_0 and q_1 are the first and the second terms of the symbol of Q.

DEFINITION 11. For any u in $\mathscr{D}(\boldsymbol{R}^n)^l$,

(92) $$\psi_j(\delta D)u(x)=(2\pi)^{-n}\int_{R^n}\psi_j(\delta\xi)\hat{u}(\xi)e^{ix\cdot\xi}d\xi .$$

LEMMA 12. *Let Q be a pseudo-differential operator of order $2s$. Assume that for any u in $\mathscr{D}(\Omega)^l$ $\operatorname{supp} Qu$ is contained in a compact set in Ω. Then there are a constant $C>0$ independent of δ and a constant C_δ dependent on δ such that*

$$\left|\sum_j \int_{R^n} \langle Q_j^\delta \psi_j(\delta D)u(x), \psi_j(\delta D)u(x)\rangle dx - \int_\Omega \langle Qu(x), u(x)\rangle dx\right| \leq C\delta \|u\|^2_{H^{s-1/2}(\Omega)} + C_\delta \|u\|^2_{H^{s-3/4}(\Omega)} \tag{93}$$

for any u in $\mathscr{D}(R^n)^l$.

PROOF. We may assume that Q is given by

$$(Qu)(x) = (2\pi)^{-n}\int_{R^n} q(x, \xi)\hat{u}(\xi)e^{ix\cdot\xi}d\xi \ . \tag{94}$$

Its Fourier transform is

$$\widehat{Qu}(\eta) = (2\pi)^{-n}\int_{R^n} \hat{q}(\eta-\xi, \xi)\hat{u}(\xi)d\xi \ , \tag{95}$$

where

$$\hat{q}(\eta, \xi) = \int_{R^n} q(x, \xi)e^{-ix\cdot\eta}dx \ .$$

The following estimate holds for any integer $N>0$ and multi-index α:

$$|D_\xi^\alpha \hat{q}(\eta, \xi)| \leq C_{\alpha,N}(1+|\eta|)^{-N}(1+|\xi|)^{2s-|\alpha|} \ . \tag{96}$$

Let

$$q^{\delta,j}(x, \xi) = \sum_{|\beta|\leq 2} \frac{1}{\beta!} q_0^{(\beta)}\left(x, \frac{\xi^j}{\delta}\right)\left(i\left(\xi - \frac{\xi^j}{\delta}\right)\right)^\beta + q_1\left(x, \frac{\xi^j}{\delta}\right) . \tag{97}$$

Then

$$(Q_j^\delta \psi_j(\delta D)u)(x) = (2\pi)^{-n}\int_{R^n} q^{\delta,j}(x, \xi)\hat{u}(\xi)\psi_j(\delta\xi)e^{ix\cdot\xi}d\xi \ , \tag{98}$$

and its Fourier transform is given by

$$\widehat{Q_j^\delta \psi_j(\delta D)u}(\eta) = (2\pi)^{-n}\int_{R^n} \widehat{q^{\delta,j}}(\eta-\xi, \xi)\hat{u}(\xi)\psi_j(\delta\xi)e^{ix\cdot\xi}d\xi \ ,$$

where

$$\widehat{q^{\delta,j}}(\eta, \xi) = \int_{R^n} q^{\delta,j}(x, \xi)e^{ix\cdot\eta}dx \ .$$

If $\psi_j(\delta\xi)\neq 0$, then $|\delta\xi - \xi^j| \leq C\sqrt{\delta|\xi|}$. Hence if $|\xi|$ is large enough, we have

$$\left|\xi - \frac{\xi^j}{\delta}\right| \leq C\delta^{-1/2}\sqrt{|\xi|} \leq \frac{|\xi|}{2} \ . \tag{99}$$

In this case Taylor's formula gives that for any $N>0$

(100)
$$|\hat{q}(\eta-\xi,\xi)-\widehat{q^{\delta,j}}(\eta-\xi,\xi)|$$
$$\leq C(1+\delta^{-3/2}+\delta^{-1/2})(1+|\xi|)^{2s-3/2}(1+|\eta-\xi|)^{-N-3}.$$

Set

(101)
$$\int\langle Qu(x),u(x)\rangle dx-\sum_j\int\langle Q_j^\delta\psi_j(\delta D)u(x),\psi_j(\delta D)u(x)\rangle dx$$
$$=(2\pi)^{-n}\iint_{R^{2n}}H(\xi,\eta)\hat{u}(\xi)\overline{\hat{u}(\eta)}d\xi d\eta$$

where

(102)
$$H(\xi,\eta)=\hat{q}(\eta-\xi,\xi)-\sum_j\psi_j(\delta\xi)\psi_j(\delta\eta)\widehat{q^{\delta,j}}(\eta-\xi,\xi)$$
$$=\frac{1}{2}\Big(\sum_j(\psi_j(\delta\xi)-\psi_j(\delta\eta))^2\Big)\hat{q}(\eta-\xi,\xi)$$
$$+\sum_j\psi_j(\delta\xi)\psi_j(\delta\eta)(\hat{q}(\eta-\xi,\xi)-\widehat{q^{\delta,j}}(\eta-\xi,\xi)).$$

