Ideal boundaries of Neumann type associated with elliptic differential operators of second order[*)]

Dedicated to Professor Kôsaku Yosida on his sixtieth birthday

By Seizô ITÔ

Introduction. The theory of Martin boundary [11] is a generalization of the boundary value problem of Dirichlet type and the theory is extended to the case of the ideal boundary associated with the elliptic differential operator $A=\operatorname{div}\nabla+\boldsymbol{b}\cdot\nabla$ with variable coefficients by many authors (for instance [6], [14]).

The purpose of the present paper is to construct a theory of ideal boundaries of Neumann type associated with the elliptic operator $A^*: A^*u=\operatorname{div}(\nabla u-\boldsymbol{b}u)$, formally adjoint to the operator A. Such an ideal boundary was originally introduced by Z. Kuramochi [9] in case A is Laplacian on a Riemann surface; the boundary is now called the Kuramochi boundary and is studied by several authors (for instance [2], [3], [10]). Recently M. Ohtsuka [12] has simplified the main part of the Kuramochi's theory. Ohtsuka's method may be applied to Laplace-Beltrami operator Δ on a Riemannian manifold of arbitrary dimension by using some results on elliptic differential equations; in this case, the formal self-adjointness of Δ is useful.

In the present paper, we do not assume the formal self-adjointness of A but we assume some condition for $\boldsymbol{b}$ (see ASSUMPTION (A) in § 1), and we shall construct a theory of ideal boundaries similar to that of Kuramochi. Though the contents of this paper are quite parallel to those of [12], we use the notion of regular mapping defined in the author's previous paper [7] instead of th Dirichlet principle used in [12], and some other changes of arguments are necessary for the reason of non self-adjointness of A. However, most of the arguments in § 6 where we discuss the classification of the boundary points and canonical representation of full superharmonic functions, can be achieved in the same way as in [12]. So, in § 6, we state only the outline of the procedure except some lemmas being influenced by non self-adjointness of A.

We may establish the imbedding of the smooth boundary into the ideal boundary. But the imbedding theorem will be shown somewhere else.

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*) The result of this paper was reported at the International Conference on Functional Analysis and Related Topics held at Tokyo in April, 1969 [8].

While the method in this paper is purely analytic, it should be noted that the theory of the same kind of ideal boundaries is being established by the probabilistic method (e.g. [3], [15]).

§1. **Preliminaries.** Let R be a non-compact orientable C^∞-manifold of dimension $m\geqq 2$, and A be an elliptic differential operator of the form:

$$\begin{aligned} Au(x) &= \operatorname{div}[\nabla u(x)]+(\boldsymbol{b}(x)\cdot\nabla u(x)) \\ &= \sum_{i,j}\frac{1}{\sqrt{a(x)}}\frac{\partial}{\partial x^i}\left\{\sqrt{a(x)}\,a^{ij}(x)\frac{\partial u(x)}{\partial x^j}\right\}+\sum_i b^i(x)\frac{\partial u(x)}{\partial x^i}\end{aligned}$$

where $\|a^{ij}(x)\|$ and $\|b^i(x)\|$ are contravariant tensors of class C^2 in R, $\|a^{ij}(x)\|$ is symmetric and strictly positive-definite for each $x\in R$ and $a(x)=\det\|a_{ij}(x)\|=\det\|a^{ij}(x)\|^{-1}$. We shall denote by dx and $dS(x)$ respectively the volume element and the $m-1$ dimensional hypersurface element with respect to the Riemannian metric defined by the tensor $\|a_{ij}(x)\|$.

Given subdomain Ω of R and positive-valued continuous function $\omega(x)$ on Ω, we define the measure $d_\omega x=\omega(x)dx$ in Ω. For any m-vector fields Φ and Ψ defined in Ω with covariant components (Φ_i) and (Ψ_i) respectively, we define

$$(\Phi\cdot\Psi)=\sum_{i,j}a^{ij}\Phi_i\Psi_j \quad \text{(as a scalar function on } \Omega) ,$$

$$(\Phi,\Psi)_{\Omega,\omega}=\int_\Omega(\Phi\cdot\Psi)d_\omega x \text{ and } \|\Phi\|_{\Omega,\omega}=(\Phi,\Phi)^{1/2}_{\Omega,\omega}$$

whenever the right-hand side of each formula makes sense. For example, if u is a function piecewise smooth in Ω in the sense defined later, then

$$\|\nabla u\|^2_{\Omega,\omega}=\int_\Omega\sum_{i,j}a^{ij}\frac{\partial u}{\partial x^i}\frac{\partial u}{\partial x^j}\,d_\omega x\ (\leqq\infty)$$

is well defined.

By definition, a subset E of R is said to be *regular* if the boundary of E consists of a finite number of simple hypersurfaces of class C^3 ($\bar{E}$ is not necessarily compact)[1]. A function u is said to be *piecewise smooth* on a domain Ω if it is continuous on $\bar{\Omega}$ and there exists a finite number of regular domains $\Omega_1,\cdots,\Omega_n$ such that u is of class C^1 in every connected component of $\Omega-(\partial\Omega_1\cup\cdots\cup\partial\Omega_n)$.

We denote by $\boldsymbol{L}^2_\omega(\Omega)$ the completion of the space of all m-vector fields Φ in Ω with finite norm $\|\Phi\|_{\Omega,\omega}$, and by $P_\omega(\Omega)$ the totality of piecewise smooth function φ on Ω such that $\nabla\varphi\in\boldsymbol{L}^2_\omega(\Omega)$. Given compact set $K\subset\Omega$, we denote by

1) $\bar{E}$ and E° respectively denote the closure and the interior of E as a subset of R.

$P_\omega(\Omega;K)$ the totality of piecewise smooth functions ψ on $\Omega-K$ such that $\psi|_{\partial K}=0$ and $\nabla\psi\in \boldsymbol{L}^2_\omega(\Omega-K)$.

Throughout this paper, we are concerned with the formally adjoint operator A^* of A:

$$A^*u=\mathrm{div}\,(\nabla u-\boldsymbol{b}u)\ .$$

A function u is said to be *harmonic* in a domain Ω if it satisfies $A^*u=0$ in Ω, and is said to be *superharmonic* in Ω if it is lower semi-continuous and locally summable in Ω with respect to the measure dx and satisfies $A^*u\leqq 0$ in Ω in the sense of distribution by L. Schwartz [13].

A superharmonic function u in Ω has the following properties:[2]

i) u is locally bounded from below;

ii) if D is a domain with compact closure $\bar{D}\subset\Omega$, if w is continuous on $\bar{D}$ and harmonic in D and if $w\leqq u$ on ∂D, then $w\leqq u$ in D.

We fix a regular compact set $K_0\subset R$ and a point $x_0\in K_0$. For any domain $\Omega\supset K_0$, we set $\Omega'=\Omega-K_0$, in particular $R'=R-K_0$.

For every relatively compact regular domain $D\supset K_0$, let ω^D be the solution of the following elliptic boundary value problem (1.1) satisfying the normalizing condition $\omega^D(x_0)=1$:

$$A^*w=0 \text{ in } D,\ \left(\frac{\partial w}{\partial \boldsymbol{n}_D}-\beta_D w\right)\Big|_{\partial D}=0 \tag{1.1}$$

where $\dfrac{\partial w}{\partial \boldsymbol{n}_D}$ and β_D respectively denote the *outer* normal derivative of w and the *outer* normal component of $\boldsymbol{b}$ on ∂D; as is shown in [5],[3] the solution w of (1.1) uniquely exists up to a multiplicative constant and does not change sign on $\bar{D}$, and accordingly, by means of the normalizing condition, ω^D is uniquely determined and $\omega^D>0$ on $\bar{D}$.

Throughout this paper, we set the following

ASSUMPTION (A): There exist functions $q\in C^1(R)$ and $w>0$ on R such that

[2] The superharmonicity of u in Ω is usually defined by the following three conditions: a) $-\infty<u\leqq\infty$ and u is not identically equal to ∞ in Ω, b) u is lower semi-continuous in Ω, and c) u has the property ii) mentioned above.

Actually we may prove that, if u satisfies a), b) and c), then d) u is locally summable in Ω, and e) $A^*u\leqq 0$ in Ω in the sense of distribution, and conversely that any function with properties d) and e) coincides almost everywhere with a function satisfying a), b) and c). So our definition is equivalent to usual one. We adopt this definition for the sake of simplicity.

