On hyperelliptic surfaces

By Tatsuo SUWA

The classification of hyperelliptic surfaces was established earlier by Italian geometers (Bagnera-Franchis [1], Enriques-Severi [2]). The purpose of this note is to take up this classification, viewing hyperelliptic surfaces as elliptic bundles.

Section 1 contains preliminary considerations about elliptic bundles, which is a special case of the theory of elliptic surfaces of Kodaira [5]. We define a hyperelliptic surface to be an elliptic bundle of which the first Betti number is equal to 2. Lemmas 1 and 2 assert that our definition coincides with that of "irregular hyperelliptic surfaces of rank $r>1$" in [2]. The classification of these surfaces is given in Theorem in Section 2. As a corollary to this theorem it is shown that the plurigenera of hyperelliptic surfaces are topological invariants. In Section 3 we give various characterizations of hyperelliptic surfaces. Especially we show that every hyperelliptic surface has two different fiberings of elliptic curves. It is this fact that Enriques and Severi used as a clue to the classification.

§1. Elliptic bundles.

By a surface we shall mean a compact complex manifold of complex dimension 2. A surface S is said to be an *elliptic surface* if there exists a holomorphic map Ψ of S onto a non-singular curve Δ such that the inverse image $\Psi^{-1}(u)$ of any general point $u \in \Delta$ is an elliptic curve. For the theory of elliptic surfaces we refer to Kodaira [5]. Let us call the elliptic surface $\Psi: S \to \Delta$ an *elliptic bundle* over Δ if Ψ is everywhere of maximal rank, i.e., S is free from singular fibres over Δ. In this case the functional invariant $\mathscr{J}(u)$ of the elliptic surface S is reduced to a constant. Hence all the fibres are complex analytically homeomorphic to one and the same elliptic curve C, and S is the total space of a complex analytic fibre bundle over Δ with fibre C.

Let $\Psi: S \to \Delta$ be an elliptic bundle over an elliptic curve Δ with fibre C. We represent C as a quotient group: $C = \boldsymbol{C}/\Gamma$, where Γ is a discontinuous subgroup of the additive group of complex numbers $\boldsymbol{C}$ generated by ω and 1, $\operatorname{Im}\omega>0$, and, for any $\zeta \in \boldsymbol{C}$, we denote by $[\zeta]$ the corresponding element of $C=\boldsymbol{C}/\Gamma$. Choose a finite covering $\{U_j\}$ of Δ by small disks U_j. Then the surface S can be described as follows: $S = \bigcup_j U_j \times C$, where $(u, [\zeta_j]) \in U_j \times C$ and $(u, [\zeta_k]) \in U_k \times C$ are identified if and only if

$$[\zeta_j]=[\varepsilon_{jk}\zeta_k+\eta_{jk}(u)]\ .$$

The constant ε_{jk} is a root of unity representing an automorphism: $[\zeta]\to[\varepsilon_{jk}\zeta]$ of C and $\eta_{jk}(u)$ is a holomorphic function of $u\in U_j\cap U_k$. Moreover we have $\varepsilon_{ik}=\varepsilon_{ij}\varepsilon_{jk}$ and $[\eta_{ik}(u)]=[\eta_{ij}(u)+\varepsilon_{ij}\eta_{jk}(u)]$, for $u\in U_i\cap U_j\cap U_k$. The automorphism $[\zeta]\to[\varepsilon_{jk}\zeta]$ of C induces a linear isomorphism $(\varepsilon_{jk})_*$ of the homology group $H_1(C, Z)\simeq\Gamma\simeq Z\oplus Z$. The homological invariant G of S is a locally constant sheaf over Δ defined as follows: $G=\bigcup_j U_j\times H_1(C, Z)$, where $(u, \gamma_j)\in U_j\times H_1(C, Z)$ and $(u, \gamma_k)\in U_k\times H_1(C, Z)$ are identified if and only if $\gamma_j=(\varepsilon_{jk})_*\gamma_k$. The basic member B of the family $\mathcal{F}(\mathcal{J}, G)$ consisting of all elliptic surfaces whose functional and homological invariants are $\mathcal{J}$ and G is defined as follows: $B=\bigcup_j U_j\times C$, where $(u, [\zeta_j])$ and $(u, [\zeta_k])$ are identified if and only if $[\zeta_j]=[\varepsilon_{jk}\zeta_k]$. Let $\Omega(B)$ denote the sheaf over Δ of germs of holomorphic sections of B. Moreover let $\mathfrak{f}$ be the line bundle over Δ defined by the 1-cocycle $\{\varepsilon_{jk}\}$. Then we have the exact sequence (cf. [5] Theorem 11.2)