Note that

$$\sum_j|\psi_j(\delta\xi)-\psi_j(\delta\eta)|^2\leq C\delta\frac{|\xi-\eta|^2}{|\xi|}\quad\text{if}\quad|\xi-\eta|\leq\frac{|\xi|}{2}$$

and that

$$|\psi_j(\delta\xi)-\psi_j(\delta\eta)|^2\leq 4\quad\text{for any}\quad\xi,\ \eta.$$

Then we obtain the estimate

(103)
$$|\hat{q}(\eta-\xi,\xi)(\sum_j(\psi_j(\delta\xi)-\psi_j(\delta\eta))^2)|$$
$$\leq C(1+|\xi-\eta|)^{-N}(1+|\xi|)^{2s}\left(\frac{\delta|\xi-\eta|^2}{1+|\xi|}+\frac{(1+|\xi-\eta|)^2}{(1+|\xi|)^2}\right).$$

This together with (100) and (102) gives

$$|H(\xi,\eta)|\leq C(1+|\xi-\eta|)^{-N+2}(\delta(1+|\xi|)^{2s-1}+(1+|\xi|)^{2s-2})$$
$$+\sum_j\psi_j(\delta\xi)\psi_j(\delta\eta)(1+\delta^{-1/2}+\delta^{-3/2})(1+|\xi|)^{2s-3/2}(1+|\xi-\eta|)^{-N-3}.$$

It follows from this and the equality (101) that

$$\left|\int_{R^n}\langle Qu(x),u(x)\rangle\,dx-\sum_j\int\langle Q_j^\delta\psi_j(\delta D)u(x),\psi_j(\delta D)u(x)\rangle\,dx\right|$$
$$\leq C\delta\|u\|^2_{H^{s-1/2}(\Omega)}+C^{-1}(1+\delta^{-1/2}+\delta^{-3/2})\|u\|^2_{H^{s-3/4}(\Omega)}.$$

Thus the lemma is now proved.

Now we prove the converse of Theorem 10.

THEOREM 13. *Let P be an $l\times l$ matrix valued pseudo-differential operator of order 1 defined by* (75) *and* (76). *Assume that $p_0(x,\xi)$ is a hermitian matrix and that for any compact set K_1 in Ω there are a constant $C_{K_1}>0$, an integer $N>0$ and a function $\varepsilon(\xi)$ with $\lim_{|\xi|\to\infty}\varepsilon(\xi)=0$ such that, for any x in K_1 and $\psi\in\mathscr{D}(\boldsymbol{R}^n)^l$,*

$$\text{(104)}\quad \mathrm{Re}\Big[\sum_{|\alpha|+|\beta|\le 2}\frac{(|\xi|)^{(|\beta|-|\alpha|)/2}}{\alpha!\,\beta!}\int_{\boldsymbol{R}^n}\langle p_{0(\alpha)}^{(\beta)}(x,\xi)(iD)^\beta\psi(y),\,(-iy)^\alpha\psi(y)\rangle\,dy$$
$$+\int_{\boldsymbol{R}^n}\langle p_1(x,\xi)\psi(y),\,\psi(y)\rangle dy+\varepsilon(\xi)\sum_{|\alpha|+|\beta|\le N}\int_{\boldsymbol{R}^n}|D^\alpha y^\beta\psi(y)|^2dy\Big]\ge C_{K_1}\int_{\boldsymbol{R}^n}|\psi(y)|^2dy\,.$$

Then for any compact set K there are positive constants C_0 and C_1 such that for any $u\in\mathscr{D}(K)^l$ we have

$$\text{(105)}\qquad \mathrm{Re}\int\langle Pu(y),\,u(y)\rangle\,dy\ge C_1\|u\|^2_{L^2(\Omega)}-C_0\|u\|^2_{H^{-1/2}(\Omega)}\,.$$

PROOF. Let φ be a $\mathscr{D}(\Omega)$ function which is identically 1 on some neighbourhood ω of K. Set $Q=\varphi P\varphi$. Then we have $Qu=\varphi Pu$ and

$$\int_\Omega\langle Qu(x),\,u(x)\rangle\,dx=\sum_j\int_{\boldsymbol{R}^n}\langle Q_j^\delta\psi_j(\delta D)u(x),\,\psi_j(\delta D)u(x)\rangle\,dx$$
$$+\int_{\boldsymbol{R}^n}\langle Qu(x),\,u(x)\rangle\,dx-\sum_j\int\langle Q_j^\delta\psi_j(\delta D)u(x),\,\psi_j(\delta D)u(x)\rangle\,dx\,.$$

Since the operator Q satisfies the condition of Lemma 12, we have

$$\text{(106)}\qquad \mathrm{Re}\int_\Omega\langle Qu(x),\,u(x)\rangle\,dx+C\delta\|u\|^2_{L^2(\Omega)}+C_\delta\|u\|^2_{H^{-1/4}(\Omega)}$$
$$\ge\mathrm{Re}\sum_j\int_{\boldsymbol{R}^n}\langle(Q_j^\delta\psi_j(\delta D)u)(x),\,\psi_j(\delta D)u(x)\rangle dx\,.$$

