A characterization of Janko's two new simple groups

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PART I
Preliminaries

1. Introduction. The object of this paper is to give a characterization of the two recently discovered simple groups of Janko of orders 604,800 and 50,232,960 (which we designate as J_2 and J_3, respectively) by means of their Sylow 2-subgroups which are known to be isomorphic groups of order 2^7. For convenience we say that a 2-group is of *Janko type* if it is isomorphic to a Sylow 2-subgroups of J_2 or J_3. More generally, for any simple group G, we say that a 2-group is of *type* G if it is isomorphic to a Sylow 2-subgroup of G.

A 2-group S of Janko type can be described as follows: it is generated by seven involutions $z_1, z_2, a_1\, a_2, b_1, b_2$ and t satisfying the relations

$$\begin{aligned} &[a_1, b_1]=[a_2, b_2]=z_1,\ [a_2, b_1]=z_2,\ [a_1, b_2]=z_1z_2 \\ &[a_1, t]=[t, b_1]=a_1b_1,\ [a_2, t]=[t, b_2]=a_2b_2,\ [t, z_2]=z_1\ , \end{aligned} \tag{1}$$

with all remaining commutators of pairs of generators being trivial.

Actually S is generated by the three involutions a_1, a_2, t. However, to describe S in terms of these generators alone requires 4-fold commutator relations. On the other hand, one can give a more conceptual picture based on the easily verified fact that the subgroup

$$T=\langle z_1, z_2, a_1\, a_2, b_1, b_2\rangle$$

of S of order 2^6 is of type $PSL(3,4)$. Indeed, if α denotes the automorphism of $PGL(3,4)$ induced by the transpose-inverse map of $GL(3,4)$ and σ denotes the automorphism of $PGL(3,4)$ induced from the generator of the Galois group of $GF(4)$, then the product $t^*=\alpha\cdot\sigma$ is an automorphism of $PGL(3,4)$ (and of $PSL(3,4)$) of order 2. If one now forms the semi-direct product of $PGL(3,4)$ (or of $PSL(3,4)$) and the group $\langle t^*\rangle$ in the usual sense, then a Sylow 2-subgroup of the resulting group is, in fact, of Janko type (see Lemma 4.8 below).

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Because a 2-group of Janko type has such a representation, the semidirect products of $PGL(3,4)$ and $PSL(3,4)$ with the group $\langle t^*\rangle$ play a role in the paper. It will be convenient to denote them by $PGL^*(3,4)$ and $PSL^*(3,4)$ respectively. We regard each as a subgroup of the automorphism group of $PSL(3,4)$, containing $PSL(3,4)$. Since $|PGL(3,4):PSL(3,4)|=3$, $PSL(3,4)$ has index 6 in $PGL^*(3,4)$ and 2 in $PSL^*(3,4)$. Clearly $PGL^*(3,4)$ and $PSL^*(3,4)$ both contain normal subgroups of index 2 with Sylow 2-subgroup of type $PSL(3,4)$.

Furthermore, one checks directly that $SCN_3(S)$ is empty if S is of Janko type.

Our main result is as follows:

THEOREM A. *If G is a finite group with Sylow 2-subgroup of Janko type, then one of the following holds:*

(i) $G=O(G)C_G(x)$ *for some involution x of G;*

(ii) $G/O(G)$ *is isomorphic to a subgroup of* $PGL^*(3,4)$;

(iii) $G/O(G)$ *is isomorphic to J_2 or J_3.*

As a corollary, we obtain the following characterization of J_2 and J_3:

COROLLARY A. *If G is a finite simple group with Sylow 2-subgroup of Janko type, G is isomorphic to J_2 or J_3.*[1]

Characterizations of J_2 and J_3 have already been established in terms of their orders. The existence of J_2 was shown by M. Hall [17] and its characterization by its order by M. Hall and D. Wales [18]. The corresponding result was obtained for J_3 by the combined work of S. K. Wong [28] and G. Higman and J. McKay [21], the latter proving as well the existence of J_3. Furthermore, in Janko's initial investigation of these groups [23], he considered a simple group G^* with an involution z^* in the center of a Sylow 2-subgroup such that $N^*=C_{G^*}(z^*)$ is an extension of a normal subgroup of order 32 by the alternating group A_5. Under these conditions, he proved that the structure of N^* was uniquely determined, that G^* had exactly two or one conjugacy classes of involutions, and that correspondingly G^* had order 604,800 or 50,232,960. Combining this with the various results stated above, it follows that G^* is necessarily isomorphic to J_2 or J_3. Thus the groups J_2 and J_3 are completely characterized by means of the centralizers of suitable involutions, these centralizers being isomorphic to N^*.

We reduce the proof of Theorem A to the case in which the given group G

[1] Under the additional assumption that G has more than one class of involutions, Richard Lyons, using the methods of modular character theory, has shown that G must be isomorphic to J_2. Use of his result would give a slight simplification of our proof. See Section 11.

is simple. Hence if z_1 is an involution in the center of a Sylow 2-subgroup of G and $N=C_G(z_1)$, we see from the preceding remarks that to establish the theorem, it will suffice to show that N is isomorphic to the group N^* above. The bulk of the paper is devoted to this task. It is divided into two distinct parts:

(I) The determination of the structure of $N/O(N)$;

(II) A proof that $O(N)=1$.

In order to solve the problem posed in (I), we are led naturally to a study of the group $\bar{N}=N/O(N)\langle z_1\rangle$, which has Sylow 2-subgroups of order 2^6 and of type A_8. Furthermore, because of the embedding of N in G, the group $\bar{N}$ satisfies some additional restrictions. In fact, we obtain a solution of (I) as a direct consequence of the following theorem:

THEOREM B. *Let G be a finite group with Sylow 2-subgroup of type A_8, which satisfies the following conditions*:

(a) *G has three conjugacy classes of involutions*;

(b) *G contains an elementary abelian subgroup A of order 16 such that $N_G(A)/O(N_G(A))$ is isomorphic to an extension of A by A_5.*

Under these conditions, $AO(G)$ is normal in G.

As an immediate corollary, this yields

COROLLARY B. *Let G be a finite group with Sylow 2-subgroups of Janko type and no normal subgroups of index 2. If z_1 is an involution in the center of a Sylow 2-subgroup of G, and $N=C_G(z_1)$ then $N/O(N)$ is isomorphic to the centralizer of an involution in the center of a Sylow 2-subgroup of J_2 or J_3.*

As with Theorem A, after making an initial reduction on G, in order to establish Theorem B it will suffice to show that $O(C_G(x))=1$ for every involution x of G. Indeed, once this is known, the desired conclusion follows from Lemma 8 of K. Harada [20].

In order to treat problem (II) above, we require a detailed knowledge of the structure of the proper subgroups of G. As we have already noted, a 2-group of Janko type contains a maximal subgroup of type $PSL(3,4)$. In particular, then, G possesses proper subgroups whose Sylow 2-subgroups are of type $PSL(3,4)$. To determine their possible structure, we shall classify all finite groups having such a Sylow 2-subgroup. Specifically we prove

THEOREM C. *If G is a finite group with Sylow 2-subgroups of type $PSL(3,4)$, then $G/O(G)$ is isomorphic to a subgroup of $PGL(3,4)$. In particular, if G is simple, then G is isomorphic to $PSL(3,4)$.*

Likewise, to establish Theorem C, after an initial reduction, it suffices to show that $O(C_G(x))=1$ for every involution x of G. Once this is accomplished, it is not difficult to show that the centralizer of every involution of G is, in fact, 2-closed, in which case the theorem follows from a known result of Suzuki [24].

The proofs of Theorems A, B and C bear considerable similarity to each other. In each case we use Theorem B of Gorenstein [14] concerning strongly flat signalizer functors in groups of 2-rank 4.[2] In fact, if A is an elementary abelian 2-subgroup of G of maximum rank, then in each case $m(A)=4$, and we prove that the mapping

$$\theta(C_G(a))=O(C_G(a))$$

for all a in $A^\sharp$ defines a strongly flat A-signalizer functor of G. On the basis of Theorem B of [14], we are then able to show that our group G satisfies the assumptions of one of two theorems of D. Gorenstein and J. Walter [14*] and [16]. These theorems then yield that $O(C_G(x))=1$ for every involution x of G and therefore the present Theorems A, B, C follow from the known classification theorems mentioned above. In Section 2, we shall state the main concepts and results of D. Gorenstein [13], [14], and D. Gorenstein and J. Walter [14*], [16] to the extent that they are needed in the paper.

We remark parenthetically that our proof that $O(C_G(x))=1$ for every involution x of G in the case of Theorem B will not generalize directly to arbitrary groups G with Sylow 2-subgroups of type A_8 (in which $O(G)=1$). Indeed, in each of Theorems A, B, C, our proof that O is an A-signalizer functor on G depends in part upon the following fact: If H is a proper subgroup of G containing A, then every element of $M_H(A)$ lies in $O(H)$. However, this condition is not true, in general, among even the known groups having Sylow 2-subgroups of type A_8, for these include the groups $PS_p(4, q)$ with $q\equiv 3,5 \pmod 8$. It may nevertheless be possible to show, by means of a more involved argument, that O is a (strongly flat) A-signalizer functor in the case of arbitrary groups with Sylow 2-subgroups of type A_8.

Finally a word about notation. We shall use the "bar" convention throughout the paper: If $\bar{H}$ is a homomorphic image of the group H, then $\bar{X}$ will always denote the image in $\bar{H}$ of a subgroup, subset, or element X of H.

In general, our notation is standard, including the following terminology:

Z_n=cyclic group of order n;

D_{2^n}=dihedral group of order 2^n;

Q_{2^n}=generalized quaternion group of order 2^n;

[2] See the final comment of paper which has been added in proof.

A_n=alternating group of degree n;

S_n=symmetric group of degree n;

E_{2^n}=elementary abelian group of order 2^n;

$Q_8 * D_8$, etc.=central product of Q_8 and D_8, etc.;

$A_5 \cdot E_{16}^{(1)}$=unique nontrivial split extension of E_{16} by A_5 in which A_5 acts nontransitively on the involutions of E_{16};

$A_5 \cdot E_{16}^{(2)}$=unique split extension of E_{16} by A_5 in which A_5 acts transitively on the involutions of E_{16}.

It is well known that E_{16} has exactly two nontrivial inequivalent split extensions by A_5.

2. **Assumed results.** In this section we state without proof the relevant results of [13], [14], [14*] and [16] that we shall need. As mentioned in the introduction, the only signalizer functor θ that we shall use is $\theta = O$; so for simplicity we restrict our statements to this case.

Let A be an elementary abelian 2-subgroup of the group G with $m(A) \geq 3$.

DEFINITION 2.1. We say that O is an A-signalizer functor on G provided for each involution a and b of A, we have

$$O(C_G(a)) \cap C_G(b) \subseteq O(C_G(b)) . \tag{1}$$

(Cf. definition 1 of [13]; note that the remaining conditions of that definition are automatically fulfilled when $\theta = O$).

DEFINITION 2.2. If O is an A-signalizer functor on G, and B is a noncyclic subgroup of A, then $\mathcal{V}_O(B)$ is the set of B-invariant subgroups K of G of odd order such that

$$K = \langle K \cap O(C_G(b)) \mid b \in B^\# \rangle . \tag{2}$$

Furthermore, for any odd prime p, $\mathcal{V}_O(B; p)$ is the set of elements of $\mathcal{V}_O(B)$ of order a power of p.

DEFINITION 2.3. If O is an A-signalizer functor on G, we say that O is *strongly flat* provided for each proper subgroup H of G such that $B = A \cap H$ is noncyclic, one of the following two conditions is satisfied:

(i) The elements of $\mathcal{V}_O(B)$ contained in H generate a subgroup of H of odd order; or

(ii) (a) B is a four group;

(b) Any two elements of $\mathcal{V}_O(B; p)$, p an odd prime, which are maximal subject to being contained in H, are conjugate by an element of $N_H(B)$;

(c) If K_i, $1 \leq i \leq m$, are the distinct elements of $\mathcal{V}_O(B)$ which are maximal subject to being contained in H, then

(1) $K_i \cap K_j / K_i \cap K_j \cap O(H)$ is cyclic for all $i \neq j$, and

(2) $K_i \cap K_k = K_j \cap K_k$ for all $i \neq j \neq k \neq i$.

If a given subgroup H of G satisfies one of these two conditions, it will be convenient to say that H is *strongly A-flat.*

Theorem B of [14] asserts in the present case:

THEOREM 2.1. *If A is an elementary abelian 2-subgroup of order 16 of the group G and O is a strongly flat A-signalizer functor on G, then the subgroup*

$$W_A = \langle O(C_G(a)) \mid a \in A^\# \rangle$$

of G has odd order.

We turn now to the results of [14*]. First of all, condition (1) above is clearly equivalent to the following condition: For each a, b in $A^\#$,

$$O(C_G(a)) \cap C_G(b) = O(C_G(b)) \cap C_G(a)\ . \tag{3}$$

Equation (3) forms the basis of the following definition:

DEFINITION 2.4. An elementary abelian 2-subgroup A of the group G is said to be *balanced* if condition (3) holds for each a, b in $A^\#$. Furthermore, G is said to be *balanced* if each of its noncyclic elementary abelian 2-subgroups is balanced.

If G is balanced, it follows at once from the definitions that O is an A-signalizer functor on G for each elementary abelian 2-subgroup A of G with $m(A) \geq 3$.

The significance of this concept can be seen by the following result, which is a consequence of the proposition of Part III, Section 1 of [14*].

LEMMA 2.2. *If the centralizer of every involution of the group G is 2-constrained, then G is balanced.*

In this paper we shall actually need a slight extension of this lemma; which we shall now prove.

LEMMA 2.3. *If for any involution x of the group G either $C_G(x)$ is 2-constrained or $C_G(x)/O(C_G(x)) \cong Z_2 \times Z_2 \times A_5$, then G is balanced.*

PROOF. Let a, b be commuting involutions of G. By symmetry we need only prove that $O(C_G(a)) \cap C_G(b) \subseteq O(C_G(b))$. Set $H = C_G(b)$ and $D = O(C_G(a)) \cap H$. If H is 2-constrained, the quoted proposition of [14*] applies, so we may assume that $\bar{H} = H/O(H) = \bar{T} \times \bar{L}$, where $\bar{T}$ is a four group and $\bar{L} \cong A_5$. If $\bar{a} \notin \bar{T}$, it is immediate that $C_{\bar{H}}(\bar{a})$ is a 2-group. Since $\bar{D} \subseteq C_{\bar{H}}(\bar{a})$, this forces $\bar{D} = 1$ and so $D \subseteq O(H)$, as required. On the other hand, if $\bar{a} \in \bar{T}$, then $C_H(a)$ maps onto $\bar{H}$. But $[D, C_H(a)]$ has odd order and hence so does $[\bar{D}, \bar{H}]$. Since $[\bar{D}, \bar{H}]$ is normal in $\bar{H}$, this yields $[\bar{D}, \bar{H}]=1$ as $O(\bar{H})=1$ and we conclude at once that $\bar{D}=1$. Thus $D\subseteq O(H)$ in this case as well.

DEFINITION 2.5. Let S be a 2-group and let A and A^* be two noncyclic elementary abelian subgroups of S. We say that A and A^* are *connected* if there is a sequence of noncyclic elementary abelian 2-subgroups $A=A_1, A_2, \cdots, A_n=A^*$ of S such that either $A_i\subseteq A_{i+1}$ or $A_{i+1}\subseteq A_i$ for all i, $1\leq i\leq n-1$. We say that S is *connected* if every pair of noncyclic elementary abelian subgroups of S is connected. Moreover, we say that an arbitrary group G is *connected* if a Sylow 2-subgroup of it is connected.

The following sufficient conditions for a 2-group to be connected are proved in Part IV, Section 1 of [14*]:

LEMMA 2.4. *If S is a 2-group in which $SCN_3(S)$ is nonempty then S is connected.*

If S is a 2-group of type A_8 or $PSL(3, 4)$, then $SCN_3(S)$ is, in fact, nonempty, so S is connected by the lemma. On the other hand, if S is of Janko type, it turns out that S is not connected (see Lemma 5.1 (i)).

In the connected case, we have the following result, which is the so-called "balanced theorem" of Part IV, Section 1 of [14*]:

THEOREM 2.5. *Let G be a group with $O(G)=1$ and $SCN_3(2)$ nonempty which satisfies the following conditions:*[3)]

(a) *G is balanced and connected;*

(b) *For some noncyclic elementary abelian 2-subgroup A of G, the subgroup*

$$W_A=\langle O(C_G(a))\mid a\in A^{\#}\rangle$$

of G has odd order.

Under these conditions, $O(C_G(x))=1$ for every involution x of G.

To state an analogous result in the non-connected case, we need one further concept:

DEFINITION 2.6. Let G be a group in which $SCN_2(2)$ is nonempty and let $U\in U(2)$. We say that G is *weakly connected* provided the following conditions hold:

(a) $N_G(U)/C_G(U)$ has order divisible by 3;

(b) If $u\in U^{\#}$, then $C_G(u)$ is 2-constrained and $U\subseteq O_{2',2}(C_G(u))$.

(c) If $H=\langle C_G(u), N_G(U)\rangle$ and H is a proper subgroup of G, then H is strongly

3) In this paper, the balanced theorem is used only in the case that G satisfies the assumptions of Lemma 2.3, in which case the centralizer of every involution of G is "2-generated". This case of the theorem is proved in Part IV, Section 3 of [14*].

embedded in G.

(Groups with a strongly embedded subgroup have been classified by Bender [3], [4]).

We shall see later in Section 12 that a simple group G with Sylow 2-subgroups of Janko type is, in fact, weakly connected.

Theorem 11.3 of [16] asserts the following:

THEOREM 2.6. *Let G be a finite group with no normal subgroups of index 2 and $O(G)=1$ which satisfies the following conditions:*

(a) *G is balanced and weakly connected;*

(b) *If $U \in U(2)$, then the subgroup*

$$W_U = \langle O(C_G(u)) \mid u \in U^{\sharp} \rangle$$

of G has odd order.

Under these conditions, $O(C_G(x))=1$ for every involution x of G.

This summary should make evident our procedure for demonstrating that $O(C_G(x))=1$ for every involution x of G in each of Theorems A, B and C. Indeed, we first derive sufficient information concerning the subgroup structure of G to enable us to prove that G is balanced and that O is a strongly flat A-signalizer functor on G, where A is an elementary abelian subgroup of G of order 16. Application of Theorem 2.1 then yields that the subgroup W_A is of odd order, so condition (b) of Theorems 2.5 and 2.6 is satisfied. (For the latter theorem, we note that A necessarily contains an element U of $U(2)$. But then $W_U \subseteq W_A$ and so W_U is of odd order.)

In the case of Theorems B and C, G is connected, so condition (a) of Theorem 2.5 is also satisfied; while in the case of Theorem A we show that G is weakly connected, so condition (a) of Theorem 2.6 is satisfied. Theorems 2.5 and 2.6 together yield that $O(C_G(x))=1$ for every involution x of G.

3. Strongly flat subgroups. We have already remarked that the concept of a strongly flat A-signalizer functor on a group G will be very important for us. In this section, we derive some preliminary lemmas which will facilitate the later verification that O is a strongly flat A-signalizer functor for suitable elementary abelian 2-subgroups A of G.

Throughout we consider a group G and an elementary abelian 2-subgroup A of G with $m(A) \geq 3$ and a subgroup H of G such that $B=A \cap H$ is noncyclic. We shall show, under suitable conditions on H, that H is strongly A-flat. We fix this notation.

We begin with a definition. If $b \in B^{\sharp}$, then clearly $O(C_G(b)) \cap H \subseteq O(C_H(b))$.

However, this inclusion may be proper, and so there may exist B-invariant subgroups K of H of odd order of the form $K=\langle K\cap O(C_H(b))\,|\,b\in B^{\#}\rangle$ with K *not* an element of $\mathcal{V}_O(B)$. It will be convenient to denote the set of such subgroups by $\mathcal{V}_O(B;H)$ and to say that H is *strongly B-flat*, if the elements of $\mathcal{V}_O(B;H)$ satisfy the conclusions of Definition 2.3. (Even if $B=A$, the context will make evident the meaning of strongly A-flat intended.)

The conditions for strong B-flatness depend upon H and B alone, while those for strong A-flatness depend also upon G. However, we have the following result:

LEMMA 3.1. *If H is strongly B-flat, then H is strongly A-flat.*

PROOF. Every element of $\mathcal{V}_O(B)$ contained in H is clearly an element of $\mathcal{V}_O(B;H)$. Hence if the set of elements of $\mathcal{V}_O(B;H)$ generate a subgroup of odd order, so do the elements of $\mathcal{V}_O(B)$ contained in H. Thus, H is strongly A-flat in this case.

Suppose then that the elements of $\mathcal{V}_O(B;H)$ satisfy condition (ii) of Definition 2.3. We make a preliminary observation: if $K\in\mathcal{V}_O(B;H)$ and L_1, L_2 are elements of $\mathcal{V}_O(B)$ contained in K, then $\langle L_1, L_2\rangle\in\mathcal{V}_O(B)$. Indeed, since $\langle L_1, L_2\rangle\subseteq K$, it has odd order and the assertion is immediate from the definition of $\mathcal{V}_O(B)$. In particular, the elements of $\mathcal{V}_O(B)$ contained in K possess a unique maximal element.

Now let K_i, $1\leq i\leq m$, be the distinct elements of $\mathcal{V}_O(B)$ which are maximal subject to being contained in H and let K_i^* be a maximal element of $\mathcal{V}_O(B;H)$ containing K_i, $1\leq i\leq m$. By the preceding remarks, K_i is clearly the unique element of $\mathcal{V}_O(B)$ which is maximal subject to being contained in K_i^*. This implies that $K_i^*\neq K_j^*$ for $i\neq j$, since otherwise $K_i=K_j$, which is not the case. Since H is strongly B-flat, we thus have that $K_i^*\cap K_j^*/K_i^*\cap K_j^*\cap O(H)$ is cyclic for $i\neq j$ and consequently $K_i\cap K_j/K_i\cap K_j\cap O(H)$ is also cyclic. Furthermore, for $i\neq j\neq k\neq i$, we have $K_i^*\cap K_j^*=K_j^*\cap K_k^*$. But then $K_i\cap K_j\subseteq K_k^*$ and as $K_i\cap K_j\in\mathcal{V}_O(B)$, we have $K_i\cap K_j\subseteq K_j$. Thus $K_i\cap K_j\subseteq K_j\cap K_k$. Similarly the reverse inclusion holds and so $K_i\cap K_j=K_j\cap K_k$. We conclude that H satisfies condition (ii-c) of the definition of strong A-flatness.

Now let P_1, P_2 be elements of $\mathcal{V}_O(B;p)$, p an odd prime, that are maximal subject to being contained in H and let Q_i be a maximal element of $\mathcal{V}_O(B;H;p)$ containing P_i, $1\leq i\leq 2$. By assumption $Q_2=Q_1^x$ for some element x in $N_H(B)$. Thus $P_1^x\subseteq Q_2$. We claim that $P_1^x\in\mathcal{V}_O(B)$. Indeed, we have

$$P_1=\langle P_1\cap O(C_G(b))\,|\,b\in B^{\#}\rangle\ ,$$

whence

$$P_1^x=\langle P_1^x\cap O(C_G(b^x))\,|\,b\in B^{\#}\rangle\ .$$

Since b^x runs over B^x as b does, the assertion follows from the definition of $\mathcal{V}_O(B)$.

But now we see that $\langle P_1^x, P_2\rangle \in \mathcal{V}_O(B; p)$ as $\langle P_1^x, P_2\rangle \subseteq Q_2$. It follows therefore from our maximal choice of P_2 that $P_1^x \subseteq P_2$. Since $Q_1 = Q_2^{x^{-1}}$, we obtain similarly that $P_2^{x^{-1}} \subseteq P_1$, whence $P_2 \subseteq P_1^x$. Thus $P_1^x = P_2$ and so condition (ii-b) of the definition of the strong A-flatness holds.

Since B is a four group by assumption, condition (ii-a) holds also, and therefore H is strongly A-flat.

Setting $\bar{H} = H/O(H)$, we define the elements of $\mathcal{V}_O(\bar{B}; \bar{H})$ in the obvious way and we say that $\bar{H}$ is *strongly $\bar{B}$-flat* if the elements of $\mathcal{V}_O(\bar{B}; \bar{H})$ satisfy the obvious analogues of the conditions of Definition 2.3. The following lemma has the effect of reducing the verification of strong B-flatness for H to properties of $H/O(H)$.

LEMMA 3.2. *If $\bar{H} = H/O(H)$ is strongly $\bar{B}$-flat, then H is strongly B-flat.*

PROOF. If $b \in B^\#$, then $O(C_H(b))$ maps onto $O(C_{\bar{H}}(\bar{b}))$. Hence the elements of $\mathcal{V}_O(B; H)$ map onto elements of $\mathcal{V}_O(\bar{B}; \bar{H})$ and each element of $\mathcal{V}_O(\bar{B}; \bar{H})$ is the image of an element of $\mathcal{V}_O(B; H)$. Since B is noncyclic, we also have that $O(H) \in \mathcal{V}_O(B; H)$. Thus the maximal elements of $\mathcal{V}_O(B; H)$ are precisely the complete inverse images of those of $\mathcal{V}_O(\bar{B}; \bar{H})$. Hence if condition (ii) of Definition 2.3 holds for $\bar{H}$, it follows at once that condition (ii-c) holds also for H. We claim that condition (ii-b) holds also. Indeed, let P_1, P_2 be two maximal elements of $\mathcal{V}_O(B; H; p)$, p an odd prime. Set $K_i = P_i O(H)$, so that $K_i \in \mathcal{V}_O(B; H)$. Hence P_i is a Sylow p-subgroup of K_i by the maximality of P_i, $1 \leq i \leq 2$. This in turn implies that $\bar{P}_i$ is a maximal element of $\mathcal{V}_O(\bar{B}; \bar{H}; p)$, $1 \leq i \leq 2$. But then by condition (ii-b) of Definition 2.3 in $\bar{H}$, $\bar{P}_2 = \bar{P}_1^{\bar{x}}$ for some element $\bar{x}$ in $N_{\bar{H}}(\bar{B})$. Since $N_H(B)$ maps onto $N_{\bar{H}}(\bar{B})$, it follows that $K_2 = K_1^x$ for some element x in $N_H(B)$. Thus P_1^x and P_2 are two B-invariant Sylow p-subgroups of K_2, whence $P_2 = P_1^{xy}$ for some y in $C_{K_2}(B)$. Since $xy \in N_H(B)$, we conclude that condition (ii-b) holds in H. Since $\bar{B}$ is a four group by assumption, so is B and hence condition (ii-a) also holds in H. Thus H is strongly B-flat.

On the other hand, if condition (i) of Definition 2.3 holds in $\bar{H}$, then obviously it also holds in H and again H is strongly B-flat.

We also have

LEMMA 3.3. *If $O(C_H(b)) \subseteq O(H)$ for every b in $B^\#$, then H is strongly A-flat.*

PROOF. Our assumption implies that every element of $\mathcal{V}_O(B; H)$ is contained in $O(H)$, so H is strongly B-flat. Hence H is strongly A-flat by Lemma 3.1.

We next prove

LEMMA 3.4. *If H is 2-constrained, then H is strongly A-flat.*

PROOF. In view of Lemmas 3.1 and 3.2, it will be enough to prove that H is strongly B-flat in the special case that $O(H)=1$. If $b \in B^{\sharp}$, set $C=C_H(b)$ and $R=O_2(H) \cap C$. Then $[O(C), R] \subseteq O(C) \cap O_2(H)=1$, so $O(C)$ centralizes $R=C_{O_2(H)}(b)$. Theorem 5.3.4 of [12] now yields that $O(C)$ centralizes $O_2(H)$. Since H is 2-constrained, it follows that $O(C)=1$. Thus 1 is the unique maximal element of $\mathcal{V}_O(B; H)$ and so H is strongly B-flat.

LEMMA 3.5. *If H has dihedral Sylow 2-subgroups, then H is strongly A-flat.*

PROOF. Again we may assume $O(H)=1$ and need only verify that H is strongly B-flat. We may also suppose H is not a 2-group, otherwise the conclusion is obvious. We apply the main theorem of D. Gorenstein and J. Walter [15] and conclude that H is either isomorphic to A_7 or to a subgroup of $P\Gamma L(2, q)$ containing $PSL(2, q)$. More specifically, in the latter case, $H=LE$, where $L \cong PSL(2, q)$ or $PGL(2, q)$, q odd, $L \triangleleft H$, E is cyclic of odd order and is induced from the Galois group of $GF(q)$, and $L \cap E=1$. We identify H with its image in A_7 or $P\Gamma L(2, q)$. We note that since B is noncyclic and H has dihedral Sylow 2-subgroups, B is necessarily a four group.

If $H=A_7$, we can identify B with $\langle(1\,2)(3\,4), (1\,3)(2\,4)\rangle$ or $\langle(1\,2)(3\,4), (3\,4)(5\,6)\rangle$ as these are representatives of the two conjugacy classes of four groups in A_7. In the first case, we see that $\langle(5\,6\,7)\rangle$ is the unique maximal element of $\mathcal{V}_O(B; H)$ and so condition (i) of Definition 2.3 is satisfied. In the second case, there are three maximal elements of $\mathcal{V}_O(B; H)$; namely, $\langle(1\,2\,7)\rangle$, $\langle(3\,4\,7)\rangle$, and $\langle(5\,6\,7)\rangle$. Since $(1\,3\,5)(2\,4\,6) \in N_H(B)$ and cyclically permutes these three subgroups, it is immediate that condition (ii) of Definition 2.3 is satisfied.

Suppose then that $H=LE \subseteq P\Gamma L(2, q)$. By Lemma 3.3(i) of [15], we can assume that E centralizes B. We let b_1, b_2, b_3 be the involutions of B. By Lemmas 3.1(iii), (vii) and 3.3(ii) of [15], $C_L(b_i)$ is a dihedral group and if $K_i=O(C_L(b_i))E$, $1 \leq i \leq 3$, then every maximal B-invariant subgroup of H of odd order is equal to some K_i. Since $C_H(b_i)$ has a normal 2-complement, the same is true of every maximal element of $\mathcal{V}_O(B; H)$. By Lemma 3.1(v) of [15], $O(C_L(b_i)) \cap O(C_L(b_j))=1$ for $i \neq j$. Hence $K_i \cap K_j=E$ for all $i \neq j, 1 \leq i, j \leq 3$. Since E is cyclic, it follows that condition (ii-c) of Definition 2.3 is satisfied.

Since B is a four group, it thus remains to verify condition (ii-b). Let p be an odd prime and let P_i be a B-invariant Sylow p-subgroup of $K_i, 1 \leq i \leq 3$. Then $P_i=[P_i, B]C_{P_i}(B)$. By Lemma 3.3(i) of [15], $C_H(B)=B \times E$, so $C_{P_i}(B) \subseteq E$. Moreover, $[P_i, B] \subseteq L$. Since the unique Sylow p-subgroup P of E normalizes the unique Sylow p-subgroup Q_i of $O(C_L(b_i))$, we conclude that, in fact, $P_i=Q_iP$,

$1 \leq i \leq 3$. In particular, each P_i is uniquely determined. If $B \subseteq L' = PSL(2, q)$, then there exists an element x of $N_L(B)$ which cyclically permutes b_1, b_2, b_3. But then x cyclically permutes K_1, K_2, K_3 and consequently also P_1, P_2, P_3, so condition (ii-b) holds in this case.