$$0\longrightarrow G\longrightarrow\Omega(\mathfrak{f})\longrightarrow\Omega(B)\longrightarrow 0$$

and the corresponding exact cohomology sequence

$$(1)\qquad \cdots\longrightarrow H^1(\Delta, \Omega(\mathfrak{f}))\xrightarrow{h} H^1(\Delta, \Omega(B))\xrightarrow{c} H^2(\Delta, G)\longrightarrow 0\ .$$

The elliptic surface S is denoted by B^η, where η is the cohomology class in $H^1(\Delta, \Omega(B))$ represented by the 1-cocycle $\{[\eta_{jk}(u)]\}$.

On the other hand we have the exact homotopy sequence

$$1\longrightarrow\pi_1(C)\longrightarrow\pi_1(S)\longrightarrow\pi_1(\Delta)\longrightarrow 1\ .$$

Since $\pi_1(\Delta)\simeq Z\oplus Z$ is abelian, we have the surjective homomorphism $H_1(S, Z)=\pi_1(S)/[\pi_1(S), \pi_1(S)]\longrightarrow\pi_1(\Delta)$. Hence we infer that the first Betti number b_1 of S satisfies the inequality $2\leqq b_1\leqq 4$. If the homological invariant G is trivial then $b_1=4$ or 3 according as $c(\eta)=0$ or $c(\eta)\neq 0$ (see [5] Theorem 11.9). On the other hand we infer readily that if G is non-trivial, then $b_1=2$.

§ 2. Classification of hyperelliptic surfaces.

DEFINITION. By a *hyperelliptic surface* we shall mean an elliptic bundle over an elliptic curve of which the first Betti number is equal to 2.

Remark. As will be shown in the next section, the above definition is equivalent to saying that a hyperelliptic surface is a surface of geometric genus zero having an Abelian variety as its finite unramified covering manifold. In Enriques-

Severi [2] the surfaces of this type are called "irregular hyperelliptic surfaces of rank $r>1$".

Let $\Psi: S \longrightarrow \Delta$ be an elliptic bundle over an elliptic curve Δ and let K be the canonical bundle of S. Then K is given by $K=\Psi^*(-\mathfrak{f})$ ([6] I, Theorem 12). Since $\varepsilon_{jk}^{12}=1$, we have $c(\mathfrak{f})=0$ and $12\mathfrak{f}=0$. Moreover $\mathfrak{f}=0$ if and only if G is trivial. Hence hyperelliptic surfaces are classified into the following four types:

I) $2K=0$ $(K\neq 0)$, II) $3K=0$ $(K\neq 0)$,
III) $4K=0$ $(2K\neq 0)$, IV) $6K=0$ $(2K, 3K\neq 0)$.

The fibres of hyperelliptic surfaces of types II) and IV) are equianharmonic elliptic curves, and that of the surfaces of type III) are harmonic elliptic curves. Note that, $S=B^{\eta}$ is a hyperelliptic surface, if and only if the basic member B is a hyperelliptic surface. Moreover S and B belong to the same class. From the general theory of classification of compact surfaces due to Kodaira we infer that every hyperelliptic surface is an algebraic surface with the following numerical characters: $p_g=0$, $q=1$, $c_1^2=c_2=0$, where p_g, q and c_ν denote, respectively, the geometric genus, the irregularity and the ν-th Chern class.