Let

$$\text{(107)}\qquad u^{jk}(x)=\varphi_k(x\sqrt{|\xi^j|})\psi_j(\delta D)u(x)\,.$$

Then

$$\text{(108)}\quad (Q_j^\delta u^{jk})(x)=\varphi_k(x\sqrt{|\xi^j|})Q_j^\delta\psi_j(\delta D)u(x)+[Q_j^\delta,\,\varphi_k(x\sqrt{|\xi^j|})]\psi_j(\delta D)u(x)$$

where $[Q_j^\delta,\,\varphi_k(x\sqrt{|\xi^j|})]$ is given by

$$\text{(109)}\quad [Q_j^\delta,\,\varphi_k(x\sqrt{|\xi^j|})]w(x)=\sum_{\nu=1}^n q_0^{(\nu)}\Big(x,\frac{\xi^j}{\delta}\Big)i\sqrt{|\xi^j|}(D_\nu\varphi_k)(x\sqrt{|\xi^j|})w(x)$$
$$+\sum_{|\beta|=2}\frac{1}{\beta!}q_0^{(\beta)}\Big(x,\frac{\xi^j}{\delta}\Big)|\xi^j|^{1/2}\Big[\frac{\xi^j_{\beta_2}}{\delta}D_{x_{\beta_1}}\varphi_k(x\sqrt{|\xi^j|})+\frac{\xi^j_{\beta_1}}{\delta}D_{\beta_2}\varphi_k(x\sqrt{|\xi^j|})\Big]w(x)$$

$$-\sum_{|\beta|=2}\frac{1}{\beta!}q_0^{(\beta)}\left(x,\frac{\xi^j}{\delta}\right)(D^\beta\varphi_k)(x\sqrt{|\xi^j|})|\xi^j|w(x)$$

$$+\sum_{|\beta|=2}\frac{\sqrt{|\xi^j|}}{\beta!}q_0^{(\beta)}\left(x,\frac{\xi^j}{\delta}\right)\{iD_{\beta_1}\varphi_k)(x\sqrt{|\xi^j|})(iD)_{\beta_2}w(x)$$
$$+(iD_{\beta_2}\varphi_k)(x\sqrt{|\xi^j|})(iD)_{\beta_1}w(x)\}\ .$$

Since $q_0(x,\xi)$ is hermitian and the function $D_\nu\varphi(x)$ is of pure imaginary valued, we have

$$\text{(110)}\qquad \operatorname{Re} i\sqrt{|\xi^j|}\int_{R^n}\langle q_0^{(\nu)}\left(x,\frac{\xi^j}{\delta}\right)(D_\nu\varphi_k)$$
$$\times(x\sqrt{|\xi^j|})\psi_j(\delta D)u(x),\ \varphi_k)x\sqrt{|\xi^j|})\psi_j(\delta D)u(x)\rangle dx=0\ .$$

$$\text{(111)}\qquad \operatorname{Re}|\xi^j|^{1/2}\int_{R^n}\langle q_0^{(\beta)}\left(x,\frac{\xi^j}{\delta}\right)D_{x_{\beta_1}}\varphi_k$$
$$\times(x\sqrt{|\xi^j|})\psi_j(\delta D)u(x),\ \varphi_k(x\sqrt{|\xi^j|})\psi_j(\delta D)u(x)\rangle\,dx=0\ .$$

And through integration by part

$$\int_{R^n}\langle q_0^{(\beta)}\left(x,\frac{\xi^j}{\delta}\right)(iD_{\beta_1}\varphi_k)(x\sqrt{|\xi^j|})iD_{\beta_2}\psi_j(\delta D)u(x),\ \varphi_k(x\sqrt{|\xi^j|})\psi_j(\delta D)u(x)\rangle\,dx$$
$$=-\int_{R^n}\langle\varphi_k(x\sqrt{|\xi^j|})\psi_j(\delta D)u(x),\ q_0^{(\beta)}\left(x,\frac{\xi^j}{\delta}\right)(iD_{\beta_1}\varphi_k)(x\sqrt{|\xi^j|})iD_{\beta_2}\psi(\delta D)u(x)\rangle dx$$
$$-\int_{R^n}\langle iD_{\beta_2}(\varphi_k(x\sqrt{|\xi^j|})q_0^{(\beta)}\left(x,\frac{\xi^j}{\delta}\right)(iD_{\beta_1}\varphi_k)(x\sqrt{|\xi^j|}))\psi_j(\delta D)u(x),\ \psi_j(\delta D)u(x)\rangle\,dx\ .$$