Suppose finally that $B \not\subseteq L'$. Then $|B \cap L'| = 2$. For definiteness, assume $b_1 \in B \cap L'$, in which case b_2 and b_3 are noncentral involutions of a Sylow 2-subgroup S of H containing B. Hence S contains an element which interchanges b_2 and b_3 and so P_2 and P_3 are conjugate in $N_H(B)$. However, by Lemma 3.1 (iii) of [15], $O(C_L(b_1))$ and $O(C_L(b_i))$ have coprime orders for $i=2$ and 3. Hence either P_2 and P_3 are the only maximal elements of $\mathrm{M}_H(B; p)$ or else $Q_2 = Q_3 = 1$, while $Q_1 \neq 1$, and P_1 is the unique maximal element of $\mathrm{M}_H(B; p)$. In either case we conclude that condition (ii-b) holds. Thus H is strongly B-flat and the lemma is proved.

LEMMA 3.6. *If H has quasi-dihedral Sylow 2-subgroups, then H is strongly A-flat.*[4)]

PROOF. Again we may assume $O(H)=1$ and need only verify that H is strongly B-flat. Since a quasi-dihedral group contains no noncyclic abelian subgroup of order 8, B is necessarily a four group. By Proposition 2.1.1 of J. Alperin, R. Brauer and D. Gorenstein [2], either H has a normal 2-complement or H is a Q-group, a D-group, or a QD-group in the sense of [2]. Since the lemma is obvious in the first case, it suffices to treat the remaining three possibilities.

Suppose first that H is a Q-group. Then by Proposition 2.3.1 of [2] and the fact that $O(H)=1$, we have that $|Z(H)|=2$. Since $\langle B, Z(H)\rangle$ is abelian, we must have $Z(H) \subseteq B$. We number the involutions b_1, b_2, b_3 of B so that $\langle b_1 \rangle = Z(H)$. Since $O(H)=1$, we have $O(C_H(b_1))=1$. Furthermore, $C_H(b_i)=C_H(B)$, $2 \leq i \leq 3$. However, as shown at the beginning of Section 2.4 of [2], $C_H(B)$ has a normal 2-complement X. Thus $O(C_H(b_i)) \subseteq X$ for all i, $1 \leq i \leq 3$, and consequently every element of $\mathrm{M}_0(B; H)$ is contained in X. Since $|X|$ is odd, we conclude that H is strongly B-flat.

Suppose next that H is a D-group. Then by Proposition 2.1.1 of [2], H has a normal subgroup K of index 2 with dihedral Sylow 2-subgroups. Clearly K contains a four group. Since a quasi-dihedral group contains only one conjugacy

4) If G satisfies the hypotheses of Theorem A, one can show directly that $H/O(H)$ is isomorphic to $PSL(3,3)$, M_{11}, $GL(2,3)$, or a Sylow 2-subgroup of H, in which cases it is trivial to verify the strong flatness of H (Cf. Lemma 11.8). Thus reference to the main results of [2] is not essential.

class of four groups, K contains every four subgroup of H and, in particular, contains B. Since $O(C_K(b))=O(C_H(b))$ for b in $B^{\sharp}$, it is clearly enough to prove that K is strongly B-flat. However, this has already been demonstrated in the preceding lemma.

Suppose finally that H is a QD-group. By the third Main Theorem of [2], H contains a normal subgroup K of odd index with $K\cong PSL(3, q)$, $q\equiv -1 (\mathrm{mod}\ 4)$, $PSU(3, q^2)$, $q\equiv 1(\mathrm{mod}\ 4)$ or M_{11}. One verifies directly in each case that B centralizes $O(C_K(b))$. But by the Frattini argument, $H=KC_H(B)$, so certainly B centralizes $O(C_H(b))$ for each b in $B^{\sharp}$ and, as noted above, $C_H(B)$ has a normal 2-complement X. Thus $O(C_H(b))\subseteq X$ for each b in $B^{\sharp}$ and it follows in this case as well that H is strongly B-flat.

We conclude this section with the following property of the groups $PSL^*(3, 4)$ and $PGL^*(3, 4)$.

LEMMA 3.7. *If $H/O(H)$ is isomorphic to $PSL^*(3, 4)$ or $PGL^*(3, 4)$, then H is strongly A-flat.*

PROOF. Once again we may assume $O(H)=1$ and verify that H is strongly B-flat. Hence we may assume that $H=PSL^*(3, 4)$ or $PGL^*(3, 4)$. Correspondingly we set $L=PSL(3, 4)$ or $PGL(3, 4)$, so that L is normal of index 2 in H. We have $H=L\langle t\rangle$, where the automorphism of t on L is induced by the product of the transpose-inverse map of $GL(3, 4)$ and the field automorphism of $GL(3, 4)$ determined by a generator of the Galois group of $GF(4)$.

We note first that L has only one conjugacy class of involutions and the centralizer of an involution of L is 2-closed and has no normal subgroups of odd order. Furthermore, if $L=PSL(3, 4)$, this centralizer is, in fact, a 2-group. These properties of L are easily verified by direct computation.

If $B\subseteq L$, it follows, in particular, that $W_O(B; H)$ is trivial and so certainly H is strongly B-flat. Thus we can assume that $B\not\subseteq L$. One checks also that every involution of $H-L$ is conjugate to t (see Lemma 4.8(iv) and Lemma 5.1(viii)) and so without loss we can assume that $t\in B$.

Regarding t as an automorphism of $GL(3, 4)$, we compute the centralizer of t in $GL(3, 4)$. This is the set of matrices X such that $(X^{\sigma})^{T}X=1$, where X^{σ} is the matrix obtained from X by applying the nontrivial automorphism of $GF(4)$ to the entries of X and where $(X^{\sigma})^{T}$ denotes the transpose of X^{σ}. Thus $X\in GU(3, 4)$. Using this observation, we obtain that $C_L(t)$ is isomorphic to $PGU(3, 4)$ or $PSU(3, 4)$ according as $L=PGL(3, 4)$ or $PSL(3, 4)$. Now $PGU(3, 4)$ is a solvable group of order $8\cdot 27$ in which $SL(2, 3)$ acts faithfully on an elementary abelian group of order 9, while $PSU(3, 4)$ is a normal subgroup of $PGU(3, 4)$ of index 3.

In particular, we see that a Sylow 2-subgroup Q of $C_L(t)$ is quaternion. Since $Q\times\langle t\rangle$ is a Sylow 2-subgroup of $C_H(t)$, it follows that $C_H(t)$ does not contain an elementary abelian subgroup of order 8. However, $B\subseteq C_H(t)$ as $t\in B$. Thus B is a four group and hence condition (ii-a) of Definition 2.3 holds. We proceed now to verify conditions (ii-b) and (ii-c).

We have $B\cap L=\langle z_1\rangle$ is of order 2 and $B=\langle z_1, t\rangle$. Moreover, z_1t and t are conjugate in $N_S(B)$, where S is a Sylow 2-subgroup of H containing B and a Sylow 2-subgroup of $C_H(t)$. Hence $C_H(t)$ and $C_H(z_1t)$ are conjugate by an element of $N_H(B)$ and therefore so are $K_1=O(C_H(t))$ and $K_2=O(C_H(z_1t))$. Furthermore, z_1t inverts K_1 and t inverts K_2, so $K_1\cap K_2=1$. We shall now argue that K_1 and K_2 are the complete set of maximal elements of $\mathit{И}_O(B; H)$, in which case conditions (ii-b) and (ii-c) of Definition 2.3 will follow at once.

Let K be a maximal element of $\mathit{И}_O(B; H)$. We have already noted that $O(C_H(z_1))=1$ as $z_1\in L$. Hence $K=\langle R_1, R_2\rangle$, where $R_1=K\cap K_1$ and $R_2=K\cap K_2$. It will clearly suffice to prove that $R_1=1$ or $R_2=1$; so without loss we can assume that $R\neq 1$ and $R_2\neq 1$. In particular, $|K|\geq 9$. But clearly $K_i\subseteq PSL(3, 4)$, $1\leq i\leq 2$, and therefore also $K\subseteq PSL(3, 4)$. Since a Sylow 3-subgroup of $PSL(3, 4)$ is elementary abelian of order 9, it follows that K is a Sylow 3-subgroup of $PSL(3, 4)$, whence $K=R_1\times R_2$ and $|R_1|=|R_2|=3$. Now set $C=C_H(R_1)$. Then $\langle t\rangle$ is a Sylow 2-subgroup of C, otherwise some involution of $PSL(3, 4)$ would centralize R_1, contrary to the above-noted fact that the centralizer of every involution of $PSL(3, 4)$ is a 2-group. Thus $C=\langle t\rangle O(C)$. Since $PSL(3, 4)$ contains no elements of order 15 or 21, we conclude at once that $O(C)$ is a 3-group. On the other hand, since K_1 is abelian of order 9 and $R_1\subseteq K_1$, we have $K_1\subseteq O(C)$, whence $\langle K_1, R_2\rangle$ is a 3-subgroup of $PSL(3, 4)$ of order at least 27, which is impossible. This completes the proof.

4. Some properties of groups with specified Sylow 2-subgroups. In this section we collect a number of results about groups with Sylow 2-subgroups of order at most 2^7 which we shall need. The discussion will be based in part upon two general concepts which will be used throughout the paper.

Let S be a Sylow p-subgroup of the group G, p a prime, and let K be a conjugacy class of p-elements of G. An element x of $S\cap K$ will be called an *extremal* element of S in G if

$$|C_S(x)|\geq|C_S(y)|$$

for every element y in $S\cap K$.

In this case, it follows directly from Sylow's theorem that $C_S(x)$ is a Sylow

p-subgroup of $C_G(x)$. Furthermore, if $y \in S \cap K$, we can choose g in G such that

$$y^g = x \quad \text{and} \quad C_S(y)^g \subseteq C_S(x) .$$

These facts will be used repeatedly in the paper.

We shall also make considerable use of Glauberman's theorem concerning isolated involutions [9]. An involution x of a Sylow 2-subgroup S of the group G is said to be *isolated* in G if it is not conjugate in G to any other involution of S. If x is an isolated involution of G, Glauberman's theorem asserts that $x \in Z^*(G)$, where $Z^*(G)$ denotes the preimage in G of $Z(\bar{G})$, where $\bar{G} = G/O(G)$. Since $C_G(\bar{x})$ maps onto $C_{\bar{G}}(\bar{x}) = \bar{G}$, it follows in this case that $G = O(G)C_G(x)$.

The following remark is useful: If the group G has a normal subgroup H of index a power of 2 and H has an isolated involution, then so does G. Indeed, we have $G = SH$, where S is a Sylow 2-subgroup of G. We may assume without loss that $O(G) = 1$, whence also $O(H) = 1$. Then $Z(H)$ is a 2-group and, as H has an isolated involution, $Z(H) \neq 1$. Since $Z(H)$ is normal in S, it follows that $Z(S) \cap Z(H)$ contains an involution x. Then $x \in Z(G)$ and so x is an isolated involution of G.

We shall also make considerable use of Thompson's well-known lemma (Lemma 5.38 of [26]) which asserts that if S is a Sylow 2-subgroup of the group G and R is a maximal subgroup of S, then either G has a normal subgroup of index 2 or any involution of $S-R$ is conjugate in G to an involution of R. In some situations we shall actually need the following extension of this result, which is proved by a similar transfer computation (compare Lemma 1 of [20]): If $x_1, x_2, \cdots, x_n$ $(n \geq 1)$ are elements of $S-R$ such that (a) x_i is not conjugate in G to an element of R and (b) $x_i^{2^j}$ is not conjugate in G to an element of $S-R$, $1 \leq i \leq n, j \geq 1$, then G possesses a normal subgroup of index 2 not containing $x_1, x_2, \cdots, x_n$.

We first prove

LEMMA 4.1. *If G is a group with Sylow 2-subgroup S of order* 16 *and G does not contain an isolated involution, then S is either elementary abelian, homocyclic of type* (4.4), *dihedral, quasi-dihedral, or the direct product of a dihedral group with a group of order* 2.

PROOF. Suppose first that S is abelian. Then by a theorem of Burnside, the fusion of S in G is determined in $N_G(S)$. Since G does not contain an isolated involution, Glauberman's theorem implies that $S \cap Z(N_G(S)) = 1$. Since $|S| = 16$, this is possible only if S is elementary abelian or homocyclic of type (4, 4). Hence we can suppose that S is not abelian.

If S is of maximal class, then S is either dihedral, quasi-dihedral, or generalized quaternion by Theorem 5.4.5 of [12]. However, the third case cannot occur, since then the unique involution of S would be isolated in G [6]. Hence we can also assume that S is not of this form.

Suppose next that S' is cyclic and let z be the involution of S'. Since z is not isolated, z is conjugate in G to an involution $x \neq z$ of S. Setting $R = C_S(x)$, we can therefore choose g in G such that $x^g = z$ and $R^g \subseteq S$. If R were nonablelian, then $\Omega_1(R') = \langle z \rangle$ and $\Omega_1((R^g)') = \langle z \rangle$, whence $z^g = z$, which is not the case. Hence R is abelian. If $|R| = 4$, it is easy to see that S must be of maximal class, contrary to our present assumption. Since S is nonabelian, $R \subset S$ and so $|R| = 8$. In particular, $S' \subseteq R$. If R were cyclic, then $\Omega_1(R) = \Omega_1(S')$, whence $x = z$, which is not the case. Thus R is either elementary abelian or of type $(4, 2)$.

Suppose R is elementary abelian. Since S is nonabelian, an element y of $S - R$ induces an automorphism of R of order 2, so we can write $R = R_1 \times R_2$, where R_1 is a four group, R_2 is of order 2, and each is y-invariant. If y can be chosen as in involution, then $\langle R_1, y \rangle$ is dihedral and $S = \langle R_1, y \rangle \times R_2$, so S has one of the forms of the lemma. In the contrary case, $R = \Omega_1(S)$ and so R is weakly closed in S with respect to G. But then two elements of R conjugate in G are conjugate in $H = N_G(R)$. Setting $C = C_G(R)$ and $\bar{H} = H/C$, we have that $|\bar{S}| = 2$ and so $\bar{H}$ has a normal 2-complement. Since $|R| = 8$, $\bar{H}$ is also isomorphic to a subgroup of $GL(3, 2)$. Together these two conditions imply that $\bar{H}$ is nonabelian of order 6. Setting $R_0 = C_R(O(\bar{H}))$, it follows that R_0 is of order 2 and is invariant under $\bar{H}$. But then $R_0 \subseteq Z(H) = Z(N_G(R))$ and consequently the involution of R_0 is isolated in G, contrary to our hypothesis on G.

Assume next that R is of type $(4, 2)$. We have $z \in R$ as $z \in Z(S)$. Hence $\langle z, x \rangle = \Omega_1(R)$ and $|\mho^1(R)| = 2$. But $\mho^1(R) \lhd S$ and so $\mho^1(R) \subseteq Z(S)$. This forces $\mho^1(R) = \langle z \rangle$, since otherwise $\langle z, x \rangle = \langle z, \mho^1(R) \rangle \subseteq Z(S)$ and $x \in Z(S)$, which is not the case. Since $z^g \neq z$, we conclude therefore that $R^g \neq R$. Thus $S = RR^g$. But then $R \cap R^g$ has order 4 and lies in $Z(S)$ as R is abelian. If $Z(S)$ were elementary, then $R = \langle Z(S), x \rangle$ would be elementary, which is not the case, so $Z(S)$ is cyclic of order 4. Since $Z(S) = R \cap R^g \subseteq R^g$, it follows that $\mho^1(R^g) = \langle z \rangle$. Since also $\mho^1(R) = \langle z \rangle$, this yields $z^g = z$, a contradiction.

Finally consider the case that S' is not cyclic, whence $|S'| \geq 4$ and $|S/S'| \leq 4$. Clearly equality must hold and so S is generated by two elements x, y. Furthermore, since S is not of maximal class, its class is 2 and hence $S' = Z(S)$. Thus $S = \langle Z(S), x, y \rangle$ and consequently $S' = \langle [x, y] \rangle$. But $[x, y]$ has order 2 as $S/Z(S)$ is elementary abelian. Hence $|S'| = 2$, which is not the case. This completes the proof.

LEMMA 4.2. *Let G be a group with elementary abelian Sylow 2-subgroup S of order 16. If $N_G(S)/C_G(S)$ has order 3 or 5 and $N_G(S)$ contains an element which acts fixed-point-free on S, then G is solvable of 2-length 1.*

PROOF. Let $x \in S^{\#}$ and set $C=C_G(x)$. Since every element of $N_C(S)$ fixes x, our assumptions imply that $N_C(S) \subseteq C_C(S)$, so C has a normal 2-complement by Burnside's transfer theorem. We conclude, in particular, that the centralizer of every involution of S, and hence of G, is solvable. But now the main result of D. Gorenstein [11] yields the lemma.

We next prove

LEMMA 4.3. *If G is a group with Sylow 2-subgroups isomorphic to $Z_2 \times D_8$, then G has a normal subgroup of index 2 with dihedral Sylow 2-subgroups of order 8.*

PROOF. It will clearly suffice to prove the lemma for $G/O(G)$, since then it will follow at once for G. Hence without loss we can assume to begin with that $O(G)=1$. We argue first that G has a normal subgroup of index 2. Since $O(G)=1$, it follows from a theorem of K. Harada [19] that either this is the case or $D=O_2(G) \neq 1$. By the structure of S, we have $N_G(S)=SC_G(S)$ and $N_G(S)$ has a normal 2-complement. Hence if $D=S$, $G=N_G(S)=SO(G)=S$ and the assertion is obvious; so we may assume $D \subset S$. Setting $\bar{G}=G/D$, we see that $N_{\bar{G}}(\bar{S})=\bar{S} \cdot C_{\bar{G}}(\bar{S})$. Hence if $\bar{S}$ is abelian, Burnside's theorem implies that $\bar{G}$ has a normal 2-complement, in which case G has a normal subgroup of index 2. The only other possibility is that $|D|=2$ and $S=D \times R$, where $R \cong D_8$. Since $D \subseteq Z(G)$, Gaschütz's theorem (see [22], p. 121) yields in this case that $G=D \times L$ for some subgroup L of G and again, G has a normal subgroup of index 2.

Let H be a subgroup of index 2 in G. Then $H \triangleleft G$ and $Q=S \cap H$ is a Sylow 2-subgroup of H and is of order 8. If Q is dihedral, the lemma holds; so we may assume that this is not the case. By the structure of S, Q is then abelian and $Z(S) \subseteq Q$. Since $N_G(S)=SC_G(S)$ distinct involutions of $Z(S)$ are not conjugate in G and hence not in H. Since $|Q|=8$, it follows that $|N_H(Q)/C_H(Q)|$ is not divisible by 7 and so is equal to 1 or 3. Now transfer yields that H has a normal subgroup K of index 2. If K has a normal 2-complement, so does H and hence so does G, in which case the lemma holds. In the contrary case, $S \cap K$ is a four group, $K \triangleleft G$, and G/K is a four group. However, in this case we see that there is a subgroup of index 2 in G containing K with dihedral Sylow 2-subgroups of order 8 and the lemma is proved.

REMARK. Using the classification of groups with dihedral Sylow 2-subgroups, one can give a more exact description of G in the above lemma. To this end, consider $PGL(2, q)$, q odd, and let E denote the cyclic group of automorphisms of it induced from the Galois group of $GF(q)$. Set $E_0=O(E)$ and, if q is a square, let E_1 be the subgroup of E of order $2|O(E_0)|$. Finally denote the semi-direct product of $PGL(2, q)$ and E_0 by $P\Gamma L_0(2, q)$ and the semi-direct product of $PSL(2, q)$ by E_1 (in the case that q is a square) by $P\Gamma L_1(2, q)$.

Using this terminology together with main theorem of J. Walter and D. Gorenstein [15], one obtains the following sharpened form of Lemma 4.3: $\bar{G}=G/O(G)$ is isomorphic to one of the following groups:

(i) a Sylow 2-subgroup of G,

(ii) $A_7\times Z_2$,

(iii) the symmetric group S_7,

(iv) $H\times Z_2$, where H is isomorphic to a subgroup of $P\Gamma L_0(2, q)$ containing $PSL(2, q)$, or

(v) a subgroup of $P\Gamma L_1(2, q)$ containing $PSL(2, q)$.

For G to have a Sylow 2-subgroup of order 16, some additional restrictions must, of course, be imposed on q above. In particular, it is easy to see that, except in the case that $\bar{G}$ is a 2-group, $\bar{G}$ possesses an involution $\bar{t}$ such that $C_{\bar{G}}(\bar{t})$ contains a subgroup isomorphic to $Z_2\times S_4$. Primarily it is this consequence of the lemma that we shall need.

We also require the following result, which is Lemma 2 of [20]:

LEMMA 4.4. *If G is a group with a Sylow 2-subgroup isomorphic to the wreathed product $(Z_2\times Z_2)\int Z_2$, then G contains a normal subgroup of index 2 with an elementary abelian Sylow 2-subgroup of order 16.*

Both a Janko 2-group and a 2-group of type $PSL(3, 4)$ contain a subgroup of order 2^5 given by generators a, b, c and relations

$$(*) \qquad a^4=b^4=c^2=1, \quad [a, b]=1, \quad a^c=a^{-1}, \quad b^c=b^{-1}a^2 .$$

This group is a special 2-group with center $\langle a^2, b^2\rangle$.

We need the following result:

LEMMA 4.5. *If S is a 2-group of the type (*) above, then*

(i) *$\mathrm{Aut}(S)$ is a 2-group;*

(ii) *If S is a Sylow 2-subgroup of the group G, then G contains an isolated involution.*

PROOF. We may assume $S=\langle a, b, c\rangle$, where $\langle a, b, c\rangle$ satisfy (*). Setting $U=\langle a, b\rangle$, we have that $U\cong Z_4\times Z_4$. Furthermore, U is characteristic in S and $\Omega_1(U)=\langle a^2, b^2\rangle=Z(S)$. If α is an automorphism of S of odd order, then $U^\alpha=U$. But $S/\langle a^2\rangle$ and $S/\langle b^2\rangle$ are not isomorphic as is easily checked. Hence α does not cyclically permute the three involutions a^2, b^2, a^2b^2 of $\Omega_1(U)$. Since $|\alpha|$ is odd, α must therefore centralize $\Omega_1(U)$, whence α centralizes U. Since $|S/U|=2$, α also acts trivially on S/U, so α stabilizes the chain $S\supset U\supset 1$. We conclude that α acts trivially on S and hence that $\alpha=1$, thus establishing (i).

To prove (ii), we first argue that no involution of $S-U$ is conjugate to an involution of U. Indeed, any involution of $S-U$ is conjugate in S to c or ca. Suppose then that c or ca is conjugate to an element of U. Since c and ca induce the same automorphism of U, our argument is the same in either case, so for definiteness assume c is conjugate to an element of U and hence of $Z(S)$. Let R be a Sylow 2-subgroup of $C_G(c)$ containing $C_S(c)=\langle Z(S), c\rangle$. Then R is a Sylow 2-subgroup of G and so $Z(S)\cap Z(R)=\langle z\rangle$ is of order 2. If $z=a^2$, then a normalizes, but does not centralize, $Z(R)=\langle a^2, c\rangle$. However, this is impossible since $|N_G(Z(R))/C_G(Z(R))|$ is odd as R is a Sylow 2-subgroup of G. We reach a similar contradiction if $z=b^2$ or a^2b^2.

It follows therefore by the extension of Thompson's fusion lemma described above that G has a normal subgroup H of index 2 not containing c or ca. Since $T=S\cap H$ is a Sylow 2-subgroup of H, we conclude by direct calculation that either $T=U$ or $T=\langle a, bc, b^2\rangle\cong Q_8\times Z_2$. In the latter case, Lemma 4.1 implies that H has an isolated involution. Since $|G:H|=2$, G also contains an isolated involution.

Suppose then that $T=U$, in which case $T\cong Z_4\times Z_4$. Set $\bar{G}=G/O(G)$. Since $O(H)=O(G)$, we have $O(\bar{H})=O(\bar{G})=1$. But now a theorem of Brauer [5] yields that $\bar{T}$ is normal in $\bar{H}$ of index 1 or 3. In the first case, $\bar{G}=\bar{S}$ and G has a normal 2-complement, so certainly G contains an isolated involution. Hence we may assume $|\bar{H}:\bar{T}|=3$. Then $\bar{S}\ntriangleleft\bar{G}$, otherwise by (i) $\bar{G}=\bar{S}C_{\bar{G}}(\bar{S})=\bar{S}O(\bar{G})=\bar{S}$, contrary to our present assumption. Thus $\bar{G}/\bar{T}\cong S_3$ and so the image of $\bar{c}$ in $\bar{G}/\bar{T}$ inverts the Sylow 3-subgroup of $\bar{G}/\bar{T}$. But $\bar{c}$ centralizes $\Omega_1(\bar{T})=\Omega_1(\bar{U})=Z(\bar{S})$, whence $\bar{H}=[\bar{H}, \bar{c}]$ centralizes $\Omega_1(\bar{T})$. Hence $\bar{H}$ centralizes $\bar{T}$, which is not the case.

Our next result is known, but we prove it for completeness.

LEMMA 4.6. *If the group G has an extra-special Sylow 2-subgroup of order 2^n, $n\geq 5$, then G has an isolated involution.*

PROOF. Let S be a Sylow 2-subgroup of G with center $\langle z\rangle$, and suppose by way of contradiction that $a\sim z$ for some a in $S-\langle z\rangle$. We can then choose x in G such that $a^x=z$ and $C_S(a)^x\subseteq S$. But as S is extraspecial, $|S:C_S(a)|=2$. Since $|S|=2^n$ with $n\geq 5$, it follows that $C_S(a)$ is nonabelian. But then $C_S(a)'=S'=\langle z\rangle$ and consequently $z^x=z$, contrary to $a^x=z$.

We conclude the section with some results of a different nature. If S is a

Sylow 2-subgroup of $PSL(3, 4)$, then S can be generated by involutions $z_1, z_2, a_1, a_2, b_1, b_2$ satisfying

$$[a_1, b_1]=[a_2, b_2]=z_1,\ [a_2, b_1]=z_2, \quad \text{and} \quad [a_1, b_2]=z_1z_2 \tag{**}$$

with all remaining commutators of pairs of generators being trivial. In fact, we can represent S as

$$S=\left\{\begin{pmatrix}1 & a & b\\ 0 & 1 & c\\ 0 & 0 & 1\end{pmatrix} \mid a, b, c \in GF(4)\right\}$$

with

$$a_1=\begin{pmatrix}1 & 1 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix},\quad a_2=\begin{pmatrix}1 & x & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix},\quad b_1=\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 1\\ 0 & 0 & 1\end{pmatrix},$$
$$b_2=\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & x^2\\ 0 & 0 & 1\end{pmatrix},\quad z_1=\begin{pmatrix}1 & 0 & 1\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix},\quad z_2=\begin{pmatrix}1 & 0 & x\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}$$

and

$$\{0, 1, x, x^2\}=GF(4)\ .$$

It is not difficult to show that the only elements of odd order in $\mathrm{Aut}(S)$ are of order 3 and that a Sylow 3-subgroup P of $\mathrm{Aut}(S)$ can be represented by the matrices

$$P=\left\{\begin{pmatrix}1 & 0 & 0\\ 0 & x & 0\\ 0 & 0 & y\end{pmatrix} \mid x, y \in GF(4)^{\#}\right\}.$$

In addition, one has

$$\begin{pmatrix}1 & 0 & 0\\ 0 & x & 0\\ 0 & 0 & y\end{pmatrix}^{-1}\begin{pmatrix}1 & a & b\\ 0 & 1 & c\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0 & 0\\ 0 & x & 0\\ 0 & 0 & y\end{pmatrix}=\begin{pmatrix}1 & xa & yb\\ 0 & 1 & x^{-1}yc\\ 0 & 0 & 1\end{pmatrix}.$$

From these results, one obtains immediately the following lemma:

LEMMA 4.7. *The following conditions hold*:

(i) $\mathrm{Aut}(S)$ *is a* $\{2, 3\}$-*group*;

(ii) *a Sylow* 3-*subgroup* P *of* $\mathrm{Aut}(S)$ *is elementary abelian of order* 9;

(iii) P *fixes* $A=\langle z_1, z_2, a_1, a_2,\rangle$ *and* $B=\langle z_1, z_2, b_1, b_2\rangle$; $C_P(S/A)$ *and* $C_P(S/B)$ *are of order* 3 *and act fixed-point-free on* A *and* B *respectively*;

(iv) *If* $P_0=C_P(Z(S))$, *then* P_0 *is of order* 3, P_0 *acts fixed-point-free on* $S/Z(S)=S/\langle z_1, z_2\rangle$, *and* P_0 *normalizes but does not centralize* $\langle a_1, a_2\rangle$.

(v) *If A or $B=Z(S)\times T$ and a nontrivial 3-element of* Aut (S), *normalizes T, then T is conjugate in S to $\langle a_1, a_2\rangle$ or $\langle b_1, b_2\rangle$, respectively.*

LEMMA 4.8. *Let K be a group with Sylow 2-subgroup S of type $PSL(3, 4)$ and assume that K satisfies one of the following conditions:*

(a) $K\cong PSL(3, 4)$ *or* $PGL(3, 4)$

(b) $|K:K'|=1$ *or* 3, $K'\cong A_5\cdot E_{16}^{(2)}$, *and* $O(K)=1$; *or*

(c) $S\lhd K$ *and* $C_K(S)\subseteq S$.

Then we have

(i) *K is isomorphic to a subgroup of $PGL(3, 4)$;*

(ii) *Any subgroup of $PGL(3, 4)$ containing a Sylow 2-subgroup of $PGL(3, 4)$ satisfies one of the above conditions;*

(iii) *If $K\cong PSL(3, 4)$ and K is contained as a normal subgroup of odd index of a group H and if $C_H(K)=1$, then $H\cong PSL(3, 4)$ or $PGL(3, 4)$.*

(iv) *If K is contained as a normal subgroup of index 2 of a group H having Sylow 2-subgroups of Janko type, then H is isomorphic to a subgroup of $PGL^*(3, 4)$. Moreover, if H is nonsolvable, then $H\cong PSL^*(3, 4)$ or $PGL^*(3, 4)$.*

PROOF. If K satisfies (a), then obviously K is isomorphic to a subgroup of $PGL(3, 4)$; while if K satisfies (c), the same holds by the preceding lemma. On the other hand, if K satisfies (b), then for each value of $|K:K'|=1$ or 3, K is uniquely determined up to isomorphism. Moreover, one verifies by direct computation that $G=PGL(3, 4)$ contains a subgroup $A\cong E_{16}$ such that $N_G(A)$ is a split extension of A by $Z_3\times A_5$ with $N_G(A)'\cong A_5\cdot E_{16}^{(2)}$. Hence K is isomorphic to a subgroup of $PGL(3, 4)$ in this case as well. Thus (i) holds.

The subgroups of G containing a Sylow 2-subgroup of G are easily determined. In fact, the centralizer of every involution of G is 2-closed, so we can use Suzuki's classification of such groups [24] to determine them. It follows easily that any such subgroup satisfies condition (a), (b) or (c).

In proving (iii) we may identify K with Inn $(PSL(3, 4))$ and H with a subgroup of Aut $(PSL(3, 4))$. One knows that Aut $(PSL(3, 4))$/Inn $(PSL(3, 4))$ is of order 12 and has a normal Sylow 3-subgroup. Since $|H:K|$ is odd, we have $|H:K|=1$ or 3 and in each case H is uniquely determined, whence $H=K$ or $H=PGL(3, 4)$.

Finally to prove (iv), set $\hat{G}=\mathrm{Aut}(PGL(3, 4))$ and $G=\mathrm{Inn}(PGL(3, 4))$. Then $\hat{G}=G\langle t_1, t_2\rangle$, where t_1 is an involution induced from a field automorphism of $GL(3, 4)$ and t_2 is an involution induced by the transpose-inverse map of $GL(3, 4)$. Furthermore, t_1 and t_2 commute. We set $G_1=\langle t_1\rangle G$, $G_2=\langle t_2\rangle G$ and $G_3=\langle t_1t_2\rangle G$. We let $\hat{S}$ be a Sylow 2-subgroup of $\hat{G}$ containing $\langle t_1, t_2\rangle$ and set $S_i=\hat{S}\cap G_i$, $1\leq i\leq 3$, and $S=\hat{S}\cap G$.