Let $\Psi: S\rightarrow\Delta$ be a hyperelliptic surface. Then $c(\mathfrak{f})=0$ and $\mathfrak{f}\neq 0$. Hence we get $H^1(\Delta, \Omega(\mathfrak{f}))=0$ and, consequently, we have, by (1), the isomorphism $H^1(\Delta, \Omega(B))\simeq H^2(\Delta, G)$. Recall that the cohomology group $H^2(\Delta, G)$ is finite, since the sheaf G is non-trivial ([5] Theorem 11.7). Take a suitable finite unramified covering $f: \tilde{\Delta}\rightarrow\Delta$ of Δ. Then the fibre space $\tilde{B}=B\times_{\Delta}\tilde{\Delta}$ induced from $\Phi: B\rightarrow\Delta$ by the covering map f is trivial: $\tilde{B}=\tilde{\Delta}\times C$. The number of sheets of f is equal to 2, 3, 4 or 6 according as S (or B) is of type I), II), III) or IV), and $\tilde{\Delta}$ is also an elliptic curve. Considering the fibre space $\tilde{S}=S\times_{\Delta}\tilde{\Delta}$ induced from $\Psi: S\rightarrow\Delta$ by the covering map f, we have an elliptic bundle $\tilde{\Psi}: \tilde{S}\rightarrow\tilde{\Delta}$. The basic member of $\tilde{S}$ is obviously $\tilde{\Phi}: \tilde{B}\rightarrow\tilde{\Delta}$, where $\tilde{\Phi}$ denotes the canonical projection. The map f induces a homomorphism $f^*: H^1(\Delta, \Omega(B))\rightarrow H^1(\tilde{\Delta}, \Omega(\tilde{B}))$, and it is easy to see that if $S=B^{\eta}$, $\eta\in H^1(\Delta, \Omega(B))$, then $\tilde{S}=\tilde{B}^{f^*\eta}$. On the other hand, the sheaf G is the quotient of the homological invariant $\tilde{G}=\tilde{\Delta}\times H_1(C, \boldsymbol{Z})$ of $\tilde{S}$ (or $\tilde{B}$) by the group induced from the group of covering transformations of $\tilde{B}$ over B. Hence we have a homomorphism $f^*: H^2(\Delta, G)\rightarrow H^2(\tilde{\Delta}, \tilde{G})$ and the commutative diagram:

$$\begin{array}{ccc} H^1(\Delta, \Omega(B)) & \xrightarrow{c} & H^2(\Delta, G) \\ {\scriptstyle f^*}\downarrow & & \downarrow{\scriptstyle f^*} \\ H^1(\tilde{\Delta}, \Omega(\tilde{B})) & \xrightarrow{c} & H^2(\tilde{\Delta}, \tilde{G}) . \end{array}$$

As was mentioned above the group $H^2(\Delta, G)$ is finite, while the group $H^2(\tilde{\Delta}, \tilde{G})$

is isomorphic to $Z\oplus Z$, hence $f^*H^2(\Delta, G)=0$. Consequently, we have $c(f^*\eta)=f^*c(\eta)=0$. In view of the exact cohomology sequence

$$\cdots\longrightarrow H^1(\tilde{\Delta}, \mathcal{O}_{\tilde{\Delta}})\longrightarrow H^1(\tilde{\Delta}, \Omega(\tilde{B}))\xrightarrow{c} H^2(\tilde{\Delta}, \tilde{G})\longrightarrow 0\ ,$$

we infer that the surface $\tilde{S}$ is a deformation of $\tilde{B}=\tilde{\Delta}\times C$. Since S is algebraic $\tilde{S}$ is also algebraic. Thus we obtain the following

LEMMA 1. *Any hyperelliptic surface has an Abelian variety as its finite unramified covering manifold.*

Now we can determine all the hyperelliptic surfaces as follows:

I) $2K=0, K\neq 0$. In this case we have $H^2(\Delta, G)\simeq Z_2\oplus Z_2$. Hence we have four hyperelliptic surfaces for fixed functional and homological invariants. Let α and β be generators of the fundamental group $\pi_1(\Delta)$. Then there are three possible choices of the homological invariant G corresponding to the representations: i) $\alpha\mapsto -1_2, \beta\mapsto 1_2$, ii) $\alpha\mapsto 1_2, \beta\mapsto -1_2$ and iii) $\alpha\mapsto -1_2, \beta\mapsto -1_2$, where 1_2 denotes the unit 2×2 matrix. But it is easy to see that these representations are essentially the same if we make a suitable change of the generators α and β of $\pi_1(\Delta)$. We may assume that the period matrix of the Abelian variety $\tilde{S}$ is of the form $\begin{pmatrix}1 & 0 & \tau & 0\\ 0 & 1 & \eta & \omega\end{pmatrix}$, where $(1, \omega)$ is the periods of the fibre C and $(1, \tau)$ is the periods of the base curve $\tilde{\Delta}$. That is $\tilde{S}$ is the quotient space of $C\times C$ by the group generated by the following four automorphisms: $(u, \zeta)\mapsto(u+1, \zeta)$, $(u, \zeta)\mapsto(u, \zeta+1)$, $(u, \zeta)\mapsto(u+\tau, \zeta+\eta)$ and $(u, \zeta)\mapsto(u, \zeta+\omega)$. η is so determined that there exists an automorphism g of $\tilde{S}$ of the form $g:(u, \zeta)\mapsto\left(u+\frac{1}{2}, -\zeta\right)$. By an elementary calculation, we can show that η is given by $\eta=\frac{n}{2}+\frac{m}{2}\omega$, where $n, m=0$ or 1. Corresponding to these four possible values of η we have four hyperelliptic surfaces with the same basic member. But it is easily seen that three surfaces corresponding to $\eta=\frac{1}{2}, \frac{\omega}{2}$ or $\frac{1+\omega}{2}$ can be expressed in the same manner if we make a suitable change of the periods representing the Abelian variety $\tilde{S}$. Thus we have only to consider the surfaces corresponding to $\eta=0$ and $\frac{1}{2}$.