Thus

$$2\operatorname{Re}\int_{R^n}\langle q_0^{(\beta)}\left(x,\frac{\xi^j}{\delta}\right)(iD_{\beta_1}\varphi_k)(x\sqrt{|\xi^j|})iD_{\beta_2}\psi_j(\delta D)u(x),\ \varphi_k(x\sqrt{|\xi^j|})\psi_j(\delta D)u(x)\rangle\,dx$$
$$=-\frac{1}{2}\int_{R^n}\langle((iD_{\beta_1}\varphi_k^2)(x\sqrt{|\xi^j|})iD_{\beta_2}q_0^{(\beta)}\left(x,\frac{\xi^j}{\delta}\right)\psi_j(\delta D)u(x),\ \psi_j(\delta D)u(x)\rangle\,dx$$
$$+\frac{1}{2}\int_{R^n}\langle((D^\beta\varphi_k^2)(x\sqrt{|\xi^j|})q_0^{(\beta)}\left(x,\frac{\xi^j}{\delta}\right)\psi_j(\delta D)u(x),\ \psi_j(\delta D)u(x)\rangle\,dx\ .$$

Adding this with respect to k, we have

$$\text{(112)}\qquad \operatorname{Re}\sum_k\int_{R^n}\langle q_0^{(\beta)}\left(x,\frac{\xi^j}{\delta}\right)(iD_{\beta_1}\varphi_k)$$
$$\times(x\sqrt{|\xi^j|})iD_{\beta_2}\psi_j(\delta D)u(x),\ \varphi_k(x\sqrt{|\xi^j|})\psi_j(\delta D)u(x)\rangle\,dx=0\ ,$$

because $\sum_k\varphi_k^2(x)\equiv 1$.

For any multi-index β with length 2, the following estimate holds:

$$(113)\quad \left|\sum_k |\xi^j| \int_{R^n} \langle q_\delta^{(\beta)}\left(x, \frac{\xi^j}{\delta}\right)(D^\beta \varphi_k)(x\sqrt{|\xi^j|})\psi_j(\delta D)u(x),\ \varphi_k(x\sqrt{|\xi^j|})\psi_j(\delta D)u(x)\rangle dx\right|$$

$$\leq C\left(1+\frac{|\xi^j|}{\delta}\right)^{-1}|\xi^j|\left[\sum_k \int_{R^n} |(D^\beta\varphi_k)(x\sqrt{|\xi^j|})\psi_j(\delta D)u(x)|^2 dx\right.$$

$$\left.+\sum_k \int_{R^n} |\varphi_k(x\sqrt{|\xi^j|})\psi_j(\delta D)u(x)|^2 dx\right] \leq C\delta \int |\psi_j(\delta D)u(x)|^2 dx .$$

(109), (110), (111), (112) and (113) imply

$$(114)\quad \left|\mathrm{Re}\sum_k \int \langle [Q_j^\delta, \varphi_k(x\sqrt{|\xi^j|})]\psi_j(\delta D)u(x),\ u_{jk}(x)\rangle dx\right| \leq C\delta \int |\psi_j(\delta D)u(x)|^2 dx .$$

Hence we have from (106), (108) and (114) that

$$(115)\quad \mathrm{Re}\int \langle Qu(x), u(x)\rangle dx + C\delta\|u\|^2_{L^2(\Omega)} + C_\delta\|u\|^2_{H^{-1/4}(\Omega)} \geq \mathrm{Re}\sum_{j,k}\int_{R^n} \langle Q_j u^{jk}(x), u^{jk}(x)\rangle dx .$$

To obtain the estimate of

$$\mathrm{Re}\sum_{j,k}\int \langle Q_j u^{jk}(x), u^{jk}(x)\rangle dx$$

from below we introduce some notations. For any j, k let x^{jk} be a point with $\varphi_k(x^{jk}\sqrt{|\xi^j|})\neq 0$. We may choose the sequence $\{x^{jk}\}_{j,k}$ in such a manner that $x^{jk}\in \mathrm{supp}\,\varphi$ if there are some points $y\in\mathrm{supp}\,\varphi$ with $\varphi_k(y\sqrt{|\xi^j|})\neq 0$.

Set

$$(116)\quad u^{jk}(x) = e^{ix\cdot\xi^j/\delta} v^{jk}((x-x^{jk})\sqrt{|\xi^j|}) ,$$

$$(117)\quad \psi_j(\delta D)u(x) = e^{ix\cdot\xi^j/\delta} v^j(x\sqrt{|\xi^j|})$$

and

$$(118)\quad \varphi(x) = \phi((x-x^{jk})\sqrt{|\xi^j|}) .$$

Then the following relations hold:

$$(119)\quad \varphi_k(y)v^j(y) = v^{jk}(y - x^{jk}\sqrt{|\xi^j|}) ,$$

$$(120)\quad \varphi(x)u^{jk}(x) = e^{ix\cdot\xi^j/\delta}(\phi v^{jk})((x-x^{jk})\sqrt{|\xi^j|}) ,$$

$$(121)\quad \phi(y) = \varphi\left(x^{jk} + \frac{y}{\sqrt{|\xi^j|}}\right) ,$$

$$(122)\quad D_y^\alpha \phi(y) = |\xi^j|^{-|\alpha|/2}(D^\alpha\varphi)\left(x^{jk}+\frac{y}{\sqrt{|\xi^j|}}\right) ,$$

$$(123)\quad \int |v^{jk}(y)|^2 dy = |\xi^j|^{n/2}\int |u^{jk}(x)|^2 dx ,$$

(124) $$\int |y^{\alpha}||v^{jk}(y)|^2 dy \leq C_{\alpha} \int |v^{jk}(y)|^2 dy \ .$$

