The conjugacy classes of Chevalley groups of type (G_2) *over finite fields of characteristic 2 or 3*

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Let G be a Chevalley group of type (G_2) over a finite field $K=\boldsymbol{F}_q$ of characteristic p. The conjugacy classes of G have been given by Chang [1] in the case $p\neq 2$, 3. It is our purpose in this paper to settle the remaining case $p=2$ or 3. We deal with the case $p=2$ in section II, and the case $p=3$ in section III. The main results are given by Theorem 1 and Theorem 2, especially the group G has q^2+2q+8 conjugacy classes in both cases and the number of classes of p-elements is 8 (resp. 9) in case $p=2$ (resp. $p=3$), while Chang [1] has shown that G has q^2+2q+7 conjugacy classes including 7 classes of p-elements in case $p\neq 2$, 3. Lemma (2.2) is crucial for the determination of the conjugacy classes of 2-elements.

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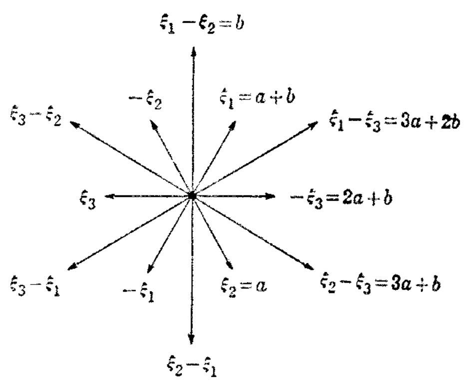
I. Preliminaries

1. We shall follow the notations used in [1]. The contents of sections 1 and 2 in [1] remain valid in our case. Especially let $\Sigma=\{\pm\xi_i, \xi_i-\xi_j \mid 1\leq i, j\leqq 3, i\neq j\}$ (where $\xi_1+\xi_2+\xi_3=0$) be the root system of type (G_2), and choose $a=\xi_2$, $b=\xi_1-\xi_2$ for a fundamental system of roots. We denote by Σ^+ the set of positive roots



Root system of type (G_2)

with respect to $\{a, b\}$. Let χ be a homomorphism of the root module P_0 (i.e. P_0 is the additive group generated by the roots) into the multiplicative group K^*. Put $\chi(\xi_i)=z_i$, $i=1, 2, 3$. Then the element $h(\chi)$ of the Cartan subgroup $\mathfrak{H}$ associated with χ will be denoted by $h(z_1, z_2, z_3)$. For each $r \in \Sigma$, there exists uniquely a homomorphism φ_r of $SL(2, K)$ into G such that

$$\varphi_r\left(\begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}\right)=x_r(t), \quad \varphi_r\left(\begin{bmatrix} 1 & 0 \\ t & 1 \end{bmatrix}\right)=x_{-r}(t) \text{ for every } t \in K.$$

We put $\omega_r=\varphi_r\left(\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\right)=x_r(1)x_{-r}(-1)x_r(1)$ and $\mathfrak{W}=\langle \mathfrak{H}, \omega_r \mid r \in \Sigma \rangle$. Then there exists a homomorphism π of $\mathfrak{W}$ onto the Weyl group W such that $\operatorname{Ker}(\pi)=\mathfrak{H}$ and $\pi(\omega_r)=w_r$ for every $r \in \Sigma$, where w_r is the reflection associated with r. In particular we have $\mathfrak{W}/\mathfrak{H} \simeq W$. Let $\bar{K}$ be the algebraic closure of K and $\bar{G}$ the Chevalley group of type (G_2) over $\bar{K}$. We may and shall identify G with a subgroup of $\bar{G}$ in the natural manner. The Cartan subgroup of $\bar{G}$ is denoted by $\bar{\mathfrak{H}}$. We summarize here several well-known results for the sake of convenience.