If S is represented as the image of the set of upper triangular matrices $\begin{pmatrix}1 & a & b\\ 0 & 1 & c\\ 0 & 0 & 1\end{pmatrix}$, a, b, c in $GF(4)$, one checks by direct computation that t_1 normalizes an elementary abelian normal subgroup of S of order 16. Thus $SCN_4(S_1)$ is nonempty and so S_1 is not of Janko type inasmuch as such a 2-group contains no elementary abelian normal subgroups of order 8.

Now set $t=t_1t_2$. As we have noted in Section 3, $C_G(t)$ has quaternion Sylow 2-subgroups and so $|C_{Z(S)}(t)|=2$. Hence t does not centralize $Z(S)\cong Z_2\times Z_2$. Furthermore, using the representation of S of the preceding paragraph, one checks also that $|C_{Z(S)}(t_1)|=2$. However, $\mathrm{Aut}(Z(S))\cong S_3$. Since $\langle t_1, t_2\rangle$ normalizes $Z(S)$ and both t_1 and $t=t_1t_2$ do not centralize $Z(S)$, we see that t_2 must centralize $Z(S)$. Hence $Z(S_2)=Z(S)\cong Z_2\times Z_2$. However, a 2-group of Janko type has a center of order 2. Thus S_2 is also not of Janko type.

Finally we consider S_3. We shall prove that, in fact, S_3 is of Janko type. By the structure of S, we see that S has exactly two elementary abelian subgroups A and B of order 16, each of which is normal in S and contains $Z(S)$. If t normalized A, then $C_A(t)$ would contain a four group, contrary to the fact that $C_G(t)$ has a quaternion Sylow 2-subgroup. Hence, $A^t=B$ and $B^t=A$. Now t normalizes, but does not centralize, $Z(S)$. We write $Z(S)=\langle z_1, z_2\rangle$ with $z_1^t=z_1$ and $z_2^t=z_1z_2$. We let $\langle a_1, a_2\rangle$ be a complement of $Z(S)$ in A and hence $\langle a_1^t, a_2^t\rangle$ is a complement of $Z(S)$ in B. We set $b_i=a_i^t$, $1\leq i\leq 2$, $a_3=a_1a_2$, and $b_3=b_1b_2$. By the structure of S, a_i does not centralize b_j and so $[a_i, b_j]\in Z(S)^{\sharp}$, $1\leq i, j\leq 3$. But t centralizes $[a_i, b_i]$ and so $[a_i, b_i]=z_1$, $1\leq i\leq 3$. This in turn implies that $[a_2, b_1]\neq z_1$, since otherwise $[a_3, b_1]=1$. Thus $[a_2, b_1]=z_2$ or z_1z_2. Interchanging z_2 and z_1z_2, if necessary, we can assume without loss that $[a_2, b_1]=z_2$. (This does not affect any of the preceding relations). Conjugating by t, we obtain $[b_2, a_1]=[a_1, b_2]=z_1z_2$. Thus S_3 is generated by the involutions $z_1, z_2, a_1, b_1, a_2, b_2, t$ which satisfy all the defining relations of a group of Janko type and so S_3 is of Janko type, as asserted.

By the notation introduced in Section 1, $G_3=PGL^*(3, 4)$. The argument of the preceding paragraph shows that G_3 does, in fact, have Sylow 2-subgroups of Janko type. But now if K satisfies (a), we can identify H with a subgroup of $\mathrm{Aut}(PSL(3, 4))=\mathrm{Aut}(PGL(3, 4))$ and our argument shows that necessarily $H\subseteq G_3$, whence $H\cong PSL^*(3, 4)$ or $PGL^*(3, 4)$. Likewise if K satisfies (c), we see that H will be isomorphic to a subgroup of $PGL^*(3, 4)$. To complete the proof of (iv), we need only show that K does not satisfy (b). However, in this case $O_2(K)$ would be normal in H and elementary abelian of order 16, whence the

Sylow 2-subgroups of H could not be of Janko type.

Finally we have the following easily verified property of A_5:

LEMMA 4.9. *If H is a subgroup of A_8 with Sylow 2-subgroup of type* (2, 2) *then either H is 2-closed or $H \cong A_5$, $Z_3 \times A_5$, $Z_2 \times S_3$, or $S_3 \times S_3$.*

PART II
Theorem A.

5. **Some properties of S.** In this part of the paper we shall carry out the proof of Theorem A *under the assumption that Theorems* B *and* C *are valid.* Theorems B and C themselves will be proved in Parts III and IV of the paper.

We let G be a group with Sylow 2-subgroup S of Janko type. Thus S is generated by involutions $z_1, z_2, a_1, a_2, b_1, b_2$, and t which satisfy the relations (1) of Section 1. In this section we list without proof a number of properties of S that we shall need. These can all be verified without difficulty by direct computation. They are given by the following omnibus lemma:

LEMMA 5.1. *The following conditions hold:*

(i) $Z(S)=\langle z_1\rangle$, $S'=\Phi(S)=\langle z_2\rangle\times\langle a_1b_1, a_2b_2\rangle\cong Z_2\times Q_8$, $S/\Phi(S)\cong Z_2\times Z_2\times Z_2$, *$\langle z_1, z_2\rangle$ is the unique element of $U(S)$, $SCN_3(S)$ is empty, and S is not connected (see Definition* 2.5).

(ii) *S has precisely two elementary abelian subgroups of order* 16; *namely, $A=\langle z_1, z_2, a_1, a_2\rangle$ and $B=A^t$.*

(iii) *S has precisely seven maximal subgroups. One is isomorphic to $T_1=\langle z_1, z_2, a_1, a_2, b_1, b_2\rangle$, which is of type $PSL(3, 4)$, three to $T_2=\langle z_1, z_2, a_1, b_1, a_2b_2, t\rangle$, and three to $T_3=\langle z_1, z_2, a_1, b_1, a_2b_2, ta_2\rangle$. $Z(T_2)=Z(T_3)=Z(S)$.*

(iv) *T_1 has precisely fifteen maximal subgroups. Six are isomorphic to $U_1=\langle z_1, z_2, a_1, a_2, b_1\rangle\cong(Z_2\times Z_2)\int Z_2$, nine to $U_2=\langle z_1, z_2, a_1, b_1, a_2b_2\rangle$, which is of type* (*) *of Section* 4.

(v) *T_2 has precisely seven maximal subgroups. One is U_2; one is $U_3=\langle z_1, z_2, a_1b_1, a_2b_2, ta_1\rangle$, one is $U_4=\langle z_1, z_2, a_1b_1, a_2b_2, t\rangle\cong D_8*Q_8$ and is characteristic in S, two are isomorphic to $U_5=\langle z_1, z_2, a_1, b_1, t\rangle$ and these are conjugate in S, two to $U_6=\langle z_1, z_2, a_1b_1, a_1a_2b_2, t\rangle\cong Z_4\int Z_2$ and each contains z_1.*

(vi) *T_3 has precisely three maximal subgroups. One is U_2 and two are isomorphic to U_3 (T_3 does not contain U_3).*

(vii) *$\bar{S}=S/\langle z_1\rangle$ is of type A_8; $\bar{U}_4$ is the unique elementary abelian subgroup of $\bar{S}$ of order* 16; *$\bar{S}$ splits over $\bar{U}_4$; and $\langle\bar{a}_1, \bar{a}_2, \bar{b}_1, \bar{b}_2\rangle\cong D_8*D_8$.*

(viii) *The representatives of the conjugacy classes of involutions in S and their cardinalities are:*

z_1	z_2	a_1	a_2	$a_3=a_1a_2$	t
1	2	8	8	8	8

(ix) $C_S(a_1)=C_S(a_2)=C_S(a_3)=A$, $C_S(z_2)=T_1$, $C_S(t)=\langle t, z_1, a_1b_1z_2, a_2b_2z_2\rangle\cong Z_2\times Q_8$, *and* $C_S(t)'=\langle z_1\rangle$.

(x) z_1 *is not the square of an element of* $S-U_4$; *every element of* T_3-U_2 *and of* $U_3-\langle z_1, z_2, a_1b_1, a_2b_2\rangle$ *has order* 8; $U_5-\langle z_1, z_2, a_1b_1, t\rangle$ *contains eight involutions and eight elements of order* 8.

(xi) *If R is an elementary abelian subgroup of S of order* 8, *then either* $C_S(R)=A$ *or* B. *In particular*, $R\subseteq A$ *or* $R\subseteq B$.

(xii) *If R is a subgroup of S containing U_4 properly, then R contains a conjugate of* a_1, a_2 *or* a_3.

We note that the given generators of T_1 above satisfy the same relations as those of equations (**) of Section 4. Hence Lemma 4.7 can be applied to T_1 with $z_1, z_2, a_1, a_2, b_1, b_2$ having the same meanings as in that lemma.

We preserve all the above notation throughout Part II.

6. Fusion of involutions and the structure of $C_G(z_1)$. In this and the next sections we assume that G has no normal subgroups of index 2 and contains no isolated involution and determine the possible fusion pattern of involutions in G. On the basis of this analysis and with the aid of Theorem B, which we are assuming in Part II, we shall then determine the structure of $C_G(z_1)$. We continue the notation introduced in the preceding section and, in addition, we set $N=C_G(z_1)$. We fix this notation throughout. In particular, S has 6 conjugacy classes of involutions, represented by z_1, z_2, t, a_1, a_2 and a_3.

We shall prove

PROPOSITION 6.1. *Either G has one or two conjugacy classes of involutions; and in the latter case,* $z_1\sim z_2\sim t$ *and* $a_1\sim a_2\sim a_3$.

We shall carry out the proof in a sequence of lemmas.

LEMMA 6.2. *t is conjugate to z_2 in $C_G(z_1)$.*

PROOF. By Thompson's fusion lemma t is conjugate in G to an element of the maximal subgroup $T_1=\langle z_1, z_2, a_1, a_2, b_1, b_2\rangle$ of S inasmuch as G has no normal subgroups of 2. By Lemma 5.1 (ix), we see that

$$|C_S(t)|=|C_S(a_1)|=|C_S(a_2)|=|C_S(a_3)|<|C_S(z_2)|<|C_S(z_1)|\ .$$

Hence if t were not conjugate to z_1 or z_2 in G, it would follow that t was conjugate to some a_i and that $C_S(t)$ was a Sylow 2-subgroup of $C_G(t)$. However, this is impossible as $C_S(t)$ and $C_S(a_i)$ are not isomorphic by Lemma 5.1 (ix). Thus t is conjugate to z_1 or z_2 in G.

Suppose next that t is not conjugate to z_1 in G, in which case $C_G(t)$ does not contain a Sylow 2-subgroup of G and t is conjugate to z_2. This implies that T_1 is a Sylow 2-subgroup of $C_G(z_2)$ and so there exists g in G such that

$$t^g=z_2 \quad \text{and} \quad C_S(t)^g \subseteq T_1=C_S(z_2) .$$

Since $C_S(t)'=\langle z_1\rangle$ and $T_1'=\langle z_1, z_2\rangle$, it follows that

$$z_1^g \subseteq \langle z_1, z_2\rangle .$$

This forces $z_1^g=z_1$ as $z_1z_2\sim z_2\sim t\not\sim z_1$ under our present assumptions. Hence $g\in C_G(z_1)$ and the lemma holds in this case.

Now assume $t\sim z_1$ in which case there is an element h of G such that $t^h=z_1$ and $C_S(t)^h\subseteq S$. As in the preceding case, this implies that $z_1^h\subseteq S'$ But $\Omega_1(S')=\langle z_1, z_2\rangle$ by Lemma 5.1 (i). Since clearly $h\in C_G(z_1)$, it follows that $z_1^h=z_2$ or z_1z_2, whence $z_1\sim z_2$. Thus t is also conjugate to z_2 in this case.

Next let R, R_1 be Sylow 2-subgroups of $C_G(t)$ and $C_G(z_2)$ containing $C_S(t)$ and T_1 respectively. Since $t\sim z_1$, R and R_1 are Sylow 2-subgroups of G. Hence by Lemma 5.1 (i), $\Omega_1(R')$ and $\Omega_1(R_1')$ are each four groups. But clearly $\langle t, z_1\rangle\subseteq\Omega_1(R')$, while $\langle z_1, z_2\rangle\subseteq\Omega_1(R_1')$. Hence

$$\langle t, z_1\rangle=\Omega_1(R') \quad \text{and} \quad \langle z_1, z_2\rangle=\Omega_1(R_1') .$$

We can choose an element k in G such that $t^k=z_2$ and $R^k=R_1$, whence also $\langle t, z_1\rangle^k=\langle z_1, z_2\rangle$. It follows that either $z_1^k=z_1$ or z_1z_2. However, $(z_1z_2)^r=z_1$ for some element $r\in R_1$ and also $t^{kr}=z_2$. Since either $z_1^k=z_1$ or $z_1^{kr}=z_1$, we obtain the desired conclusion of the lemma in either case.

For simplicity, set $U=U_4=\langle z_1, z_2, a_1b_1, a_2b_2, t\rangle$. By Lemma 5.1 (v), we have $U\cong D_8*Q_8$. We also set $K=N_G(U)$ and $C=C_G(U)$.

LEMMA 6.3. $K/C\cong A_5\cdot E_{16}^{(1)}$.

PROOF. Since $U\lhd S$ and $\langle z_1\rangle=Z(U)$, we have $S\subseteq K\subseteq N=C_G(z_1)$. Hence if $\bar{N}=N/\langle z_1\rangle$, the preceding lemma implies that $\bar{z}_2$ and $\bar{t}$ are conjugate in $\bar{N}$. But $\bar{U}$ is the unique elementary abelian subgroup of $\bar{S}$ of order 16 by Lemma 5.1 (vii) and so $\bar{U}$ is weakly closed in $\bar{S}$ with respect to $\bar{N}$. Therefore any two elements of $\bar{U}$ which are conjugate in $\bar{N}$ are conjugate in $N_{\bar{N}}(\bar{U})$. This implies that K contains SC properly and also that z_2 and t are conjugate in K.

We use this to prove that K/UC is isomorphic to A_5. Indeed, $\mathrm{Aut}(U)/\mathrm{Inn}(U)$ is isomorphic to the symmetric group S_5 (B. Huppert [22] pp. 356–357, L. Dickson

[7], §§197, 198). Since S/U is a four group, it follows that K/UC is isomorphic to A_4, $Z_2\times D_6$ or A_5. Suppose by way of contradiction that it is isomorphic to A_4 or $Z_2\times D_6$. Correspondingly we have $SC\lhd K$ or $RC\lhd K$, where R is maximal in S. It follows easily from Lemma 5.1 (i) and (viii) that $\langle z_1, z_2\rangle$ is characteristic in S or R respectively, whence $K\subseteq CN_G(\langle z_1, z_2\rangle)$. This inclusion implies that z_2 is conjugate only to involutions of $\langle z_1, z_2\rangle$ in K, contrary to the fact that z_2 is conjugate to t in K. This proves the assertion.

Since $\langle z_1\rangle = Z(U)$, $\langle z_1\rangle$ is a Sylow 2-subgroup of C. Hence if we now set $\bar{K}=K/C$, it follows that $\bar{U}$ is elementary abelian of order 16 and that $\bar{K}/\bar{U}\cong A_5$. Furthermore, $\bar{S}$ splits over $\bar{U}$ by Lemma 5.1 (vii) and so $\bar{K}$ splits over $\bar{U}$ by Gaschütz's theorem (see [22], p. 121). In addition, the involutions $\bar{z}_2$ and $\overline{a_1b_1}$ of $\bar{U}$ are not conjugate in $\bar{K}$, otherwise z_2 or z_1z_2 would be conjugate to a_1b_1 in K. However, this is impossible as z_2, z_1z_2 are involutions, while a_1b_1 has order 4. Thus $\bar{K}$ does not act transitively on $\bar{U}^{\sharp}$ and we conclude now from the definition that $\bar{K}\cong A_5\cdot E_{16}^{(1)}$.

As a consequence, we obtain

LEMMA 6.4. *$K/O(K)$ is isomorphic to a split extension of D_8*Q_8 by A_5.*

PROOF. We set $\bar{K}=K/O(K)$. Since $\langle z_1\rangle$ is a Sylow 2-subgroup of C, C has a normal 2-complement. But $O(C)\lhd K$ as $O(C)$ char $C\lhd K$. Hence $O(C)\subseteq O(K)$ and so $O(\bar{C})=1$. Thus $\bar{C}=\langle\bar{z}_1\rangle$ and $\bar{U}\bar{C}=\bar{U}$. It follows now from the preceding lemma that $\bar{K}/\langle\bar{z}_1\rangle=\bar{U}\bar{L}/\langle\bar{z}_1\rangle$, where $\bar{L}\supseteq\langle\bar{z}_1\rangle$, $\bar{U}\cap\bar{L}=\langle\bar{z}_1\rangle$, and $\bar{L}/\langle\bar{z}_1\rangle\cong A_5$. If $\bar{L}$ splits over $\langle\bar{z}_1\rangle$, then $\bar{K}$ splits over $\bar{U}$ and the lemma holds. However, in the contrary case, the results of I. Schur imply that $\bar{L}\cong SL(2,5)$, in which case $\bar{S}-\bar{U}$ contains an element $\bar{x}$ such that $\bar{x}^2=\bar{z}_1$. But then $S-U$ would contain an element x such that $x^2=z_1$, contrary to Lemma 5.1 (x).

We also have

LEMMA 6.5. *U contains exactly ten noncentral involutions and they are all conjugate on K.*

PROOF. Since $U\cong D_8*Q_8$, U has exactly ten noncentral involutions. A 5-element of $K-C$ acts regularly on $U/\langle z_1\rangle$ and so permutes these involutions in two orbits of length five. Since the noncentral involutions z_2 and z_1z_2 of U are conjugate in S and have the same image in $U/\langle z_1\rangle$, we conclude at once that K acts transitively on the noncentral involutions of U.

We next prove

LEMMA 6.6. *The involutions a_1, a_2, a_3 of S are conjugate in K.*

PROOF. By Lemma 6.4, $N_K(S)$ contains a 3-element x with $x\in C$. By Lemma 5.1 (iii), T_1 char S and so x normalizes T_1. Since x centralizes z_1 and normalizes $Z=\langle z_1, z_2\rangle=Z(T_1)$, x centralizes $Z(T_1)$. We conclude now from Lemma 4.7 (iv) that a suitable conjugate of x in $N_K(T_1)$ normalizes but does not centralize $\langle a_1, a_2\rangle$; and the lemma follows.

Finally we prove

LEMMA 6.7. *Any element of* $A=\langle z_1, z_2, a_1, a_2\rangle$ *which is conjugate to* z_1 *in* G *is conjugate to* z_1 *in* $N_G(A)$.

PROOF. Suppose $a^g=z_1$ for some a in A and g in G. Since A is abelian, $A\subseteq C_G(a)$ and so without loss we can assume that $A^g\subseteq S$. Application of Lemma 5.1 (ii) now yields that either $A^g=A$ or $A^g=A^t$. In the first case, $g\in N_G(A)$, while in the second case, $gt\in N_G(A)$. Since $a^{gt}=z_1^t=z_1$, the lemma follows in either case.

We can now easily complete the proof of Proposition 6.1. By Lemmas 6.2 and 6.6, We have

$$(1) \qquad z_2\sim t \quad \text{and} \quad a_1\sim a_2\sim a_3 .$$

If $z_1\sim z_2$, then G has either one or two conjugacy classes of involutions and in the latter case, these satisfy the fusion pattern asserted in the proposition. Hence we can suppose that $z_1\not\sim z_2$. However, by our assumptions on G, no involution of S is isolated in G and so by Glauberman's theorem [9], z_1 is conjugate to some noncentral involution of S. Our conditions force $z_1\sim a_1$. But now using (1) together with the fusion of involutions within S, it follows that the fifteen involutions of A divide into two conjugacy classes C_1, C_2 in G with $C_1=\{z_2, z_1z_2\}$ and $|C_2|=13$. Lemma 6.7 now yields that C_1, C_2 are the conjugacy classes of involutions of A within $N_G(A)$. However, $N_G(A)/C_G(A)$ is isomorphic to a subgroup of $GL(4, 2)$. Since $|GL(4, 2)|$ is not divisible by 13, we reach a contradiction at once.

We shall now determine the structure of $N/O(N)$. We prove

PROPOSITION 6.8. *$N/O(N)$ is isomorphic to a split extension of D_8*Q_8 by A_5.*

PROOF. It will suffice to prove that $O(N)U\lhd N$. Indeed, assume this is the case. Then by the Frattini argument, $N=O(N)(N\cap K)$, where, as above, $K=N_G(U)$. But $K\subseteq N$ as $\langle z_1\rangle=Z(U)$ and so $N=O(N)K$. This implies that $O(K)\subseteq O(N)$, whence $N/O(N)\cong K/O(K)$ and the proposition now follows from Lemma 6.4.

Setting $\bar{N}=N/O(N)\langle z_1\rangle$, it will therefore be enough to show that $\bar{U}$ is normal in $\bar{N}$. We have that $\bar{S}$ is of type A_8 by Lemma 5.1 (vii). Moreover, it is immediate that $\bar{K}=N_{\bar{N}}(\bar{U})$ and that $\overline{O(K)}=O(N_{\bar{N}}(\bar{U}))$. It follows therefore from Lemmas 6.3 and 6.4 that $N_{\bar{N}}(\bar{U})/O(N_{\bar{N}}(\bar{U}))\cong A_5\cdot E_{16}^{(1)}$. We see then that $\bar{N}$ satisfies condition (b) of Theorem B. Suppose $\bar{N}$ also satisfies condition (a) of the theorem. Since we are proving Theorem A on the assumption that Theorems B and C hold, we can apply Theorem B to conclude that $\bar{U}=O(\bar{N})\bar{U}\lhd\bar{N}$, as required.

Thus to complete the proof, it remains only to show that $\bar{N}$ has three conjugacy classes of involutions. Since $A_5 \cdot E_{16}^{(1)}$ has exactly three conjugacy classes of involutions, as is easily checked, so does $\bar{K}$ and representatives of these can be taken as $\bar{z}_2, \overline{a_1 b_1}$ and $\bar{a}_1$. (Here we have used the previously noted fact that z_2 is an involution, while $a_1 b_1$ has order 4.) Since $a_1 b_1, a_1 b_1 z_1$ are both of order 4, while z_2, a_1 are involutions, we see that neither $\bar{z}_2$ nor $\bar{a}_1$ is conjugate to $\overline{a_1 b_1}$ in $\bar{N}$. We argue finally that also $\bar{a}_1$ and $\bar{z}_2$ are not conjugate in $\bar{N}$. Since $\bar{z}_2 \in Z(\bar{S})$, it follows in the contrary case that $\bar{a}_1^{\bar{x}} = \bar{z}_2$ and $C_{\bar{S}}(\bar{a}_1)^{\bar{x}} \subseteq \bar{S}$ for some element $\bar{x}$ of $\bar{N}$. Then $\bar{x}$ maps $C_{\bar{S}}(\bar{a}_1)' = \langle \bar{z}_2 \rangle$ into $\bar{S}' = \langle \bar{z}_2, \overline{a_1 b_1}, \overline{a_2 b_2}, \rangle$. However, the only involutions in the inverse image of $\bar{S}'$ in S are $z_2, z_1 z_2$ and z_1. This forces $\bar{z}_2^{\bar{x}} = \bar{z}_2$, contrary to the fact that $\bar{a}_1^{\bar{x}} = \bar{z}_2$, and the proposition is proved.

Proposition 6.8 has a number of direct consequences we shall need in our subsequent analysis, and which we combine into the following proposition:

PROPOSITION 6.9. *The following conditions hold:*

(i) *N does not involve $SL(2, q), q \geq 5$, and $PSL(2, q), q > 5$, and $N/O(N)$ does not contain subgroups isomorphic to $Z_2 \times Z_2 \times PSL(2, q), q \geq 5$, or to $Z_2 \times S_4$.*

(ii) *Any A-invariant subgroup of N of odd order lies in $O(N)$.*

(iii) *The three maximal subgroups of S isomorphic to T_2 are conjugate in $N_N(S)$.*

(iv) *If $\bar{N} = N/O(N)$, then $N_{\bar{N}}(\bar{A}) = \bar{T}_1 \bar{X}$, where $\bar{X}$ is of order 3, $\bar{X}$ normalizes both $\bar{T}_1$ and $\langle \bar{a}_1, \bar{a}_2 \rangle$, and $C_{\bar{N}}(\bar{X}) = \bar{X} \times \langle \bar{z}_1, \bar{z}_2, \bar{t}' \rangle$, where $\bar{t}' \sim \bar{t}$ in $\bar{S}$.*

(v) *If x is a 3-element of $N - O(N)$ and K is a proper normal subgroup of N containing $O(N)$, then the image of x in N/K normalizes, but does not centralize, some four subgroup of N/K and the normal closure of x in N covers N/K.*

(vi) *If H is a non 2-constrained subgroup of N, then $H/O(H) \cong Z_2 \times A_5$ or A_5.*

PROOF. If N involves $SL(2, q), q \geq 5$, or $PSL(2, q), q > 5$, then clearly so does $\bar{N} = N/O(N)$. Since $\bar{N}/O_2(\bar{N}) \cong A_5$, obviously $\bar{N}$ does not involve $PSL(2, q)$ or $SL(2, q)$ for $q > 5$. Suppose that $\bar{N}$ contains subgroups $\bar{H}, \bar{K}$ with $\bar{K} \lhd \bar{H}$ and $\bar{H}/\bar{K} \cong SL(2, 5)$. Then $\bar{H}$ contains a normal subgroup $\bar{L}$ with $\bar{K} \subset \bar{L}$ such that $\bar{H}/\bar{L} \cong PSL(2, 5) \cong A_5$. Furthermore, $\bar{N} = O_2(\bar{N})\bar{H}$ and consequently $\bar{K} \subset \bar{L} \subseteq O_2(\bar{N}) = \bar{U}_4$. Thus $\bar{K}, \bar{L}$ are subgroups of $\bar{U}_4$ that are invariant under a Sylow 5-subgroup of $\bar{N}$ and $|\bar{L} : \bar{K}| = 2$. However, by the action of A_5 on $\bar{U}_4$, the only such subgroups of $\bar{U}_4$ are $1, \langle \bar{z}_1 \rangle$, and $\bar{U}_4$. Hence we must have $\bar{K} = 1$ and $\bar{L} = \langle \bar{z}_1 \rangle$, whence $\bar{H} \cong SL(2, 5)$. Without loss we can suppose that $\bar{S} \cap \bar{H}$ is a Sylow 2-subgroup of $\bar{H}$. Thus $\bar{S} \cap \bar{H}$ is a quaternion group and so contains an element $\bar{x}$ such that $\bar{x}^2 = \bar{z}_1$, which contradicts Lemma 5.1 (x).

Assume next that $\bar{N}$ contains a subgroup $\bar{H} \cong Z_2 \times Z_2 \times PSL(2, q), q \geq 5$. As above, we must have $q=5$. But if $\bar{x}$ is an element of $\bar{N}$ of order 5, then $C_{\bar{N}}(\bar{x})=\langle\bar{x}\rangle\times\langle\bar{z}_1\rangle$ and so $\bar{x}$ does not centralize a four subgroup of $\bar{N}$. Clearly then no such subgroup $\bar{H}$ of $\bar{N}$ can exist.

To complete the proof of (i), it remains to show that $\bar{N}$ does not contain a subgroup $\bar{H}\cong Z_2\times S_4$. Indeed, assume $\bar{H}$ is such a subgroup of $\bar{N}$. Since $O_2(\bar{N})=\bar{U}_4\cong Q_8*D_8$ does not contain a subgroup isomorphic to $Z_2\times Z_2\times Z_2$ and since $\bar{N}/O_2(\bar{N})\cong A_5$, $\bar{H}\cap O_2(\bar{N})\cong Z_2$ or $Z_2\times Z_2$. Correspondingly the image of $\bar{H}$ in $\bar{N}/O_2(\bar{N})$ is isomorphic to S_4 or $Z_2\times S_3$. However, A_5 does not possess subgroups of either of these forms.

(ii) is immediate from the structure of $\bar{N}$. As for (iii), Proposition 6.8 implies that $|N_N(S):SC_N(S)|=3$. Let x be a 3-element of $N_N(S)-C_N(S)$. By Lemma 5.1 (i), $S/\Phi(S)\cong Z_2\times Z_2\times Z_2$. Clearly $T_1/\Phi(S)$ is a four subgroup of $S/\Phi(S)$, and it is left invariant by x as T_1 is characteristic in S. It follows that x permutes the maximal subgroups of $S/\Phi(S)$, other than $T_1/\Phi(S)$, in orbits of length 3 and hence permutes the maximal subgroups of S, other than T_1, in orbits of length 3. Thus (iii) holds.

To prove (iv), observe first that as $SCN_3(S)$ is empty, $A \ntriangleleft S$. Since $A\triangleleft T_1$, it follows that $\bar{T}_1$ is a Sylow 2-subgroup of $N_{\bar{N}}(\bar{A})$. Furthermore, if x is as in (iii), $\bar{x}$ normalizes $\bar{T}_1$ as T_1 is characteristic in S. But then $\bar{x}$ normalizes $\bar{A}$ by Lemma 4.7 (iii). Since $\bar{A}/\langle\bar{z}_1\rangle$ is elementary abelian of order 8, any 5-element of $\bar{N}$ which normalizes $\bar{A}$ necessarily centralizes it. Hence $|N_{\bar{N}}(\bar{A})|$ is not divisible by 5 and we conclude that $N_{\bar{N}}(\bar{A})=\bar{T}_1\bar{X}$, where $\bar{X}=\langle\bar{x}\rangle$. Replacing $\bar{X}$ by a suitable conjugate in $N_{\bar{N}}(\bar{A})$, we can assume without loss, in view of Lemma 4.7 (v), that $\bar{X}$ normalizes $\langle\bar{a}_1,\bar{a}_2\rangle$. Since $\bar{X}$ centralizes $\bar{z}_1$ and normalizes $Z(\bar{T}_1)=\langle\bar{z}_1,\bar{z}_2\rangle$, $\bar{X}$ centralizes $\langle\bar{z}_1,\bar{z}_2\rangle$. But $\bar{X}$ also normalizes, but does not centralize $\bar{U}_4$. Since $\bar{U}_4/\langle\bar{z}_1\rangle\cong E_{16}$, we conclude at once that $|C_{\bar{U}_4}(\bar{X})|=8$. Furthermore, $\bar{U}_4$ contains 10 noncentral involutions and $\bar{X}$ centralizes two of them; namely, $\bar{z}_2$ and $\bar{z}_1\bar{z}_2$. Hence it centralizes at least one more involution $\bar{t}'\neq\bar{z}_1$. By Lemma 5.1 (viii) $\bar{t}'\sim\bar{t}$ in $\bar{S}$. Thus $C_{\bar{U}_4}(\bar{X})=\langle\bar{z}_1,\bar{z}_2,\bar{t}'\rangle$, where $\bar{t}'\sim\bar{t}$ in $\bar{S}$. Finally the image $\bar{X}$ in $\bar{N}/\bar{U}_4=\bar{N}/O_2(\bar{N})$ is self-centralizing and consequently $C_{\bar{N}}(\bar{X})=C_{\bar{U}_4}(\bar{X})$. Thus all parts of (iv) are proved.

Next let x and K be as in (v). By the structure of $\bar{N}=N/O(N)$, we see that either $K=O(N)$, $K=O(N)\langle z_1\rangle$, or $K=O(N)U_4$. In the first case, $N/K=\bar{N}$ and $\bar{x}$ normalizes a conjugate of $\bar{S}$, which without loss we may assume to be $\bar{S}$ itself. Then, as above, $\bar{x}$ normalizes both $\bar{T}_1$ and $\bar{A}$. By Lemma 4.7, $\bar{x}$ does not centralize $\bar{A}$ and so normalizes but does not centralize, a four subgroup of $\bar{A}$ and hence of N/K. In the remaining two cases $N/K\cong A_5\cdot E_{16}^{(1)}$ or A_5 and the corresponding assertion is clear. Since no proper normal subgroup of $\bar{N}$ contains $\bar{x}$, the final assertion of (v) is obvious.