The inequality (124) follows from the fact that supp v^{jk} is contained in a fixed ball $\{x; |x| \leq C\}$. It is proved in Hörmander [11] that for arbitrary multi-indices α and β there is a constant C_{δ} such that

(125) $$\sum_k \int |x^{\beta} D^{\alpha} v^{jk}(x)|^2 dx \leq C_{\delta} |\xi^j|^{n/2} \int_{R^n} |\psi_j(\delta D) u|^2 dx \ .$$

Since $Q = \varphi P \varphi$, the first two terms of its symbol are

(126) $$q_0(x, \xi) = \varphi(x)^2 p_0(x, \xi) \ ,$$

(127) $$q_1(x, \xi) = \varphi(x)^2 p_1(x, \xi) + 2i\varphi(x) \sum_{\nu} D_{\nu}\varphi(x) p_0^{(\nu)}(x, \xi) \ .$$

The second term of the right side of (127) is a skew hermitian matrix for any $x \in \Omega$ and $0 \neq \xi \in \boldsymbol{R}^n$, so we have

(128) $$\begin{aligned} \mathrm{Re} \int \langle Q_j^{\delta} u^{jk}(x), u^{jk}(x) \rangle dx &= \mathrm{Re} \int \varphi(x)^2 \langle P_j^{\delta}(x) u^{jk}(x), u^{jk}(x) \rangle dx \\ &= \mathrm{Re} \int \langle P_j^{\delta} \varphi(x) u^{jk}(x), \varphi(x) u^{jk}(x) \rangle dx \\ &\quad - \mathrm{Re} \int \langle [P_j^{\delta}, \varphi] u^{jk}(x), \varphi(x) u^{jk}(x) \rangle dx \ . \end{aligned}$$

Just as we did to prove (113), we can prove that for any w in $\mathscr{D}(\boldsymbol{R}^n)$

(129) $$\left| \mathrm{Re} \int \langle [P_j^{\delta}, \varphi] w(x), \varphi(x) w(x) \rangle dx \right| \leq C \left(1 + \frac{|\xi^j|}{\delta}\right)^{-1} \int_{\sigma} |w(x)|^2 dx \ ,$$

where $\sigma = \mathrm{supp}\, \varphi \cap \omega^C$. In fact we have

(130) $$\begin{aligned} [P_j^{\delta}, \varphi] w(x) &= \sum_{\nu=1}^{n} p_0^{(\nu)} \left(x, \frac{\xi^j}{\delta}\right) (iD_{\nu}\varphi) w(x) \\ &\quad - \sum_{|\beta|=2} \frac{1}{\beta!} p_0^{(\beta)} \left(x, \frac{\xi^j}{\delta}\right) \left(\frac{\xi^j_{\beta_2}}{\delta} D_{\beta_1}\varphi + \frac{\xi^j_{\beta_1}}{\delta} D_{\beta_2}\varphi \right) w(x) \\ &\quad - \sum_{|\beta|=2} \frac{1}{\beta!} p_0^{(\beta)} \left(x, \frac{\xi^j}{\delta}\right) D^{\beta}\varphi(x) w(x) \\ &\quad - \sum_{|\beta|=2} \frac{1}{\beta!} p_0^{(\beta)} \left(x, \frac{\xi^j}{\delta}\right) (D^{\beta_1}\varphi(x) D^{\beta_2} w(x) + D^{\beta_2}\varphi(x) D^{\beta_1} w(x)) \ . \end{aligned}$$

Since φ is real and $p_0\left(x, \frac{\xi^j}{\delta}\right)$ is a hermitian matrix, it is clear that for any $\nu = 1, 2, \cdots, n$,

(131) $$\mathrm{Re}\,\langle p_0^{(\nu)}\left(x, \frac{\xi^j}{\delta}\right)(iD_\nu\varphi)(x)w(x), \varphi(x)w(x)\rangle=0\ ,$$

and that, for any multi-index β with $|\beta|=2$,

(132) $$\mathrm{Re}\,\langle p_0^{(\beta)}\left(x, \frac{\xi^j}{\delta}\right)\left(\frac{\xi^j_{\beta_2}}{\delta}\right)(D_{\beta_1}\varphi)w(x), \varphi(x)w(x)\rangle=0\ .$$

For any multi-index β with $|\beta|=2$, integrating by part, we have

$$\begin{aligned}&\int_{R^n}\langle p_0^{(\beta)}\left(x, \frac{\xi^j}{\delta}\right)D^{\beta_1}\varphi(x)D^{\beta_2}w(x), \varphi(x)w(x)\rangle\, dx\\ &\quad=-\int_{R^n}\langle w(x), D^{\beta_2}(D^{\beta_1}\varphi(x)p_0^{(\beta)}\left(x, \frac{\xi^j}{\delta}\right)\varphi(x))w(x)\rangle\, dx\\ &\quad=-\int_{R^n}\langle w(x), D^{\beta_1}\varphi(x)p_0^{(\beta)}\left(x, \frac{\xi^j}{\delta}\right)\varphi(x)D^{\beta_2}w(x)\rangle\, dx\\ &\qquad-\int_{R^n}\langle w(x), D^{\beta_2}(D^{\beta_1}\varphi(x)p_0^{(\beta)}\left(x, \frac{\xi^j}{\delta}\right)\varphi(x))w(x)\rangle\, dx\ .\end{aligned}$$