II) $3K=0, K\neq 0$. In this case, the fibre C is an equianharmonic curve and we may let the periods of C be $(1, \rho^2)$, where $\rho=\exp\frac{\pi i}{3}$. We have $H^2(\Delta, G)\simeq Z_3$. Hence there exist three hyperelliptic surfaces for a fixed homological invariant. There are eight possible choices of the homological invariant G. But as in I) we have only to consider the sheaf G corresponding to the representation: $\alpha\mapsto$

$\begin{pmatrix}0 & -1\\ 1 & -1\end{pmatrix}$, $\beta\mapsto 1_2$. We may assume that the period matrix of the Abelian variety $\tilde{S}$ is of the form $\begin{pmatrix}1 & 0 & \tau & 0\\ 0 & 1 & \eta & \rho^2\end{pmatrix}$. η is so determined that there exists an automorphism g of $\tilde{S}$ of the form $g:(u,\zeta)\mapsto\left(u+\frac{1}{3}, \rho^2\zeta\right)$. By an elementary calculation, we have $\eta=\frac{n}{3}(1-\rho^2)$, where $n=0, 1$ or 2. Corresponding to these three possible values of η, we have three hyperelliptic surfaces with the same basic member. But it is easily seen that two surfaces corresponding to $\eta=\frac{1}{3}(1-\rho^2)$ and $\eta=\frac{2}{3}(1-\rho^2)$ can be expressed in the same manner if we make a suitable change of the periods representing the Abelian variety $\tilde{S}$. Thus we have only to consider the surfaces corresponding to $\eta=0$ and $\eta=\frac{1}{3}(1-\rho^2)$.

III) $4K=0, 2K\neq 0$. In this case the fibre C is a harmonic elliptic curve and we may let the periods of C be $(1, i)$. We have $H^2(\Delta, G)\simeq Z_2$. Hence there exist two different hyperelliptic surfaces for a fixed homological invariant. There are 12 possible choices of the homological invariant G. But essentially we have only to consider the sheaf G corresponding to the representation: $\alpha\mapsto\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$, $\beta\mapsto 1_2$. We may assume that the period matrix of the Abelian variety $\tilde{S}$ is of the form $\begin{pmatrix}1 & 0 & \tau & 0\\ 0 & 1 & \eta & i\end{pmatrix}$. η is so determined that there exists an automorphism g of $\tilde{S}$ of the form $g:(u,\zeta)\mapsto\left(u+\frac{1}{4}, i\zeta\right)$. By an elementary calculation we have $\eta=\frac{n}{2}(1+i)$, where $n=0$ or 1.

IV) $6K=0, 2K, 3K\neq 0$. In this case the fibre C is an equianharmonic elliptic curve and we may let the periods of C be $(1, \rho^2)$. We have $H^2(\Delta, G)=0$, hence all the elliptic surfaces of this type are basic members. There are 24 possible choices of the homological invariant G. But essentially we have only to consider the sheaf G corresponding to the representation: $\alpha\mapsto\begin{pmatrix}0 & 1\\ -1 & 1\end{pmatrix}$, $\beta\mapsto 1_2$. S is the quotient space of the Abelian variety $\tilde{S}$ with the period matrix $\begin{pmatrix}1 & 0 & \tau & 0\\ 0 & 1 & 0 & \rho^2\end{pmatrix}$ by the group generated by an automorphism $g:(u,\zeta)\mapsto\left(u+\frac{1}{6}, -\rho^2\zeta\right)$ of $\tilde{S}$.