PROPOSITION 1.1 (Lang [3]). *Let Ω be an algebraically closed field and n a positive integer. Let Γ be a connected algebraic subgroup of $GL(n, \Omega)$ defined over a finite field F_q. Let $x \rightarrow x^{(q)}$ be the automorphism of Γ induced from the field automorphism $t \rightarrow t^q$ of Ω. Then the map $x \rightarrow x^{-1}x^{(q)}$ is surjective, i.e. given any y in Γ, there exists x such that $y=x^{-1}x^{(q)}$.*

PROPOSITION 1.2. *Suppose that two elements x and y of G are conjugate in $\bar{G}$. Suppose furthermore that the centralizer $C_{\bar{G}}(x)$ of x in $\bar{G}$ is a connected algebraic group. Then x and y are conjugate in G.*

PROOF. We apply Proposition 1.1 to $\Gamma=C_{\bar{G}}(x)$. Suppose $\alpha x \alpha^{-1}=y, \alpha \in \bar{G}$, then $\alpha^{(q)}x\alpha^{(q)-1}=y$ and $\alpha^{-1}\alpha^{(q)} \in \Gamma$. Hence there exists an element $z \in \Gamma$ such that $\alpha^{-1}\alpha^{(q)}=z^{-1}z^{(q)}$. Now if we put $\beta=\alpha z^{-1}$, we have $\beta x \beta^{-1}=y$ and $\beta=\beta^{(q)}$, i.e. $\beta \in G$.

PROPOSITION 1.3. *An element $x \in \bar{G}$ is conjugate to an element in G if and only if x and $x^{(q)}$ are conjugate in $\bar{G}$.*

PROOF. First, suppose $zxz^{-1} \in G, z \in \bar{G}$. Then $z^{(q)}x^{(q)}z^{(q)-1}=zxz^{-1}$, and therefore x and $x^{(q)}$ are conjugate by $z^{-1}z^{(q)}$. Conversely, suppose $x^{(q)}=\alpha x \alpha^{-1}, \alpha \in \bar{G}$. By Proposition 1.1 there exists an element $z \in \bar{G}$ such that $z^{-1}z^{(q)}=\alpha^{-1}$. Then $(zxz^{-1})^{(q)}=zxz^{-1}$, i.e. $zxz^{-1} \in G$.

We put $\Gamma(\alpha)=\{x \in \bar{G} \mid x^{(q)}=\alpha x \alpha^{-1}\}$ for $\alpha \in \bar{G}$. Then the above proof shows that G and $\Gamma(\alpha)$ are conjugate by z, hence isomorphic. Especially $C_G(zxz^{-1})$ and $C_{\Gamma(\alpha)}(x)$ are isomorphic for every $x \in \Gamma(\alpha)$.

PROPOSITION 1.4. *Two elements* h_1, h_2 *of* $\mathfrak{H}$ *are conjugate in* G *if and only if they are conjugate under the action of the Weyl group* W, *i.e.* $\omega(w)h_1\omega(w)^{-1}=h_2$ *for some* $w\in W$, *where* $\omega: W\to\mathfrak{W}$ *is a fixed mapping such that* $\pi\circ\omega=\mathrm{id}_W$.

PROOF. Suppose $xh_1x^{-1}=h_2$, $x\in G$ and let $x=uh\omega(w)v$ be the Bruhat factorization of x. Then we have

$$uh\omega(w)vh_1=h_2uh\omega(w)v$$

and from the uniqueness of the Bruhat factorization, we have

$$u\in C_G(h_2),\ v\in C_G(h_1) \text{ and } \omega(w)h_1\omega(w)^{-1}=h_2\ .$$

II. The Case $\mathrm{ch}(K)=2$

2. Conjugacy classes of 2-elements.

In the present case, Theorem 3.1 (i) in [1] no longer holds and the elements of S_1 split into two conjugacy classes (cf. Proposition 2.3 below). In order to determine the conjugacy classes of 2-elements in G, it is essential to determine the conjugacy classes in a parabolic subgroup $P=\langle B, \omega_a\rangle$. In Proposition 2.5 we shall prove that any element of $S_3-\{1\}$ is conjugate in P to one and only one of y_i, $i=1, \cdots, 6$. Namely it is shown that no two of y_i's are conjugate in P and that $|S_3-\{1\}|$ is equal to $\sum_{i=1}^{6}|P|/|C_P(y_i)|$, which is the sum of the numbers of the elements in $S_3-\{1\}$ conjugate to y_i. Lemma 2.2 is used for the calculation of $|C_P(y_6)|$.

First we shall prove some preliminary lemmas. Let $K[X]$ be the polynomial ring over K in the indeterminate X. For any element t of K, put $\theta(t)=-1$ if X^2+X+t is irreducible in $K[X]$ and $\theta(t)=1$ if X^2+X+t is reducible in $K[X]$. If we define $f(u)=u^2+u$ for every $u\in K$, then f is an endomorphism of the additive group K. The kernel of f consists of 0 and 1, hence $f(K)$ is a subgroup of index 2 in K. Let η be an element in $K-f(K)$, then we have

$$K=f(K)\cup(f(K)+\eta)\ .$$

Note that $\theta(t)=1$ if and only if t belongs to $f(K)$.