Thus

(133) $$\begin{aligned}2\,\mathrm{Re}&\int_{R^n}\langle p_0^{(\beta)}\left(x, \frac{\xi^j}{\delta}\right)D^{\beta_1}\varphi(x)D^{\beta_2}w(x), \varphi(x)w(x)\rangle\, dx\\ &=-\int_{R^n}\langle w(x), D^{\beta_2}(D^{\beta_1}\varphi(x)p_0^{(\beta)}\left(x, \frac{\xi^j}{\delta}\right)\varphi(x))w(x)\rangle\, dx\ .\end{aligned}$$

Since

$$\left|D^{\beta_2}(D^{\beta_1}\varphi(x)p_0^{(\beta)}\left(x, \frac{\xi^j}{\delta}\right)\varphi(x))\right|\le C\left(1+\frac{|\xi^j|}{\delta}\right)^{-1},$$

the estimate (129) holds.

Now it follows from (128) and (129) that

(134) $$\begin{aligned}\mathrm{Re}\sum_k\int_{R^n}\langle Q_j^\delta u^{jk}(x), u^{jk}(x)\rangle\, dx &\ge \mathrm{Re}\sum_k\int_{R^n}\langle P_j^\delta\varphi(x)u^{jk}(x), \varphi(x)u^{jk}(x)\rangle\, dx\\ &\quad -C\left(1+\frac{|\xi^j|}{\delta}\right)^{-1}\int_{R^n}|\psi_j(\delta D)u(x)|^2dx\ .\end{aligned}$$

By change of variables, we have

(135) $$\begin{aligned}&\int_{R^n}\langle P_j^\delta\varphi(x)u^{jk}(x), \varphi(x)u^{jk}(x)\rangle\, dx\\ &=|\xi^j|^{-n/2}\sum_{|\beta|\le 2}\frac{|\xi^j|^{|\beta|/2}}{\beta!}\int_{R^n}\langle p_0^{(\beta)}\left(x^{jk}+\frac{y}{\sqrt{|\xi^j|}}, \frac{\xi^j}{\delta}\right)(iD)^\beta\phi v^{jk}(y), \phi v^{jk}(y)\rangle\, dy\\ &+|\xi^j|^{-n/2}\int_{R^n}\langle p_1\left(x^{jk}+\frac{y}{\sqrt{|\xi^j|}}, \frac{\xi^j}{\delta}\right)\phi v^{jk}(y), \phi v^{jk}(y)\rangle\, dy\ .\end{aligned}$$

We obtain from Taylor's formula that for any $y \in \operatorname{supp}(v^{jk})$ and for sufficiently large $|\xi^j|$:

$$\left| p_0\left(x^{jk}+\frac{y}{\sqrt{|\xi^j|}}, \frac{\xi^j}{\delta}\right) - \sum_{|\alpha|\le 2} \frac{1}{\alpha!}\left(\frac{iy}{\sqrt{|\xi^j|}}\right)^{\alpha} p_{0(\alpha)}\left(x^{jk}, \frac{\xi^j}{\delta}\right)\right| \le C\left(1+\frac{|\xi^j|}{\delta}\right)\left(\frac{|y|}{\sqrt{|\xi^j|}}\right)^3, \tag{136}$$

$$\left| p_0^{(\nu)}\left(x^{jk}+\frac{y}{\sqrt{|\xi^j|}}, \frac{\xi^j}{\delta}\right) - \sum_{|\alpha|\le 1} \frac{1}{\alpha!}\left(\frac{iy}{\sqrt{|\xi^j|}}\right)^{\alpha} p_{0(\alpha)}^{(\nu)}\left(x^{jk}, \frac{\xi^j}{\delta}\right)\right| \le C\left(\frac{|y|}{\sqrt{|\xi^j|}}\right)^2. \tag{137}$$

For multi-index α with $|\alpha|=2$

$$\left| p_0^{(\alpha)}\left(x^{jk}+\frac{y}{\sqrt{|\xi^j|}}, \frac{\xi^j}{\delta}\right) - p_0^{(\alpha)}\left(x^{jk}, \frac{\xi^j}{\delta}\right)\right| \le C\left(1+\frac{|\xi^j|}{\delta}\right)^{-1}\frac{|y|}{\sqrt{|\xi^j|}}, \tag{138}$$

$$\left| p_1\left(x^{jk}+\frac{y}{\sqrt{|\xi^j|}}, \frac{\xi^j}{\delta}\right) - p_1\left(x^{jk}, \frac{\xi^j}{\delta}\right)\right| \le C\frac{|y|}{\sqrt{|\xi^j|}}. \tag{139}$$