Finally let H be a non-2-constrained subgroup of N. Then the image $\bar{H}$ of H in $\bar{N}$ is also not 2-constrained and so is certainly non-solvable. Hence $\bar{H}/O_2(\bar{H}) \cong \bar{N}/O_2(\bar{N}) \cong A_5$ and $C_{\bar{H}}(O_2(\bar{H})) \not\subseteq O_2(\bar{H})$. Thus a nontrivial 5-element of $\bar{H}$ centralizes $O_2(\bar{H})$ and consequently $O_2(\bar{H})=1$ or $\langle z_1 \rangle$. Since $\bar{N}$ does not involve $SL(2, 5)$ by (i), we conclude that $\bar{H} \cong A_5$ or $Z_2 \times A_5$, proving (vi).

Our analysis also yields

PROPOSITION 6.10. *If a is an involution of G which is not conjugate to z_1, then a Sylow 2-subgroup of $C_G(a)$ is elementary abelian of order* 16.

PROOF. By Proposition 6.1, if $a \not\sim z_1$, then $a \sim a_1$ and hence $C_G(a)$ contains a conjugate of A. If R is a Sylow 2-subgroup of $C_G(a)$, it follows that $|R| \geq 16$ and if equality holds, then R is elementary abelian. However, in the contrary case, it follows from Lemma 5.1 (viii) and (ix) that $a \sim z_1, z_2$ or $z_1 z_2$, whence $a \sim z_1$ by Proposition 6.1, contrary to assumption.

7. **Preliminary results on the subgroup structure of** G. We shall ultimately give a complete description of the structure of the proper subgroups of G (when G is a minimal counterexample to the theorem). Here we shall derive some necessary preliminary results.

We first prove

PROPOSITION 7.1. *Let $Z=\langle z_1, z_2 \rangle$. Then we have*

(i) *$T_1 O(N_G(Z))$ is a normal subgroup of $N_G(Z)$ of index* 18.

(ii) *$N_G(Z)/C_G(Z)$ is of order* 6.

PROOF. Set $M=N_G(Z)$ and $L=C_G(Z)$, so that S and T_1 are Sylow 2-subgroups of M and L respectively. By Proposition 6.1, $z_2 \sim z_1$. Since $T_1 \subseteq C_G(z_2)$, we can choose $g \in G$ such that $z_2^g = z_1$ and $T_1^g \subseteq S$. By Lemma 5.1 (iii), $T_1^g = T_1$ and so $g \in N_G(T_1)$. Since $Z=Z(T_1)$, $g \in M$. Clearly $g \notin SL$ and so $M \supset SL$, whence $|M/L|=6$, proving (ii).

To establish (i), it will suffice to show that L is solvable of 2-length 1. Indeed, if this is the case, then $O(L)T_1$ is characteristic in L and hence is normal in M. But also $O(L) \lhd M$ and so $O(L) \subseteq O(M)$. On the other hand, clearly $O(M) \subseteq L$, whence $O(M) \subseteq O(L)$. Thus $O(L)=O(M)$ and consequently $O(M)T_1 \lhd M$. Furthermore, by Lemma 6.4, there is a 3-element x in $K-C=N_G(U_4)-C_G(U_4)$ which normalizes S and centralizes z_1. Then x normalizes T_1 by Lemma 5.1 (iii) and so x normalizes Z. Since x centralizes z_1, it centralizes Z and hence $x \in L$. Thus $[N_L(T_1): C_L(T_1)] \geq 3$. But now Lemma 4.7 (iv) yields that $[N_L(T_1): C_L(T_1)]=3$. Since $O(M)T_1 \lhd L$, it follows that $|L : O(M)T_1|=3$. Combining this with (ii), we conclude that $|M:O(M)T_1|=18$.

We now prove that L is solvable of 2-length 1. Lemma 4.7 (iv) also implies that the element x above acts fixed-point-free on T_1/Z. Hence if we set $\bar{L}=L/Z$, we see that $\bar{T}_1$ is a Sylow 2-subgroup of $\bar{L}$, $\bar{T}_1$ is elementary abelian of order 16, $[N_{\bar{L}}(\bar{T}_1):C_{\bar{L}}(\bar{T}_1)]=3$, and $\bar{x}$ acts fixed-point-free on $\bar{T}_1$. Hence $\bar{L}$ is solvable of 2-length 1 by Lemma 4.2 and we conclude at once that also L is solvable of 2-length 1.

We next analyze the normalizer of A and B in G.

PROPOSITION 7.2. *$N_G(A)/C_G(A)$ is isomorphic to $Z_3\times A_5$ or $Z_3\times A_4$ according as the number of conjugacy classes of involutions in G is one or two. Furthermore, $N_G(A)/O(N_G(A))$ splits over the image of A. Similar statements hold for $N_G(B)$.*

PROOF. Since $B=A^t$, it obviously will suffice to establish the proposition for A. We set $H=N_G(A)$ and $C=C_G(A)$. As a consequence of Proposition 6.1, the element z_1 of A has either 15 or 3 conjugates in G which lie in A. Correspondingly z_1 has 15 or 3 conjugates in H which lie in A by Lemma 6.7. Thus $[H:H\cap N]=15$ or 3. Furthermore, by Proposition 6.9 (iv), $|H\cap N|=3\cdot 64|H\cap O(N)|$, a Sylow 2-subgroup of $H\cap N$ is T_1, and a 3-element of $H\cap N$ normalizes T_1 and does not centralize T_1/A. Since $H\cap O(N)$ clearly centralizes A, we also have $H\cap O(N)\subseteq C$. We therefore conclude that

$$|H/C|=15\cdot 3\cdot 4 \quad \text{or} \quad 3\cdot 3\cdot 4\,.$$

But H/C is isomorphic to a subgroup of $GL(4,2)\cong A_8$. By the preceding paragraph H/C is not 3-closed. Since T_1/A is a four group, a Sylow 2-subgroup of H/C is a four group and it follows now directly from Lemma 4.9 that $H/C\cong Z_3\times A_5$ or $Z_3\times A_4$, respectively. Since T_1 splits over A, $N_G(A)/O(N_G(A))$ does as well by Gaschütz's theorem.

As a corollary we have

LEMMA 7.3. *If R is an elementary abelian subgroup of G of order 8, then $|N_G(R)/C_G(R)|$ is not divisible by 7.*

PROOF. Replacing R by a suitable conjugate, if necessary, we may assume without loss that $C_S(R)$ is a Sylow 2-subgroup of $C_G(R)$. By Lemma 5.1 (xi), $C_S(R)=A$ or B, say $C_S(R)=A$ for definiteness. Hence if $H=N_G(R)$ and $C=C_G(R)$, we have that A is a Sylow 2-subgroup of C and hence $H=CN_H(A)$ by the Frattini argument. By Proposition 7.2, $|N_H(A)/C_H(A)|$ is not divisible by 7. Since $C_H(A)\subseteq C$, it follows that $|H/C|$ is not divisible by 7, as asserted.

We also have

LEMMA 7.4. *G does not contain a subgroup H such that $H/O(H)\cong PSL(2, q_1)\times PSL(2, q_2)$ with q_i odd and $q_i\geq 5, 1\leq i\leq 2$.*

PROOF. We can assume without loss that $S\cap H$ is a Sylow 2-subgroup of H. Then $S\cap H$ is elementary abelian of order 16 and so $S\cap H=A$ or B, by Lemma 5.1 (ii); say, $S\cap H=A$, for definiteness.

If G has only one conjugacy class of involutions, we know from Lemma 6.7 that the involutions of A are all conjugate in $N_G(A)$. But then replacing H by a conjugate by a suitable element of $N_G(A)$ we can assume without loss that the image of z_1 in $\bar{H}=H/O(H)$ lies in one of the two factors $\bar{L}_1$ or $\bar{L}_2$ in $\bar{H}$. If G has two conjugacy classes of involutions, we claim that this condition necessarily holds. Indeed, by the structure of $\bar{H}$, we have $\bar{A}\cap\bar{L}_i=\bar{A}_i$ is a four group, $N_{\bar{L}_i}(\bar{A})\cong A_4, 1\leq i\leq 2$, and $N_{\bar{H}}(\bar{A})=N_{\bar{L}_1}(\bar{A}_1)\times N_{\bar{L}_2}(\bar{A}_2)$. If $\bar{z}_1\notin\bar{A}_1$ or $\bar{A}_2$, then clearly $\bar{z}_1$ would have 9 conjugates in $\bar{A}$ under the action of $N_{\bar{H}}(\bar{A})$. But then z_1 would have 9 conjugates in A. However, as we have already noted in Proposition 7.2, Proposition 6.1 implies in this case that z_1 has only 3 conjugates in A. Thus $\bar{z}_1\in\bar{A}_i\subseteq\bar{L}_i$, $i=1$ or 2, as asserted.

For definiteness, assume $\bar{z}_1\in\bar{L}_1$. Since $C_H(z_1)$ maps onto $C_{\bar{H}}(\bar{z}_1)$, it follows that $C_H(z_1)=H\cap N$ contains a subgroup K such that $K/O(K)\cong Z_2\times Z_2\times PSL(2,q_2)$. However, since $K\subseteq N$ and $q_2\geq 5$, this conflicts with Proposition 6.9 (i).

In the next three lemmas we use the extension of Thompson's fusion lemma stated in Section 4.

LEMMA 7.5. *If H is a subgroup of G having a Sylow 2-subgroup isomorphic to $U_3=\langle z_1, z_2, a_1b_1, a_2b_2, ta_1\rangle$, then H contains an isolated involution.*

PROOF. Let R be a Sylow 2-subgroup of H. As in the case of U_2 in Lemma 4.5, the proof depends only upon the structure of R and not upon the embedding of H in G. Without loss, we may assume that $O(H)=1$.

Since $R\cong U_3$, R contains a subgroup $W\cong\langle z_1, z_2, a_1b_1, a_2b_2\rangle\cong Z_2\times Q_8$. By Lemma 5.1 (x), every element of $R-W$ has order exactly 8. Since every element of W has order at most 4, it follows therefore from the extension of Thompson's fusion lemma that H contains a normal subgroup K of index 2 with Sylow 2-subgroup W. We have $O(K)\subseteq O(H)$ and hence $O(K)=1$. By Lemma 4.1 and the structure of W, K has an isolated involution. As noted at the beginning of Section 4, this implies that H also has an isolated involution.

LEMMA 7.6. *If H is a subgroup of G having a Sylow 2-subgroup isomorphic to $U_5=\langle z_1, z_2, a_1, b_1, t\rangle$, then H contains an isolated involution.*

PROOF. In this case, we use the embedding of H in G to establish our lemma. Replacing H by a suitable conjugate in G, if necessary, we may assume that $R=S\cap H$ is a Sylow 2-subgroup of H. By Proposition 6.9 (iii), the three maximal subgroups of S isomorphic to T_2 are conjugate in $N_G(S)$. But by Lemma 5.1 (iv), (v) and (vi), R lies in one of these three maximal subgroups. Hence without loss we may assume that $R\subset T_2$. Furthermore, T_2 contains two subgroups isomorphic to U_5 and these are conjugate in S by Lemma 5.1 (v). Hence without loss we may also assume that $R=U_5$.

Set $W=\langle z_1, z_2, a_1b_1, t\rangle$. Then W is a central product of Z_4 and Q_8; that is, $W=\langle a_1b_1z_2\rangle\cdot\langle a_1b_1, tz_2\rangle\cong Z_4*Q_8$. By Lemma 5.1 (x), $R-W$ contains eight elements of order 8 and eight involutions. We shall argue that H possesses a normal subgroup of index 2 which contains none of the elements of $R-W$ of order 8. Consider the element ta_1 which is of order 8. Hence ta_1 is not conjugate to an element of W, since the elements of W have order at most 4, and the element $(ta_1)^2$ of order 4 in W is not conjugate to an element of $R-W$. Therefore by the extended fusion lemma, either H contains a normal subgroup K of index 2 with $ta_1\in K$ or else $(ta_1)^4=z_1$ is conjugate to an involution of $R-W$.

Consider the latter possibility. Representatives of the conjugacy classes of involutions of $R-W$ in R are a_1 and a_1z_2. Suppose $z_1\sim a_1$ in H. Put $V=C_R(a_1)$, so that by Lemma 5.1 (ix), $V=\langle z_1, z_2, a_1\rangle$. We can choose an element x in H such that $a_1^x=z_1$ and $V^x\subseteq R$. Now R contains exactly two elementary abelian subgroups of order 8: namely, V and $V'=\langle z_1, z_2, b_1\rangle$. Replacing x by xt, if necessary, we may assume that x normalizes V. We claim that $N_H(V)/C_H(V)\cong S_3$. Indeed, as $SCN_3(R)$ is empty and R is a Sylow 2-subgroup of H, $N_H(V)$ does not contain a Sylow 2-subgroup of H. Hence $N_R(V)=\langle V, b_1\rangle$ of order 16 is a Sylow 2-subgroup of H, and so $N_H(V)/C_H(V)$ has a normal 2-complement. Furthermore, this factor group does not have order divisible by 7 by Lemma 7.3. Since b_1 centralizes $\langle z_1, z_2\rangle$ and conjugates a_1 into a_1z_1, b_1 and x do not commute $(\mathrm{mod}\ C_H(V))$. Hence $N_H(V)/C_H(V)\cong S_3$, as asserted.

We shall now contradict this conclusion. By Lemma 5.1 (xi), A is a Sylow 2-subgroup of $C_G(V)$, $V\lhd T_1$, and T_1/A is a four group. Since $SCN_3(S)$ is empty by Lemma 5.1 (i), it follows that T_1 is a Sylow 2-subgroup of $N_G(V)$ and consequently a Sylow 2-subgroup of $N_G(V)/C_G(V)$ is a four group. However, this factor group is a subgroup of $GL(3,2)\cong PSL(2,7)$ and so it must be isomorphic to $Z_2\times Z_2$ or to A_4. Since neither of these groups contains a subgroup isomorphic to S_3, we obtain the desired contradiction. Since also $C_R(a_1z_2)=V$, we reach a similar contradiction if $z_1\sim a_1z_2$ in H.

An entirely analogous argument applies to each of the remaining seven elements of order 8 in $R-W$. We conclude therefore from the extended fusion lemma that H possesses a normal subgroup K of index 2 which contains none of the elements of order 8 in $R-W$, as asserted.

We have that $R\cap K$ is a maximal subgroup of $R=U_5$ and that $R\cap K$ contains none of the elements of order 8 in $R-W$. One checks directly that either $R\cap K=\langle z_1, z_2, a_1, b_1\rangle\cong Z_2\times D_8$ or $R\cap K=W\cong Z_4*Q_8$. In the latter case, Lemma 4.1 implies that K has an isolated involution, whence H does as well. In the remaining case, either K has a normal 2-complement and hence H contains an isolated involution or else, by the remark following Lemma 4.4, $\bar{K}=K/O(K)$ contains an involution $\bar{u}$ such that $C_{\bar{K}}(\bar{u})$ contains a subgroup isomorphic to $Z_2\times S_4$. Let u be an involution of K whose image is $\bar{u}$. Since a Sylow 2-subgroup of $C_K(u)$ is nonabelian, $u\sim z_1$ in G by Proposition 6.10. However, $N/O(N)$ does not contain a subgroup isomorphic to $Z_2\times S_4$ by Proposition 6.9 (i) and we reach a contradiction. This completes the proof of the lemma.

LEMMA 7.7. *If H is a subgroup of G having a Sylow 2-subgroup isomorphic to $T_2=\langle z_1, z_2, a_1, b_1, a_2b_2, t\rangle$ or $T_3=\langle z_1, z_2, a_1, b_1, a_2b_2, ta_2\rangle$, then H contains an isolated involution.*

PROOF. Let R be a Sylow 2-subgroup of H. The proof depends only upon the structure of R and not upon the embedding of H in G. We may therefore assume that $R=T_2$ or T_3.

Suppose first that $R=T_3$. Then R contains U_2 which has exponent 4. Moreover, every element of T_3-U_2 has order 8 by Lemma 5.1 (x). Hence by the extension of Thompson's lemma, H has a normal subgroup K of index 2 with Sylow 2-subgroup U_2. Since U_2 is of type (*) of Section 4 by Lemma 5.1 (iv), K has an isolated involution by Lemma 4.5 and hence so does H.

Assume then that $R=T_2$. We want to prove that z_1 is an isolated involution of H. Consider first the case $z_1\sim z_2$ in H. We can then choose an element x in H such that $z_2^x=z_1$ and $C_R(z_2)^x=U_2^x\subseteq R$. Since U_2 is the only maximal subgroup of T_2 isomorphic to U_2 by Lemma 5.1 (v), x normalizes U_2. But Aut(U_2) is a 2-group by Lemma 4.5, so $N_H(U_2)=C_H(U_2)R$. Since $z_1\in Z(R)$ and $x\in N_H(U_2)$, it follows that $z_1^x=z_1$, contrary to $z_2^x=z_1$. We conclude that $z_1\not\sim z_2$ in H.

Suppose next that $z_1\sim t$ in H. In this case, we can choose an element x in H such that $t^x=z_1$ and $C_R(t)^x=\langle z_1, z_2a_1b_1, z_2a_2b_2, t\rangle^x\subseteq R$. But one checks directly that $C_R(t)\cong Z_2\times Q_8$ and that $\Omega_1(C_R(t)')=\langle z_1\rangle$. On the other hand, one also checks that $\Omega_1(R')=\Omega_1(T_2')=\langle z_1, z_2\rangle$. Thus $z_1^x\in\langle z_1, z_2\rangle$. Since $z_2\not\sim z_1z_2$ in R, it follows therefore from the preceding paragraph that $z_1^x=z_1$, contrary to $t^x=z_1$. Hence z_1 is not conjugate to t in H. Similarly z_1 is not conjugate to a_2b_2t in H.

Representatives of the conjugacy classes of involutions of R in R can be taken as $z_1, z_2, a_1, t, a_2b_2t$. Hence to complete the proof of the lemma, it remains only to show that z_1 is not conjugate to a_1 in H, so assume the contrary. Setting $V=C_R(a_1)$, we have $V=\langle z_1, z_2, a_1\rangle$. Furthermore, R contains exactly two elementary abelian subgroups of order 8; namely, V and $V'=\langle z_1, z_2, b_1\rangle$. As in the preceding lemma, it is possible to choose x in H such that $a_1^x=z_1$ and $V^x=V$. On the other hand, $SCN_3(R)$ is empty, $C_R(V)=V$ and $N_R(V)/V$ is a four group. This implies that $N_R(V)$ is a Sylow 2-subgroup of $N_H(V)$ and that a Sylow 2-subgroup of $N_H(V)/C_H(V)$ is a four group. Since this factor group is isomorphic to a subgroup of $GL(3, 2)$, it must be isomorphic to $Z_2\times Z_2$ or A_4. In either case, $C_H(V)N_R(V)\triangleleft N_H(V)$. But one also checks that $Z(N_R(V))=\langle z_1, z_2\rangle$. It follows therefore that $\langle z_1, z_2\rangle$ is normal in $N_H(V)$. Since $x\in N_H(V)$, $z_1^x\in\langle z_1, z_2\rangle$, contrary to $a_1^x=z_1$. This completes the proof of the lemma.

8. **The automorphism group of J_2 and J_3.** In order to show that a minimal counterexample to Theorem A is simple, we require some knowledge of the automorphism groups of J_2 and J_3. In this section we shall determine their automorphism groups completely by proving:

PROPOSITION 8.1. *The outer automorphism group of J_2 and J_3 is of order* 2.[5]

In this section, G will denote the group J_2 or J_3. We let S be a Sylow 2-subgroup of G and use the notation of Sections 1 and 5. In particular, the results of Sections 6 and 7 are applicable.

We need a preliminary lemma.

LEMMA 8.2. *$C_G(z_1)$ is a maximal subgroup of G.*

PROOF. Suppose false and let H be a proper subgroup of G which contains $C_G(z_1)$ as a maximal subgroup. We know that $z_1\sim z_2\sim z_1z_2$ in G and we know the exact structure of $C_G(z_1)$. In particular, $O(C_G(z_1))=1$. We have $T_1\subseteq C_G(\langle z_1, z_2\rangle)$. Hence if $x=z_1$, z_2 or z_1z_2, we have $[C_{O(H)}(x), T_1]$ has odd order. The structure of $C_G(x)$ now forces $C_{O(H)}(x)\subseteq O(C_G(x))=1$. Since $\langle z_1, z_2\rangle$ is a four group, we conclude that $O(H)=1$. Since $C_G(z_1)\subset H$, it follows that z_1 is not an isolated involution of H. Since $C_G(z_1)$ has no normal subgroups of index 2, neither does H. Hence Proposition 6.1 applies to H and tells us that $z_1\sim t$ in H.

We shall argue now that H is simple. Let K be a minimal normal subgroup of H and set $R=S\cap K$. Since $O(H)=1$ and $S\subseteq H$, R is a nontrivial normal subgroup of S and so $\langle z_1\rangle=Z(S)\subseteq R$. Since $K\triangleleft H$, $t\in K$ by the preceding paragraph. Hence the normal closure U of $\langle z_1, t\rangle$ in $C_G(z_1)$ is contained in K. But by Lemma 6.2 and Proposition 6.8, $U=O_2(C_G(z_1))$ is equal to U_4 and U is extra-special of

[5] The case of J_2 is also treated in [18].

order 32. If U were a Sylow 2-subgroup of K, then z_1 would be isolated in K by Lemma 4.6. However, $O(K)=1$ as $O(K)\subseteq O(H)=1$. Since $\langle z_1\rangle=Z(R)$ as $R\supseteq U$, it would follow that $\langle z_1\rangle=Z(K)$. But then $z_1\in Z(H)$, which is not the case. Thus $R\supset U$ and so $K\cap C_G(z_1)\supset U$. Since $C_G(z_1)/U$ is simple, we conclude that $C_G(z_1)\subseteq K$. It follows therefore from our choice of H that either $K=C_G(z_1)$ or $K=H$. However, the first case cannot occur as H does not have an isolated involution. Hence $H=K$. Since K is a minimal normal subgroup of H, this implies that K has no nontrivial normal subgroups and we conclude at once that H is simple, as asserted.

But now by a theorem of Janko [23], $|H|=|J_2|$ or $|J_3|$. However, 7 divides $|J_2|$ and does not divide $|J_3|$. Hence, whether $G=J_2$ or J_3, we see that G does not contain a proper subgroup of the required order and the lemma is proved.

We now prove the proposition. For simplicity, we identify G with $\mathrm{Inn}(G)$. It is known that $|\mathrm{Aut}(G)/G|$ is divisible by 2 [25], [21]. Suppose that Proposition 8.1 is false. Then we can choose a subgroup $\hat{G}$ of $\mathrm{Aut}(G)$ containing G as a normal subgroup of index $2p$, where p is prime (p may be 2, of course). By the Frattini argument, we have $\hat{G}=GN_{\hat{G}}(S)$. But $U=U_4$ is a characteristic subgroup of S by Lemma 5.1 (v). Hence $\hat{G}=GN_{\hat{G}}(U)$ and it follows that $|N_{\hat{G}}(U)/N_G(U)|=2p$. Since $|N_G(U)|=2^7\cdot3\cdot5$, we have $|N_{\hat{G}}(U)|=2^8\cdot3\cdot5\cdot p$. On the other hand, we have already noted that $\mathrm{Aut}(U)/\mathrm{Inn}(U)\cong S_5$. Since $\langle z_1\rangle=C_G(U)$, we conclude that $|C_{\hat{G}}(U)|=2p$ or $4p$. We have therefore shown that $\hat{G}-G$ contains an element x which centralizes U. If p is odd, we can assume $x^p=1$; while if $p=2$, we can assume $x^2\in G$, in which case $x^2\in\langle z_1\rangle=C_G(U)$.

We set $K=G\langle x\rangle$, so that $C_K(U)$ is an abelian group of order $2p$. Since $N_G(U)$ acts on $C_K(U)$, and since $N_G(U)/U$ is simple, we see that $N_G(U)=C_G(z_1)$ centralizes $C_K(U)$. In particular, x centralizes $C_G(z_1)$ and consequently centralizes $Z=\langle z_1, z_2\rangle$. Thus x normalizes $N_G(Z)$. Furthermore, $O(N_G(Z))\subseteq N$. Since $G=J_2$ or J_3, $O(N)=1$ and it is immediate that $O(N_G(Z))=1$. Hence by Proposition 7.1, $N_G(Z)$ is of order $2^7\cdot3^2$, $N_G(Z)$ has T_1 as a normal subgroup and $C_G(T_1)=Z$. We see also that $N_K(Z)=N_G(Z)\langle x\rangle$.

Consider first the case that p is odd. If $p=3$, x is contained in a Sylow 3-subgroup R of $N_K(Z)$ and $Q=R\cap N_G(Z)$ is a Sylow 3-subgroup of $N_G(Z)$ normalized by x. On the other hand, if $p\neq3$, $\langle x\rangle$ is a Sylow p-subgroup of $N_K(Z)$, so by the Frattini argument, if Q_1 is a Sylow 3-subgroup of $N_G(Z)$, then $N_K(Q_1)\cap N_K(Z)$ contains a conjugate of x, hence x normalizes a suitable conjugate Q of Q_1 by an element of $N_K(Z)$ and Q is a Sylow 3-subgroup of $N_G(Z)$. But x centralizes T_1 as $T_1\subseteq C_G(z_1)$, so $C_K(T_1)=\langle Z, x\rangle$. Since $C_K(T_1)\triangleleft N_K(Z)$, we have

$[Q, x] \subseteq Q \cap \langle Z, x \rangle = 1$. Therefore x centralizes the group $\langle C_G(z_1), Q \rangle$. Since $|Q| = 9$ and $|C_G(z_1)|$ is not divisible by 9, this group contains $C_G(z_1)$ properly and hence $G = \langle C_G(z_1), Q \rangle$ by Lemma 8.1. Thus x centralizes G, which is a contradiction.

Assume next that $x^2 = z_1$. Then $C_K(T_1) = \langle Z, x \rangle \cong Z_4 \times Z_2$. Hence a Sylow 3-subgroup of $N_G(Z)$ centralizes $\langle Z, x \rangle$, contrary to the fact that $N_G(Z)$ contains a 3-element which does not centralize Z.

Suppose finally that $x^2 = 1$. We shall argue that x or xz_1 centralizes $O_2(C_G(z_2))$. By Proposition 7.1, z_1 and z_2 are conjugate in $N_G(\langle z_1, z_2 \rangle)$, so $C_G(z_2) = V_1 F_1$, where $V_1 \cong D_8 * Q_8$, $F_1 \cong A_5$, $V_1 \lhd V_1 F_1$, and $z_1 \in V_1$. Clearly x normalizes $C_G(z_2)$. Since x centralizes $T_1 = C_S(z_2)$ and since T_1 is maximal in a Sylow 2-subgroup S_1 of $C_G(z_2)$, x centralizes $W_1 = T_1 \cap V_1$ and $|V_1 : W_1| \leq 2$. Since T_1 does not contain a subgroup isomorphic to $D_8 * Q_8$, we obtain $|V_1 : W_1| = 2$. Since $|S_1 : T_1| = 2$, it follows that $S_1 = T_1 V_1$. Now set $M_1 = C_G(z_2)\langle x \rangle$, and let $\bar{M}_1 = M_1 / V_1$. Then $\bar{M}_1 = \bar{F}_1 \langle \bar{x} \rangle$ and $\bar{T}_1$ is a Sylow 2-subgroup of $\bar{F}_1$. Since $\bar{F}_1 \cong A_5$ and $\bar{x}$ centralizes $\bar{T}_1$ as well as a nontrivial 3-element of $\overline{C_G(z_1)} \cap \overline{C_G(z_2)}$, we have, in fact, $\bar{M}_1 = \bar{F}_1 \times \langle \bar{x} \rangle$. On the other hand, we know that $\mathrm{Aut}(V_1)/\mathrm{Inn}(V_1) \cong S_5$, which implies that $C_{M_1}(V_1) \not\subseteq V_1$. Since $C_{M_1}(V_1) \lhd M_1$, we conclude that $x \in V_1 C_{M_1}(V_1)$. Thus $C_{M_1}(V_1) = \langle z_2, xv \rangle$ for some element v of V_1. Since x and xv both centralize W_1, so also does v. Since $|V_1 : W_1| = 2$ and $V_1 \cong D_8 * Q_8$, this forces $v \in W_1$. Hence $v \in Z(W_1)$. However, $W_1 = C_{V_1}(z_1) \cong Z_2 \times Q_8$ as $z_1 \in V_1 \cong D_8 * Q_8$, whence $Z(W_1) = \langle z_1, z_2 \rangle$ and so $v = 1, z_1, z_2$, or $z_1 z_2$. Since z_2 centralizes V_1, we conclude that either x or xz_1 centralizes V_1.

Since xz_1 centralizes $C_G(z_1)$, it follows now that either x or xz_1 centralizes $\langle C_G(z_1), V_1 \rangle$. Since this group contains $C_G(z_1)$ properly, Lemma 8.1 now yields that x or xz_1 centralizes G. Since $x \notin \mathrm{Inn}(G)$, we reach a contradiction and the proposition is proved.

9. Reduction to the simple case. For the balance of Part II, G will denote a minimal counterexample to Theorem A. In this section we prove

PROPOSITION 9.1. *G is a simple group.*

PROOF. Clearly $O(G) = 1$, since otherwise the theorem would hold for $G/O(G)$ by the minimality of G and would then follow for G. If G has an isolated involution x, then $G = C_G(x)$ by Glauberman's theorem [9] and part (i) of Theorem A holds, contrary to our choice of G. Hence G does not contain such an involution.

Suppose next that G contains a normal subgroup K of index 2. Then $S \cap K$ is a Sylow 2-subgroup of K and is of order 2^6. Hence either $S \cap K = T_1$ or $S \cap K \cong T_2$ or T_3 by Lemma 5.1 (iii). However, in the latter two cases, K would contain an isolated involution by Lemma 7.7. (The lemma is applicable as it depends only upon the structure of T_2 and T_3.) But then G would contain an isolated involution as well, contrary to our choice of G.

Thus $S\cap K=T_1$. But T_1 is of type $PSL(3, 4)$ by Lemma 5.1 (iii). Since we are assuming Theorem C and since $O(K)=1$, it follows that K is isomorphic to a subgroup of $PGL(3, 4)$. We conclude therefore from Lemma 4.8 (iv) that G is isomorphic to a subgroup of $PGL^*(3, 4)$ and so part (ii) of Theorem A holds, contrary to our choice of G. Thus G has no normal subgroups of index 2. In particular, G satisfies the assumptions of Section 5 and so all the results of Sections 6, 7 hold for G.

Now let H be a minimal normal subgroup of G and set $R=S\cap H$. Then R is a Sylow 2-subgroup of H and $R\neq 1$. Since $R\lhd S$, $z_1\in R$. As in the proof of Lemma 7.2, Proposition 6.1 implies that $t\in R$ and then Lemma 6.2 and Proposition 6.8 yield that $U=\langle z_1, t\rangle^N\subseteq R$. Since U is extra-special of order 32, $U\subset R$; otherwise H would have an isolated involution by Lemma 4.6, in which case G would as well. But then by Lemma 5.1 (xii) R contains a conjugate of a_1, a_2 or a_3. Now Proposition 6.1 shows that every involution of S is contained in R. Since $S=\Omega_1(S)$, we conclude that $S=R\subseteq H$.