[3] Differential operators A and A^* in the present paper respectively correspond to A^* and A in [5] (and also those in [4] cited in §2).

$$\text{(1.2)} \qquad \boldsymbol{b}-\nabla q\in \boldsymbol{L}^2_\omega(R) \quad\text{and}\quad \lim_{D\uparrow R}\sup_{x\in D}\left|\log\frac{\omega^D(x)}{w(x)}\right|<\infty\,.$$

It may easily be seen that the existence of such functions q and w does not depend on the choice of x_0. The second condition in (1.2) is equivalent to the following one: there exists a monotone increasing sequence $\{D_n\}$ of relatively compact regular domains such that

$$\text{(1.3)} \qquad \lim_{n\to\infty} D_n=R \quad\text{and}\quad \sup_n \sup_{x\in D_n}\left|\log\frac{\omega^{D_n}(x)}{w(x)}\right|<\infty\,.$$

If especially $\boldsymbol{b}=\nabla p$ for some $p\in C^3(R)$, we have $\omega^D=e^{p-p(x_0)}$ for any D by the uniqueness of the solution of (1.1), whence Assumption (A) is satisfied by $q=p$ and $w=e^p$.

§ 2. **Regular mapping and kernel function.** In his previous paper [7], the author defined the regular mapping L associated with the elliptic operator A^* under the assumption (A) (§ 1) and the kernel function $N(x, y)$ for the elliptic boundary value problem with *vanishing normal flux at the point at infinity.* The mapping L and the function N play important roles in the present paper. In this section, we sketch the main results of the previous paper [7] without proof.

THEOREM 2.1. *There exists unique function ω on R satisfying that*

$$\text{(B)} \qquad \begin{cases} \omega(x_0)=1,\ \omega>0 \text{ on } R,\ \boldsymbol{b}-\nabla p\in \boldsymbol{L}^2_\omega(R) \quad \text{and} \\ (\boldsymbol{b}-\nabla p,\ \nabla\psi)_{R,\omega}=0 \text{ for any } \psi\in P_\omega(R) \end{cases}$$

where $p=\log\omega$. The function ω is harmonic in R, and $\omega=\lim_{n\to\infty}\omega^{D_n}$ uniformly on every compact subset of R for any sequence $\{D_n\}$ satisfying (1.3). (See Theorem 3.1 in [7].)

THEOREM 2.2. *For any regular compact set $K\subset R'$ and any function $\varphi\in C^1(\partial K)$, there exists unique function u on $R-(K_0)^\circ-K^\circ$ satisfying that*

$$\text{(C)} \qquad \begin{cases} u|_{\partial K_0}=0,\ u|_{\partial K}=\varphi,\ \left\|\nabla\dfrac{u}{\omega}\right\|_{R'-K,\omega}<\infty,\ \sup\limits_{R'-K}\left|\dfrac{u}{\omega}\right|<\infty \text{ and} \\ \left(\nabla\dfrac{u}{\omega}-[\boldsymbol{b}-\nabla p]\dfrac{u}{\omega},\ \nabla\psi\right)_{R'-K,\omega}=0 \text{ for any } \psi\in P_\omega(R;\,K+K_0)\,. \end{cases}$$

The function u is harmonic in $R'-K$ and satisfies $\sup\limits_{R'-K}\left|\dfrac{u}{\omega}\right|\le\max\limits_{\partial K}\left|\dfrac{\varphi}{\omega}\right|$. If we denote by v^D, for any $D\supset K+K_0$, the solution of the boundary value problem: $A^*v=0$ in $D'-K$, $v|_{\partial K_0}=0$, $v|_{\partial K}=\varphi$, $\left.\left(\dfrac{\partial v}{\partial n_D}-\beta_D v\right)\right|_{\partial D}=0$, then we have $u=\lim\limits_{n\to\infty}v^{D_n}$ uniformly on every compact subset of $R-(K_0)^\circ-K^\circ$ for any sequence $\{D_n\}$ satis-

fying (1.3).

This theorem is obtained by replacing K in Theorem 3.2 in [7] by K_0+K.

By virtue of the above two theorems, we can define a mapping $L=L_K^0$ of $C^1(\partial K)$ into the space of harmonic functions in $R'-K$ in such a way that $u=L_K^0\varphi$ satisfies the condition (C). The mapping L_K^0 is called a *regular mapping*; this is the restriction of the mapping L_{K+K_0} in the previous paper [7] to the set of all functions $\varphi\in C^1(\partial K+\partial K_0)$ satisfying that $\varphi|_{\partial K_0}=0$.

For any fixed $y\in R-(K_0)^\circ-K^\circ$, we have

$$\left|\frac{(L_K^0\varphi)(y)}{\omega(y)}\right|\leqq\max_{\partial K}\left|\frac{\varphi}{\omega}\right| \qquad \text{for any } \varphi\in C^1(\partial K) \tag{2.1}$$

by Theorem 2.2. Hence the mapping $\dfrac{\varphi}{\omega}\to\dfrac{(L_K^0\varphi)(y)}{\omega(y)}$ is uniquely extended to a bounded positive linear functional on $C(\partial K)$ and accordingly there exists a Borel measure μ_K^y on ∂K such that $\mu_K^y(\partial K)\leqq 1$ and that

$$(L_K^0\varphi)(y)=\omega(y)\int_{\partial K}\frac{\varphi(x)}{\omega(x)}d\mu_K^y(x) \qquad \text{for any } \varphi\in C(\partial K)\,. \tag{2.2}$$

For any lower semi-continuous function φ on ∂K, we define $(L_K^0\varphi)(y)$ by the formula (2.2). Thus the regular mapping L_K^0 is extended to a mapping defined on the space of all lower semi-continuous functions on ∂K, and we have the following (see Theorems 4.1 and 4.2 in [7])

THEOREM 2.3. *For any lower semi-continuous function φ on ∂K, $L_K^0\varphi$ is harmonic in any connected component of $R'-K$ in which $L_K^0\varphi$ is not identically equal to ∞.*

THEOREM 2.4. *Let K_1 and K_2 be regular compact sets such that $K_1\subset K_2\subset R'-K_0$, and φ be a lower semi-continuous functinn on ∂K_1. Then $L_{K_2}^0(L_{K_1}^0\varphi)=L_{K_1}^0\varphi$ in $R'-(K_2)^\circ$.*

For any relatively compact regular domain $D\supset K_0$, we denote by $N^D(x,y)$ the kernel function of the boundary value problem:

$$Av=-f \text{ in } D',\quad v|_{\partial K_0}=\varphi_0,\quad \left.\frac{\partial v}{\partial \boldsymbol{n}_D}\right|_{\partial D}=\varphi_1'\,; \tag{2.3}$$

$N^D(x,y)$ is also the kernel function of the adjoint boundary value problem:

$$A^*v=-f \text{ in } D',\quad v|_{\partial K_0}=\varphi_0,\quad \left.\left(\frac{\partial v}{\partial \boldsymbol{n}_D}-\beta_D v\right)\right|_{\partial D}=\varphi_1\,. \tag{2.4}$$

(See [7; §2] and [4; §10].[4]) For any $x\in R'$, we denote by $\boldsymbol{K}(x)$ the totality of

[4] See the foot-note 3).

regular compact subsets K of R' such that $x \in K^{\circ}$. The kernel function $N(x, y)$ is defined by the following (see Theorem 5.1 in [7])

THEOREM 2.5. *There exists unique function* $N(x, y)$ *continuous on*

$$(R'+\partial K_0)\times(R'+\partial K_0)-\{(z, z);\ z\in(R'+\partial K_0)\} \tag{2.5}$$

with the following properties i) *and* ii):

i) *For any Hölder-continuous function f whose support is a compact subset of R', the function*

$$v(y)=\int_{R'} f(x)N(x, y)dx \tag{2.6}$$

satisfies that

$$A^{*}v=-f \text{ in } R' \text{ and } v|_{\partial K_0}=0\ . \tag{2.7}$$

ii) *For any fixed $x \in R'$, it holds that*

$$L_K^0 N(x, \cdot)=N(x, \cdot) \text{ in } R'-K \text{ for any } K\in \boldsymbol{K}(x)\ . \tag{2.8}$$

Further we have

$$\lim_{n\to\infty} N^{D_n}(x, y)=N(x, y) \quad \text{uniformly on every compact subset of the set (2.5)} \tag{2.9}$$

for any sequence $\{D_n\}$ of relatively compact regular domains satisfying (1.3).

COROLLARY 2.5.1 (See Corollary 5.1.1 in [7]). *Let f be a C^2 function on R' whose support is a compact subset of R'. Then*

$$\int_{R'} A^{*}f(x)\cdot N(x, y)dx=-f(y) \tag{2.10}$$

and

$$\int_{R'} N(x, y)\cdot Af(y)dy=-f(x)\ . \tag{2.11}$$

COROLLARY 2.5.2. $\displaystyle\int_{\partial K_0}\frac{\partial N(x, y)}{\partial \boldsymbol{n}_{K_0}(y)}dS(y)=1$ *for any* $x\in R'$.