Applying these to (135), we have

$$\begin{aligned}
&\Bigg| |\xi^j|^{n/2}\operatorname{Re}\int \langle P_j^\delta \varphi(x)u^{jk}(x), \varphi(x)u^{jk}(x)\rangle\, dx \\
&\quad -\operatorname{Re}\sum_{|\alpha|+|\beta|\le 2}\frac{(|\xi^j|)^{(|\beta|-|\alpha|)/2}}{\alpha!\beta!}\int \langle p_{0(\alpha)}^{(\beta)}\left(x^{jk}, \frac{\xi^j}{\delta}\right)(iD)^{\beta}\phi v^{jk}(y), (-iy)^{\alpha}\phi v^{jk}(y)\rangle\, dy \\
&\quad -\operatorname{Re}\int \langle p_1\left(x^{jk}, \frac{\xi^j}{\delta}\right)\phi v^{jk}(y), \phi v^{jk}(y)\rangle dy \Bigg| \\
&\le C\Bigg\{\left(1+\frac{|\xi^j|}{\delta}\right)|\xi^j|^{-3/2}\int_{R^n}|y|^3|v^{jk}(y)|^2 dy \\
&\quad +\sum_{\nu=1}^{n}|\xi^j|^{-1}\int_{R^n}|y|^2|D_\nu(\phi v^{jk}(y))||\phi v^{jk}(y)|dy \\
&\quad +\left(1+\frac{|\xi^j|}{\delta}\right)^{-1}|\xi^j|^{-1/2}\sum_{|\beta|=2}\int_{R^n}|y||D^{\beta}\phi v^{jk}(y)||\phi v^{jk}(y)|dy \\
&\quad +|\xi^j|^{-1/2}\int_{R^n}|y||v^{jk}(y)|^2 dy\Bigg\} \\
&\le C_\delta|\xi^j|^{-1/2}\left\{\sum_{|\beta|\le 2}\int |D^{\beta}v^{jk}(y)|^2 dy+\sum_{|\alpha|\le 2}\int |y^{\alpha}v^{jk}(y)|^2 dy\right\}.
\end{aligned} \tag{140}$$

Hence

$$\begin{aligned}
&|\xi^j|^{n/2}\operatorname{Re}\int \langle P_j^\delta \varphi(x)u^{jk}(x), \varphi(x)u^{jk}(x)\rangle\, dx \\
&\ge \operatorname{Re}\sum_{|\alpha|+|\beta|\le 2}\frac{|\xi^j|^{(|\beta|-|\alpha|)/2}}{\alpha!\beta!}\int_{R^n}\langle p_{0(\alpha)}^{(\beta)}\left(x^{jk}, \frac{\xi^j}{\delta}\right)(iD)^{\beta}\phi v^{jk}(y), (-iy)^{\alpha}\phi v^{jk}(y)\rangle dy \\
&\quad +\operatorname{Re}\int_{R^n}\langle p_1\left(x^{jk}, \frac{\xi^j}{\delta}\right)\phi v^{jk}(y), \phi v^{jk}(y)\rangle\, dy
\end{aligned} \tag{141}$$

$$-C_\delta|\xi^j|^{-1/2}\sum_{|\alpha|+|\beta|\leq 2}\int|y^\beta D^\alpha v^{jk}(y)|^2dy\ .$$

Applied for $K_1=\text{supp}\,\varphi$, the inequality (104) gives, by change of variables, for any $x\in\text{supp}\,\varphi$ and for any ϕ in $\mathscr{D}(\boldsymbol{R}^n)$,

$$\text{Re}\sum_{|\alpha|+|\beta|\leq 2}\frac{|\xi^j|^{(|\beta|-|\alpha|)/2}}{\alpha!\beta!}\int_{R^n}\langle p_{0(\alpha)}^{(\beta)}\left(x,\frac{\xi^j}{\delta}\right)(iD)^\beta\phi(y),(-iy)^\alpha\phi(y)\rangle dy \tag{142}$$
$$+\text{Re}\int_{R^n}\langle p_1\left(x,\frac{\xi^j}{\delta}\right)\phi(y),\phi(y)\rangle\,dy+\varepsilon_\delta(\xi^j)\sum_{|\alpha|+|\beta|\leq N}\int|y^\alpha D^\beta\phi(y)|^2dy$$
$$\geq C_{K_1}\int|\phi(y)|^2\,dy\ ,$$

where $\varepsilon_\delta(\xi)$ is the function depending on ξ and δ for which $\lim_{|\xi|\to\infty}\varepsilon_\delta(\xi)=0$ if δ is fixed.