Since H is characteristically simple and $|Z(S)|=2$, H is necessarily simple. Hence if H is a proper subgroup of G, Theorem A holds for H and so $H\cong J_2$ or J_3. But $\mathrm{Aut}(H)/\mathrm{Inn}(H)$ is a 2-group by Proposition 8.1. Since $S\subset H$, it follows that $G=HC_G(H)$ and that $|C_G(H)|$ is odd. Since $C_G(H)\lhd G$, we have $C_G(H)\subseteq O(G)=1$, whence $G=H$, contrary to the fact that H is a proper subgroup of G. Thus G is simple and the proposition is proved.

10. Subgroup structure of G. In this section we shall determine the possible structures of the proper subgroups H of G.

We break up the analysis in terms of the order of a Sylow 2-subgroup R of H. We fix this notation. Without loss we may assume that $R\subseteq S$. We first prove

PROPOSITION 10.1. *If $|R|\leq 16$, then one of the following holds:*

(i) *H contains an isolated involution;*

(ii) *H is solvable;*

(iii) *a Sylow 2-subgroup of H is either dihedral, quasi-dihedral, or isomorphic to $Z_2\times D_8$.*

(iv) *$H/O(H)$ contains a normal subgroup of odd index isomorphic to $Z_2\times Z_2\times PSL(2, q)$ with $q\equiv 3, 5 \pmod 8)$, $q\geq 5$.*

(v) *$H/O(H)$ is isomorphic to $PSL(2, 16)$.*

PROOF. We apply Lemma 4.1 and conclude at once that either (i) or (iii) holds or else R is elementary abelian of order 8 or 16 or homocyclic of type $(4, 4)$.

However, in the latter case, Brauer's theorem [5] yields that $RO(H)$ is normal in H of index 1 or 3. But then H is solvable and (ii) holds.

Suppose next that R is elementary abelian of order 8. By Lemma 7.3, $|N_H(R):C_H(R)|$ is not divisible by 7 and so is equal to 1 or 3. But then $Z(N_H(R))$ contains an involution and consequently H possesses an isolated involution, so (i) holds.

Suppose finally that R is elementary abelian of order 16. We can assume that H is nonsolvable and does not contain an isolated involution, otherwise (i) or (ii) holds. The latter condition implies that $R\subseteq[N_H(R), N_H(R)]$. Furthermore, $N_H(R)/C_H(R)$ is not 3 or 5, otherwise H would have to contain an element which acted fixed-point-free on R and H would be solvable by Lemma 4.2. Thus the only possibilities are that $N_H(R)/C_H(R)$ is cyclic of order 15 or abelian of type $(3, 3)$. In these cases we apply the main theorem of Walter [27]. Since H is nonsolvable, it follows that $\bar{H}=H/O(H)$ contains a normal subgroup $\bar{L}$ of odd index such that $\bar{L}$ is isomorphic to $PSL(2, 16)$, to $PSL(2, q_1)\times PSL(2, q_2)$ with q_i odd and $q_i\geq 5, 1\leq i\leq 2$, or to $Z_2\times Z_2\times PSL(2, q)$ with $q\equiv 3, 5 \pmod 8$ and $q\geq 5$. Since $PSL(2, 16)$ does not possess an outer automorphism of odd order, it follows in the first case that $\bar{H}=\bar{L}$, whence (v) holds. Furthermore, the second case is excluded by Lemma 7.4, while in the third case, (iv) holds. This completes the proof of the proposition.

For our next result it will be convenient to introduce the term $PSL^*(2, 16)$ to denote the subgroup of $P\Gamma L(2, 16)$ containing $PSL(2, 16)$ as a subgroup of index 2; equivalently the extension of $PSL(2, 16)$ by the automorphism of order 2 induced from an automorphism of $GF(16)$ of order 2. A Sylow 2-subgroup of $PSL^*(2, 16)$ is isomorphic to $(Z_2\times Z_2)\int Z_2$.

PROPOSITION 10.2. *If* $|R|=32$, *then one of the following holds*:

(i) *H contains an isolated involution*;

(ii) *H is solvable*;

(iii) *$H/O(H)$ is isomorphic to $PSU(3, 3^2)$*;

(iv) *$H/O(H)$ is isomorphic to $PSL^*(2, 16)$*.

PROOF. Since R is a subgroup of S of order 32, $R\simeq U_i$ for some $i, 1\leq i\leq 6$, by Lemma 5.1 (iv), (v) and (vi). If $i=3$, or 5, then H contains an isolated involution by Lemmas 7.5 and 7.6. Since U_4 is extra-special of order 32 by Lemma 5.1 (v), the same conclusion holds by Lemma 4.6 if $i=4$. Furthermore, U_2 is of type (*) of Section 4 by Lemma 5.1 (iv) and so H has an isolated involution by Lemma 4.5. Hence in each of these cases, (i) or (ii) holds.

It remains therefore to treat the cases $i=1$ or 6, in which correspondingly

$R \simeq (Z_2 \times Z_2) \int Z_2$ or $Z_4 \int Z_2$. Suppose first that $R \cong (Z_2 \times Z_2) \int Z_2$. Then H contains a normal subgroup K of index 2 with $R \cap K$ elementary abelian of order 16 by Lemma 4.4. Clearly (i) or (ii) holds for H if it holds for K. In the contrary case Proposition 10.1 implies that $K/O(K)$ is either isomorphic to $PSL(2, 16)$ or else contains a normal subgroup of odd index isomorphic to $Z_2 \times Z_2 \times PSL(2, q)$ with $q \equiv 3, 5 \pmod 8$ and $q \geq 5$. Clearly $O(K) = O(H)$. Hence if the first possibility holds, then $\bar{H} = H/O(H)$ contains a normal subgroup $\bar{L}$ of index 2 isomorphic to $PSL(2, 16)$. Since $C_{\bar{H}}(\bar{L}) = 1$, $\bar{H}$ is isomorphic to a subgroup of $\mathrm{Aut}(\bar{L})$. However, the automorphism group of $PSL(2, 16)$ is isomorphic to $P\Gamma L(2, 16)$ and we conclude that $\bar{L}$ is isomorphic to $PSL^*(2, 16)$. Thus (iv) holds in this case.

We shall show next that the second alternative cannot occur. Indeed if $\bar{K}$ contains a normal subgroup $\bar{X}$ of odd index isomorphic to $Z_2 \times Z_2 \times PSL(2, q)$ with $q \equiv 3, 5 \pmod 8$ and $q \geq 5$, then clearly $\bar{X}$ is characteristic in $\bar{K}$ and so is normal in $\bar{H}$. We have $\bar{X} = \bar{Y} \times \bar{L}$, where $\bar{Y}$ is a four subgroup of $\bar{R}$ and $\bar{L} \cong PSL(2, q)$. Since $\bar{R}$ normalizes $\bar{X}$, it follows that $\bar{Y} \lhd \bar{R}$, whence $\bar{Y} \cap \bar{Z}(\bar{R}) \neq 1$. Hence there exists an involution $\bar{y}$ in $Z(\bar{R})$ such that $C_{\bar{H}}(\bar{y}) \supseteq \bar{X}$ and consequently there is an involution y in $Z(R)$ such that $C_H(y)$ contains a subgroup X with $X/O(X) \cong \bar{X}$. However, a Sylow 2-subgroup of $C_G(y)$ contains R and so has order at least 32. But then $y \sim z_1$ by Proposition 6.10 and therefore N contains a conjugate of X, contrary to Proposition 6.9 (i).

Assume finally that $R \cong U_6 \cong Z_4 \int Z_2$. By Proposition 2.1.2 of [2], one of the following holds: H has no normal subgroup of index 2 and one conjugacy class of involutions, H has a normal subgroup K of index 4 with quaternion Sylow 2-subgroup, or H has a normal subgroup K of index 2 with abelian Sylow 2-subgroup of type (4, 4). In the second case, the Brauer-Suzuki theorem [6] implies that K has an isolated involution and then H does as well. In the third case, Brauer's theorem [5] implies that K is solvable and hence H is as well. Thus (i) or (ii) holds in these cases.

Suppose then that H has no normal subgroup of index 2 and only one conjugacy class of involutions, in which case H is a QD-group in the sense of [2]. Since $R \subseteq S$, Lemma 5.1 (v) implies that $z_1 \in R$. But then by Propositions 2.2.1 and 2.3.2 of [2], if $C = C_H(z_1) = H \cap N$, then $\bar{C} = C/O(C)$ contains a normal subgroup $\bar{L}$ isomorphic to $SL(2, q)$ for some odd q. Thus $C = H \cap N$ contains a subgroup L such that $L/O(L) \simeq SL(2, q)$. Proposition 6.9 (i) now forces $q = 3$. We conclude therefore that $C_H(z_1)$ is solvable. But now a theorem of Fong [8] applies and yields that $H/O(H) \cong PSU(3, 3^2)$. Thus (iii) holds and the proposition is proved.

We next prove

PROPOSITION 10.3. *If* $|R|=64$, *then one of the following holds*:

(i) *H contains an isolated involution*;

(ii) *R is of type* $PSL(3, 4)$ *and* $H/O(H)$ *is isomorphic to a subgroup of* $PGL(3, 4)$.

PROOF. If R is of type $PSL(3, 4)$, then (ii) holds by Theorem C, which we are assuming. We can therefore assume that R is not of this type. Then by Lemma 5.1 (iii) $R \cong T_2$ or T_3. In either of these cases, H contains an isolated involution by Lemma 7.7, whence (i) holds.

Finally we prove

PROPOSITION 10.4. *If* $|R|=128$, *then one of the following holds*:

(i) *H contains an isolated involution*;

(ii) $H/O(H)$ *is isomorphic to a subgroup of* $PGL^*(3, 4)$;

(iii) $H/O(H)$ *is isomorphic to* J_2 *or* J_3.

PROOF. In this case $R=S$ is of Janko type. Since H is a proper subgroup of G, Theorem A holds for H by our minimal choice of G and the proposition follows.

In the next section, when G has two conjugacy classes of involutions, we shall prove that

$$C_G(a_1)/O(C_G(a_1)) \cong Z_2 \times Z_2 \times A_4 \quad \text{or} \quad Z_2 \times Z_2 \times A_5 .$$

Once this is established, it is possible to sharpen the preceding results on the subgroup structure of G. To preserve the continuity of exposition, we shall *assume* that $C_G(a_1)$ has this structure and examine its implications.

Specifically, we shall prove on the basis of this assumption:

PROPOSITION 10.5. *If H is a proper subgroup of G, then one of the following conditions holds*:

(i) *H is 2-constrained*;

(ii) *H has dihedral or quasi-dihedral Sylow 2-subgroups*;

(iii) $H/O(H) \cong Z_2 \times A_5$, $Z_2 \times Z_2 \times A_5$; *or* $A_4 \times A_5$;

(iv) $H/O(H) \cong PSL(2, 16)$ *or* $PSL^*(2, 16)$;

(v) $H/O(H) \cong PSU(3, 3^2)$;

(vi) $H/O(H) \cong PSL(3, 4)$, $PGL(3, 4)$, $PSL^*(3, 4)$, *or* $PGL^*(3, 4)$;

(vii) $H/O(H) \cong J_2$ *or* J_3.

PROOF. Suppose H has an isolated involution a. If $a \sim z_1$, we can assume without loss that $a=z_1$, in which case $H=O(H)(H \cap N)$. But now Proposition 6.9(vi) yields that either $H \cap N$ is 2-constrained or $H \cap N/O(H \cap N) \cong Z_2 \times A_5$. Correspondingly we conclude that H satisfies (i) or (iii). On the other hand, if a is not conjugate to z_1, then $a\sim a_1$ and $C_G(a)/O(C_G(a))\cong Z_2\times Z_2\times A_4$ or $Z_2\times Z_2\times A_5$ by our assumption on the structure of $C_G(a_1)$. Since $H=O(H)(H\cap C_G(a))$, it follows at once in this case as well that (i) or (iii) holds. Hence we may assume that H does not contain an isolated involution. We may also assume that H is nonsolvable, otherwise (i) holds.

The structure of H is given by Propositions 10.1–10.4. We examine the various possibilities in succession. Suppose H satisfies Proposition 10.1 (iii). Then either (ii) of the present proposition holds or a Sylow 2-subgroup of H is isomorphic to $Z_2\times D_8$. However, by the remark following Lemma 4.4, it would follow in this case that for some involution u in H, $C_H(u)$ contains $Z_2\times S_4$. Since a Sylow 2-subgroup of $C_H(u)$ is then non-abelian, $u\sim z_1$ by Proposition 6.10, contrary to Proposition 6.9 (i). Suppose H satisfies Proposition 10.1 (iv), in which case $\bar{H}=H/O(H)$ contains a normal subgroup $\bar{K}$ of odd index with $\bar{K}\cong Z_2\times Z_2\times PSL(2, q)$, $q\equiv 3, 5 \pmod 8$ and $q\geq 5$. In this case $C_H(u)$ involves a subgroup isomorphic to $\bar{K}$ for some involution u of H. Since $u\sim z_1$ or a_1, Proposition 6.9 (i) together with our assumption on $C_G(a_1)$ forces u to be conjugate to a_1, but not to z_1, and q to be equal to 5. But then $\bar{K}$ is necessarily of index 1 or 3 in $\bar{H}$ and correspondingly $\bar{H}\cong Z_2\times Z_2\times A_5$ or $A_4\times A_5$. Thus (iii) holds in this case. Finally if H satisfies Proposition 10.1 (v), then (iv) holds.

If H satisfies Proposition 10.2 (iii) or (iv), then correspondingly (v) or (iv) holds. If H satisfies Proposition 10.3 (ii), then $\bar{H}=H/O(H)$ is isomorphic to a subgroup of $PGL(3, 4)$ and has a Sylow 2-subgroup of type $PSL(3, 4)$. In this case we apply Lemma 4.8 to obtain that either $\bar{H}\cong PSL(3, 4)$ or $PGL(3, 4)$ or else $|\bar{H}:\bar{H}'|=1$ or 3 and $\bar{H}'\cong A_5\cdot E_{16}^{(2)}$. Since the latter group is 2-constrained, it follows at once that either (i) or (vi) holds.

Suppose finally that H satisfies Proposition 10.4, in which case either $\bar{H}$ is isomorphic to a subgroup of $PGL^*(3, 4)$ or to J_2 or J_3. In the latter two cases, (vii) holds; while in the first case, $\bar{H}$ contains a normal subgroup $\bar{K}$ of index 2 with $\bar{K}$ isomorphic to a subgroup of $PGL(3, 4)$ and having Sylow 2-subgroups of type $PSL(3, 4)$. Since $\bar{H}$ is nonsolvable, Lemma 4.8 (iv) yields that $\bar{H}\cong PSL^*(3, 4)$ or $PGL^*(3, 4)$. Thus (vi) holds and the proposition is proved.

11. The structure of $C_G(a_1)$. We shall now determine the structure of $C_G(a_1)$ in the case that G has two conjugacy classes of involutions.[6] We set $M=C_G(a_1)$. Then by Proposition 6.10 and Lemma 5.1 (ix) $A=C_S(a_1)=\langle z_1, z_2, a_1, a_2\rangle$ is a Sylow 2-subgroup of M and is elementary abelian of order 16. We know already from Proposition 7.2 that if $H=N_G(A)$, then $\bar{H}=H/O(H)$ is a split extension of $\bar{A}$ by

[6] Use of Lyons characterization of J_2 would eliminate the need for this section.

a group $\bar{X}\times\bar{Y}$, where $\bar{X}\cong Z_3$ and $\bar{Y}\cong A_4$. In particular, $\bar{H}$ is 2-closed with $\bar{T}_1$ as its Sylow 2-subgroup and a Sylow 3-subgroup $\bar{P}$ of $\bar{H}$ is elementary abelian of order 9. It follows now from Lemma 4.7 that for a suitable choice of $\bar{P}$, $\bar{P}$ leaves both $\langle\bar{z}_1, \bar{z}_2\rangle$ and $\langle\bar{a}_1, \bar{a}_2\rangle$ invariant. This implies that $C_{\bar{H}}(\bar{a}_1)=\bar{A}\bar{R}$, where $|\bar{R}|=3$, $\bar{R}$ centralizes $\langle\bar{a}_1, \bar{a}_2\rangle$, and $\bar{R}$ acts fixed-point-free on $\langle\bar{z}_1, \bar{z}_2\rangle$. Since $C_H(a_1)=H\cap M$ maps onto $C_{\bar{H}}(\bar{a}_1)$, we conclude at once that

$$|N_M(A):C_M(A)|=3 \tag{1}$$

and that M contains a 3-element which centralizes $W=\langle a_1, a_2\rangle$ and acts fixed-point-free on $Z=\langle z_1, z_2\rangle$.

In particular, if M is nonsolvable, Proposition 10.1 (iv) applies and yields that $\bar{M}=M/O(M)$ contains a normal subgroup $\bar{D}$ of odd index of the form

$$\bar{D}=\bar{W}\times\bar{L}\ , \tag{2}$$

where $\bar{L}\cong PSL(2, q)$, $q\equiv 3, 5\ (\mathrm{mod}\ 8)$ and $q\geq 5$ with $\bar{Z}\subseteq\bar{L}$. If M is solvable, then clearly the same conclusion holds with $q=3$.

Our goal will be to show that, in fact, $q=3$ or 5. Once this is established, it will clearly follow that $\bar{D}=\bar{M}$ and hence will yield our desired objective; namely, a proof of the following assertion:

PROPOSITION 11.1. *$C_G(a_1)/O(C_G(a_1))$ is isomorphic to $Z_2\times Z_2\times A_4$ or $Z_2\times Z_2\times A_5$.*

We shall carry out the proof in a sequence of lemmas, preserving the above notation. By way of contradiction, we assume that $q>5$. Our argument is based on a method that was first used in Chapters IV and V of [2].

We first introduce some additional notation. Since $\bar{a}_1\in Z(\bar{M})$ and $\bar{D}$ has odd index in $\bar{M}$, clearly $\bar{W}=Z(\bar{M})$ and $C_{\bar{M}}(\bar{L})=\bar{W}$. In particular, $\bar{L}\lhd\bar{M}$ and $\bar{M}/\bar{L}$ is a direct product. Hence

$$\bar{M}=\bar{W}\times\bar{K}\ , \tag{3}$$

where $\bar{K}\supseteq\bar{L}$. Moreover $C_{\bar{K}}(\bar{L})=1$ and so $\bar{K}$ is isomorphic to a subgroup of $P\Gamma L(2, q)$. Since $C_M(W)$ maps onto $C_{\bar{M}}(\bar{W})$, if follows that

$$C_M(W)=W\times K\ , \tag{4}$$

where $K/O(K)\cong\bar{K}$. We let L be the normal subgroup of K containing $O(K)$ such that $L/O(K)\cong\bar{L}$. In addition, we have $M=O(M)C_M(W)$ and consequently

$$M=O(M)(W\times K)\ . \tag{5}$$

Note also that as $a_1\in W$, $C_G(W)=C_M(W)$.

Finally consider $C_{\bar{L}}(\bar{z}_1)$ which by Lemma 3.1 (iii) of [15] is a dihedral group of order $q-\varepsilon$, where $q\equiv\varepsilon\pmod 4$ and $\varepsilon=\pm1$. Since $q\equiv 3, 5\pmod 8$ and $q>5$ by assumption, $O(C_{\bar{L}}(\bar{z}_1))\neq 1$. Since $\bar{Z}$ is a Sylow 2-subgroup of $\bar{L}$, it follows that for some odd prime p, $O(C_{\bar{L}}(\bar{z}_1))$ contains a nontrivial p-element inverted by $\bar{z}_2$. For such a choice of p, there exists a nontrivial p-element y of L such that

$$(6) \qquad y^{z_1}=y \quad\text{and}\quad y^{z_2}=y^{-1}\,.$$

We fix such an odd prime p for the proof and preserve all the notation introduced above. We wish to apply Glauberman's ZJ-theorem [10] to certain collections of subgroups of G. To do this, we must show that each subgroup in question is both p-constrained and p-stable. We begin with p-constraint.

Let H be a subgroup of G containing S. *We shall say that* H is p-constrained *within* $Z(S)$ if H is p-constrained and if $O_{p'}(H)\subseteq O(H)Z(S)$.

LEMMA 11.2. *N is p-constrained within $Z(S)$.*

PROOF. Since $\langle y\rangle$ is A-invariant, $y\in O(N)$ by Proposition 6.9 (ii). Hence z_2 does not centralize some S-invariant Sylow p-subgroup P of $X=O(N)Z(S)=O(N)\langle z_1\rangle$. Setting $R=P\cap O_{p',p}(X)$, then R is an S-invariant Sylow p-subgroup of $O_{p',p}(X)$. Since X is solvable, it is p-constrained and so $C_P(R)\subseteq R$. We claim also that z_2 does not centralize R. Indeed, if $[z_2, R]=1$, then $[z_2, R, P]=1$. Also $[R, P]\subseteq R$, so $[R, P, z_2]=1$ and hence $[P, z_2, R]=1$ by the three subgroup lemma. Thus z_2 centralizes P/R and consequently z_2 stabilizes the chain $P\supseteq R\supseteq 1$. We conclude that z_2 centralizes P, which is not the case and our assertion is proved.

Now set $C=C_N(R)$, so that $XC\triangleleft XN_N(R)$. Since $N=XN_N(R)$ by the Frattini argument, it follows that $\bar{C}$ is normal in $\bar{N}=N/X$. But $\bar{z}_2\notin\bar{C}$ and so by the structure of $\bar{N}$, we have $\bar{C}=1$. Thus $C\subseteq X$ and, as X is solvable, we have, in fact, $C\subseteq O_{p',p}(X)\subseteq O_{p',p}(N)$.

Hence to complete the proof, we need only show that $O_{p'}(N)\subseteq X$, for then R will be a Sylow p-subgroup of $O_{p',p}(N)$ and, as $C_N(R)\subseteq O_{p',p}(N)$, N will be p-constrained within $Z(S)$. But by the structure of N, either this is the case or $z_2\in O_{p'}(N)$. However, in that case $[z_2, R]\subseteq O_{p'}(N)\cap R=1$ and so z_2 centralizes R, a contradiction. Thus $O_{p'}(N)\subseteq X$, as required.

We now define the family $\mathcal{N}$ as the set of proper subgroups H of G which satisfy the following conditions:

(a) H contains S;

(b) H contains an S-invariant Sylow p-subgroup of $O(N)$;

(c) H covers $N/O(N)$.

Clearly N itself is in $\mathcal{N}$.

LEMMA 11.3. *If $H\in\mathcal{N}$, then the following conditions hold:*

(i) *Either $H/O(H)$ is isomorphic to $N/O(N)$ or to J_2;*

(ii) *H is p-constrained within $Z(S)$;*

(iii) *If Q is a nontrivial S-invariant p-subgroup of H such that $O_{p'}(H)Q$ is normal in H, then $N_G(Q)\in\mathcal{N}$.*

PROOF. Since H covers $N/O(N)$ and N has no normal subgroups of index 2, neither does H. But H is a proper subgroup of G containing S and a_1, z_1 are not conjugate in H as they are not conjugate in G. Proposition 10.4 together with the fact that J_3 has only one conjugacy class of involutions now yields that either $H/O(H)\cong J_2$ or H has an isolated involution. In the latter case, $H=O(H)C_H(z_1)=O(H)(H\cap N)$ and, as H covers $N/O(N)$ by assumption, it follows that $H/O(H)\cong N/O(N)$. Thus (i) holds.

As for (ii), suppose first that $H/O(H)\cong N/O(N)$. Then $H=O(H)(H\cap N)$ and $X=O(H)Z(S)\lhd H$. If P is an S-invariant Sylow p-subgroup of $O(N)$ contained in H, then clearly $P\subseteq O(H)\subseteq X$. Moreover, the previous lemma shows that z_2 does not centralize P. But now (ii) follows exactly as in the preceding lemma.

Suppose, on the other hand, that $H/O(H)\cong J_2$. This time we set $X=O(H)$ and note that either $O_{p'}(H)\subseteq X$ or $O_{p'}(H)\supseteq S$. Setting $\bar{H}=H/X\cong J_2$, we have that $\bar{P}\subseteq C_{\bar{H}}(\bar{z}_1)$ and that $\bar{P}$ is $\bar{S}$-invariant. This forces $\bar{P}=1$ and again $P\subseteq X$. But now it follows as in the preceding lemma that $O_{p'}(H)\subseteq X$ and if R is an S-invariant Sylow p-subgroup of $O_{p',p}(X)$, then $C_H(R)\subseteq O_{p',p}(X)\subseteq O_{p',p}(H)$. Thus H is p-constrained within $Z(S)$ in this case as well.

Finally let Q be as in (iii). We have already seen that $\langle O_{p'}(H), P\rangle\subseteq O(H)Z(S)$. Since the latter group has a normal 2-complement, $P\subseteq R$ and $Q\lhd R^*$ for suitable S-invariant Sylow p-subgroups R and R^* of $O(H)$. Since $R^*=R^x$ for some element x in $C_{O(H)}(S)$, R^* also contains an S-invariant Sylow p-subgroup of $O(N)$. Thus $N_G(Q)$ does as well. Since $Q\neq 1$ and Q is S-invariant, $N_G(Q)$ is a proper subgroup of G containing S. Furthermore, $H=O_{p'}(H)N_H(Q)=O(H)Z(S)N_H(Q)=O(H)N_H(Q)$. Since H covers $N/O(N)$, this implies that $N_G(Q)$ does as well. Hence $N_G(Q)\in\mathcal{N}$ and (iii) holds.

In order to carry out a similar analysis on the subgroup M, we need one further property of N:

LEMMA 11.4. *If P is an S-invariant Sylow p-subgroup of $O(N)$, then the following conditions hold:*

(i) *Either $[C_P(W), z_2]$ is noncyclic or $[C_P(a_i), a_j, z_2]\neq 1$ for some i, j, $1\leq i, j\leq 3$;*

(ii) *$[C_P(a_i), a_j]\neq 1$ for some i, j, $1\leq i, j\leq 3$.*

PROOF. We shall use the action of $\bar{S}=S/\langle z_1\rangle$ on $\bar{P}=P/\Phi(P)$ to establish the lemma. Since N is p-constrained within $Z(S)$ by Lemma 11.2, $C_{\bar{S}}(P)=1$ and hence $C_{\bar{S}}(\bar{P})=1$ by a theorem of Burnside.

Suppose (i) is false. Since $W=\langle a_1, a_2\rangle$ is a four group, we have by Lemma 10.5.1 of [12] that

$$P=P_1P_2P_3=P_0P_1'P_2'P_3', \tag{7}$$

where $P_0=C_P(W)$, $P_i=C_P(a_i)$, and P_i' is the subset of P_i inverted by a_j, $1\leq i, j\leq 3$, $i\neq j$. By our assumption, z_2 centralizes each P_i' and $[P_0, z_2]$ is cyclic. Hence by (7), we have that $[P, z_2]$ is cyclic and consequently $[\bar{P}, \bar{z}_2]=\bar{Q}$ is cyclic. But $\bar{S}$ leaves $\bar{Q}$ invariant as $\bar{z}_2\in Z(\bar{S})$. Since $\bar{z}_2\in\bar{S}'$, it follows that $\bar{z}_2$ centralizes $\bar{Q}$, whence $\bar{Q}=1$. Thus $\bar{z}_2$ centralizes $\bar{P}$, contrary to the fact that $C_{\bar{S}}(\bar{P})=1$. Hence (i) holds.

If (ii) were false, (7) would reduce to $P=P_0$, whence W would centralize P, contrary to $C_S(P)=\langle z_1\rangle$. Thus (ii) also holds.

If H is a subgroup of G containing A, we shall say that H is p-constrained *within* $\langle a_1\rangle$ if H is p-constrained and $O_{p'}(H)\subseteq O(H)\langle a_1\rangle$.

LEMMA 11.5. *M is p-constrained within $\langle a_1\rangle$.*

PROOF. We shall first argue that z_2 does not centralize some A-invariant Sylow p-subgroup T of $O(M)$. By Proposition 6.9 (iv), there exists a 3-element x of G which cyclically permutes a_1, a_2, a_3 and centralizes $Z=\langle z_1, z_2\rangle$. In particular, $x\in N$. Hence if P is as in the preceding lemma, it follows by the Frattini argument that we can choose x to normalize P. But then either $[C_P(a_1), a_2, z_2]\neq 1$ or $[C_P(a_i), a_j, z_2]\neq 1$ for all $i, j, 1\leq i, j\leq 3$. Consider the first case. We see from equation (5) that $X=[C_P(a_1), a_2]\subseteq O(M)$. Since X is A-invariant and z_2 does not centralize X, we can take T to be any A-invariant Sylow p-subgroup of $O(M)$ containing X.

In the second case, we apply Lemma 11.4 (i) to obtain that $R=[Q, z_2]$ is noncyclic, where $Q=C_P(W)$. This implies that z_2 does not centralize $R\cap O(M)$. Indeed, suppose the contrary. If $\bar{M}=M/O(M)$, then $\bar{M}=\bar{W}\times\bar{K}$, $\bar{L}\lhd\bar{K}$, $\bar{L}\cong PSL(2, q)$, and $\bar{Z}\subseteq\bar{L}$. Hence $\bar{R}=[\bar{Q}, \bar{z}_2]\subseteq C_{\bar{L}}(\bar{z}_1)$ and so $\bar{R}$ is cyclic and is inverted by $\bar{z}_2$ by Lemma 3.1 (iii) of [15]. Since z_2 centralizes $R\cap O(M)$ by assumption, this implies that $R=(R\cap O(M))R_1$, where R_1 is cyclic and inverted by z_2. Hence $[R, z_2]=R_1$. However, $[R, z_2]=[Q, z_2, z_2]=[Q, z_2]=R$, so $R=R_1$ is cyclic, which is not the case.

Thus T exists in all cases. Setting $X=T\cap O_{p',p}(O(M))$, it now follows that z_2 does not centralize X. Since any normal subgroup of M, not contained in $O(M)W$, necessarily contains z_2, we must have that $O_{p'}(M)\subseteq O(M)W$; otherwise $z_2\in O_{p'}(M)$ and z_2 would be forced to centralize X, which is not the case. For the same reason, $C_M(X) \subseteq O(M)W$. Since X is a Sylow p-subgroup of $O_{p',p}(O(M)W)$ and $O(M)W$ is solvable, $C_M(X) \subseteq O_{p',p}(O(M)W) \subseteq O_{p',p}(M)$. Thus M is p-constrained.

To complete the proof, it remains to show that $O_{p'}(M) \subseteq O(M)\langle a_1 \rangle$. Since $O_{p'}(M) \subseteq O(M)W$ by the preceding argument and since $a_1 \in O_{p'}(M)$, either this is true or $W \subseteq O_{p'}(M)$. However, in the latter case W centralizes T inasmuch as T is W-invariant. Since $M = O(M)(W \times K)$ and T is a Sylow p-subgroup of $O(M)$, W centralizes every p-subgroup of M that it normalizes. However, by Lemma 11.4 (ii) and the fact that a_1, a_2, a_3 are cyclically permuted by an element of $N_G(P)$, W does not centralize $C_P(a_1) = P \cap M$.

We now define a second family $\mathscr{M}$ as the set of proper subgroups H of G which satisfy the following conditions:

(a) H contains A;

(b) H contains an A-invariant Sylow p-subgroup of $O(M)$;

(c) H covers $M/O(M)$.

Clearly M itself is in $\mathscr{M}$.