This fact is not explicitly shown in [7]. But it is proved as follows. By means of Green's formula we may show that $\displaystyle\int_{\partial K_0}\frac{\partial N^{D_n}(x, y)}{\partial \boldsymbol{n}_{K_0}}dS(y)=1$ for every n. Letting $n\to\infty$, we obtain this corollary in virtue of (5.9) and (2.4) in [7].

THEOREM 2.6 (See Theorem 5.2 in [7]). *For any fixed $x\in R'+\partial K_0$ and any regular compact set $K\subset R'$, it holds that $L_K^0 N(x, \cdot)\leqq N(x, \cdot)$ in $R'-K$; the equality holds if $x \in K^{\circ}$.*

We can take $N(x, y)$ as a kernel of potential. For any Borel measure μ in R', we define the potential μN by

$$\mu N(y)=\int_{R'} N(x, y)d\mu(x) \tag{2.12}$$

whenever the right-hand side as a function of y is not identically equal to ∞ in R'.

For any lower semi-continuous function v on R', and any regular compact set $K\subset R'$, we define v_K by

$$v_K=\begin{cases} L_K^0\varphi & \text{in } R'-K+\partial K_0 \text{ with } \varphi=v|_{\partial K} \\ v & \text{in } K. \end{cases} \tag{2.13}$$

We also define, for any fixed $x\in R'+\partial K_0$,

$$N_K(x, y)=[N(x, \cdot)]_K(y) . \tag{2.14}$$

Here we show some lemmas which will be necessary in the following sections.

LEMMA 2.1 (Maximum-minimum principle). *Let Ω be a subdomain of R and assume that u is continuous on $\bar{\Omega}$ and harmonic in Ω. Then $\frac{u}{\omega}$ takes its maximum and minimum at some points on $\partial\Omega$.*

This fact may be shown from the fact that $\frac{u}{\omega}$ satisfies

$$\operatorname{div}\left\{\omega\left(\nabla\frac{u}{\omega}\right)\right\}-\omega\left([\boldsymbol{b}-\nabla p]\cdot\nabla\frac{u}{\omega}\right)=\operatorname{div}\{\nabla u-\boldsymbol{b}u\}=0 .$$

LEMMA 2.2. i) *The potential μN defined above is superharmonic in R'.* ii) *Any superharmonic function v in a subdomain Ω of R' is uniquely expressible in the form $v=\mu N+h$ where μ is a Borel measure in Ω and h is a harmonic function in Ω* (Riesz decomposition).

This lemma may be proved by the same argument as in the theory of Schwartz distribution [13; Chap. VI] in virtue of (2.11).

LEMMA 2.3. *If u and v are superharmonic in a domain Ω and if $u=v$ a.e. in Ω, then $u\equiv v$ in Ω.*

PROOF. For any fixed $y_0\in\Omega$ and any $\alpha<u(y_0)$, there exists a neighborhood $\Omega_\alpha\subset\Omega$ of y_0 such that $u>\alpha$ in Ω_α (lower semi-continuity). Hence, from the assumption and by Fubini's theorem, there exists a regular neighborhood $\Omega_0\subset\Omega_\alpha$ of y_0 such that $v>\alpha$ dS-a.e. on $\partial\Omega_0$, and accordingly $v(y_0)>\alpha$. Hence we get $v(y_0)\geqq u(y_0)$. Similarly we have $u(y_0)\geqq v(y_0)$.

LEMMA 2.4. $([\boldsymbol{b}-\nabla p]\psi, \nabla\psi)_{R',\omega}=0$ *for any $\psi\in P_\omega(R;K_0)$ which is bounded on R'.*

This is an immediate consequence of Lemma 3.2 in [7] since $\Phi=\boldsymbol{b}-\nabla p$ satisfies (3.1) in [7]. Though the definition of $P_\omega(R; K_0)$ in the present paper is slightly different from that in [7] in the point that functions in $P_\omega(R; K_0)$ in the present paper are not necessarily smooth but may be piecewise smooth, the modification of the proof for this point is quite easy.

§3. **Construction of the ideal boundary.** Let $N(x, y)$ be the kernel function defined in §2, Ω be a relatively compact regular domain in R containing K_0 and $G^{\Omega-K_0}(x, y)$ be the Green function for Dirichlet boundary value problem in $\Omega-K_0$. Then, by means of Green's formula, we may show that

$$N(x, y)=-\int_{\partial\Omega} N(x, z)\frac{\partial G^{\Omega-K_0}(z, y)}{\partial \boldsymbol{n}_\Omega(z)}dS(z) \tag{3.1}$$

for any $x\in R-\Omega$ and any $y\in\Omega'$.

We have also that (see Lemma A in [7; Appendix])

$$\sup_{D\supset\Omega}\left\{\sup_{x\in D-\bar{\Omega}}\int_{\partial\Omega} N^{D-K_0}(x, z)dS(z)\right\}<\infty$$

and accordingly, by means of (2.9) that

$$\sup_{x\in R-\bar{\Omega}}\int_{\partial\Omega} N(x, z)dS(z)<\infty . \tag{3.2}$$

Combining (3.1) with (3.2), we obtain the following

LEMMA 3.1. *Let F be a compact subset of $R-(K_0)^\circ$ and Ω be a relatively compact regular domain containing $K_0\cup F$. Then*

$$\sup_{x\in R-\bar{\Omega}, y\in F}\left\{N(x, y)+|\nabla_y N(x, y)|\right\}<\infty . \tag{3.3}$$

Let ρ_0 be a metric in R which defines the topology of the one-point compactification of R, and D_0 be a relatively compact domain containing K_0. We define $N(x, y)=0$ for any $x\in K_0$ and $y\in R'$, and put

$$\rho_1(x_1, x_2)=\int_{D_0'}\frac{|N(x_1, y)-N(x_2, y)|}{1+|N(x_1, y)-N(x_2, y)|}dy \tag{3.4}$$

and

$$\rho(x_1, x_2)=\rho_0(x_1, x_2)+\rho_1(x_1, x_2) \quad \text{for} \quad x_1, x_2\in R . \tag{3.5}$$

Then, using Lemma 3.1, we may show the following two lemmas in the same way as proofs of Lemmas 3.2 and 3.3 in the authors previous paper [6] on Martin boundary.

LEMMA 3.2. *The function $\rho(x_1, x_2)$ is a metric in R, which defines the same topology as the original one in R.*

LEMMA 3.3. *R is totally bounded with respect to the metric ρ.*

Let $\hat{R}$ be the completion of R with respect to ρ and put $\hat{S}=\hat{R}-R$; the function $\rho(x_1, x_2)$ naturally extended to $\hat{R}\times\hat{R}$ is denoted by the same notation. Then we may also prove the following two theorems and corollaries in the same way as corresponding theorems and corollaries in [6] cited above.

THEOREM 3.1. *$\hat{R}$ is compact and $\hat{S}$ is a closed subset of $\hat{R}$ with respect to the metric ρ. The relative topology in R arising from the metric is equivalent to the original one.*

THEOREM 3.2. *The function $N(x, y)$ is extended to a function continuous on $\hat{R}\times(R'+\partial K_0)-\{(z, z);\ z\in R'+\partial K_0\}$, and the extended function $N(x, y)$ is harmonic in $y\in R'-\{x\}$ for any fixed $x\in\hat{R}$.*

COROLLARY 3.2.1. *For any closed subset E of $\hat{R}$ and any compact subset F of $R'+\partial K_0-E$, the function $N(x, y)$ is uniformly continuous on $E\times F$ with respect to the metric ρ.*

COROLLARY 3.2.2. *If $\xi, \eta\in\hat{S}$ and $N(\xi, y)=N(\eta, y)$ for any $y\in R'$ then $\xi=\eta$.*

Here we prove the following

THEOREM 3.3. *$\hat{R}$ is independent of the choice of $\{K_0, \rho_0, D_0\}$ in the following sense: Let K_0 and $\tilde{K}_0$ be regular compact sets in R, ρ_0 and $\tilde{\rho}_0$ be metrics defining the topology of the one-point compactification of R, D_0 and $\tilde{D}_0$ be relatively compact domains containing K_0 and $\tilde{K}_0$ respectively. Let $N(x, y)$ and $\tilde{N}(x, y)$ be kernel functions defined as in § 2 corresponding to K_0 and $\tilde{K}_0$ respectively. Define ρ_1 and ρ (resp. $\tilde{\rho}_1$ and $\tilde{\rho}$) by* (3.4) *and* (3.5) *with the kernel N (resp. $\tilde{N}$), and let $\hat{R}$ (resp. $\tilde{R}$) be the completion of R with respect to the metric ρ (resp. $\tilde{\rho}$). Then compact metric spaces $\hat{R}(\rho)$ and $\tilde{R}(\tilde{\rho})$ are mutually homeomorphic, and the homeomorphism restricted to R is the identity mapping.*

PROOF. It is sufficient to prove this theorem in case $\tilde{D}_0\subset K_0$. The mutual equivalence of metrics ρ_0 and $\tilde{\rho}_0$ in R is evident, and the mutual equivalence of ρ and $\tilde{\rho}$ in any compact subset of R may easily be shown. Hence, if we verify that ρ_1 and $\tilde{\rho}_1$ are mutually equivalent metrics in $R-D$ for a certain relatively compact domain D in R, then we may conclude that ρ and $\tilde{\rho}$ are mutually equivalent in R and we obtain Theorem 3.3 by virtue of the uniqueness of the completion of a metric space. So we have only to prove the mutual equivalence of ρ_1 and $\tilde{\rho}_1$ in $R-D$.