If there are some points $y\in\text{supp}\,\varphi$ with $\varphi_k(y\sqrt{|\xi^j|})\neq 0$ then $x^{jk}\in\text{supp}\,\varphi$. Thus applying (142) to (141) and adding in k, we have

$$|\xi^j|^{n/2}\,\text{Re}\sum_{k=1}^\infty\int_{R^n}\langle P_j^\delta\varphi(x)u^{jk}(x),\varphi(x)u^{jk}(x)\rangle dx$$
$$\geq C_{K_1}\sum_k{}'\int_{R^n}|\phi v^{jk}(y)|^2dy-\varepsilon_\delta(\xi^j)\sum_{|\alpha|+|\beta|\leq N}\sum_{k=1}^\infty\int|y^\alpha D^\beta\phi v^{jk}(y)|^2dy$$
$$-C_\delta|\xi^j|^{-1/2}\sum_{k=1}^\infty\sum_{|\alpha|+|\beta|\leq 2}\int|y^\beta D^\alpha v^{jk}(y)|^2dy\ .$$

Here the summation $\sum_k{}'$ means the summation over those k's for which $\text{supp}\,u^{jk}\subset\omega$.

After applying (125) to this, divide both sides by $|\xi^j|^{n/2}$ and sum in j. Then we have

$$\text{Re}\sum_{j,k}\int_{R^n}\langle P_j^\delta\varphi(x)u^{jk}(x),\varphi(x)u^{jk}(x)\rangle dx \tag{143}$$
$$\geq C_{K_1}\sum_{j,k}{}'\int|u^{jk}(x)|^2dx-\sum_j\varepsilon_\delta(\xi^j)\int_{R^n}|\phi_j(\delta D)u(x)|^2dx$$
$$-\sum C_\delta|\xi^j|^{-1/2}\int|\phi_j(\delta D)u(x)|^2dx\ ,$$

where $\sum_{j,k}{}'$ means the summation over those (j,k)'s for which $\text{supp}\,u^{jk}\subset\omega$.

Let ω' be any open neighbourhood of K which has compact closure in ω. Then Hörmander proved in [11] that

$$\sum_j{}''\int|u^{jk}(x)|^2dx\leq C_\delta\|u\|^2_{H^{-1/2}(\Omega)}\ , \tag{144}$$

where $\sum''$ means the summation over those (j, k)'s for which $\operatorname{supp} u^{jk} \cap \omega' = \phi$. For large j or k the support of u^{jk} must either lie in ω or in the complement of $\bar{\omega}'$ so that either (143) or (144) is applicable. Any single u^{jk} can be estimated in terms of $\|u\|_{H^{-1/2}(\Omega)}$. Hence the estimates (143) and (144) together give, with another function $\varepsilon'_\delta(\xi)$ which is similar to $\varepsilon_\delta(\xi)$,

$$\text{(145)} \qquad \operatorname{Re} \sum_{j,k} \int_{R^n} \langle P^0_j \varphi(x) u^{jk}(x), \varphi(x) u^{jk}(x) \rangle dx$$
$$\geq C_{K'} \int_{R^n} |u(x)|^2 dx - \sum_j \varepsilon'_\delta(\xi^j) \int |\psi_j(\delta D) u(x)|^2 dx - C \|u\|^2_{H^{-1/2}(\Omega)} .$$

It follows from (115), (134) and (145) that

$$\text{(146)} \qquad \operatorname{Re} \int_{R^n} \langle Qu(x), u(x) \rangle dx + C\delta \int_{R^n} |u(x)|^2 dx + C_\delta \|u\|^2_{H^{-1/4}(\Omega)}$$
$$+ C \|u\|^2_{H^{-1/2}(\Omega)} + \sum_j \varepsilon''_\delta(\xi^j) \int_{R^n} |\psi_j(\delta D) u(x)|^2 dx \geq C_{K'} \int_{R^n} |u(x)|^2 dx ,$$

where $\varepsilon''_\delta(\xi)$ is a function similar to $\varepsilon_\delta(\xi)$. Now fix δ so that $C\delta < \frac{1}{4} C_{K_1}$, then choose J so that $\varepsilon''_\delta(\xi^j) < \frac{1}{4} C_K$ when $j \geq J$.

Note that for any fixed j, we have

$$\int_{R^n} |\psi_j(\delta D) u(x)|^2 dx \leq C_\delta \|u\|^2_{H^{-1/2}(\Omega)} .$$

So we have

$$\text{(147)} \qquad \operatorname{Re} \int_{R^n} \langle Qu(x), u(x) \rangle dx + C(\|u\|^2_{H^{-1/4}(\Omega)} + \|u\|^2_{H^{-1/2}(\Omega)}) \geq \frac{1}{2} C_{K_1} \int_{R^n} |u(x)|^2 dx .$$

For any $\varepsilon > 0$, there is a constant $C_\varepsilon > 0$ such that

$$\|u\|^2_{H^{-1/4}(\Omega)} \leq \varepsilon \int_{Rn} |u(x)|^2 dx + C_\varepsilon \|u\|^2_{H^{-1/2}(\Omega)} .$$

Taking $\varepsilon C < \frac{1}{4} C_{K_1}$ we have

$$\operatorname{Re} \int_\Omega \langle Qu(x), u(x) \rangle dx + C \|u\|^2_{H^{-1/2}(\Omega)} \geq \frac{1}{4} C_{K'} \int_\Omega |u(x)|^2 dx$$

for any $u \in \mathcal{D}(K)^l$.

Thus Theorem 13 has been proved.

Theorem 8 follows from Theorem 10 and Theorem 13.

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