LEMMA 11.6. *If $H \in \mathscr{M}$, then the following conditions hold:*

(i) *$O(H)(H \cap M)$ is normal in H of index 1 or 3;*

(ii) *H is p-constrained within $\langle a_1 \rangle$;*

(iii) *If Q is a nontrivial A-invariant p-subgroup of H such that $O_{p'}(H)Q$ is normal in H, then $N_G(Q) \in \mathscr{M}$.*

PROOF. Since H covers $M/O(M)$, $C_H(a_1)$ involves $PSL(2, q)$. For the same reason, if H contains an isolated involution, a_1 is necessarily isolated in H and so $H = O(H)(H \cap M)$. Suppose then that a_1 is not isolated in H. If A were not a Sylow 2-subgroup of H, the structure of $\bar{H} = H/O(H)$ would be given by Propositions 10.2, 10.3 or 10.4. However, in none of these cases does $C_{\bar{H}}(\bar{a}_1)$ involve $PSL(2, q)$ with $q > 5$. Hence A is a Sylow 2-subgroup of H and, as A is abelian, $\bar{H}$ has the structure given in Proposition 10.1 (iv) and consequently $\overline{H \cap M}$ is normal in $\bar{H}$ of index 1 or 3. Since a_1 is not isolated, this index is, in fact, 3. We conclude at once that (i) holds.

Since H contains an A-invariant Sylow p-subgroup T of $O(M)$, it follows directly from (i) that $T \subseteq O(M)$. As in the preceding lemma, neither a_2 nor z_2 centralizes T and we conclude by the same argument that H is p-constrained within $O(H)\langle a_1 \rangle$. Thus (ii) holds. Likewise (iii) follows exactly as in part (iii) of Lemma 11.3.

We turn now to the question of p-stability. Since every element of $\mathscr{M}$ has A as a Sylow 2-subgroup and A is abelian, Theorem 3.8.3 of [12] yields at once:

LEMMA 11.7. *Every element of $\mathscr{M}$ is p-stable.*

In order to obtain the analogous result for the elements of $\mathscr{N}$, we need two preliminary lemmas.

LEMMA 11.8. *If G involves $SL(2, q)$, q odd, then $q=3$.*

PROOF. Assume by way of contradiction that there are subgroups H, F of G with $F \lhd H$ and $H/F \cong SL(2, q)$ for some odd $q \geq 5$. Observe first that N does not involve $SL(2, q)$ by Proposition 6.9(v) and that M does not involve $SL(2, q)$ as a Sylow 2-subgroup of M is abelian. Since every involution of G is conjugate to z or a_1, the centralizer of no involution of G involves $SL(2, q)$. This implies that $C_H(x)$ does not cover H/F for any involution x of H. In particular, this forces $|F|$ to be even, otherwise $H=FC_H(x)$ for any involution x of H and $C_H(x)$ would cover H/F.

Let Y be a Sylow 2-subgroup of F. Since $N_H(Y)$ covers H/F by the Frattini argument and so involves $SL(2, q)$, we can assume without loss that $H=N_H(Y)$, in which case $Y \lhd H$. Suppose Y contains a characteristic subgroup X of order at most 4. Then any element of H of prime order exceeding 3 centralizes X and, as H/F is perfect, it follows that $C_H(X)$ covers H/F and hence so does $C_H(x)$ for any involution x of X, a contradiction. We conclude that Y possesses no such characteristic subgroup X. On the other hand, $|H/F|$ is divisible by 8. Since a Sylow 2-subgroup of G has order 2^7, this forces $|Y| \leq 16$. The only possibilities therefore are that Y is elementary abelian of order 8 or 16.

If $|Y|=16$, then Y is conjugate to A in G by Lemma 5.1(ii). However, by Proposition 7.2, $N_G(A)$ does not involve $SL(2, q)$. On the other hand, if $|Y|=8$, we must have $q=7$ and $|N_H(Y)/C_H(Y)|$ must be divisible by 7, otherwise $C_H(Y)$ would cover H/F. However, this is impossible by Lemma 7.3.

LEMMA 11.9. *Every non-identity 3-element of J_2 normalizes, but does not centralize, a four subgroup of J_2.*

PROOF. We set $H=J_2$ and for simplicity of notation, we assume $S \subset H$. We note that the results of Sections 5, 6 and 7 hold for H. By Lemma 6.5 of [23] a Sylow 3-subgroup of H is nonabelian of order 27 and exponent 3 and H has exactly two conjugacy classes of elements of order 3. One of these classes has a representative x contained in $C_H(z_1)$. But now Proposition 6.9(v) shows that x normalizes, but does not centralize a four subgroup of $C_H(z_1)$. Thus it will be enough to prove the same assertion when x is a representative of the remaining conjugacy class.

If $F=N_H(A)$, then $O(F)=\langle C_{O(F)}(a) \mid a \in A^{\#} \rangle$. Since $C_H(a) \cong Z_2 \times Z_2 \times A_5$ or a split extension of $D_8 * Q_8$ by A_5, it is immediate that $C_{O(F)}(a)=1$ for all a in $A^{\#}$, whence $O(F)=1$. Now Proposition 7.2 yields that $F=A(L \times Y)$, where $L \cong A_4$ and $Y \cong Z_2$.

Without loss we can assume $\langle b_1, b_2\rangle \subseteq L$. Since Y normalizes, but does not centralize a four subgroup of A, we have only to show that Y does not centralize a conjugate w of z_1, so assume the contrary.

Set $C=C_H(Y)$ and let R be a Sylow 2-subgroup of C containing w. We can assume without loss that $w \in Z(R)$. Indeed, this is the case if R is abelian. On the other hand, if R is nonabelian and v is an involution of $Z(R)$, then $R \subseteq C_H(v)$, and so a Sylow 2-subgroup of $C_H(v)$ is nonabelian, whence $v \sim z_1$. Hence we can take $v=w$ in this case. Thus a suitable conjugate of R lies in $C_H(z_1)$ and centralizes some Sylow 3-subgroup of $C_H(z_1)$. However, by Proposition 6.9 (iv), there is a Sylow 3-subgroup X of $C_H(z_1)$ such that $C_H(z_1) \cap C_H(X) = \langle z_1\rangle \times \langle t', z_2\rangle$, where $t' \sim t$ in S. Since every involution of $\langle z_1\rangle \times \langle t', z_2\rangle$ is conjugate to z_1 in H by Proposition 6.1, it follows that every involution of R is conjugate to z_1 in H. But R contains a conjugate of b_1 as $b_1 \in C_H(Y)$ and R is a Sylow 2-subgroup of $C_H(Y)$. Since $b_1 \sim a_1 \not\sim z_1$ in J_2, we reach a contradiction. This establishes the lemma.

We can now prove

LEMMA 11.10. *Every element of $\mathcal{N}$ is p-stable.*

PROOF. Suppose false for some element H of $\mathcal{N}$. Then for some nontrivial p-subgroup P of H and some p-element $x \neq 1$ of H, we have $O_{p'}(H)P \lhd H$, $x \in N_H(P)$, $\langle x\rangle C_H(P)/C_H(P) \not\subseteq O_p(N_H(P)/C_H(P))$ and $[P, x, x]=1$. Setting $F=N_H(P)$ and regarding F as an operator group on P, it follows now by a standard argument (cf. Theorem 6.5.3 of [12]) that there exists a composition factor V of P under the action of F such that, if $\bar{F}=F/C_F(V)$, then $\bar{x} \neq 1$ and $[V, \bar{x}, \bar{x}]=1$. Thus $\bar{F}$ is represented faithfully and irreducibly, but not p-stably, on V regarded as a vector space over $GF(p)$.

By Theorem 3.8.3 of [12], the normal closure $\bar{X}$ of $\bar{x}$ in $\bar{F}$ involves $SL(2, p)$. Hence G involves $SL(2, p)$ and so $p=3$ by Lemma 11.8. Furthermore, obviously $\bar{x} \in O(\bar{F})$. On the other hand, $O_{p'}(H) \subseteq O(H)\langle z_1\rangle$ and hence $(O_{p'}(H) \cap O(H))P \lhd H$. It follows therefore by the Frattini argument that $H=(O_{p'}(H) \cap O(H))F$, whence F covers $H/O(H)$. By definition of $\mathcal{N}$, we know that $H/O(H) \cong N/O(N)$ or to J_2. It follows therefore that either $\bar{F}/O(\bar{F})$ is isomorphic to a nontrivial homomorphic image of $N/O(N)$ or to J_2. In the first case, Proposition 6.9 (v) implies that $\bar{X}$ covers $\bar{F}/O(\bar{F})$. In the second case, this conclusion is clear as J_2 is simple.

Since $\bar{F}$ is irreducibly represented on V, we have $O_3(\bar{F})=1$. We set $\bar{R}=O_{3'}(O(\bar{F}))$ and $\bar{C}=C_{\bar{F}}(\bar{R})$. Since $O(\bar{F})$ is solvable, Theorem 6.3.2 of [12] implies that $\bar{C} \cap O(\bar{F}) \subseteq \bar{R}$. On the other hand, $\bar{R}\langle \bar{x}\rangle$ is 3-stable as it is of odd order. Since

$[V, \bar{x}, \bar{x}]=1$, this is possible only if $\bar{x}$ centralizes $\bar{R}$. Thus $\bar{x} \in \bar{C}$. Since $\bar{C} \lhd \bar{F}$, it follows that $\bar{X} \subseteq \bar{C}$ and therefore $\bar{C}$ also covers $\bar{F}/O(\bar{F})$. Furthermore, $O(\bar{C}) \subseteq O(\bar{F})$ as $O(\bar{C})$ is characteristic in $\bar{C}$ and hence $O(\bar{C}) \subseteq \bar{C} \cap O(\bar{F}) \subseteq \bar{R}$. Since $\bar{C}=C_{\bar{F}}(\bar{R})$, we conclude that $O(\bar{C})$ is a $3'$-group in the center of $\bar{C}$ and that $\bar{C}/O(\bar{C}) \cong \bar{F}/O(\bar{F})$.

Finally by Proposition 6.9 (v) and Lemma 11.9, the image of $\bar{x}$ in $\bar{C}/O(\bar{C})$ normalizes, but does not centralize, a four subgroup of $\bar{C}/O(\bar{C})$. Since $O(\bar{C})$ is a $3'$-group, it follows that $\bar{x}$ normalizes, but does not centralize, a four subgroup $\bar{T}$ of $\bar{C}$. But $\bar{T}\langle\bar{x}\rangle$ is a 3-stable group and consequently $[V, \bar{x}, \bar{x}] \neq 1$. This contradiction establishes the lemma.

We can now establish the principal goal of our analysis.

LEMMA 11.11. *There exist Sylow p-subgroups P_1 and P_2 of G such that $N_G(Z(J(P_1))) \in \mathscr{N}$ and $N_G(Z(J(P_2))) \in \mathscr{M}$.*

PROOF. Choose H in $\mathscr{N}$ such that a Sylow p-subgroup R of H has maximal order. Since we know that H is both p-constrained and p-stable, we can apply Glauberman's ZJ-theorem [10] (or Theorem 8.2.11 of [12]) to conclude that

$$O_{p'}(H)Q \lhd H\,,$$

where $Q=Z(J(R))$. But $O_{p'}(H) \subseteq O(H)Z(S)$ as H is p-constrained within $Z(S)$, whence $O_{p'}(H) \subseteq O_{p'}(O(H))Z(S)$. Since $|Z(S)|=2$, Q thus centralizes $O_{p'}(H)/O_{p'}(O(H))$ and consequently $O_{p'}(O(H))Q \lhd H$. Hence by the Frattini argument some Sylow 2-subgroup of H normalizes Q. Since $S \subseteq H$ as $H \in \mathscr{N}$, we can therefore assume, upon replacing R by a suitable conjugate in H, that Q is S-invariant. Hence $N_G(Q) \in \mathscr{N}$ by Lemma 11.3 (iii). Since Q is characteristic in R, we conclude from our maximal choice of H that R is a Sylow p-subgroup of G. Taking $R=P_1$, we have thus established that $N_G(Z(J(P_1))) \in \mathscr{N}$.

Now choose H in $\mathscr{M}$ so that a Sylow p-subgroup of H has maximal order. Arguing now with A and $\langle a_1\rangle$ in place of S and $Z(S)$ and using the fact that H is p-constrained within $\langle a_1\rangle$ together with Lemma 11.6 (iii), we conclude by the same argument that for a suitable choice of R we have $N_G(Z(J(R))) \in \mathscr{M}$ and that R is a Sylow p-subgroup of G. Taking $P_2=R$, we obtain the second assertion of the lemma.

We can now easily establish Proposition 11.1. Indeed, let P_1, P_2 be as in the preceding lemma and set $H_i=N_G(Z(J(P_i)))$, $1 \leq i \leq 2$, so that $H_1 \in \mathscr{N}$ and $H_2 \in \mathscr{M}$. By definition of $\mathscr{N}$, S is a Sylow 2-subgroup of H_1; while by Lemma 11.6 (i), A is a Sylow 2-subgroup of H_2. However, this is impossible as H_1 and H_2 are clearly conjugate in G.

12. Proof of Theorem A. Now that we know the subgroup structure of

G, we can complete the proof of Theorem A on the basis of the results of Section 2 without much difficulty. Recall that the concept of a weakly connected group is given in Definition 2.6. We shall first prove

PROPOSITION 12.1. *G is balanced and weakly connected.*

PROOF. By the structure of $N=C_G(z_1)$ given in Proposition 6.8, $C_G(z_1)$ is 2-constrained and so the same holds for any conjugate of z_1. If G contains an involution a not conjugate to z_1, then $a\sim a_1$ and so by Proposition 11.1, $C_G(a)/O(C_G(a))\cong Z_2\times Z_2\times A_4$ or $Z_2\times Z_2\times A_5$. In the first case $C_G(a)$ is solvable and so is 2-constrained. We see then that G satisfies the assumptions of Lemma 2.3, so G is balanced by that lemma.

To prove that G is weakly connected, note first that $Z=\langle z_1, z_2\rangle$ is the unique element of $U(S)$ by Lemma 5.1 (i). Furthermore, by Proposition 7.1, $|N_G(Z)/C_G(Z)|$ is divisible by 3. This implies that $z_2=z_1^x$ and $z_2z_1=z_2^x$ for some element x in $N_G(Z)$. Since $Z\subseteq U_4=S\cap O_{2',2}(C_G(z_1))$, it follows that $Z=Z^x\subseteq(O_{2',2}(C_G(z_1)))^x=O_{2',2}(C_G(z_1^x))\subseteq O_{2',2}(C_G(z_2))$. Similarly $Z\subseteq O_{2',2}(C_G(z_1z_2))$. Since $C_G(z_1)$ is 2-constrained by Proposition 6.9, we conclude that conditions (a) and (b) in the definition of weak connection given in Section 2 are satisfied.

It remains to verify condition (c). Set $H=\langle C_G(z_1), N_G(Z)\rangle$. It will suffice to prove that if $H\subset G$, then H is strongly embedded in G. Since $C_G(z_1)$ has no normal subgroups of index 2, neither does H. Since $z_1\sim z_2$ in H, H does not contain an isolated involution. Since $S\subset H$, Proposition 10.4 applies to H and yields that $H/O(H)\cong J_2$ or J_3. In the latter case, H has only one conjugacy class of involutions. Since $C_G(z_1)\subseteq H$, it follows that $C_G(a)\subseteq H$ for every involution a of H. Furthermore, $N_G(S)\subseteq N_G(Z(S))=C_G(z_1)\subseteq H$. Since G is simple, $G-H$ contains an involution and we conclude from the definition that H is strongly embedded in G.

Suppose, on the other hand, that $H/O(H)\cong J_2$. Then H has two conjugacy classes of involutions and, by Lemma 3.3 of [23], $C_H(a_1)/O(C_H(a_1))\cong Z_2\times Z_2\times A_5$. Since $C_G(z_1)/O(C_G(z_1))$ does not contain a subgroup isomorphic to $Z_2\times Z_2\times A_5$ by Proposition 6.9 (i), a_1 is not conjugate to z_1 in G. It follows therefore from Proposition 11.1 that $C_G(a_1)/O(C_G(a_1))\cong C_H(a_1)/O(C_H(a_1))$, whence

$$C_G(a_1)=O(C_G(a_1))C_H(a_1)\ .$$

But $Z\subseteq C_G(a_1)$ and so Z acts on $O(C_G(a_1))$. However, H contains a 3-element which cyclically permutes $z_1, z_2, z_3=z_1z_2$, and, as $C_G(z_1)\subseteq H$, we have $C_G(z_i)\subseteq H, 1\leq i\leq 3$. Thus

$$O(C_G(a_1))=\langle C_{O(C_G(a_1))}(z_i)\,|\,1\leq i\leq 3\rangle\subseteq H$$

and therefore $C_G(a_1)\subseteq H$. Since any involution a of H is conjugate in H to a_1 or z_1, it follows that $C_G(a)\subseteq H$ and we conclude in this case as well that H is strongly embedded in G.

PROPOSITION 12.2. *O is a strongly flat A-signalizer functor on G.*

PROOF. Here, of course, $A=\langle z_1, z_2, a_1, a_2\rangle$ is elementary abelian of order 16. As pointed out in Section 2, the fact that G is balanced implies that O is an A-signalizer functor on G.

Now let H be a proper subgroup of G such that $A\cap H$ is noncyclic. To prove the proposition we must show that H is strongly A-flat. If H is 2-constrained or has dihedral or quasi-dihedral Sylow 2-subgroups, this follows from Lemmas 3.4, 3.5 and 3.6 respectively. Likewise, if $\bar{H}=H/O(H)\cong PSL^*(3,4)$ or $PGL^*(3,4)$, H is strongly A-flat by Lemma 3.7. Hence we may assume that $\bar{H}$ is of none of these forms.

Proposition 10.5 shows that $\bar{H}\cong Z_2\times A_5$, $Z_2\times Z_2\times A_5$, $A_4\times A_5$, $PSL(2,16)$, $PSL^*(2,16)$, $PSU(3,3^2)$, $PSL(3,4)$, $PGL(3,4)$, J_2 or J_3. In each of these cases, $O(C_{\bar{H}}(\bar{a}))=1$ for every involution $\bar{a}$ of $\bar{A}$ and hence $O(C_H(a))\subseteq O(H)$ for every involution a of A. It follows therefore from Lemma 3.3 that H is strongly A-flat in each case.

We conclude from the definition that O is a strongly flat A-signalizer functor on G.

We now prove Theorem A. Since O is a strongly flat A-signalizer functor on G and A is elementary abelian of order 16, Theorem 2.1 yields that $\langle O(C_G(a))\mid a\in A^{\#}\rangle$ has odd order. Since $Z=\langle z_1, z_2\rangle\subset A$, this implies that

$$W_Z=\langle O(C_G(a))\mid a\in Z^{\#}\rangle$$

has odd order.

By Proposition 9.1, G is simple. By Proposition 12.1, G is balanced and weakly connected. Since Z is the unique element of $U(S)$ by Lemma 5.1 (i), and since W_Z has odd order, we see that all the hypotheses of Theorem 2.6 are satisfied and we conclude from that theorem that $O(C_G(x))=1$ for every involution x of G. In particular, $O(C_G(z_1))=1$. Thus $C_G(z_1)$ is isomorphic to an extension of D_8*Q_8 by A_5, by Proposition 6.8. The combined results of Janko [23], Hall and Wales [18], Higman and McKay [21], and Wong [28] now yield that $G\cong J_2$ or J_3.

Therefore a minimal counterexample to Theorem A does not exist and the theorem is proved.

PART III

Theorem B

13. The centralizers of involutions. We now begin the proof of Theorem B. Thus G is a group with Sylow 2-subgroup S of type A_8, G contains exactly three conjugacy classes of involutions and an elementary abelian group A of order 16 such that $N_G(A)/O(N_G(A))$ is an extension of $AO(N_G(A))$ by A_5. To establish Theorem B, we must show under these conditions that $AO(G)$ is normal in G.

We proceed by induction on $|G|$ and assume that G is a minimal counter-example. Since our assumptions clearly carry over to $G/O(G)$, the minimality of G implies that $O(G)=1$. We may assume that $A\subseteq S$. By the structure of A_8, S is a 2-group of order 64 and is generated by involutions a_i, $1\leq i\leq 6$, with the following relations:

$$(1) \qquad [a_3, a_5]=[a_2, a_6]=a_1, \ [a_4, a_5]=a_2, \ [a_4, a_6]=a_3 ,$$

with all other commutators of two generators being trivial. From these relations we see that S possesses exactly one elementary abelian subgroup of order 16; namely, $\langle a_1, a_2, a_3, a_4\rangle$. This subgroup must then be A. Furthermore, S splits over A: we have $S=\langle a_5, a_6\rangle A$ with $\langle a_5, a_6\rangle$ a four group. One sees also that S has nine conjugacy classes of involutions, represented by:

$$(2) \qquad a_1, a_2, a_2a_3, a_3, a_4, a_4a_1, a_5, a_5a_6, a_6 .$$

Since S splits over A, $N_G(A)/O(N_G(A))$ is a split extension by Gaschütz's theorem and so is isomorphic to an extension of E_{16} by A_5. There are three possible ways for A_5 to act on an elementary abelian group A of order 16, one of which is trivial and a second is transitive on the involutions of A. Since $\langle a_5, a_6\rangle$ does not centralize A, our action is certainly nontrivial. Furthermore, we can assume that the image of $\langle a_5, a_6\rangle$ in $N_G(A)/O(N_G(A))$ lies in the subgroup isomorphic to A_5 since every complement of A in S is conjugate to $\langle a_5, a_6\rangle$ in S. Thus we see that a_5, a_6 and a_5a_6 are conjugate in G. But then if our action were transitive, it would follow from (2) that G had at most two conjugacy classes of involutions, contrary to hypothesis. Thus A_5 acts nontransitively on the involutions of A. We conclude therefore that

$$N_G(A)/O(N_G(A))\cong A_5\cdot E_{16}^{(1)} .$$

At the same time, checking the group $A_5\cdot E_{16}^{(1)}$ directly, we know the fusion of involutions in G; namely,

$$(3) \qquad a_1\sim a_4, \ a_2\sim a_3\sim a_2a_3\sim a_4a_1, \quad \text{and} \quad a_5\sim a_5a_6\sim a_6$$

(interchanging a_4 and a_4a_1, if necessary).

We begin the proof of Theorem B by analyzing the centralizers of involutions in G. We shall prove

PROPOSITION 13.1. *The centralizer of every involution in G is solvable.*

As usual, we carry out the proof in a sequence of lemmas. We set $N=N_G(A)$ and $\bar{N}=N/O(N)$, so that $\bar{N}$ is isomorphic to $A_5\cdot E_{16}^{(1)}$. The various parts of the following lemma are easily obtained from the structure of S and $\bar{N}$ and its proof is left to the reader.

LEMMA 13.2. *The following conditions hold:*

(i) $Z(S)=\langle a_1\rangle$ *and* $S/\langle a_1\rangle\cong(Z_2\times Z_2)\int Z_2$.

(ii) *S has precisely seven maximal subgroups. One is* $U_1=\langle a_1, a_2, a_3, a_5, a_6\rangle\cong Q_8*Q_8$, *three are isomorphic to* $U_2=\langle a_1, a_2, a_3, a_4, a_5\rangle\cong(Z_2\times Z_2)\int Z_2$ *and the remaining three to* $U_3=\langle a_1, a_2, a_3, a_5, a_4a_6\rangle$.

(iii) $C_{\bar{N}}(\bar{a}_1)$ *is 2-closed and* $[C_{\bar{N}}(\bar{a}_1):\bar{S}]=3$.

(iv) $C_{\bar{N}}(\bar{a}_2)$ *contains a normal subgroup of index 2 of the form* $\langle\bar{a}_2, \bar{a}_4\rangle\times\bar{F}$, *where* $\bar{F}\cong A_4$.

(v) $C_{\bar{N}}(\bar{a}_5)=\langle\bar{a}_1, \bar{a}_2, \bar{a}_5, \bar{a}_6\rangle$.

(vi) $\bar{N}$ *contains an element of order 3 which normalizes* $\bar{U}_1$ *and acts fixed-point-free on* $\bar{U}_1/\langle\bar{a}_1\rangle$.

We first prove

LEMMA 13.3. $C_G(a_1)$ *is solvable.*

PROOF. Set $C=C_G(a_1)$ and $\bar{C}=C_G(a_1)/\langle a_1\rangle$, so that by Lemma 13.2(i), the Sylow 2-subgroup $\bar{S}$ of $\bar{C}$ is isomorphic to $(Z_2\times Z_2)\int Z_2$. Hence by Lemma 4.4, $\bar{C}$ has a normal subgroup of index 2 with Sylow 2-subgroup $\bar{U}_1$ and so C has a normal subgroup D of index 2 with Sylow 2-subgroup U_1. By Lemma 13.2(vi), G possesses a 3-element which normalizes U_1 and acts fixed-point-free on $\bar{U}_1=U_1/\langle a_1\rangle$. Furthermore, since $U_1\cong Q_8*Q_8$, it follows from the structure of Aut(Q_8*Q_8) that $[N_D(U_1):U_1C_D(U_1)]=3$, or 9. However, if this index were 9, it would follow directly that the 18 noncentral involutions of U_1 were conjugate in $N_D(U_1)$, which would imply that $a_3\sim a_5$, a contradiction. Since $N_D(U_1)$ and $U_1C_D(U_1)$ map onto $N_{\bar{D}}(\bar{U}_1)$ and $C_{\bar{D}}(\bar{U}_1)$ respectively, our argument shows that $|N_{\bar{D}}(\bar{U}_1):C_{\bar{D}}(\bar{U}_1)|=3$ and that $\bar{D}$ contains a 3-element which acts fixed-point-free on $\bar{U}_1$. But now Lemma 4.2 yields that $\bar{D}$ is solvable of 2-length 1. We conclude that $\bar{C}$, and hence also C, is solvable.

LEMMA 13.4. $C_G(a_2)$ *is solvable.*

PROOF. Now set $C=C_G(a_2)$. Then $U_2=\langle a_1, a_2, a_3, a_4, a_5\rangle\subseteq C$. Since a_2 is not conjugate to a_1, C does not contain a Sylow 2-subgroup of G and so U_2 is a Sylow

2-subgroup of C. Since $U_2 \cong (Z_2 \times Z_2) \int Z_2$, C contains a normal subgroup D of index 2 with Sylow 2-subgroup A, by Lemma 4.4.

We shall show that $\bar{D} = D/O(D) \cong \langle a_2, a_4 \rangle \times A_4$. We know from Lemma 13.2 (iv) that $D \cap N/O(D) \cap N$ has this structure. Since $a_2 \in Z(D)$, it follows now that $A \cap Z(D \cap N) = A \cap Z(N_D(A))$ is of order 4. Hence by a transfer theorem, D contains a normal subgroup K of index 4. Since $D \cap N$ does not have a normal 2-complement, neither does K. Since $A \cap K$ is a four group and is a Sylow 2-subgroup of K, we conclude from the main theorem of [15] that K possesses a characteristic subgroup L with $L \supseteq O(K) = O(D)$ such that $L/O(D) \cong PSL(2, q)$ with $q \equiv 3, 5 \pmod 8$. Moreover, $L \lhd D$ and $AL \cap N/O(D) \cap N$ is also isomorphic to $Z_2 \times Z_2 \times A_4$.

Now consider $\bar{A}\bar{L} = AL/O(D)$. Because of our congruence on q, $\mathrm{Aut}(\bar{L}) \cong P\Gamma L(2, q)$ has dihedral Sylow 2-subgroups of order 8. Hence $\bar{A}\bar{L} = \bar{A}_0 \times \bar{L}$, where $\bar{A}_0$ is a four subgroup of $\bar{A}$. Furthermore, by the structure of $AL \cap N/O(D) \cap N$, $\bar{L}$ contains an element $\bar{x}$ of order 3 which normalizes $\bar{A}$ and has the property $C_{\bar{A}}(\bar{x}) = \langle \bar{a}_2, \bar{a}_4 \rangle$. Since $\bar{x}$ centralizes $\bar{A}_0$, it follows that $\langle \bar{a}_2, \bar{a}_4 \rangle = \bar{A}_0$. Thus $\bar{a}_4$ centralizes $\bar{L}$ and consequently $C_G(a_4)$ involves $PSL(2, q)$. On the other hand, $a_4 \sim a_1$ by (3) and so $C_G(a_4)$ is solvable by the preceding lemma. This forces $q = 3$ and we conclude that $\bar{D} = \bar{A}\bar{L} \cong \langle \bar{a}_2, \bar{a}_4 \rangle \times \bar{L}$ with $\bar{L} \cong A_4$, as asserted. In particular, D and therefore also C, is solvable, and the lemma is proved.

Finally we prove

LEMMA 13.5. *$C_G(a_5)$ is solvable.*

PROOF. Now set $C = C_G(a_5)$. Since a_5 is not conjugate to a_1 or a_2, it follows from the structure of S that a Sylow 2-subgroup of C has order at most 16. But $W = \langle a_1, a_2, a_5, a_6 \rangle = C_S(a_5) \subseteq C$ and has order 16 and hence W is a Sylow 2-subgroup of C. Clearly $W = \langle a_5 \rangle \times \langle a_6, a_2 \rangle \cong Z_2 \times D_8$. Furthermore, by (3) a_1 is not conjugate to any other involution of W and so is isolated in C. Hence $O(C)\langle a_1 \rangle$ is normal in C. Setting $\bar{C} = C/O(C)\langle a_1 \rangle$, we have that $\bar{W}$ is a Sylow 2-subgroup of $\bar{C}$ and is elementary abelian of order 8. On the other hand, since $W \cong Z_2 \times D_8$, it is immediate that $N_C(W) = WC_C(W)$, which in turn implies that $N_{\bar{C}}(\bar{W}) = C_{\bar{C}}(\bar{W})$. Now Burnside's transfer theorem yields that $\bar{C}$ has a normal 2-complement and hence so also does C. This proves the lemma.

Proposition 13.1 follows at once from Lemmas 13.3, 13.4 and 13.5 inasmuch as every involution of G is conjugate to either a_1, a_2 or a_5 by (3).

14. Subgroup structure of G. We shall now analyze the structure of the proper subgroups of G. Our main result is as follows:

PROPOSITION 14.1. *If H is a proper subgroup of G, then one of the follow-*

ing holds:

(i) *H is solvable;*

(ii) *H has dihedral Sylow 2-subgroups;*

(iii) *$H/O(H)$ is isomorphic to $A_5 \cdot E_{16}^{(1)}$.*

We carry out the proof in a sequence of lemmas. Throughout H will denote a proper subgroup of G and R a Sylow 2-subgroup of H.

LEMMA 14.2. *If H contains an isolated involution, then H is solvable.*

PROOF. If x is an isolated involution of H, Glauberman's theorem implies that $H=O(H)C_H(x)$. Since $C_G(x)$ is solvable by Proposition 13.1, the lemma follows at once.

LEMMA 14.3. *If E is an elementary abelian subgroup of H of order 8, then $|N_G(E)/C_G(E)|$ is not divisible by 7.*

PROOF. We may assume $E \subset S$. One checks directly from the structure of S and the fusion of involutions in G, given in (13.3) that S does not possess an elementary abelian subgroup of order 8 whose involutions are all conjugate in G, which immediately implies the lemma.

LEMMA 14.4. *If $|R| \leq 8$, then either H is solvable or R is dihedral.*

PROOF. Suppose then that R is not dihedral, in which case R is necessarily either quaternion or abelian and R is not a four group. In the first case, H has an isolated involution and so is solvable by Lemma 14.2. Thus R is abelian. If R is cyclic or of type $(4, 2)$, then clearly H has a normal 2-complement and again H is solvable. The only other possibility is that R is elementary abelian of order 8, in which case Lemma 14.3 implies that H has an isolated involution. Hence by Lemma 14.2, H is solvable in this case as well.

LEMMA 14.5. *If $|R|=16$, then H is solvable.*

PROOF. We may assume that H is not solvable. Lemmas 4.1 and 14.2 together imply that either $R \simeq E_{16}$, $Z_2 \times D_8$, $Z_4 \times Z_4$, D_{16}, or R is quasi-dihedral. However, examination of the structure of S shows that S does not possess subgroups of any of the last three types. Thus $R \simeq E_{16}$ or $Z_2 \times D_8$.