Let Ω and D be relatively compact regular domains such that $\tilde{D}_0\subset\Omega\subset\bar{\Omega}\subset D$. Since $\tilde{D}_0\subset K_0$, we may show by Green's formula that

$$\tilde{N}(x_1, y)-\tilde{N}(x_2, y)=N(x_1, y)-N(x_2, y) + \int_{\partial K_0}\left\{\frac{\partial N(x_1, z)}{\partial n_{K_0}}-\frac{\partial N(x_2, z)}{\partial n_{K_0}}\right\}\tilde{N}(z, y)dS(z) \tag{3.6}$$

for any $x_1, x_2 \in R-K_0$ and $y \in \tilde{D}_0'$ $(\tilde{D}_0'=D-\tilde{K}_0)$

and

$$N(x_1, y)-N(x_2, y)=\tilde{N}(x_1, y)-\tilde{N}(x_2, y) - \int_{\partial K_0}\{\tilde{N}(x_1, z)-\tilde{N}(x_2, z)\}\frac{\partial N(z, y)}{\partial n_{K_0}(z)}dS(z) \tag{3.7}$$

for any $x_1, x_2 \in R-K_0$ and $y \in D_0'$.

We put

$$C=\sup_{y\in\tilde{D}_0'}\omega(y)\,/\inf_{y\in\partial\tilde{D}_0}\omega(y)\,,\quad \tilde{M}=\sup_{y\in\partial D_0}\int_{\partial K_0}\tilde{N}(z, y)dS(z)$$

and

$$\delta(\varepsilon)=\sup_{\substack{x_1,x_2\in R-D\\ \rho_1(x_1,x_2)<\varepsilon\\ y\in\partial\tilde{D}_0,\, z\in\partial K_0}}\left\{|N(x_1, y)-N(x_2, y)|+\tilde{M}\left|\frac{\partial N(x_1, z)}{\partial n_{K_0}(z)}-\frac{\partial N(x_2, z)}{\partial n_{K_0}(z)}\right|\right\}$$

for any $\varepsilon>0$. Since $\tilde{N}$ satisfies $A^*\tilde{N}=0$ in $R-\tilde{K}_0$ and $\tilde{N}|_{\partial\tilde{K}_0}=0$ as a function of y, we have by Lemma 2.1

$$\sup_{y\in\tilde{D}_0'}\left|\frac{\tilde{N}(x_1, y)-\tilde{N}(x_2, y)}{\omega(y)}\right|\leq\max_{y\in\partial\tilde{D}_0}\left|\frac{\tilde{N}(x_1, y)-\tilde{N}(x_2, y)}{\omega(y)}\right|.$$

Hence we obtain from the formulas (3.4) and (3.6) that

$$\tilde{\rho}_1(x_1, x_2)\leq|\tilde{D}_0'|\sup_{y\in\tilde{D}_0'}|\tilde{N}(x_1, y)-\tilde{N}(x_2, y)| \qquad \left(|\tilde{D}_0'|=\int_{\tilde{D}_0'}dy\right)$$

$$\leq C|\tilde{D}_0'|\max_{y\in\partial D_0'}|\tilde{N}(x_1, y)-\tilde{N}(x_2, y)|\leq C|\tilde{D}_0'|\delta(\varepsilon)$$

whenever $x_1, x_2 \in R-D$ and $\rho(x_1, x_2)<\varepsilon$. We may also show from (3.1) that $\lim_{\varepsilon\downarrow 0}\delta(\varepsilon)=0$ since $N(x, y)$ is uniformly continuous on $(R-D)\times(K_0-\tilde{D}_0)$ by Corollary 3.2.1. Similarly we may obtain from (3.7) that

$$\rho(x_1, x_2)\leq C_1\tilde{\delta}(\varepsilon) \quad \text{whenever} \quad x_1, x_2\in R-D \quad \text{and} \quad \tilde{\rho}(x_1, x_2)<\varepsilon$$

where C_1 is a suitable constant and

$$\tilde{\delta}(\varepsilon)=\sup_{\substack{x_1, x_2 \in R-D \\ \tilde{\rho}_1(x_1, x_2)<\varepsilon \\ y \in D_0-\bar{D}_0}} |\tilde{N}(x_1, y)-\tilde{N}(x_2, y)| ,$$

and also that $\lim_{\varepsilon \downarrow 0} \tilde{\delta}(\varepsilon)=0$. From these result, we may conclude that ρ_1 and $\tilde{\rho}_1$ are mutually equivalent. Theorem 3.3 is thus proved.

DEFINITION. The set $\hat{S}$ defined above is called the *ideal boundary of Neumann type* (or *Kuramochi boundary*) *of* R associated with the elliptic operator A^*.

By virtue of Theorem 3.2, we can extend the definition (2.12) of the potential μN to the case where μ is a Borel measure in $R'+\hat{S}$.

THEOREM 3.4. *For any regular compact set $K\subset R'$ and any Borel measure μ in $R'+\hat{S}$, we have $(\mu N)_K \leqq \mu N$ in $R'-K$; the equality holds if the support of μ is contained in K°.*

PROOF. It is mentioned in Theorem 2.6 that $N_K(x, \cdot)\leqq N(x, \cdot)$ in $R-(K_0)^\circ-K^\circ$ for any $x\in R'$ and the equality holds if $x\in K^\circ$. For any $x\in \hat{S}$, we take a sequence $\{x_n\}\subset R'-K$ which tends to x. Then $N_K(x_n, \cdot)\leqq N(x_n, \cdot)$ and hence $N_K(x, \cdot)\leqq N(x, \cdot)$ in $R-(K_0)^\circ-K^\circ$. If μ is a measure whose support is contained in K°, we have

$$\begin{aligned}(\mu N)_K(y)&=\int_{\partial K}\frac{\omega(y)}{\omega(z)}d\mu_K^y(z)\int_K N(x, z)\,d\mu(x)=\int_{K^\circ}d\mu(x)\int_{\partial K}\frac{\omega(y)}{\omega(z)}N(x, z)d\mu_K^y(z)\\ &=\int_{K^\circ}N_K(x, y)\,d\mu(x)=\int_{K^\circ}N(x, y)d\mu(x)=(\mu N)(y)\ .\end{aligned}$$

For general μ, we get $(\mu N)_K(y)\leqq(\mu N)(y)$ by means of the inequality $N_K\leqq N$ for $x\in R'-K^\circ+\hat{S}$.

§ 4. FH functions and FSH functions.

In Definitions below, v always denotes a non-negative and lower semi-continuous function in R' which is not identically equal to ∞, and v_K denotes the function defined by (2.13).

DEFINITIONS. i) If $v_K\leqq v$ in R' for any regular compact set $K\subset R'$, then v is called an *FSH function* (or a *full superharmonic function*) in R'; if in addition v is harmonic in R', then it is called an *FH function* (or a *full harmonic function*) in R'.

ii) If v is an FSH function and if $\lim_{n\to\infty} v_{\partial K_n}(x)=0$ in R' for any sequence $\{K_n\}_{n\geqq 1}$ of regular compact sets such that $(K_n)^\circ\supset K_{n+1}$ for any $n\geqq 1$ and $\lim_{n\to\infty} K_n=K_0$, then v is called an *FSH_0 function* in R'; if in addition v is harmonic in R', then it is called an *FH_0 function* in R'.

REMARK. Any FH_0 function takes the boundary value zero on ∂K_0 (Corollary

4.2.2 below), but not every FSH_0 function does; we may easily construct counter examples.

LEMMA 4.1. *Any FSH function is superharmonic in R'.*

PROOF. If v is an FSH function and if Ω is a regular domain with compact closure $\bar{\Omega}\subset R'$, then $v_{\partial\Omega}\leqq v$ in Ω; from this fact, we may see the superharmonicity of v.