LEMMA 2.1. *We have*

$$\theta(t+u)=\theta(t)\cdot\theta(u)$$

for every $t, u\in K$, *i.e.* θ *is a linear character of the additive group* K.

Next, choose an element ζ of K such that $X^3+X+\zeta$ is irreducible in $K[X]$. Then we have the following lemma.

LEMMA 2.2. (i) *ζX^3+X^2+1 is an irreducible polynomial of $K[X]$.*
(ii) *$X^2+\zeta X+\zeta^2+1$ is a reducible polynomial of $K[X]$.*

PROOF. (i) By way of contradiction, suppose ζX^3+X^2+1 is reducible in $K[X]$ and $\zeta t^3+t^2+1=0, t\in K$. Then $t\neq 0$ and $\zeta+t^{-1}+t^{-3}=0$. This means $X^3+X+\zeta$ is reducible in $K[X]$ and this contradicts the choice of ζ.
(ii) We distinguish two cases according as $q\equiv 1$ or -1 (mod. 3).

Case 1. $q\equiv -1$ (mod. 3)

We shall show that $X^2+\zeta X+1$ is irreducible in this case. By way of contradiction, suppose $X^2+\zeta X+1$ is reducible in $K[X]$, and $t^2+\zeta t+1=0$, $t\in K$. In this case there exists a cubic root u of t in K, and then $u+u^{-1}$ is a root of $X^3+X+\zeta=0$. This contradicts the choice of ζ, and therefore $X^2+\zeta X+1$ is irreducible. This means $\theta(\zeta^{-2})=1$. The polynomial X^2+X+1 is also irreducible in this case, i.e. $\theta(1)=-1$. By Lemma 2.1 we have $\theta(1+\zeta^{-2})=1$.

Case 2. $q\equiv 1$ (mod. 3)

We shall show that $X^2+\zeta X+1$ is reducible in this case. By way of contradiction, suppose $X^2+\zeta X+1$ is irreducible. Then the equation $X^2+\zeta X+1=0$ has no root in K, but has a root t in the quadratic extension K_2 of K. Then

$$t+t^{-1}=\zeta=\zeta^q=t^q+t^{-q}$$

and therefore $t^q=t$ or t^{-1}. But $t^q=t$ means that t is an element of K, which is not the case. Therefore we have $t^{q+1}=1$, and there is a cubic root u of t such as $u^{q+1}=1$, since $q+1$ is prime to 3. Then $u+u^{-1}\in K$, and $u+u^{-1}$ is a root of $X^3+X+\zeta=0$. This contradicts the choice of ζ, and therefore $X^2+\zeta X+1$ is reducible, i.e. $\theta(\zeta^{-2})=1$. In this case X^2+X+1 is also reducible, i.e. $\theta(1)=1$, hence we have $\theta(1+\zeta^{-2})=1$.

Thus we have $\theta(1+\zeta^{-2})=1$ in any case, which means that $X^2+\zeta X+\zeta^2+1$ is reducible, and the lemma is proved.

Note that in this case the commutator relations 2.1 in [1] are reduced to:

$$\begin{aligned}
&[x_a(t), x_b(u)]=x_{a+b}(tu)x_{2a+b}(t^2u)x_{3a+b}(t^3u)\ ,\\
&[x_a(t), x_{a+b}(u)]=x_{3a+b}(t^2u)x_{3a+2b}(tu^2)\ ,\\
&[x_a(t), x_{2a+b}(u)]=x_{3a+b}(tu)\ ,\\
&[x_b(t), x_{3a+b}(u)]=x_{3a+2b}(tu)\ ,\\
&[x_{a+b}(t), x_{2a+b}(u)]=x_{3a+2b}(tu)\ ,\\
&[x_r(t), x_s(u)]=1, \quad \text{for all other pairs of } r, s\in\Sigma^+\ .
\end{aligned}$$

Now we divide the set of elements of $\mathfrak{U}$ into the following three subsets as in [1]:

$$S_1=\{x_a(z_1)x_b(z_2)u \mid z_1, z_2\in K^*, u\in \mathfrak{U}_2\} ,$$
$$S_2=\{x_a(z)u \mid z\in K^*, u\in \mathfrak{U}_2\} ,$$
$$S_3=\{x_b(t)u \mid t\in K, u\in \mathfrak{U}_2\} ,$$

where $\mathfrak{U}_2=\mathfrak{X}_{a+b}\mathfrak{X}_{2a+b}\mathfrak{X}_{3a+b}\mathfrak{X}_{3a+2b}$. (Note that $\mathfrak{U}_2=[\mathfrak{U}, \mathfrak{U}]$ in case $q>2$, $[\mathfrak{U}, \mathfrak{U}]$ being the commutator subgroup of $\mathfrak{U}$.)