Summarizing the above results, we obtain the following

THEOREM (Compare [1] p. 596 and [2] §57). *Any hyperelliptic surface can be expressed as the quotient space of an Abelian variety A by the group generated by an automorphism g of A. The period matrix of A and the automorphism g are given as follows:*

I) i) $\begin{pmatrix}1 & 0 & \tau & 0\\ 0 & 1 & 0 & \omega\end{pmatrix}$, ii) $\begin{pmatrix}1 & 0 & \tau & 0\\ 0 & 1 & \frac{1}{2} & \omega\end{pmatrix}$,

$$g:(u,\zeta)\longmapsto\left(u+\frac{1}{2}, -\zeta\right),$$

II) i) $\begin{pmatrix}1 & 0 & \tau & 0\\ 0 & 1 & 0 & \rho^2\end{pmatrix}$, ii) $\begin{pmatrix}1 & 0 & \tau & 0\\ 0 & 1 & \frac{1}{3}(1-\rho^2) & \rho^2\end{pmatrix}$,

$$g\colon (u, \zeta) \longmapsto \left(u+\frac{1}{3},\ \rho^2\zeta\right),$$

III) i) $\begin{pmatrix}1 & 0 & \tau & 0\\ 0 & 1 & 0 & i\end{pmatrix}$, ii) $\begin{pmatrix}1 & 0 & \tau & 0\\ 0 & 1 & \frac{1}{2}(1+i) & i\end{pmatrix}$,

$$g\colon (u, \zeta) \longmapsto \left(u+\frac{1}{4},\ i\zeta\right),$$

IV) $\begin{pmatrix}1 & 0 & \tau & 0\\ 0 & 1 & 0 & \rho^2\end{pmatrix}$, $g\colon (u, \zeta) \longmapsto \left(u+\frac{1}{6},\ -\rho^2\zeta\right)$,

where τ and ω denote arbitrary constants with non-zero imaginary parts.

REMARK 1. If we write the period matrices in normal forms, we obtain the table of Enriques-Severi [2] §57.

REMARK 2. The surfaces of any one of seven types of the above table form a complex analytic family, with parameters τ and ω in the case I) i) and ii) and with a parameter τ in other cases. Moreover, these families are everywhere effectively parametrized and complete. The dimension of the cohomology groups with coefficients in the sheaf Θ of germs of holomorphic vector fields of a hyperelliptic surface S is computed easily. The result is as follows:

$$\dim H^0(S, \Theta)=1,$$

$$\dim H^1(S, \Theta)=\begin{cases}2, & \text{in the case I)},\\ 1, & \text{in other cases},\end{cases}$$

$$\dim H^2(S, \Theta)=\begin{cases}1, & \text{in the case I)},\\ 0, & \text{in other cases}.\end{cases}$$

REMARK 3. The 1-dimensional homology groups $H_1(S, Z)$ of the above surfaces are given, respectively, by I) i) $Z\oplus Z\oplus Z_2\oplus Z_2$, ii) $Z\oplus Z\oplus Z_2$, II) i) $Z\oplus Z\oplus Z_3$, ii) $Z\oplus Z$, III) i) $Z\oplus Z\oplus Z_2$, ii) $Z\oplus Z$ and IV) $Z\oplus Z$. Thus two hyperelliptic surfaces of different classes may have the same homology group. But S. Iitaka remarked that the four classes I), II), III) and IV) can be distinguished by the fundamental group (see [4] Table II). Hence the seven types of hyperelliptic surfaces are completely classified topologically, and *the plurigenera of hyperelliptic surfaces are topological invariants.*

REMARK 4. The surfaces of the types I) i), II) i), III) i) and IV) are topologically homeomorphic, respectively, to $S^1\times\mathscr{M}(D_4')$, $S^1\times\mathscr{M}(E_6')$, $S^1\times\mathscr{M}(E_7')$ and $S^1\times\mathscr{M}(E_8')$, where S^1 is a circle and $\mathscr{M}(\Gamma)$ denotes the tree-manifold associated with a tree Γ (see [3] §10).

§3. Various characterizations of hyperelliptic surfaces.

Let S_1 be a surface with $p_g=0$ which has an Abelian variety as its finite unramified covering manifold. Then we infer readily that the Chern numbers c_1^2 and c_2 of S_1 vanish, and that S_1 contains no exceptional curve (of the first kind). The Neother formula $12(p_g-q+1)=c_1^2+c_2$ implies that $q=1$. On the other hand S_1 is algebraic. Hence we have $b_1=2$.