LEMMA 4.2. *If $\{v_n\}$ is a sequence of FSH functions such that $v=\lim_{n\to\infty} v_n$ exists in R', and if v is lower semi-continuous and is not identically equal to ∞, then v is an FSH function in R'.*

In fact, for any regular compact set $K\subset R'$ and any $x\in R'-K$, we have, by the Lebesgue-Fatou lemma, $v_K(y)=\omega(y)\int_{\partial K}\frac{v(x)}{\omega(x)}d\mu_K^y(x)\leqq\varliminf_{n\to\infty}\omega(y)\int_{\partial K}\frac{v_n(x)}{\omega(x)}d\mu_K^y(x)$ $\leqq\lim_{n\to\infty} v_n(y)=v(y)$.

LEMMA 4.3. *If $\{v_n\}$ is a sequence of FSH_0 functions such that $v=\lim_{n\to\infty} v_n$ exists in R', and if v is lower semi-continuous and dominated by an FSH_0 function u in R', then v is an FSH_0 function in R'.*

This may be proved by Lemma 4.2 and from the fact that $v_{\partial K_n}\leqq u_{\partial K_n}$ for any K_n mentioned in Definition ii).

Theorems 4.1, 4.2 and 4.3 mentioned below may be proved by the same argument as proofs of Theorems 6, 7 and 8 in [12; § 4]; Corollary 4.2.1 is an immediate consequence of Theorems 4.1 and 4.2, and Corollary 4.2.2 may be shown by the same way as the proof of Lemma 4 in [12; § 7].

THEOREM 4.1. *For any Borel measure μ in $R'+\hat{S}$, the potential μN is an FSH_0 function.*

THEOREM 4.2. *For any FSH function v in R' and any regular compact set $K\subset R'$, v_K is equal to the potential of a measure supported by K.*

COROLLARY 4.2.1. *v_K in the above theorem is an FSH_0 function.*

COROLLARY 4.2.2. *Every FH_0 function takes the boundary value zero on ∂K_0.*

THEOREM 4.3. *Every FSH function is equal to the sum of an FH function and a potential.*

For any FSH function v in R' and any open subset Ω of R', we define

$$v_\Omega=\sup_{K\in \boldsymbol{K}(\Omega)} v_K(\leqq v) \qquad \text{in } R' \tag{4.1}$$

where $\boldsymbol{K}(\Omega)$ denotes the totality of regular compact sets contained in Ω. Then

LEMMA 4.4. *If $\{K_n\}\subset \boldsymbol{K}(\Omega)$, $K_n\subset(K_{n+1})^\circ$ $(n\geqq 1)$ and $\bigcup_{n=1}^{\infty} K_n=\Omega$, then $\{v_{K_n}\}$ is monotone increasing in n and $\lim_{n\to\infty} v_{K_n}=v_\Omega$ in R'. v_Ω is an FSH function and, in particular, v_Ω is an FSH_0 function if $\bar{\Omega}$ does not intersect K_0. $(u+v)_\Omega=u_\Omega+v_\Omega$ for any FSH functions u and v.*

PROOF. For $m>n$, we have $v_{K_n}=(v_{K_n})_{K_m}\leqq v_{K_n}\leqq v_\Omega$ by Theorem 2.4. Hence $\{v_{K_n}\}$ is monotone increasing and $\lim_{n\to\infty} v_{K_n}\leqq v_\Omega$. For any fixed $x\in R'$ and any $\alpha<v_\Omega(x)$, there exists $K\in \boldsymbol{K}(\Omega)$ such that $v_K(x)>\alpha$, and accordingly $v_{K_n}(x)\geqq v_K(x)>\alpha$ whenever $K\subset(K_n)^\circ$. Hence we get $\lim_{n\to\infty} v_{K_n}(x)>\alpha$; accordingly we may conclude $\lim_{n\to\infty} v_{K_n}=v_\Omega$ in R'. This result implies that $(u+v)_\Omega=u_\Omega+v_\Omega$ and also that v_Ω is an FSH functiun by Lemma 4.2. If $\bar{\Omega}\cap K_0$ is empty, then there exists a regular compact set F such that $F^\circ\supset K_0$ and $F\cap\bar{\Omega}$ is empty. Hence $0\leqq v_{K_n}\leqq v_{\partial F}$ in $F-(K_0)^\circ$, and accordingly $0\leqq v_\Omega\leqq v_{\partial F}$ in $F-(K_0)^\circ$. Since $v_{\partial F}$ takes the boundary value zero on ∂K_0 by Corollaries 4.2.1 and 4.2.2, we may see that v_Ω is an FSH_0 function.

LEMMA 4.5. *If Ω_1 and Ω_2 are open subsets of R' and $\Omega_1\subset\Omega_2$, then $(v_{\Omega_1})_{\Omega_2}=v_{\Omega_1}$ in R' for any FSH function v.*

PROOF. v_{Ω_1} is an FSH function by the above lemma. Since $v_{\Omega_1}=v$ in Ω_1 by (4.1), we have $(v_{\Omega_1})_K=v_K$ in R' for any $K\in\boldsymbol{K}(\Omega_1)$, and accordingly $(v_{\Omega_1})_{\Omega_2}\geqq(v_{\Omega_1})_{\Omega_1}=v_{\Omega_1}$ in R'. On the other hand, since $(v_{\Omega_1})_K\leqq v_\Omega$ in R' for any $K\in\boldsymbol{K}(\Omega_2)$, we have $(v_{\Omega_1})_{\Omega_2}\leqq v_{\Omega_1}$ in R'. Thus we get $(v_{\Omega_1})_{\Omega_2}=v_{\Omega_1}$.

Hereafter E^a denotes the closure of E as a subset of the compact metric space $\hat{R}$ ($\bar{E}$ denotes, as before, the closure of E as a subset of the original manifold R).

For any FSH function v in R' and any closed subset Γ of $\hat{S}$, we define

$$v_\Gamma=\inf_{\Omega\in \boldsymbol{O}(\Gamma)} v_\Omega \tag{4.2}$$

where $\boldsymbol{O}(\Gamma)$ denotes the totality of regular open sets Ω in R' such that $\Omega^a\supset\Gamma$. Then we may prove the following lemma from Lemmas 4.3, 4.4 and 4.5 (the proof is similar to that of Lemma 4.4).

LEMMA 4.6. *If $\{\Omega_n\}\subset\boldsymbol{O}(\Gamma)$, $\Omega_n\supset\overline{\Omega_{n+1}}$ $(n\geqq1)$ and $\bigcap_{n=1}^\infty\Omega_n^a=\Gamma$, then $\{v_{\Omega_n}\}$ is monotone decreasing in n and $\lim_{n\to\infty} v_{\Omega_n}=v_\Gamma$ in R'. v_Γ is an FH_0 function, and $(u+v)_\Gamma=u_\Gamma+v_\Gamma$ for any FSH functions u and v.*

THEOREM 4.4. *$v_{\hat{S}}=v$ in R' for any FH_0 function v.*

PROOF. Let $\{D_n\}$ be a sequence of relatively compact regular domains in R such that $K_0\subset D_n\subset\bar{D}_n\subset D_{n+1}(n\geqq1)$ and $\bigcup_{n=1}^\infty D_n=R$, and put $\Omega_n=R-\bar{D}_n$ and $K_n=\bar{D}_{n+2}-D_{n+1}$ $(n\geqq1)$. Then $v_{K_n}\leqq v_{\Omega_n}\leqq v$ in R' since $K_n\in\boldsymbol{K}(\Omega_n)$ for every n. On the other hand, $v_{K_n}=v$ on ∂D_{n+1} $(\subset\partial K_n)$, $v_{K_n}=v=0$ on ∂K_0 by Corollary 4.2.2, and both v_{K_n} and v are harmonic in D'_{n+1}. Hence we get $v_{K_n}=v_{\Omega_n}=v$ in D'_{n+1}. Since $\{\Omega_n\}$ satisfies the assumption of Lemma 4.6 with $\Gamma=\hat{S}$, we have $\lim_{n\to\infty} v_{\Omega_n}=v_{\hat{S}}$ and accordingly $v_{\hat{S}}=v$ in R'.

§5. **Integral representations of FH_0 and FSH_0 functions.** Most of proofs of theorems and lemmas in this § are similar to or essentially the same as proofs of those is [12; §7]. So we partly skip the detail of those proofs.