PROPOSITION 2.3. (i) *Any element of* S_1 *is conjugate in* B *to either* $x_1=x_a(1)x_b(1)$ *or* $x_2=x_a(1)x_b(1)x_{2a+b}(\eta)$.

(ii) *Let* $g^{-1}xg=y$, $x\in S_1$, $y\in\mathfrak{U}$, $g\in G$. *Then one has* $g\in\mathfrak{U}\mathfrak{H}=B$ *and* $y\in S_1$.

(iii) $|C_G(x_i)|=2q^2$, $i=1, 2$.

(iv) x_1 *and* x_2 *are not conjugate in* G.

PROOF. (i) Any element $x=x_a(z_1)x_b(z_2)u$ of S_1 can be transformed into

$$x'=x_a(1)x_b(1)x_{a+b}(t_2)x_{2a+b}(t_3)x_{3a+b}(t_4)x_{3a+2b}(t_5), \quad t_i\in K ,$$

by an element of $\mathfrak{H}$. Then we can transform x' into

$$x''=x_a(1)x_b(1)x_{2a+b}(u^2+u+t_2+t_3), \quad u\in K ,$$

by an element of $\mathfrak{U}$, and we may choose u in such a way that $u^2+u+t_2+t_3=0$ or η according as $\theta(t_2+t_3)=1$ or -1.

(ii) The proof is the same as that of (ii) of Theorem 3.1 in [1].

(iii) We know $C_G(x_i)=C_B(x_i)$, $i=1, 2$ from (ii). Suppose x_1 is centralized by an element $y\in B$, and let

$$y=x_a(t_0)x_b(t_1)x_{a+b}(t_2)x_{2a+b}(t_3)x_{3a+b}(t_4)x_{3a+2b}(t_5)h(\chi),\ t_i\in K,\ h(\chi)\in\mathfrak{H} .$$

Then we must have $h(\chi)=1$, since we have $\chi(a)=\chi(b)=1$ from

$$y=x_1^{-1}yx_1=x_a(1+t_0+\chi(a))x_b(1+t_1+\chi(b))\cdots .$$

Then

$$\begin{aligned} y&=x_a(t_0)x_b(t_1)x_{a+b}(t_2)x_{2a+b}(t_3)\cdots \\ &=x_a(t_0)x_b(t_1)x_{a+b}(t_0+t_1+t_2)x_{2a+b}(t_0^2+t_1+t_2)\cdots \end{aligned}$$

and we have $t_0+t_1=t_0^2+t_1=0$. It follows that $t_0=t_1=0$ or 1. In case $t_0=t_1=0$, we have

$$\begin{aligned} y&=x_{a+b}(t_2)x_{2a+b}(t_3)x_{3a+b}(t_4)x_{3a+2b}(t_5) \\ &=x_{a+b}(t_2)x_{2a+b}(t_3)x_{3a+b}(t_2+t_3+t_4)x_{3a+2b}(t_2^2+t_4+t_5) \end{aligned}$$

and this implies $t_2+t_3=0$ and $t_2^2+t_4=0$. It is easily checked that the same equations hold in case $t_0=t_1=1$. Thus we have

$$C_B(x_1)=\{x_a(\delta)x_b(\delta)x_{a+b}(t)x_{2a+b}(t)x_{3a+b}(t^2)x_{3a+2b}(u) \mid \delta=0 \text{ or } 1, \quad t, u\in K\} ,$$

and therefore

$$|C_G(x_1)|=|C_B(x_1)|=2q^2 .$$

Also we have

$$|C_G(x_2)|=|C_B(x_2)|=2q^2$$

by the same calculation.

(iv) This is an immediate consequence from (ii), since x_1 and x_2 are not conjugate in B.