LEMMA 2. *S_1 is an elliptic bundle over an elliptic curve.*

PROOF. Let S be an algebraic surface with $p_g=c_1^2=0$ and $q=1$. Considering the Albanese map of S, we have a holomorphic map Ψ of S onto an elliptic curve Δ such that $\Psi^{-1}(u)$ is connected for every point $u\in\Delta$. Let π be the genus of a general fibre of Ψ. Then the proof of Theorem 51 of [6] IV implies that the universal covering manifold $\mathscr{U}$ of S is given as follows:

$$\mathscr{U}=\begin{cases} \boldsymbol{P}^1\times\boldsymbol{C}\,, & \text{if } \pi=0\,,\\ \left.\begin{matrix}\boldsymbol{C}\times\boldsymbol{C}\\ \boldsymbol{C}\times\boldsymbol{D}\end{matrix}\right\}, & \text{if } \pi=1\,,\\ \boldsymbol{C}\times\boldsymbol{D}\,, & \text{if } \pi\geqq 2\,,\end{cases}$$

where $\boldsymbol{D}$ denotes the unit disk $\{z\in\boldsymbol{C}\,|\,|z|<1\}$. Moreover in the case where $\pi=1$, $\mathscr{U}=\boldsymbol{C}\times\boldsymbol{C}$ if and only if S is free from singular fibres over Δ, q.e.d.

Now any Abelian variety which appears in the theorem of the previous section is an elliptic bundle $\tilde{S}\to\tilde{\Delta}$ over an elliptic curve $\tilde{\Delta}$ with fibre C. $\tilde{S}$ has as a finite covering the direct product $\Delta_1\times C$, where Δ_1 is a finite unramified covering of $\tilde{\Delta}$. Obviously Δ_1 is an elliptic curve. Thus any hyperelliptic surface S is expressed as $\Delta_1\times C/\mathscr{G}$, where $\mathscr{G}$ is a finite group of automorphisms of $\Delta_1\times C$. We have the natural homomorphism $\varphi\colon\mathscr{G}\to\operatorname{Aut}C$. Hence S can be viewed as an elliptic fibre space over the curve $C/\varphi(\mathscr{G})$ whose general fibre is Δ_1. Moreover $C/\varphi(\mathscr{G})$ is a non-singular rational curve. Hence we see that any hyperelliptic surface S has the following structure (Compare [7] §4):

(*) 1) S is an elliptic surface over a projective line $\boldsymbol{P}^1$, 2) $b_1=2$, 3) S has no singular fibres over the base curve other than that of the form $m\Theta$, where Θ is a non-singular elliptic curve, 4) the multiplicities m_i of the multiple fibres $m_i\Theta_i$, $i=1,2,\cdots,r$ of S satisfy the equality $\sum_{i=1}^{r}\left(1-\frac{1}{m_i}\right)=2$.

REMARK. Possible combinations of natural numbers $(m_1, m_2, \cdots, m_r)$ satisfying the equality in 4) are (2, 2, 2, 2), (3, 3, 3), (2, 4, 4) and (2, 3, 6), which correspond, respectively, to the types I), II), III) and IV) of hyperelliptic surfaces.

Conversely let S be a surface satisfying the condition (*), then from the theory of classification of surfaces we infer that S is an algebraic surface with $p_g=c_1^2=0$ and $q=1$. Moreover there exists an elliptic curve C_1 and a branched covering map $h: C_1 \to \boldsymbol{P}^1$ such that the fibre space S_1 of elliptic curves induced from $S \to \boldsymbol{P}^1$ by the map h is free from singular fibres and S_1 is an unramified covering manifold of S. Hence we see that the universal covering of S is $\boldsymbol{C} \times \boldsymbol{C}$.

From Lemma 2 together with its proof, Lemma 1 and the above consideration, we have the following four equivalent characterizations of hyperelliptic surfaces:

A) a surface with $p_g=0$ having an Abelian variety as its finite unramified covering manifold,

B) an elliptic bundle over an elliptic curve of which the first Betti number b_1 is equal to 2,

C) a surface satisfying the condition (*),

D) an algebraic surface with $p_g=c_1^2=0$ and $q=1$ of which the universal covering manifold is $\boldsymbol{C} \times \boldsymbol{C}$.

University of Tokyo

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