THEOREM 5.1. *Any FSH_0 (resp. FH_0) function is expressible as the potential μN of a measure μ on $R'+\hat{S}$ (resp. $\hat{S}$), and vice versa.*

PROOF. By virtue of Theorems 4.1 and 4.3, it suffices to prove this theorem for an FH_0 function v. Let $\{D_n\}$ be a sequence of domains such as mentioned in the proof of Theorem 4.4. Then, by Theorem 4.2, we have $v=v_{\partial D_n}=\mu_n N$ in D_n' for a suitable measure μ_n on ∂D_n for every n, and $\mu_n(\partial D_n)=\int_{\partial K_0}\frac{\partial v}{\partial \boldsymbol{n}_{K_0}}dS<\infty$ for any n by Corollary 2.5.2. Hence a subsequence of $\{\mu_n\}$ converges vaguely to a measure μ on $\hat{S}$, and we get $v=\mu N$ in R'. The converse follows from Theorem 4.1.

LEMMA 5.1. *Let v be an FSH function and Ω be an open set in R' such that $\bar{\Omega}\cap K_0$ is empty. Then there exists a measure μ supported by Ω^a such that $v_\Omega=\mu N$ in R'.*

PROOF. Let $\{K_n\}$ be a sequence of compact sets such as mentioned in Lemma 4.4. Then, by Theorem 4.2, we have $v_{K_n}=\mu_n N$ for a suitable measure μ_n on K_n for every n, and $\mu_n(K_n)=\int_{\partial K_0}\frac{\partial v_{K_n}}{\partial \boldsymbol{n}_{K_0}}dS\leqq\int_{\partial K_0}\frac{\partial v_{\partial K}}{\partial \boldsymbol{n}_{K_0}}dS<\infty$ where K is a fixed regular compact set such that $K_0\subset K^\circ\subset K\subset R-\bar{\Omega}$. Hence a subsequence of $\{\mu_n\}$ converges vaguely to a measure μ supported by Ω^a, and we get $v_\Omega=\mu N$ in $R'-\bar{\Omega}$ by Lemma 4.4 since $N(x,y)$ is continuous in $x\in\Omega^a$ for any fixed $y\in R'-\bar{\Omega}$. Denote the subsequence by $\{\mu_n\}$ again and put $\nu_n=\mu_n|_{(K_n)^\circ}$ and $\nu=\mu|_\Omega$. Then ν_n (resp. ν) is the measure which gives the potential part of the Riesz decomposition (Lemma 2.2) of v in $(K_n)^\circ$ (resp. Ω). Hence ν_n increase to ν as $n\to\infty$ and accordingly $\mu_n-\nu_n$ converges vaguely to $\mu-\nu$. Hence we have $\mu N=\nu N+(\mu-\nu)N=\lim_{n\to\infty}\{\nu_n N+(\mu_n-\nu_n)N\}=\lim_{n\to\infty}\mu_n N=\lim_{n\to\infty}v_{K_n}=v_\Omega$ in Ω. Thus we have proved that $v_\Omega=\mu N$ in $R'-\partial\Omega$. Hence we get $v_\Omega=\mu N$ in R' in virtue of Lemm 2.3.

Considering $N(x,y)$ as a function of $y\in R'$ for any fixed $x\in\hat{R}$, we define $N_K(x,y)=[N(x,\cdot)]_K(y)$ for any regular compact set K. $N_\Omega(x,y)$ and $N_\Gamma(x,y)$ are defined analogously (see (4.1) and (4.2)). Then we have the following

LEMMA 5.2. *Let Ω be as in Lemma 5.1. Then $(\mu N)_\Omega=\mu N_\Omega$ in R' for any measure μ in $R'+\hat{S}$.*

PROOF. By means of the same computation as that in the proof of Theorem 3.4, we may show that $(\mu N)_K=\mu N_K$ for any $K\in \boldsymbol{K}(\Omega)$. Hence we get $(\mu N)_\Omega=\mu N_\Omega$ by virtue of Lemma 4.4.

THEOREM 5.2. *For any FSH function v and any closed subset Γ of $\hat{S}$, there* *exists a measure* μ *supported by* Γ *such that*

$$v_\Gamma(y)=\int_\Gamma N(\xi, y)d\mu(\xi) \text{ and } \mu(\Gamma)=\int_{\partial K_0}\frac{\partial v_\Gamma}{\partial \boldsymbol{n}_{K_0}}dS .$$

If v is an FH_0 function and $\Gamma=\hat{S}$, this theorem gives an integral representation of v by virtue of Theorem 4.4.

PROOF. Let $\{\Omega_n\}$ be a sequence of open subsets such as mentioned in Lemma 4.6. Then, by Lemma 5.1, we have $v_{\Omega_n}=\mu_n N$ for a suitable measure μ_n on Ω_n^a for every n, and it may be seen from the proof of Lemma 5.1 that $\mu_n(\Omega_n^a)\leqq \int_{\partial K_0}\frac{\partial v_{\partial K}}{\partial \boldsymbol{n}_{K_0}}dS<\infty$ for any n, where K is a regular compact set such that $K_0\subset K^\circ \subset K\subset R-\bar{\Omega}_1$. Hence a subsequence of $\{\mu_n\}$ converges vaguely to a measure μ supported by $\Gamma(=\bigcap_{n=1}^{\infty}\Omega_n^a)$, and we get $v_\Gamma=\mu N$ in R' by means of Lemma 4.6, and accordingly we have $\mu(\Gamma)=\int_{\partial K_0}\frac{\partial v_\Gamma}{\partial \boldsymbol{n}_{K_0}}dS$ by Corollary 2.5.2.

THEOREM 5.3. *Let* μ *be a measure in* $R'+\hat{S}$ *and* Γ *be a closed subset of* $\hat{S}$. *Then* $(\mu N)_\Gamma=\mu N_\Gamma$ *in* R'.

PROOF. Take a sequence $\{\Omega_n\}$ such as mentioned above. Then, on account of Lemmas 4.6 and 5.2, we have

$$(\mu N)_\Gamma=\lim_{n\to\infty}(\mu N)_{\Omega_n}=\lim_{n\to\infty}\mu N_{\Omega_n}=\mu N_\Gamma .$$

§6. Classification of the boundary points, canonical representation.

By definition, a function v on R' is called a *function of class* $\mathscr{D}$ if it is piecewise smooth in R' and satisfies

$$\left\|\frac{v}{\omega}\right\|_{R',\omega}<\infty \text{ and } \sup_{R'}\left|\frac{v}{\omega}\right|<\infty .$$

If v is a function of class $\mathscr{D}$ satisfying $v|_{\partial K_0}=0$ and K is a regular compact subset of R', then

$$\left([\boldsymbol{b}-\nabla p]\frac{v-v_K}{\omega},\ \nabla\frac{v-v_K}{\omega}\right)_{R',\omega}=0 \quad \text{(by Lemma 2.4)} \tag{6.1}$$

and

$$\left(\nabla\frac{v_K}{\omega}-[\boldsymbol{b}-\nabla p]\frac{v_K}{\omega},\ \nabla\frac{v-v_K}{\omega}\right)_{R',\omega}=0 \quad \text{(by Theorem 2.2)} . \tag{6.2}$$

Hence

$$\left\|\nabla\frac{v-v_K}{\omega}\right\|^2_{R',\omega}=\left(\nabla\frac{v-v_K}{\omega}-[\boldsymbol{b}-\nabla p]\frac{v-v_K}{\omega},\ \nabla\frac{v-v_K}{\omega}\right)_{R',\omega} \tag{6.3}$$

$$=\left(\nabla\frac{v}{\omega}-[\boldsymbol{b}-\nabla p]\frac{v}{\omega},\ \nabla\frac{v-v_K}{\omega}\right)_{R',\omega}\leqq\left\|\nabla\frac{v}{\omega}-[\boldsymbol{b}-\nabla p]\frac{v}{\omega}\right\|_{R',\omega}\left\|\nabla\frac{v-v_K}{\omega}\right\|_{R',\omega},$$

consequently

$$(6.4)\qquad \left\|\nabla\frac{v_K}{\omega}\right\|_{R',\omega}\leqq 2\left\|\nabla\frac{v}{\omega}\right\|_{R',\omega}+\left\|[\boldsymbol{b}-\nabla p]\frac{v}{\omega}\right\|_{R',\omega}<\infty\ .$$