In the first case, we may assume $R=A$. Then $N_H(A)/C_H(A)$ is isomorphic to a subgroup of odd order of $N_G(A)/C_G(A) \simeq A_5$. If the order is 1, H has a normal 2-complement and so is solvable. If the order is 3 or 5 and H contains a 3- or 5-element which acts fixed-point-free on A, then H is solvable by Lemma 4.2. Since H is not solvable by assumption, the only possibility is that $|N_H(A)/C_H(A)|=3$ and $A \cap Z(N_H(A)) \neq 1$. But then H has an isolated involution and so is solvable in this case as well by Lemma 14.2.

Assume next that $R \simeq Z_2 \times D_8$. By Lemma 4.3 H has a normal subgroup K

of index 2 with dihedral Sylow 2-subgroups of order 8. An examination of S shows that S does not possess a dihedral subgroup of order 8 in which all the involutions are conjugate in G. This implies that K itself has a normal subgroup of index 2. But now the remark following Lemma 4.3 yields that $H/O(H)$ has a direct factor of order 2, in which case H has an isolated involution and so is solvable by Lemma 14.2.

LEMMA 14.6. *If $|R|=32$, then H is solvable.*

PROOF. By Lemma 13.2(ii), $R\cong U_1, U_2$ or U_3. If $R\cong U_1$, then R is extra-special of order 32 and so H is solvable by Lemmas 4.6 and 14.2. If $R\cong U_2$, then $R\cong(Z_2\times Z_2)\int Z_2$ and so H has a normal subgroup K of index 2 by Lemma 4.4. But then K is solvable by Lemma 14.5 and hence so is H. Finally if $R\cong U_3$, we can apply Lemma 4 of [20] to conclude that H contains an isolated involution. But then H is solvable by Lemma 14.2 in this case as well.

Finally, we prove

LEMMA 14.7. *If $|R|=64$, then either H is solvable or $H/O(H)$ is isomorphic to $A_5\cdot E_{16}^{(1)}$.*

PROOF. We may assume that $R=S$ and that H is nonsolvable. If $N_H(A)/C_H(A)\cong N_G(A)/C_G(A)\cong A_5$, then $N_H(A)/O(N_H(A))\cong N_G(A)/O(N_G(A))\cong A_5\cdot E_{16}^{(1)}$. But then $N_H(A)$ has only three conjugacy classes of involutions and it follows from our hypothesis on G that H has exactly three conjugacy classes of involutions. Thus H satisfies the hypotheses of Theorem B. Since H is a proper subgroup of G, we conclude from the minimality of G that $AO(H)\lhd H$. Thus $H/O(H)\cong A_5\cdot E_{16}^{(1)}$ and the lemma holds in this case.

We can therefore assume that $N_H(A)/C_H(A)$ is not isomorphic to A_5, in which case it is either isomorphic to $Z_2\times Z_2$ or to A_4. In either case, it follows by the Frattini argument that $N_H(A)\subseteq O(N_H(A))N_H(S)$. Since $O(N_H(A))$ centralizes A and $\langle a_1\rangle=Z(S)$, it follows that a_1 is isolated in $N_H(A)$. However, A is weakly closed in S with respect to H as it is the unique elementary abelian subgroup of S of order 16. Hence if a_1 is conjugate in H to an involution x of A, it is already conjugate to x in $N_H(A)$. We conclude therefore that a_1 is not conjugate in H to any other involution of A. But now (13.2) and (13.3) show that a_1 is not conjugate in H to any involution of S. Thus a_1 is isolated in H and so H is solvable by Lemma 14.2.

Since $|S|=64$, Proposition 14.1 now follows from Lemmas 14.4 to 14.7.

15. Proof of Theorem B. We can now easily establish Theorem B in essentially the same way as we derived Theorem A. We first prove

PROPOSITION 15.1. *G is balanced and connected.*

PROOF. By Proposition 14.1, the centralizer of every involution of G is solvable and hence is 2-constrained; so G is balanced by Lemma 2.2. Since $A \lhd S$ and A is elementary abelian of order 16, $SCN_3(S)$ is nonempty; so S and hence also G is connected by Lemma 2.4.

PROPOSITION 15.2. *O is a strongly flat A-signalizer functor on G.*

PROOF. As pointed out in Section 2, the fact that G is balanced implies that O is an A-signalizer functor on G.

Let H be a proper subgroup of G such that $A \cap H$ is noncyclic. We argue that H is strongly A-flat. If H is solvable, then H is 2-constrained and the assertion follows from Lemma 3.4; while if H has dihedral Sylow 2-subgroups, it follows from Lemma 3.5. However, by Proposition 14.1, either H satisfies one of these two conditions or else $H/O(H) \cong A_5 \cdot E_{16}^{(1)}$. But clearly $A_5 \cdot E_{16}^{(1)}$ is 2-constrained and hence so is H. Thus H is strongly A-flat in this case as well, by Lemma 3.4. It follows therefore from the definition that O is a strongly flat A-signalizer functor on G.

We now prove Theorem B. Since O is a strongly flat A-signalizer functor on G and A is elementary abelian of order 16, Theorem 2.1 yields that

$$W_A = \langle O(C_G(a)) \mid a \in A^{\#} \rangle \tag{1}$$

has odd order.

Since $O(G)=1$ and $SCN_3(2)$ is nonempty, Proposition 15.1 together with (1) shows that the hypotheses of Theorem 2.5 are satisfied and we conclude from that theorem that $O(C_G(x))=1$ for every involution x of G. In particular, $O(C_G(a_1))=1$. But now Lemma 8 of [20] is applicable and yields that $G \cong A_5 \cdot E_{16}^{(1)}$, contrary to our choice of G. Thus Theorem B is proved.

PART IV

Theorem C

16. Reduction to the case $O_{2',2}(G)=1$. We now begin the proof of Theorem C. Thus G is a group with Sylow 2-subgroup S of type $PSL(3,4)$ and we must show that $G/O(G)$ is isomorphic to a subgroup of $PGL(3,4)$. Since our conditions clearly carry over to $G/O(G)$, we can assume without loss that $O(G)=1$.

We have already discussed the structure of S in Section 4 preceding Lemma 4.7. We use the same notation. In particular, S is generated by the involutions $z_1, z_2, a_1, a_2, b_1, b_2$ satisfying

$$[a_1, b_1]=[a_2, b_2]=z_1, \quad [a_2, b_1]=z_2, \quad [a_1, b_2]=z_1z_2\,, \tag{1}$$

with all other commutators of two generators being trivial. We also set $z_3=z_1z_2$, $a_3=a_1a_2$, and $b_3=b_1b_2$.

We preserve this notation throughout. We note, as with T_1 in Section 5, that the given generators of S satisfy the same relations as those of equations (**) of Section 4 and so Lemma 4.7 can be applied to S.

In addition to the properties of S described in Lemma 4.7 we need the following elementary facts about S, proofs of which are left to the reader.

LEMMA 16.1. *The following conditions hold:*

(i) $Z(S)=S'=\Phi(S)=\langle z_1, z_2\rangle$.

(ii) $S/\langle z_1\rangle\cong S/\langle z_2\rangle\cong S/\langle z_1z_2\rangle\cong D_8*D_8$.

(iii) *S has precisely two elementary abelian subgroups of order* 16: $A=\langle z_1, z_2, a_1, a_2\rangle$ *and* $B=\langle z_1, z_2, b_1, b_2\rangle$. *Moreover, A and B are conjugate in* Aut(S) *and each is normal in S.*

(iv) *S has precisely fifteen maximal subgroups. Six are isomorphic to* $U_1=\langle z_1, z_2, a_1, a_2, b_1\rangle\cong(Z_2\times Z_2)\int Z_2$; *and nine to* $U_2=\langle z_1, z_2, a_1, b_1, a_2b_2\rangle$. *Moreover, U_2 is isomorphic to the group described in Lemma* 4.5.

(v) *If R is a subgroup of S of order* 8, *then* $R\cap Z(S)\neq 1$; *if R is elementary abelian, then* $C_S(R)=A$ *or* B.

(vi) *S has nine conjugacy classes of involutions, represented by* $z_1, z_2, z_3, a_1, a_2, a_3, b_1, b_2, b_3$.

(vii) $C_S(a_1)=C_S(a_2)=C_S(a_3)=A$ *and* $C_S(b_1)=C_S(b_2)=C_S(b_3)=B$.

We shall first prove

PROPOSITION 16.2. *If* $O_2(G)\neq 1$, *then one of the following holds*:

(i) *S is normal in G.*

(ii) $O_2(G)=A$ *or* B, $G=O_2(G)K$, *where* $K\cap O_2(G)=1$, $K\cong Z_3\times A_5$ *or* A_5, $G'=K'O_2(G)\cong A_5\cdot E_{16}^{(2)}$, *and correspondingly K contains* $\langle b_1, b_2\rangle$ *or* $\langle a_1, a_2\rangle$.

Furthermore, in each case G is isomorphic to a subgroup of $PGL(3, 4)$.

We divide the proof into several lemmas.

LEMMA 16.3. *If $Z(S)$ is normal in G, then S is normal in G.*

PROOF. Clearly $C_G(Z(S))\lhd G$. Put $C=C_G(Z(S))$ and $\bar{C}=C/Z(S)$. We need only prove that $\bar{S}\lhd\bar{C}$, for then $S\lhd C$, and, as $C\lhd G$, the desired conclusion $S\lhd G$ will follow at once. We examine $N_{\bar{C}}(\bar{S})$, which is the image of $N_C(S)$ in $\bar{C}$. By Lemma 4.7 (iv), $N_C(S)/SC_C(S)$ has order 1 or 3 and in the latter case a 3-element of C acts fixed-point-free on $S/Z(S)$. Corresponding statements hold therefore for $N_{\bar{C}}(\bar{S})/C_{\bar{C}}(\bar{S})$. Application of either Burnside's transfer theorem or Lemma 4.2 yields that $\bar{C}$ is solvable of 2-length 1. Since $O(G)=1$, also $O(C)=1$ and hence $O(\bar{C})=1$. Thus $\bar{S}\lhd\bar{C}$, as required.

LEMMA 16.4. *If G contains an isolated involution, then S is normal in G.*

PROOF. Let z be an isolated involution of G. Since $O(G)=1$, Glauberman's theorem implies that $z\in Z(G)$. In particular, $z=z_1$, z_2, or z_1z_2. Setting $\bar{G}=G/\langle z\rangle$, we have that $\bar{S}$ is extra-special of order 32 by Lemma 16.1 (ii). Hence $\bar{G}$ also has an isolated involution by Lemma 4.6. Since clearly $O(\bar{G})=1$, it follows that $Z(\bar{S})\subseteq Z(\bar{G})$. But $Z(\bar{S})$ is the image of $Z(S)$ in $\bar{G}$ and so $Z(S)$ is normal in G. Lemma 16.3 now implies the lemma.

LEMMA 16.5. *If A or B is normal in G, then Proposition* 16.2 (i) *or* (ii) *holds for G.*

PROOF. Since A and B are conjugate in $\mathrm{Aut}(S)$ by Lemma 16.1 (iii), the argument will be the same for A as for B; so for definiteness assume $A\lhd G$. We may also suppose that S is not normal in G. The Sylow 2-subgroup $\bar{S}$ of $\bar{G}=G/A$ is a four group. Furthermore, A is a Sylow 2-subgroup of $C_G(A)$ and so $C_G(A)$ has a normal 2-complement. Since $O(C_G(A))\subseteq O(G)=1$, it follows that $C_G(A)=A$. Thus $\bar{G}$ is isomorphic to a subgroup of $GL(4,2)\cong A_8$. But now Lemma 4.9 is applicable and, as $\bar{S}$ is not normal in $\bar{G}$, either $\bar{G}\cong A_5$, $Z_3\times A_5$, $Z_2\times S_3$ or $S_3\times S_3$.

Consider either of the first two cases. Since S splits over A, Gaschütz's theorem implies that G splits over A. If K is one of the complements of A in G, then $K\cong Z_3\times A_5$ or A_5, $K'\cong A_5$, and K' acts faithfully on A. Hence $G'=AK'$ and so G' has index 1 or 3 in G. If K' acted nontransitively on $A^{\#}$, then the Sylow 2-subgroup S of AK' would be isomorphic to one of $A_5\cdot E_{16}^{(1)}$ and hence would be of type A_8. However, a Sylow 2-subgroup of A_8 has a center of order 2, contrary to the fact that $|Z(S)|=4$. Hence K' acts transitively on the involutions of A and so $G'\cong A_5\cdot E_{16}^{(2)}$. Finally we can choose K so that $S\cap K$ is a Sylow 2-subgroup of K. Thus $Z(S)(S\cap K)$ is elementary abelian of order 16 and so $Z(S)(S\cap K)=B$. But K contains an element x of order 3 which normalizes, but does not centralize $S\cap K$. Since x normalizes $O_2(G)=A$ and $AB=S$, x normalizes S. Now Lemma 4.7 (v) implies that $S\cap K\sim\langle b_1, b_2\rangle$ in S, so replacing K by a conjugate we can assume $\langle b_1, b_2\rangle\subseteq K$.

To complete the proof, we shall argue now that the cases $\bar{G}\cong Z_2\times S_3$ or $S_3\times S_3$ are impossible. Indeed, in either case, $\bar{G}$ contains a normal subgroup $\bar{X}$ of order 3. We have $\bar{S}=\langle\bar{b}_1, \bar{b}_2\rangle$.

$\bar{X}$ centralizes $\bar{b}_i$ for some i, $1\leq i\leq 3$. Hence if X is a subgroup of order 3 in G, whose image is $\bar{X}$, we see that X normalizes $U=\langle A, b_i\rangle$. However, $U\cong(Z_2\times Z_2)\int Z_2$ and so $Z(U)=\langle z_1, z_2\rangle=Z(S)$. Thus X normalizes $Z(S)$. If $\bar{G}=\bar{S}\bar{X}$, then $G=SX$ and so $Z(S)\lhd SX=G$. In the contrary case, $\bar{G}\cong S_3\times S_3$ and $\bar{G}$ contains a second normal subgroup $\bar{Y}$ of order 3. If Y is a subgroup of order 3 in G

whose image is $\bar{Y}$, the same reasoning shows that Y normalizes $Z(S)$. Hence $Z(S) \lhd XYS = G$ in this case as well. But now $S \lhd G$ by Lemma 16.3, contrary to our present assumption.

LEMMA 16.6. *If G contains a normal subgroup of index 2, then S is normal in G.*

PROOF. Let H be a normal subgroup of index 2 of G. Then $U = S \cap H$ is a Sylow 2-subgroup of H and is a maximal subgroup of S. If $U \cong U_2$, then H has an isolated involution by Lemma 4.5 and hence so does G. Now Lemma 16.4 yields that S is normal in G.

We can therefore assume that $U \cong U_1$. In this case Lemma 4.4 implies that H has a normal subgroup K of index 2 with Sylow 2-subgroup $R = S \cap K$ which is elementary abelian of order 16. By Lemma 16.1 (iii), $R = A$ or B. Setting $N = N_G(R)$, we know that $S \subseteq N$ and that N has a normal subgroup of index 4. It follows therefore from Lemma 16.5, applied to $N/O(N)$, that $O(N)S \lhd N$, whence $N = O(N)N_N(S)$ by the Frattini argument. If x is an element of odd order in $N_N(S)$, then $x \in K$ and so $[S, x] \subseteq S \cap K = R$. Thus x centralizes S/R. We can therefore apply Lemma 4.7 (iii), to obtain that $N_N(S)/SC_N(S)$ has order 1 or 3 and in the latter case a 3-element of N acts fixed-point-free on R. It follows directly from these conditions that $N_K(R)/C_K(R)$ has the same properties. If its order is 1, K has a normal 2-complement and hence so does G, whence $G = S$. On the other hand, if its order is 3, Lemma 4.2 implies that $O(K)R$ is normal of index 3 in K. Clearly also in this case $K = O^2(K)$. But then we see that $K = O^2(G)$. Thus K is characteristic in G and so $O(K) \subseteq O(G) = 1$. We conclude that $R \lhd G$ and now Lemma 16.5 yields the desired conclusion that $S \lhd G$.

LEMMA 16.7. *If G is solvable, then S is normal in G.*

PROOF. Setting $H = O^{2'}(G)$, we have that $S \subseteq H$ and that H is characteristic in G. Hence $O(H) \subseteq O(G) = 1$. But $O^{2'}(H) = H$ and, as H is solvable, this implies that H has a normal subgroup of index 2. Thus H satisfies all the conditions of Lemma 16.6 and consequently $S \lhd H$. Since H is characteristic in G, we conclude that S is normal in G.

Finally, we prove

LEMMA 16.8. *If G is nonsolvable and $O_2(G) \neq 1$, then A or B is normal in G.*

PROOF. Set $X = O_2(G)$ and let Y be a nontrivial characteristic subgroup of X. Then $Y \lhd G$. If $|Y| \leq 4$, then $G/C_G(Y)$ is solvable. But Y contains a central involution of S and consequently $C = C_G(Y)S$ has an isolated involution. Applying Lemma 16.4 to $C/O(C)$, we conclude at once that C, and hence also $C_G(Y)$, is solvable. Thus G is solvable, contrary to hypothesis. Hence X does not possess a characteristic subgroup of order 2 or 4. On the other hand, $|S/X|\geq 4$ since G/X is nonsolvable. The only possibility therefore is that X is elementary abelian of order 8 or 16.

Taking $X=Y$, the argument of the preceding paragraph shows that $C_G(X)$ is solvable. Since $C_G(X)\lhd G$, $C_G(X)\subseteq S(G)$, the largest normal solvable subgroup of G. But $S(G)$, being solvable, is 2-constrained. Since $O(S(G))\subseteq O(G)=1$ and $O_2(S(G))=O_2(G)=X$, it follows that $C_G(X)\subseteq X$. In particular, $C_S(X)\subseteq X$. Now Lemma 16.1 (v) shows that $|X|\neq 8$. Thus $|X|=16$ and so $X=A$ or B by Lemma 16.1 (iii).

Proposition 16.2 now follows at once from Lemmas 16.5, 16.7, 16.8, and 4.8.

As a corollary, we have

PROPOSITION 16.9. *If z is a central involution of S, then $C_G(z)$ is solvable of 2-length 1.*

PROOF. Setting $H=C_G(z)$, we have $S\subseteq H$ and so $\bar{H}=H/O(H)$ has a Sylow 2-subgroup of type $PSL(3,4)$. Since $\langle\bar{z}\rangle$ is normal in $\bar{H}$, $O_2(\bar{H})\neq 1$ and so Proposition 16.2 applies to $\bar{H}$. Because $\langle\bar{z}\rangle$ is normal in $\bar{H}$, we conclude that $\bar{S}$ is normal in $\bar{H}$. Thus $\bar{H}$, and hence also H, is solvable of 2-length 1.

17. Subgroup structure of G. Now let G be a minimal counterexample to Theorem C. Clearly $O(G)=1$. By Proposition 16.2, we also have that $O_2(G)=1$. We shall determine the possible structure of every proper subgroup of G. We shall prove

PROPOSITION 17.1. *If H is a proper subgroup of G, then one of the following conditions holds:*

(i) *H has an isolated involution;*

(ii) *H is a solvable group;*

(iii) *H has a dihedral Sylow 2-subgroup;*

(iv) *$H/O(H)$ is isomorphic to $PSL(2,16)$ or $PSL^*(2,16)$;*

(v) *$H/O(H)$ contains a normal subgroup of odd index isomorphic $Z_2\times Z_2\times PSL(2,q)$, $q\equiv 3,5 \pmod 8$, $q\geq 5$;*

(vi) *$H/O(H)$ is isomorphic to a subgroup of $PGL(3,4)$ with Sylow 2-subgroup of order 2^6.*

We break up the proof into several lemmas.

LEMMA 17.2. *If $N=N_G(A)$ and $\bar{N}=N/O(N)$, then either*

(i) *$\bar{N}$ contains a normal subgroup of index 1 or 3 isomorphic to $A_5\cdot E_{16}^{(2)}$, or*

(ii) *$\bar{S}$ is normal of index 1, 3, or 9 in $\bar{N}$.*

Similar statements hold for $N_G(B)$.

PROOF. Since $S\subseteq N$, $\bar{N}$ has Sylow 2-subgroups of type $PSL(3,4)$. Since

$\bar{A} \lhd \bar{N}$, Proposition 16.2 yields the lemma.

As an immediate corollary we have

LEMMA 17.3. *The following conditions hold*:

(i) z_1 *has* 1, 3, *or* 15 *conjugates in* $N_G(A)$;

(ii) $|N_G(A)/C_G(A)|$ *is not divisible by* 7.

Similar statements hold for $N_G(B)$.

LEMMA 17.4. *If* R *is an elementary abelian subgroup of* G *of order* 8, *then* $|N_G(R)/C_G(R)|$ *is not divisible by* 7.

PROOF. Set $K=N_G(R)$ and $C=C_G(R)$. Without loss we can assume that $K \cap S$ is a Sylow 2-subgroup of K. By Lemma 16.1 (v), A or B, say A, is a Sylow 2-subgroup of C. Then $K=CN_K(A)$ by the Frattini argument. By Lemma 17.3 (ii), $|N_K(A)/C_K(A)|$ is not divisible by 7. Since $C_K(A) \subseteq C$, the lemma follows.

Now let R denote a Sylow 2-subgroup of H. Without loss we may assume $R \subseteq S$. We fix this notation.

LEMMA 17.5. *If* $|R| \leq 8$, *then either*

(i) H *has an isolated involution*;

(ii) H *is solvable*; *or*

(iii) R *is dihedral*.

PROOF. We have that either R is abelian, dihedral, or quaternion. The lemma follows in the usual way unless R is elementary abelian of order 8. However, in this case the preceding lemma shows that H has an isolated involution.

LEMMA 17.6. *If* $|R|=16$, *then either*

(i) H *has an isolated involution*;

(ii) H *is solvable*;

(iii) $H/O(H)$ *is isomorphic to* $PSL(2, 16)$;

(iv) $H/O(H)$ *contains a normal subgroup of odd index isomorphic to* $Z_2 \times Z_2 \times PSL(2, q)$, $q \equiv 3, 5 \pmod 8$, $q \geq 5$.

PROOF. Assume (i) is false. Then by Lemma 4.1, either $R \cong E_{16}, Z_2 \times D_8$, $Z_4 \times Z_4$, D_{16}, or R is quasi-dihedral of order 16. However, one checks directly that S does not contain subgroups of the last two types. Furthermore, Brauer's theorem [5] shows that H is solvable when $R \cong Z_4 \times Z_4$.

If $R \cong E_{16}$, we apply [27]. In view of Lemma 17.4, it follows that either (ii), (iii) or (iv) holds or else $\bar{H}=H/O(H)$ contains a normal subgroup $\bar{L}$ of the form $\bar{L}_1 \times \bar{L}_2$ with $\bar{L}_i \cong PSL(2, q_i)$, $q_i \equiv 3, 5 \pmod 8$, $1 \leq i \leq 2$, and $q_i \geq 5$, $i=1$ or 2. Since $R=A$ or B in this case, we can assume without loss that $R=A$. By Lemma 17.3 (i), z_1 has 1, 3 or 15 conjugates in $N_G(A)$. Since $N_{\bar{L}}(\bar{A}) \cong A_4 \times A_4$, z_1 must have 3 or 15 conjugates in $N_G(A)$ under the present assumptions. In the first case, it follows that $\bar{z}_1 \in \bar{L}_1$ or $\bar{L}_2$; while in the second, $\bar{z}_1$ is conjugate to an involution of $\bar{A} \cap \bar{L}_1$. Hence, in either case, there is an involution a in A which is conjugate to z_1 such that $\bar{a} \in \bar{L}_1$ or $\bar{L}_2$. However, $C_G(a)$ is nonsolvable, so $C_G(z_1)$ is as well, contrary to Proposition 16.9.

Suppose finally that $R \cong Z_2 \times D_8$. We may assume that H is nonsolvable. By the remark following Lemma 4.4, there exists an involution $\bar{t}$ of $\bar{H}$ such that $C_{\bar{H}}(\bar{t})$ contains a subgroup of the form $\langle \bar{t} \rangle \times \bar{K}$, where $\bar{K} \cong S_4$. Hence H contains an involution t such that $C_H(t)$ involves S_4 with t centralizing a dihedral subgroup of H of order 8. By Lemma 16.1 (vii), t is conjugate to an involution of $Z(S)$. But then $C_G(t)$ must be solvable of 2-length 1 by Proposition 16.9, contrary to the fact that $C_G(t)$ involves S_4.

LEMMA 17.7. *If* $|R|=32$, *then either*

(i) *H has an isolated involution;*

(ii) *H is solvable; or*

(iii) *$H/O(H)$ is isomorphic to* $PSL^*(2, 16)$.

PROOF. Then $R \cong U_1$ or U_2. In the latter case, H has an isolated involution by Lemma 4.5; while in the former case, H has a normal subgroup K of index 2 with $R \cap K$ elementary abelian of order 16. Setting $\bar{H}=H/O(H)$, the structure of $\bar{K}$ is given by Lemma 17.5. If (i) or (ii) of that lemma holds, then so does (i) or (ii) of the present lemma. If $\bar{K} \cong PSL(2, 16)$, then $\bar{H}$ must be isomorphic to $PSL^*(2, 16)$. Finally if $\bar{K}$ contains a normal subgroup of odd index of the form $\bar{T} \times \bar{L}$, where $\bar{T}$ is a four group and $\bar{L} \cong PSL(2, q)$ for some odd $q \geq 5$, then $\bar{T} \lhd \bar{R}$ and hence $\bar{T} \cap Z(\bar{R}) \neq 1$. But in this case, $Z(R)=Z(S)$, so there is an involution t in $Z(S)$ such that $\bar{t} \in \bar{T}$. However, $C_H(t)$ is nonsolvable as it involves $PSL(2, q)$ and this contradicts Proposition 16.9.

Finally we have

LEMMA 17.8. *If* $|R|=64$, *then* $H/O(H)$ *is isomorphic to a subgroup of* $PGL(3, 4)$.

PROOF. In this case $R=S$. Since $H \subset G$, Theorem C thus holds for H by our minimal choice of G and the lemma follows.

Now Proposition 17.1 follows Lemmas 17.5–17.8.

As a direct corollary of Proposition 17.1, we have

PROPOSITION 17.9. *The group G is simple.*

PROOF. Let H be a minimal normal subgroup of G, so that H is the direct product of isomorphic simple groups. If $H=G$, then H has only one factor and G is simple; so assume $H \subset G$. Since $O_{2',2}(G)=1$, the factors of H are nonsolvable.

The structure of H is given by Proposition 17.1 and it is immediate that part (iii), (iv) or (vi) of that proposition holds. Using the classification of groups with dihedral Sylow 2-subgroups together with Lemma 4.8, we conclude that $H \cong PSL(3, 4)$, $PSL(2, 16)$, A_7, or $PSL(2, q)$, q odd, $q \geq 5$.

In the first case, H contains a Sylow 2-subgroup of G and $C_G(H) \subseteq O(G)=1$, so G is isomorphic to a subgroup $PGL(3, 4)$ by Lemma 4.8 (iii), contrary to our choice of G. Since G does not contain an abelian subgroup of order 2^5, it follows in the second case as well that $C_G(H) \subseteq O(G)=1$, so G is isomorphic to a subgroup of $P\Gamma L(2, 16)$. Since $P\Gamma L(2, 16)/PSL(2, 16)$ is cyclic of order 4, we have, in fact, $G \cong P\Gamma L(2, 16)$ and $S/S\cap H$ cyclic of order 4. However, $S \cap H=A$ or B and $S/A \cong S/B \cong Z_2 \times Z_2$, giving a contradiction.

If $H \cong A_7$, $C_S(H) \neq 1$ and, as $C_S(H) \lhd S$, some involution t of $Z(S)$ centralizes H, contrary to Proposition 16.9. Consider the final possibility. As in the proof of Lemma 17.6, no involution of S induces a field automorphism of H. Hence $HS/C_S(H) \cong PSL(2, q)$ or $PGL(2, q)$. Since S is not dihedral, this forces $C_S(H) \neq 1$ and we reach the same contradiction as in the preceding case.

In Section 19, we shall show that if a is an involution of G which is not conjugate to a central involution of S, then

$$C_G(a)/O(C_G(a)) \cong Z_2 \times Z_2 \times K ,$$

where

$$K \cong Z_2 \times Z_2, \ A_4 \quad \text{or} \quad A_5 .$$

As in Section 10, once this result is established, Proposition 17.1 can be considerably sharpened. Assuming this result, we obtain, by an argument entirely similar to that of Proposition 10.5:

PROPOSITION 17.10. *If H is a proper subgroup of G, then one of the following conditions holds*:

(i) *H is 2-constrained;*

(ii) *H has dihedral Sylow 2-subgroups;*

(iii) *$H/O(H) \cong Z_2 \times A_5$, $Z_2 \times Z_2 \times A_5$, $A_4 \times A_5$;*

(iv) *$H/O(H) \cong PSL(2, 16)$ or $PSL^*(2, 16)$;*

(v) *$H/O(H) \cong PSL(3, 4)$ or $PGL(3, 4)$.*

18. Fusion of involutions. We shall also need some further information concerning the fusion of involutions in G.

LEMMA 18.1. *A and B are not conjugate in G. In particular, A and B are each weakly closed in S with respect to G.*

PROOF. If $A \sim B$ in G, then by Alperin's fusion theorem ([1] or Theorem 7.2.6

of [12]), there exists a tame intersection $Q=S\cap R$, R a Sylow 2-subgroup of G, with $A\subseteq Q$ and $A^x\neq A$ for some element x in $N_G(Q)$. Since A and B are the only elementary subgroups of order 16 in S, we must have $B=A^x$. Since $A^x\subseteq Q$ and $\langle A, B\rangle=S$, it follows that $Q=S$, so $x\in N_G(S)$. However, by Lemma 4.7(iii), $A\triangleleft N_G(S)$, so $A^x=A$, a contradiction.

As an immediate corollary, we have

LEMMA 18.2. *Two elements of A or B are conjugate in G if and only if they are conjugate in $N_G(A)$ or $N_G(B)$ respectively.*

We next prove

LEMMA 18.3. *$a_1\sim a_2\sim a_3$ and $b_1\sim b_2\sim b_3$.*

PROOF. Assume that $a_1\sim z_i$ for some $i, 1\leq i\leq 3$, in which case a_1 and z_i are conjugate in $N=N_G(A)$. Then $SO(N)$ is not normal in N, otherwise $Z(S)O(N)$ would be as well and then $z_i\in Z(S)$ would not be conjugate to a_1 in N. Hence by Proposition 16.2, $\bar{N}=N/O(N)\simeq Z_3\times A_5$ or A_5 and $\bar{N}$ acts transitively on the involutions of $\bar{A}$. In particular, $a_1\sim a_2\sim a_3$. Furthermore, $\bar{N}$ splits over $\bar{A}$ and $\langle\bar{b}_1, \bar{b}_2\rangle$ is contained in a complement $\bar{K}$ of $\bar{A}$ in $\bar{N}$. By the structure of $\bar{K}$, a 3-element $\bar{x}$ of $\bar{K}$ permutes $\bar{b}_1, \bar{b}_2, \bar{b}_3$ cyclically. Clearly $\bar{x}\in N_{\bar{N}}(\bar{S})$ and so there is a 3-element x in $N_N(S)$ which cyclically permutes b_1, b_2, b_3. Thus also $b_1\sim b_2\sim b_3$ and we are done in this case. A similar argument applies if $b_1\sim z_i$ for any $i, 1\leq i\leq 3$.

Thus we may assume $a_i\not\sim z_j$, $b_k\not\sim z_l$ for any $i, j, k, l, 1\leq i, j, k, l\leq 3$. Proposition 16.2 implies that both $N_G(A)$ and $N_G(B)$ have 2-length 1.