LEMMA 6.1. *Let v be an FSH function of class $\mathscr{D}$, and Ω be an open subset of R' such that $\bar{\Omega}\cap K_0$ is empty. Then*

$$(6.5)\qquad \lim_{n\to\infty}\left\|\nabla\frac{v_\Omega-v_{K_n}}{\omega}\right\|_{R',\omega}=0$$

for a suitable sequence $\{K_n\}$ such as mentioned in Lemma 4.4. Furthermore, if Ω_1 is another open set such that $\Omega\subset\Omega_1$ and $\bar{\Omega}_1\cap K_0$ is empty, then

$$(6.6)\qquad \left(\nabla\frac{v_\Omega}{\omega}-[\boldsymbol{b}-\nabla p]\frac{v_\Omega}{\omega},\ \nabla\frac{v_{\Omega_1}-v_\Omega}{\omega}\right)_{R',\omega}=0$$

and

$$(6.7)\qquad \left\|\nabla\frac{v_\Omega}{\omega}\right\|_{R',\omega}\leqq 2\left\|\nabla\frac{v_{\Omega_1}}{\omega}\right\|_{R',\omega}+\left\|[\boldsymbol{b}-\nabla p]\frac{v_{\Omega_1}}{\omega}\right\|_{R',\omega}.$$

PROOF. It is clear that $(v_\Omega)_K=v_K$ for any $K\in\boldsymbol{K}(\Omega)$, and v_Ω is an FSH$_0$ function of class $\mathscr{D}$. Hence, substituting v_Ω for v in (6.3) resp. (6.4), we obtain that

$$(6.8)\qquad \left\|\nabla\frac{v_\Omega-v_K}{\omega}\right\|^2_{R',\omega}=\left(\nabla\frac{v_\Omega}{\omega}-[\boldsymbol{b}-\nabla p]\frac{v_\Omega}{\omega},\ \nabla\frac{v_\Omega-v_K}{\omega}\right)_{R',\omega}$$

resp.

$$(6.9)\qquad \left\|\nabla\frac{v_K}{\omega}\right\|_{R',\omega}\leqq 2\left\|\nabla\frac{v_\Omega}{\omega}\right\|_{R',\omega}+\left\|[\boldsymbol{b}-\nabla p]\frac{v_\Omega}{\omega}\right\|_{R',\omega}$$

for any $K\in\boldsymbol{K}(\Omega)$. It follows from (6.9) that there exist a sequence $\{K_n\}$ such as mentioned in Lemma 4.4 and $\Phi\in\boldsymbol{L}^2_\omega(R')$ satisfying that $\lim_{n\to\infty}\nabla\frac{v_{K_n}}{\omega}=\Phi$ weakly in $\boldsymbol{L}^2_\omega(R')$. On the other hand, it follows from (4.1) and Lemma 4.4 that $\lim_{n\to\infty}\nabla\frac{v_{K_n}}{\omega}=\frac{v_\Omega}{\omega}$ boundedly in R'. Hence, for any smooth $\Psi\in\boldsymbol{L}^2_\omega(R')$ whose support is a compact subset of R', we have

$$\lim_{n\to\infty}\left(\nabla\frac{v_{K_n}}{\omega},\ \Psi\right)_{R',\omega}=\lim_{n\to\infty}\left(\frac{v_{K_n}}{\omega},\ \operatorname{div}\Psi\right)_{R',\omega}=\left(\frac{v_\Omega}{\omega},\ \operatorname{div}\Psi\right)_{R',\omega}=\left(\nabla\frac{v_\Omega}{\omega},\ \Psi\right)_{R',\omega}$$

and the totality of such Ψ's are dense in $\boldsymbol{L}^2_\omega(R')$. Therefore we get

$$(6.10)\qquad \lim_{n\to\infty}\nabla\frac{v_{K_n}}{\omega}=\nabla\frac{v_\Omega}{\omega}\qquad\text{weakly in } \boldsymbol{L}^2_\omega(R')\ .$$

Hence, putting $K=K_n$ in (6.8) and letting $n\to\infty$, we obtain (6.5). Accordingly, replacing v resp. v_K by v_{Ω_1} resp. v_{K_n} in (6.2) and (6.4), and letting $n\to\infty$, we get (6.6) and (6.7).

LEMMA 6.2. *Let v be as in Lemma 6.1 and Γ be a closed subset of $\hat{S}$. Then*

$$\lim_{n\to\infty}\left\|\nabla\frac{v_{\Omega_n}-v_\Gamma}{\omega}\right\|^2_{R',\omega}=0 \tag{6.11}$$

for a suitable sequence $\{\Omega_n\}$ such as mentioned in Lemma 4.6.

PROOF. It follows from (6.7) that

$$\lim_{n\to\infty}\nabla\frac{v_{\Omega_n}}{\omega}=\nabla\frac{v_\Gamma}{\omega} \qquad \text{weakly in } L^2_\omega(R') \tag{6.12}$$

for a suitable $\{\Omega_n\}$ such as mentioned above; the proof is similar to that of (6.10). Since $\lim_{n\to\infty}\frac{v_{\Omega_n}}{\omega}=\frac{v_\Gamma}{\omega}$ boundedly in R', we get

$$\lim_{n\to\infty}\left\|[\boldsymbol{b}-\nabla p]\frac{v_{\Omega_n}-v_\Gamma}{\omega}\right\|_{R',\omega}=0\,. \tag{6.13}$$

(6.12) and (6.13) imply that

$$\lim_{n,m\to\infty}\left([\boldsymbol{b}-\nabla p]\frac{v_{\Omega_n}}{\omega},\nabla\frac{v_{\Omega_m}}{\omega}\right)_{R',\omega}=\left([\boldsymbol{b}-\nabla p]\frac{v_\Gamma}{\omega},\nabla\frac{v_\Gamma}{\omega}\right)_{R',\omega}. \tag{6.14}$$

On account of (6.6), we get for any n

$$\begin{aligned}&\left\|\nabla\frac{v_{\Omega_n}}{\omega}\right\|^2_{R',\omega}-\left([\boldsymbol{b}-\nabla p]\frac{v_{\Omega_n}}{\omega},\nabla\frac{v_{\Omega_n}}{\omega}\right)_{R',\omega}=\left(\nabla\frac{v_{\Omega_n}}{\omega}-[\boldsymbol{b}-\nabla p]\frac{v_{\Omega_n}}{\omega},\nabla\frac{v_{\Omega_n}}{\omega}\right)_{R',\omega}\\ &=\left(\nabla\frac{v_{\Omega_n}}{\omega}-[\boldsymbol{b}-\nabla p]\frac{v_{\Omega_n}}{\omega},\nabla\frac{v_{\Omega_1}}{\omega}\right)_{R',\omega}=\left(\nabla\frac{v_{\Omega_n}}{\omega},\nabla\frac{v_{\Omega_1}}{\omega}\right)_{R',\omega}-\left([\boldsymbol{b}-\nabla p]\frac{v_{\Omega_n}}{\omega},\nabla\frac{v_{\Omega_1}}{\omega}\right)_{R',\omega}.\end{aligned}$$

Applying (6.14), (6.12) and (6.13) to the above relation, we may see the existence of $\lim_{n\to\infty}\left\|\nabla\frac{v_{\Omega_n}}{\omega}\right\|^2_{R',\omega}$. On the other hand, if $n>m$, we obtain by means of (6.6) that

$$\begin{aligned}&\left(\nabla\frac{v_{\Omega_n}}{\omega},\nabla\frac{v_{\Omega_m}}{\omega}\right)_{R',\omega}-\left([\boldsymbol{b}-\nabla p]\frac{v_{\Omega_n}}{\omega},\nabla\frac{v_{\Omega_m}}{\omega}\right)_{R',\omega}\\ &\qquad-\left\|\nabla\frac{v_{\Omega_n}}{\omega}\right\|^2_{R',\omega}+\left([\boldsymbol{b}-\nabla p]\frac{v_{\Omega_n}}{\omega},\nabla\frac{v_{\Omega_n}}{\omega}\right)_{R',\omega}=0\,.\end{aligned}$$

Applying (6.14) to this relation, we get

$$\lim_{n>m\to\infty}\left(\nabla\frac{v_{\Omega_n}}{\omega},\nabla\frac{v_{\Omega_m}}{\omega}\right)_{R',\omega}=\lim_{n\to\infty}\left\|\nabla\frac{v_{\Omega_n}}{\omega}\right\|^2_{R',\omega}. \tag{6.15}$$

Since $\left\|\nabla\frac{v_{\Omega_n}-v_{\Omega_m}}{\omega}\right\|^2_{R',\omega}=\left\|\nabla\frac{v_{\Omega_n}}{\omega}\right\|^2_{R',\omega}-2\left(\nabla\frac{v_{\Omega_n}}{\omega},\nabla\frac{v_{\Omega_m}}{\omega}\right)_{R',\omega}+\left\|\nabla\frac{v_{\Omega_m}}{\omega}\right\|^2_{R',\omega}$, we obtain $\lim_{n>m\to\infty}\left\|\nabla\frac{v_{\Omega_n}-v_{\Omega_m}}{\omega}\right\|^2_{R',\omega}=0$. From this result and (6.12), we may conclude (6.11).

LEMMA 6.3. *Let v and Γ be as in Lemma 6.2. Then $(v_\Gamma)_\Gamma=v_\Gamma$ in R'. In particular $(\omega_\Gamma)_\Gamma=\omega_\Gamma$ in R'.*

PROOF. Let Ω and Ω_1 be open subsets of R' such that $\Gamma\subset\Omega^a$ and $\Omega\subset\Omega_1\subset\bar{\Omega}_1\subset R'$, and let $\{K_n\}$ be a sequence of compact sets for which $\lim_{n\to\infty}\left\|\nabla\frac{v_{\Omega_1}-v_{K_n}}{\omega}\right\|_{R',\omega}=0$ holds (Lemma 6.1). Since $v_\Omega-v_\Gamma$ is a function of class $\mathscr{D}$ satisfying $(v_\Omega-v_\Gamma)|_{\partial K_0}=0$, we have by (6.4)

$$\left\|\nabla\frac{(v_\Omega-v_\Gamma)_{K_n}}{\omega}\right\|_{R',\omega}\leqq 2\left\|\nabla\frac{v_\Omega-v_\Gamma}{\omega}\right\|_{R',\omega}+\left\|[\boldsymbol{b}-\nabla p]\frac{v_\Omega-v_\Gamma}{\omega}\right\|_{R',\omega}\quad\text{for any } n\,.$$

Letting $n\to\infty$, we obtain by Lemma 4.5 that

$$\left\|\nabla\frac{v_\Omega-(v_\Gamma)_{\Omega_1}}{\omega}\right\|_{R',\omega}\leqq 2\left\|\nabla\frac{v_\Omega-v_\Gamma}{\omega}\right\|_{R',\omega}+\left\|[\boldsymbol{b}-\nabla p]\frac{v_\Omega-v_\Gamma}{\omega}\right\|_{R',\omega}.$$

Let $\{\Omega_n\}$ be a sequence of open sets for which (6.11) holds. Then, from the above inequality, we have for $n>m$

$$\left\|\nabla\frac{v_{\Omega_n}-(v_\Gamma)_{\Omega_m}}{\omega}\right\|_{R',\omega}\leqq 2\left\|\nabla\frac{v_{\Omega_n}-v_\Gamma}{\omega}\right\|_{R',\omega}+\left\|[\boldsymbol{b}-\nabla p]\frac{v_{\Omega_n}-v_\Gamma}{\omega}\right\|_{R',\omega}.$$

Letting $n\to\infty$ and $m\to\infty$, we obtain $\left\|\nabla\frac{v_\Gamma-(v_\Gamma)_\Gamma}{\omega}\right\|_{R',\omega}=0$, which implies $v_\Gamma=(v_\Gamma)_\Gamma$ in R' since $v_\Gamma=(v_\Gamma)_\Gamma=0$ on ∂K_0. In particular, since ω is an FSH function of class $\mathscr{D}$, we have $(\omega_\Gamma)_\Gamma=\omega_\Gamma$ in R'.

This lemma corresponds to Theorem 19 in [12]. The argument to derive this lemma is not quite the same as the corresponding procedure in [12]; the modification is necessary by reason of the existence of term $\boldsymbol{b}$ in the differential operator A^*. However, once Lemma 6.3 is established, we may achieve the essentially same arguments for the classification of the boundary points and canonical representation in our case as those in the case of usual Laplacian in [12]; only some minor modifications may be necessary.[5] So it seems not to be necessary to mention the arguments in detail. We shall state only outline of the process.

Using Lemma 6.3, we may prove the following (Cf. Theorem 20 in [12])

LEMMA 6.4. *Let v be an FSH function in R', and Γ be a closed subset of*

5) For instance, maximum-minimum principle should be used in the form given in Lemma 2.1.

$\hat{S}$ *such that* $\omega_\Gamma=0$. *Then* $(v_\Gamma)_\Gamma=v_\Gamma$.

Applying Theorem 5.2 to $v=N(\xi,\cdot)$ and $\Gamma=\{\xi\}$[6] for arbitrary $\xi\in\hat{S}$, we obtain a function $\alpha(\xi)$ on $\hat{S}$ such that

$$(6.16)\qquad N_{\{\xi\}}(\xi,y)=\alpha(\xi)N(\xi,y) \text{ and } \alpha(\xi)=\int_{\partial K_0}\frac{\partial N_{\{\xi\}}(\xi,y)}{\partial n_{K_0}(y)}dS(y)\,.$$

LEMMA 6.5. *$\alpha(\xi)=0$ or 1 for any $\xi\in\hat{S}$* (Cf. Theorem 21 in [12]).

We put

$$\hat{S}_0=\{\xi\in\hat{S};\ \alpha(\xi)=0\} \text{ and } \hat{S}_1=\{\xi\in\hat{S};\ \alpha(\xi)=1\}\,.$$

Then, from (6.16), we have the following

THEOREM 6.1. *$N_{\{\xi\}}(\xi,y)=0$ or $=N(\xi,y)$ according as $\xi\in\hat{S}_0$ or $\xi\in\hat{S}_1$.*

By definition, $\hat{S}_1$ is called the *essential part of the ideal boundary* $\hat{S}$.

THEOREM 6.2. *$\hat{S}_0$ is an F_σ-set: $\hat{S}_0=\bigcup_{n=1}^{\infty}\Gamma_n$ where Γ_n's are closed subsets of $\hat{S}$* (see Theorem 22 in [12]).

LEMMA 6.6. *If v is an FSH function, then $v_\Gamma\equiv 0$ for any closed subset Γ of $\hat{S}$ contained in $\hat{S}_0$* (see Theorem 23 in [12]).

THEOEM 6.3. *If v is an FSH function and Γ is a closed subset of $\hat{S}$, then v_Γ is expressible in the form $\int_{\Gamma\cap\hat{S}_1}Nd\mu$; in particular, any FH_0 function is expressible in the form $\int_{\hat{S}_1}Nd\mu$.*

In fact, the measure μ in Theorem 5.2 satisfies $\mu(\Gamma\cap\Gamma_n)=0$ for every Γ_n (in Theorem 6.2) by Lemma 6.6. (See Theorem 24 in [12].) As for the case of FH_0 functions, see Theorem 4.4 in the present paper.

DEFINITIONS. i) A Borel measure μ on $\hat{S}$ satisfying $\mu(\hat{S}_0)=0$ is called a *canonical measure*, and the representation $\int_{\hat{S}_1}Nd\mu$ by means of a canonical measure μ is called a *canonical representation.*

ii) An FH_0 function u is said to be *extremal* if $v=cu$ (c: non-negative constant) whenever both v and $u-v$ are FH_0 functions.

LEMMA 6.7. *Let u be extremal and Γ be a closed subset of $\hat{S}$. If $u_\Gamma>0$ and $u-u_\Gamma$ is an FH_0 function, then there exists a unique point $\xi_0\in\Gamma$ such that $u=cN(\xi_0,\cdot)$ with $c=\int_{\partial K_0}\frac{\partial u}{\partial n_{K_0}}dS$.*

In fact, the measure μ in the representation of u (Theorem 6.3) is supported by a point $\xi_0\in\Gamma\cap\hat{S}_1$ since u is extremal.

The following theorem characterizes the extremal FH_0 functions and shows the relation between the set $\hat{S}_1$ and the set of all extremal FH_0 functions. (As

[6] $\{\xi\}$ denotes the set consisting of one point ξ.

for the proofs in detail of Lemmas 6.7 and Theorem 6.4, see Theorem 26 in [12].)

THEOREM 6.4. i) *Any extremal FH_0 function u is expressible by $u=cN(\xi, \cdot)$ with a positive constant c and a point $\xi \in \hat{S}_1$ uniquely determined by u.*

ii) *$N(\xi, y)$ is an extremal FH_0 function of y if and only if $\xi \in \hat{S}_1$.*

Part i) is obtained by putting $\Gamma=\hat{S}$ in Lemma 6.7 and by Corollary 3.2.2. Proof of Part ii): if $\xi \in \hat{S}_1$ and if $N(\xi, \cdot)=u+v$ where u and v are FH_0 functions, we have $u_{(\xi)}+v_{(\xi)}=N_{(\xi)}=N=u+v$ by Theorem 6.1, accordingly $v=v_{(\xi)}=cN(\xi, \cdot)$ for some $c\geqq 0$; the converse follows from i).

THEOREM 6.5. *Canonical representation of any FH_0 function is unique. For any FH_0 function v and any closed subset Γ of $\hat{S}$, the canonical measure for v_Γ is supported by Γ.* (See Theorem 27 in [12].)

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