Suppose $a_1\sim b_i$ for some $i, 1\leq i\leq 3$. If A were a Sylow 2-subgroup of $C_G(a_1)$, then B would have to be one of $C_G(b_i)$ and then A would be conjugate to B in G, contrary to Lemma 18.1. Since $A\subseteq C_G(a_1)$, the only other possibility is that $C_G(a_1)$ contains a Sylow 2-subgroup R of G. But then as $a_1\in Z(R)$, we have $a_1\sim z_i$ for some $i, 1\leq i\leq 3$, contrary to assumption. Hence $a_1\not\sim b_i$ for any $i, 1\leq i\leq 3$. However, since G is simple, Thompson's lemma implies that a_1 is conjugate to an involution of $\langle B, a_2\rangle$. Our conditions together with Lemma 16.1(vi) force $a_1\sim a_2$. Now Lemma 18.2 implies that a_1 and a_2 are conjugate in $N_G(A)$. Since $N_G(A)$ has 2-length 1, we conclude that $N_G(A)$ contains a 3-element which cyclically permutes a_1, a_2, a_3. Thus $a_1\sim a_2\sim a_3$. Similarly $b_1\sim b_2\sim b_3$.

A portion of the above argument gives

LEMMA 18.4. *If a_i and b_j are not conjugate to z_k for any $i, j, k, 1\leq i, j, k\leq 3$, then*

(i) *a_i is not conjugate to b_j for any $i, j, 1\leq i, j\leq 3$.*

(ii) *$z_1\sim z_2\sim z_3$.*

PROOF. Under the given assumptions, A is a Sylow 2-subgroup of $C_G(a_i)$

and B is a Sylow 2-subgroup of $C_G(b_j)$. Hence if $a_i \sim b_j$, then $A \sim B$, contrary to Lemma 18.1. Thus (i) holds.

Since G does not contain an isolated involution, our assumptions together with Lemma 16.1 (vi) imply that for each i, $z_i \sim z_j$ for some $j \neq i$, $1 \leq i, j \leq 3$. Hence $z_1 \sim z_2 \sim z_3$ and so (ii) also holds.

As a corollary of Lemmas 18.3 and 18.4, we obtain

PROPOSITION 18.5. *One of the following holds:*

(i) *G has one conjugacy class of involutions;*

(ii) *G has two conjugacy classes of involutions; represented by either z_1 and a_1 or z_1 and b_1;*

(iii) *G has three conjugacy classes of involutions represented by z_1, a_1 and b_1.*

Finally we prove

LEMMA 18.6. *If $C_G(a_i)/O(C_G(a_i))$ contains a normal subgroup of odd index isomorphic to $Z_2 \times Z_2 \times PSL(2, q)$, $q \equiv 3, 5 \pmod 8$ for some i, $1 \leq i \leq 3$, then*

(i) $|N_G(S) : SC_G(S)| = 9$;

(ii) *$Z(S)$ is a Sylow 2-subgroup of $C_G(a_i)'$.*

(iii) *$N_G(S)$ contains a 3-element which centralizes $\langle a_1, a_2 \rangle$ and cyclically permutes z_1, z_2 and z_3.*

Similar statements hold for $C_G(b_i)/O(C_G(b_i))$, $1 \leq i \leq 3$.

PROOF. By our assumptions on $C_G(a_i)$, A is a Sylow 2-subgroup of $C_G(a_i)$. In particular, a_i is not conjugate to z_1 in G. Proposition 16.2 now yields that $N = N_G(A)$ has 2-length 1. Again by the structure of $C_G(a_i)$, $N - C_G(A)$ contains a 3-element x which centralizes a_i. Since N has 2-length 1, we can take x to normalize S. If $N = O(N)S\langle x \rangle$, then a_1 would not be conjugate to a_2 in N, which it is by Lemmas 18.2 and 18.3. Thus $|N_N(S) : SC_N(S)| > 3$ and now Lemma 4.7 (ii) implies (i).

Furthermore, x normalizes $Z(S)$ as it normalizes S, x centralizes a_i, and x normalizes but does not centralize A. Since $|A : \langle Z(S), a_i \rangle| = 2$, it follows that x does not centralize $Z(S)$. But then $[Z(S), x] = Z(S)$ and so $Z(S) \subseteq C_G(a_i)'$. On the other hand, it is immediate from the structure of $C_G(a_i)$ that a Sylow 2-subgroup of $C_G(a_i)'$ is a four group. Thus $Z(S)$ is a Sylow 2-subgroup of $C_G(a_i)'$ proving (ii). Moreover, x leaves $\langle a_1, a_2 \rangle$ invariant by our choice of a_1, a_2, so x centralizes $\langle a_1, a_2 \rangle$, proving (iii).

19. The structure of $C_G(a_1)$ and $C_G(b_1)$. In this section we shall determine the structure of $C_G(a_1)$ and $C_G(b_1)$ in the cases that a_1 or b_1 are not conjugate to z_1 in G. Our argument will be entirely similar to that of Section 11. We shall prove

PROPOSITION 19.1. *If a_1 is not conjugate to z_1 in G, then $C_G(a_1)/O(C_G(a_1))$ is* *isomorphic to* E_{16}, $Z_2\times Z_2\times A_4$, *or* $Z_2\times Z_2\times A_5$. *A similar statement holds for* $C_G(b_1)$ *if* b_1 *is not conjugate to* z_1 *in* G.

As usual, we carry out the proof in a sequence of lemmas. It will be enough to prove the result for $C_G(a_1)$. We assume throughout that a_1 is not conjugate to z_1 in G. We set $M=C_G(a_1)$ and $I=N_G(A)$. Our conditions imply that I does not act transitively on the involutions of A, whence $O(I)S\lhd I$ by Proposition 16.2. But now if we apply Lemma 4.7, it follows at once that

(1) $$|N_M(A):C_M(A)|=1 \quad \text{or} \quad 3$$

and that if this index is 3, then M contains a 3-element which centralizes $W=\langle a_1, a_2\rangle$ and acts fixed-point-free on $Z=\langle z_1, z_2\rangle$. Furthermore, since $a_1\not\sim z_1$ in G, a_1 is not in the center of a Sylow 2-subgroup of G and it follows that A is a Sylow 2-subgroup of M.

If M is nonsolvable, Proposition 17.1 (v) is applicable and yields that $\bar{M}=M/O(M)$ contains a normal subgroup $\bar{D}$ of odd index of the form

(2) $$\bar{D}=\bar{W}\times\bar{L}\,,$$

where $\bar{L}\cong PSL(2, q)$, $q\equiv 3, 5 \pmod 8$ and $q\geq 5$ with $\bar{Z}\subseteq\bar{L}$. If M is solvable, then the same conclusion holds with $\bar{L}=\bar{Z}$ or $\bar{L}\cong A_4$. In either of the latter cases or in the former case with $q=5$, we clearly have $\bar{M}=\bar{L}$ and $\bar{M}$ has one of the structures asserted in the proposition. Hence we suppose by way of contradiction that $q>5$.

We argue now as in the derivation of equations (3)-(6) of Section 11 and obtain

(3) $$\bar{M}=\bar{W}\times\bar{K}\,,$$

where $\bar{K}\supseteq\bar{L}$, $C_{\bar{K}}(\bar{L})=1$, and $\bar{K}$ is isomorphic to a subgroup of $P\Gamma L(2, q)$. Also

(4) $$C_M(W)=W\times K\,,$$

where $K/O(K)\cong\bar{K}$, and

(5) $$M=O(M)(W\times K)\,.$$

Once again we let L be the normal subgroup of K containing $O(K)$ such that $L/O(K)\cong\bar{L}$. We also have $C_M(W)=C_G(W)$. Furthermore, for some odd prime p there exists a nontrivial p-element y of L such that

(6) $$y^{z_1}=y \quad \text{and} \quad y^{z_2}=y^{-1}\,.$$

Again we fix such a prime p.

In addition, we set $N=C_G(z_1)$. Finally if H is a subgroup of G containing

S, we say that H is p-constrained *within* $\langle z_1\rangle$ if H is p-constrained and if $O_{p'}(H)\subseteq O(H)\langle z_1\rangle$. With all this notation fixed, we first prove

LEMMA 19.1. *N is p-constrained within $\langle z_1\rangle$.*

PROOF. Since N is solvable of 2-length 1 by Proposition 16.9, it is p-constrained. Since $y^{z_2}=y^{-1}$, it follows for the same reason that $y\in O(N)$. Hence z_2 does not centralize some S-invariant Sylow p-subgroup P of $O(N)$. Since $\bar{S}=S/\langle z_1\rangle$ is extra-special with center $\langle\bar{z}_2\rangle$, we see that $C_S(P)=\langle z_1\rangle$. Since $O_{p'}(N)\cap S$ clearly centralizes P, it follows that $O_{p'}(N)\cap S=\langle z_1\rangle$. In particular, $O_{p'}(N)$ thus has a normal 2-complement. Since $O(O_{p'}(N))$ is characteristic in $O_{p'}(N)$, $O(O_{p'}(N))\subseteq O(N)$ and we conclude that $O_{p'}(N)\subseteq O(N)\langle z_1\rangle$, which establishes the lemma.

As in Section 11, we now define the family $\mathcal{N}$ as the set of proper subgroups H of G which satisfy the following conditions:

(a) H contains S;

(b) H contains an S-invariant Sylow p-subgroup of $O(N)$;

(c) H covers $N/O(N)$.

Clearly $N\in\mathcal{N}$.

LEMMA 19.2. *If $H\in\mathcal{N}$, then the following conditions hold:*

(i) $O_{2',2}(H)\supset O(H)$;

(ii) *H is p-constrained within $\langle z_1\rangle$;*

(iii) *If Q is a nontrivial S-invariant p-subgroup of H such that $O_{p'}(H)Q$ is normal in H, then $N_G(Q)\in\mathcal{N}$.*

PROOF. By Proposition 17.1 $\bar{H}=H/O(H)$ is isomorphic to a subgroup of $PGL(3,4)$. Since G has more than one class of involutions, so does $\bar{H}$ and hence $\bar{H}$ does not contain a normal subgroup isomorphic to $PSL(3,4)$. But now (i) follows from Lemma 4.8.

If H has 2-length 1, then (ii) follows by essentially the same argument as in Lemma 19.1. In the contrary case, Lemma 4.8 implies that $O_2(\bar{H})\cong E_{16}$ and that $\bar{H}/O_2(\bar{H})\cong A_5$ or $Z_3\times A_5$. Clearly $\bar{z}_2\in O_2(\bar{H})$ and every nontrivial normal subgroup of $\bar{H}$ contains $\bar{z}_2$. Since z_2 does not centralize an S-invariant Sylow p-subgroup R of $O(H)$, it follows by the Frattini argument that $C_H(R)\subseteq O(H)$. This in turn implies that $O_{p'}(H)\subseteq O(H)$ and we conclude at once that H is p-constrained within $\langle z_1\rangle$. Thus (ii) holds in all cases.

Finally (iii) is established in essentially the same way as Lemma 11.3 (iii).

We also have

LEMMA 19.3. *Every element of $\mathcal{N}$ is p-stable.*

PROOF. We proceed as in Lemma 11.10. With the notation $P, x, F, V, \bar{F}$ as in that lemma, we obtain that $\bar{x}\in O(\bar{F})$, that $\bar{F}$ is faithfully and irreducibly represented on V regarded as a vector space over $GF(p)$, that $[V, \bar{x}, \bar{x}]=1$, and that $\bar{F}/O(\bar{F})$ is isomorphic to a nontrivial homomorphic image of $H/O(H)$. By the argument of the preceding lemma, either H has 2-length 1 and $H/O(H)$ is a $\{2, 3\}$-group or $O_{2',2}(H)/O(H)\cong E_{16}$, $H/O_{2',2}(H)\cong A_5$ or $Z_3\times A_5$, and $H/O(H)$ is a $\{2, 3, 5\}$-group. Since $\bar{x}\notin O(\bar{F})$, we conclude, in particular, that $p=3$ or 5.

As in Lemma 11.10, we set $\bar{R}=O_{p'}(O(\bar{F}))$ and $\bar{C}=C_{\bar{F}}(\bar{R})$ and obtain that $\bar{x}\in\bar{C}$ and that $\bar{C}\cap O(\bar{F})\subseteq\bar{R}$. Since $\bar{C}\lhd\bar{F}$, we also have that $\bar{C}\cap O(\bar{F})=O(\bar{C})$ and that $\bar{C}\cap O_{2',2}(\bar{F})=O_{2',2}(\bar{C})$. Thus $O(\bar{C})\subseteq\bar{R}$. Since $\bar{C}$ centralizes $\bar{R}$, it follows that $O_{2',2}(\bar{C})=O_2(\bar{C})\times Z(\bar{R})$.

Let $\bar{S}$ be the image of S in $\bar{F}$. Consider first that H, and hence also F, has 2-length 1, in which case $\bar{S}\subseteq O_{2',2}(\bar{F})$ and $p=3$. Since $\bar{x}\in\bar{C}$, we see that $[\bar{S}, \bar{x}]\subseteq O_{2',2}(\bar{F})\cap\bar{C}=O_{2',2}(\bar{C})=O_2(\bar{C})\times Z(\bar{R})$. Thus $\bar{x}$ centralizes the image of $\bar{S}$ in $\bar{C}/O_{2',2}(\bar{C})$ and, as $O_2(\bar{C})\subseteq\bar{S}$, it follows that $\bar{X}=Z(\bar{R})\bar{S}\langle\bar{x}\rangle$ is a group and that $Z(\bar{R})\bar{S}\lhd\bar{X}$. Since $\bar{R}$ is a $3'$-group and $\langle\bar{x}\rangle$ is a 3-group, a Hall-$\{2, 3\}$-subgroup of $\bar{X}$ containing $\langle\bar{x}\rangle$ possesses a normal Sylow 2-subgroup. We conclude therefore that $\bar{x}$ normalizes some Sylow 2-subgroup of $\bar{X}$. Without loss we can suppose that $\bar{x}$ normalizes $\bar{S}$. Since $\bar{x}\notin O(\bar{F})$, $\bar{x}$ does not centralize $\bar{S}$.

Since $[V, \bar{x}, \bar{x}]=1$ and $\bar{F}$ is faithfully represented on V, $\bar{x}$ must be of order 3. We need only show that $\bar{x}$ normalizes, but does not centralize, a four subgroup $\bar{T}$ of $\bar{S}$, for then $[V, \bar{x}, \bar{x}]\neq 1$ as $\bar{T}\langle\bar{x}\rangle$ is a 3-stable group, a contradiction. Since $O(H)S\lhd H$ in the present case, there exists an element x in $N_H(S)$ whose image in $\bar{F}$ is $\bar{x}$. But $S=AB$ and, by Lemma 4.7 (iii), x leaves both A and B invariant. Hence $\bar{S}=\bar{A}\bar{B}$ and $\bar{x}$ leaves both $\bar{A}$ and $\bar{B}$ invariant. Since $\bar{x}$ does not centralize $\bar{S}$, it does not centralize $\bar{A}$ or $\bar{B}$. Since $\bar{A}$ and $\bar{B}$ are abelian, the existence of the required four subgroup $\bar{T}$ follows at once.

Suppose finally that F is not of 2-length 1. Since $\bar{C}\lhd\bar{F}$ and $\bar{C}\cap O(\bar{F})=O(\bar{C})$, $\bar{C}/O(\bar{C})$ is isomorphic to a homomorphic image of a subgroup of $H/O(H)$. We know that $O_{2',2}(\bar{C})=O_2(\bar{C})\times Z(\bar{R})$ and $\bar{x}\notin O(\bar{C})$. Hence if $O_2(\bar{C})\neq 1$, it follows from the structure of $H/O(H)$ that $O_2(\bar{C})\cong E_{16}$ and that $\bar{x}$ does not centralize $O_2(\bar{C})$. But then $O_2(\bar{C})\langle\bar{x}\rangle$ is a p-stable group and consequently $[V, \bar{x}, \bar{x}]\neq 1$, a contradiction. On the other hand, if $O_2(\bar{C})=1$, then $O_{2',2}(\bar{C})=O(\bar{C})=Z(\bar{R})$ and so $\bar{C}/O(\bar{C})\cong A_5$, $Z_3\times A_5$, or Z_3. However, the representation of $\bar{C}$ on V is not p-stable and so $\bar{C}$ must involve $SL(2, p)$ by Theorem 3.8.3 of [12], contrary to the fact that a Sylow 2-subgroup of $\bar{C}$ is abelian in these cases. This completes the proof of the lemma.

We also have the following analogue of Lemma 11.4.

LEMMA 19.4. *If P is an S-invariant Sylow p-subgroup $O(N)$, then the* *following conditions hold*:

(i) *Either* $[C_P(W), z_2]$ *is noncyclic or* $[C_P(a_i), a_j, z_2] \neq 1$ *for some* i, j, $1 \leq i, j \leq 3$;

(ii) $[C_P(a_i), a_j] \neq 1$ *for some* i, j, $1 \leq i, j \leq 3$.

PROOF. As in Lemma 11.4, we consider the action of $\bar{S} = S/\langle z_1 \rangle$ on $\bar{P} = P/\Phi(P)$. We again have $C_{\bar{S}}(\bar{P}) = 1$. Since $\bar{z}_2 \in \bar{S}'$, the proof of (i) is identical to that of Lemma 11.4 (i). Likewise (ii) follows exactly as Lemma 11.4 (ii).

As in Section 11, if H is a subgroup of G containing A, we shall say that H is p-constrained *within* $\langle a_1 \rangle$ if H is p-constrained and $O_{p'}(H) \subseteq O(H)\langle a_1 \rangle$.

LEMMA 19.5. *M is p-constrained within* $\langle a_1 \rangle$.

PROOF. The proof is essentially identical to that of Lemma 11.5. However, we now use Lemma 18.6 (iii) in place of Proposition 6.9 (iv) to obtain the existence of a 3-element x of G which cyclically permutes a_1, a_2, a_3 and centralizes $Z = \langle z_1, z_2 \rangle$. Then using Lemma 19.4 (i) in place of Lemma 11.4 (i), we argue in the same way that z_2 does not centralize some A-invariant Sylow p-subgroup T of $O(M)$, and it again follows that M is p-constrained. Finally, We use Lemma 19.4 (ii) in place of Lemma 11.4 (ii) to establish that M is p-constrained within $\langle a_1 \rangle$.

We now define a second family $\mathscr{M}$ as the set of proper subgroups H of G which satisfy the following conditions:

(a) H contains A;

(b) H contains an A-invariant Sylow p-subgroup of $O(M)$;

(c) H covers $M/O(M)$.

Clearly $M \in \mathscr{M}$.

LEMMA 19.6. *If* $H \in \mathscr{M}$, *then the following conditions hold*:

(i) $O(H)(H \cap M)$ *is normal in H of index* 1 *or* 3;

(ii) *H is p-constrained within* $\langle a_1 \rangle$;

(iii) *If Q is a nontrivial A-invariant p-subgroup of H such that* $O_{p'}(H)Q$ *is normal in H, then* $N_G(Q) \in \mathscr{M}$.

PROOF. The proof is essentially identical to that of Lemma 11.6. One uses Proposition 17.1 and Lemma 4.8 in place of Propositions 10.2, 10.3 and 10.4 to conclude in the present case that A must be a Sylow 2-subgroup of H.

We have just seen that every element of $\mathscr{M}$ has A as a Sylow 2-subgroup. Since A is abelian, we thus have

LEMMA 19.7. *Every element of $\mathscr{M}$ is p-stable.*

We have now established all the analogues of the results of Section 11 which are needed to derive the following analogue of Lemma 11.11 by the identical argument:

LEMMA 19.8. *There exist Sylow p-subgroups P_1 and P_2 of G such that $N_G(Z(J(P_1)))\in \mathscr{N}$ and $N_G(Z(J(P_2)))\in \mathscr{M}$.*

Lemma 19.8 leads to the same contradiction as in Section 11 and therefore Proposition 19.1 is proved.

20. Proof of Theorem C. We first prove

PROPOSITION 20.1. *G is balanced and connected.*

PROOF. By Lemma 16.1 (iii), $SCN_3(S)$ is nonempty, so G is connected by Lemma 2.4. The structure of the centralizer of an involution of G is given by Propositions 16.9 and 19.1. Using Lemma 2.3, as we did in the proof of Proposition 12.1, it follows that G is balanced.

PROPOSITION 20.2. *O is a strongly flat A-signalizer functor on G.*

PROOF. Since G is balanced, O is an A-signalizer functor on G. Using Proposition 17.10 together with the various lemmas of Section 3, it follows just as in the proof of Proposition 12.2 that O is strongly flat.

PROPOSITION 20.3. *$O(C_G(x))=1$ for every involution x of G.*

PROOF. Proposition 20.2 and Theorem 2.1 together yield that $W_A=\langle O(C_G(a))\mid a\in A^{\#}\rangle$ has odd order. Combined with Proposition 20.1 and the fact that G is simple and $SCN_3(2)$ is nonempty, we see that the hypotheses of Theorem 2.5 are satisfied. But now the present proposition follows from that theorem.

In contrast with Theorems A and B, there is no classification theorem which can now be immediately invoked to complete the proof of Theorem C. One requires a further argument to reach the assumptions of Suzuki's characterization of $PSL(3,4)$, [24], which depends critically upon Proposition 20.3.

PROPOSITION 20.4. *The centralizer of every involution of G is 2-closed.*

PROOF. Let a be an involution of G, so that by Proposition 18.5, a is conjugate to z_1, a_1, or b_1. Propositions 16.9, 19.1 and 20.3 together imply that either $C_G(a)$ is 2-closed or else $a\not\sim z_1$ and $C_G(a)\cong Z_2\times Z_2\times A_5$. Since our argument will be the same whether $a\sim a_1$ or $a\sim b_1$, we may assume for definiteness that $a\sim a_1$. Hence, by Proposition 19.1 and equation (5) of Section 19,

$$C_G(a_1)=\langle a_1, a_2\rangle\times F, \quad \text{where} \quad F\cong A_5\,. \tag{1}$$

We shall examine the various possibilities for $C_G(b_1)$ in succession and shall derive a contradiction in each case. We carry out the argument in a sequence of lemmas. Note that by equation (1), a_1 is not conjugate to z_i for any i, $1\leq i\leq 3$.

LEMMA 20.5. *A Sylow 5-subgroup P of $C_G(a_1)$ is a Sylow 5-subgroup of G and $\langle a_1, a_2\rangle$ is a Sylow 2-subgroup of $C_G(P)$.*

PROOF. By Lemma 18.6 (ii), $\langle z_1, z_2\rangle$ is a Sylow 2-subgroup of $C_G(a_1)'\cong A_5$.

Since A_5 has only one conjugacy class of involutions and since a Sylow 5-subgroup of A_5 is inverted by an involution, we can choose a Sylow 5-subgroup Q of $C_G(a_1)$ to be inverted by z_1. If the lemma holds for Q, then it clearly also holds for P. Hence without loss we may assume $Q=P$.

We set $C=C_G(P)$ and $H=\langle z_1\rangle C$. Now $S\lhd C_G(z_1)$ and $O(C_G(z_1))=1$. Hence $C_G(z_1)$ is a $\{2, 3\}$-group by Lemma 4.7 and consequently no involution of C is conjugate to a central involution. Since every subgroup of S of order at least 8 contains a central involution of S by Lemma 16.1 (v) and since $\langle a_1, a_2\rangle\subseteq C$, we conclude that $\langle a_1, a_2\rangle$ is a Sylow 2-subgroup of C. We shall argue that C is solvable, so assume the contrary.

Clearly $O(H)=O(C)$. Hence if we set $\bar{H}=H/O(H)$, it follows from the main theorem of [15] that $\bar{H}$ contains a normal subgroup $\bar{K}=\bar{X}\times\bar{L}$, where $|\bar{X}|=2$ and $\bar{L}\cong PSL(2, q)$, $q\equiv 3, 5 \pmod 8$ and $q\geq 5$. We set $V=\langle z_1, a_1, a_2\rangle$, so that V is a Sylow 2-subgroup of H and $\bar{V}$ is a Sylow 2-subgroup of $\bar{K}$. Thus $\bar{X}\subseteq\bar{V}$ and there exists an element $\bar{y}$ in $\bar{L}$ of order 3 which normalizes $\bar{V}$ and satisfies $C_{\bar{V}}(\bar{y})=\bar{X}$. Let y be an element of $N_H(V)$ which maps onto $\bar{y}$. It will suffice to prove that $N_G(V)$ normalizes S, for then y will normalize S and hence also $Z(S)$, whence y will centralize $Z(S)\cap V=\langle z_1\rangle$. But then $\bar{z}_1\in C_{\bar{V}}(\bar{y})$, forcing $\bar{X}=\langle\bar{z}_1\rangle$. This in turn will imply that $C_G(z_1)$ is nonsolvable, contrary to the fact that $C_G(z_1)$ is 2-closed.

Since $C_G(V)\subseteq C_G(z_1)$, $C_G(V)$ is 2-closed. But A is a Sylow 2-subgroup of $C_G(V)$ by Lemma 16.1 (v), so A is characteristic in $C_G(V)$. Since $N_G(V)$ normalizes $C_G(V)$, it follows that $N_G(V)\subseteq N_G(A)$. Thus we need only show that $S\lhd N_G(A)$. We have $C_G(A)\subseteq C_G(z_1)$, $O(C_G(z_1))=1$ and $S\lhd C_G(z_1)$. Hence any nontrivial element of $C_G(A)$ of odd order must induce a nontrivial automorphism of S, which contradicts Lemma 4.7 (iv). Thus $C_G(A)=A$ and consequently $O(N_G(A))=1$. Furthermore, $N_G(A)$ acts nontransitively on the involutions of A as a_1 and z_1 are not conjugate. But now the desired conclusion $S\lhd N_G(A)$ follows from Proposition 16.2. We conclude finally that $C=C_G(P)$ is solvable.

We can now quickly complete the proof of the lemma. By equation (1) above, we have $C_G(a_1)=C_G(a_2)=C_G(a_3)$, so P is a maximal subgroup of odd order in $C_G(a_i)$ for all i, $1\leq i\leq 3$. Since $O(C)=\langle C_{O(C)}(a_i)\,|\,1\leq i\leq 3\rangle$, it follows that $O(C)=P$. But $C/O(C)\cong Z_2\times Z_2$ or A_4 as it is solvable, so P is a Sylow 5-subgroup of G. Since $\langle a_1, a_2\rangle$ is a Sylow 2-subgroup of C, the lemma is proved.

LEMMA 20.6. *$C_G(b_1)$ is not isomorphic to $C_G(a_1)$.*

PROOF. Assume false. Then by the identical argument as above, we can conclude that $\langle b_1, b_2\rangle$ is a Sylow 2-subgroup of the centralizer of some Sylow 5-

subgroup P^* of G. Since $P \sim P^*$ in G, this would imply that $\langle b_1, b_2 \rangle \sim \langle a_1, a_2 \rangle$. This conflicts with Lemma 18.4.

LEMMA 20.7. *$C_G(b_1)$ is not isomorphic to E_{16} or $Z_2 \times Z_2 \times A_4$.*

PROOF. Assume false, in which case $C_G(b_1)$ is 2-closed and b_1 is conjugate to neither z_1 nor a_1. Let b_1' be an arbitrary conjugate of b_1 in G. Then since b_1' and z_1 are not conjugate in G, $b_1' \cdot z_1$ must have even order, so there exists an involution t in G which centralizes both b_1' and z_1. Since $C_G(z_1)$ is 2-closed, $t \in S$. If $t \in Z(S)$, then $b_1' \in S$ as $C_G(t)$ would then also be 2-closed. If $t \in A - \langle z_1, z_2 \rangle$, then $t \sim a_1$ in S by Lemmas 16.1 (vi) and 18.3 and so A is a Sylow 2-subgroup of $C_G(t)$. But then, as $b_1' \in C_G(t)$, b_1' is conjugate to an involution of A and so to either z_1 or a_1, which is not the case. Finally if $t \in B - \langle z_1, z_2 \rangle$, then $t \sim b_1$ in S by Lemma 16.1 (vi) and 18.3, whence $C_G(t)$ is 2-closed and has B as Sylow 2-subgroup. Thus $b_1' \in B \subseteq S$ and we conclude that $b_1' \in S$. Hence S contains every conjugate of b_1 and so $O_2(G) \neq 1$, contrary to the fact that G is simple.

Finally we prove

LEMMA 20.8. *b_1 is not conjugate to z_1 in G.*

PROOF. Assume the contrary. Then by Lemma 18.2 and Proposition 16.2 $N_G(B)$ acts transitively on the involutions of B and $N_G(B)/C_G(B) = N_G(B)/B$ is a split extension of B by a group $K \cong Z_3 \times A_5$ or A_5, and $\langle a_1, a_2 \rangle \subseteq K$. Furthermore, by Lemma 18.6 (i), $|N_G(S) : S| = 9$. Since $N_G(S)$ normalizes B by Lemma 4.7 (iii), we have, in fact, that $N_G(B)/B$ and hence also K is isomorphic to $Z_3 \times A_5$. We set $D = Z(K)$, so that $|D| = 3$. By the structure of $C_G(a_1)$, D is a Sylow 3-subgroup of $C_G(a_1)$ and is inverted by an involution w of $C_G(a_1)$.

We set $C = C_G(D)$, $H = \langle w \rangle C$, and $V = \langle a_1, a_2, w \rangle$. In particular, V is elementary abelian of order 8. Let R be a Sylow 2-subgroup of G containing V such that $R \cap H$ is a Sylow 2-subgroup of H. Then $R \cap C$ is a Sylow 2-subgroup of C as $C \triangleleft H$ and $R \cap C \supseteq \langle a_1, a_2 \rangle$. If $R \cap C \supset \langle a_1, a_2 \rangle$, then $R \cap C$ contains an involution t of $Z(R)$ by Lemma 16.1 (v). But then $t \sim z_1$ and so $C_G(t)$ is 2-closed. Since $\langle R, D \rangle \subseteq C_G(t)$, we see that D normalizes R. But $R \cap H$ normalizes D as $D \triangleleft H$, so $R \cap H$ centralizes D, contrary to the fact that $w \in R \cap H$ and w inverts D. Thus $R \cap C = \langle a_1, a_2 \rangle$, $\langle a_1, a_2 \rangle$ is a Sylow 2-subgroup of C, and $R \cap H = V$.

On the other hand, C contains K and so C involves A_5. Hence if we apply the main theorem of [15] to $\bar{H} = H/O(H)$ as we did in Lemma 20.5, it follows that $\bar{H}$ contains a normal subgroup of the form $\bar{X} \times \bar{L}$ with $|\bar{X}| = 2$ and $\bar{L} \cong PSL(2, q)$, $q \geq 5$. Reasoning now exactly as we did in Lemma 20.5, but with R in place of S, we conclude that $V \cap Z(R) = \langle v \rangle$ is of order 2 and $\langle \bar{v} \rangle = \bar{X}$. But then $C_G(v)$ involves $PSL(2, q)$ and so is nonsolvable, contrary to the fact that $v \sim z_1$. The lemma is proved.

This establishes Proposition 20.4. Indeed, since $b_1 \not\sim z_1$ by Lemma 20.8, $C_G(b_1) \cong Z_2 \times Z_2 \times F$, where $F \cong Z_2 \times Z_2$, A_4 or A_5 by Propositions 19.1 and 20.3. However, this contradicts Lemmas 20.6 and 20.7.

Because of Proposition 20.4, the main theorem of [24] can be applied to G. Since G is simple with Sylow 2-subgroups of type $PSL(3, 4)$, we conclude that $G \cong PSL(3, 4)$. This completes the proof of Theorem C.

Added in Proof: Recently David Goldschmitt, using ideas of Bender's proof of Thompson's so-called "Uniqueness Theorem", has extended the results of [13] and [14] to *arbitrary* A-signalizer functors of rank 4. This means that our Theorem 2.1 above holds *without* the assumption of *strong flatness*.

This result gives a considerable simplification in the proofs of Theorems A, B, and C. In each case, the complete analysis of the subgroup structure of the given group G which we have carried out is no longer needed. Instead all we now need is a knowledge of the structure of the centralizers of the involutions of G.

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