PROPOSITION 2.4. *Every element of S_2 is conjugate in G to an element in S_3.*

PROOF. Every element $x=x_a(z)u$ of S_2 can be transformed into

$$x'=x_a(1)x_{2a+b}(t_3)x_{3a+b}(t_4)x_{3a+2b}(t_5)$$

by an element of B. Then

$$\omega_b x'\omega_b^{-1}=x_{a+b}(1)x_{2a+b}(t_3)x_{3a+b}(t_5)x_{3a+2b}(t_4)\in S_3 .$$

We note here that S_3 is normalized by $\mathfrak{X}_a$ and ω_a, hence by the parabolic subgroup $P=\langle B, \omega_a\rangle=B\cup B\omega_a\mathfrak{X}_a$.

PROPOSION 2.5. *Any element of $S_3-\{1\}$ is conjugate in P to one and only one of the following six elements:*

$$\begin{aligned}
y_1&=x_{3a+2b}(1) ,\\
y_2&=x_{3a+b}(1) ,\\
y_3&=x_{2a+b}(1) ,\\
y_4&=x_{a+b}(1)x_{2a+b}(1) ,\\
y_5&=x_{a+b}(1)x_{2a+b}(1)x_{3a+b}(\eta) ,\\
y_6&=x_b(1)x_{2a+b}(1)x_{3a+b}(\zeta) .
\end{aligned}$$

PROOF. First, we determine the orders of the centralizers of y_i's in P. We know at once

$$\begin{aligned}
C_P(y_1)&=\langle\{h(z^{-1}, z^2, z^{-1})\mid z\in K^*\}, \mathfrak{U}, \omega_a\rangle ,\\
C_P(y_2)&=C_B(y_2)=\langle\{h(z^2, z^{-1}, z^{-1})\mid z\in K^*\}, \mathfrak{X}_a\mathfrak{X}_{a+b}\mathfrak{X}_{2a+b}\mathfrak{X}_{3a+b}\mathfrak{X}_{3a+2b}\rangle ,\\
C_P(y_3)&=C_B(y_3)=\langle\{h(z, z^{-1}, 1)\mid z\in K^*\}, \mathfrak{X}_b\mathfrak{X}_{2a+b}\mathfrak{X}_{3a+b}\mathfrak{X}_{3a+2b}\rangle ,
\end{aligned}$$

and therefore we have

$$\begin{aligned}
|C_P(y_1)|&=q^6(q-1)(q+1) ,\\
|C_P(y_2)|&=q^5(q-1) ,\\
|C_P(y_3)|&=q^4(q-1) .
\end{aligned}$$

Suppose y_4 is centralized by $uh\omega_a x_a(t)$, $u \in \mathfrak{U}$, $h \in \mathfrak{H}$, $t \in K$. Then we have

$$\begin{aligned} uh\omega_a x_a(t) &= x_{a+b}(1)x_{2a+b}(1)uh\omega_a x_a(t)x_{2a+b}(1)x_{a+b}(1) \\ &= x_{a+b}(1)x_{2a+b}(1)uh\omega_a x_{2a+b}(1)x_{3a+b}(t)x_{a+b}(1)x_{3a+b}(t^2)x_{3a+2b}(t)x_a(t) \\ &= x_{a+b}(1)x_{2a+b}(1)uhx_{a+b}(1)x_{2a+b}(1)x_b(t+t^2)x_{3a+2b}(t)\omega_a x_a(t)\ , \end{aligned}$$

and so t^2+t must be equal to zero because of the uniqueness of the Bruhat factorization, hence $t=0$ or 1. In case $t=0$, we have

$$uh\omega_a = x_{a+b}(1)x_{2a+b}(1)uhx_{a+b}(1)x_{2a+b}(1)\omega_a\ ,$$

and so

$$(uh)^{-1}x_{a+b}(1)x_{2a+b}(1)(uh) = (x_{a+b}(1)x_{2a+b}(1))^{-1}\ .$$

The number of the elements of the form uh satisfying the above equation is equal to $|C_B(y_4)|$. In case $t=1$, we have $uh \in C_B(y_4)$ and therefore

$$|C_P(y_4)| = 3 \cdot |C_B(y_4)|\ .$$

It is easily checked that

$$C_B(y_4) = \mathfrak{X}_b \mathfrak{X}_{3a+b} \mathfrak{X}_{3a+2b} \cdot \{x_a(\delta)x_{a+b}(t)x_{2a+b}(\delta t) \mid \delta = 0 \text{ or } 1,\quad t \in K\}\ ,$$

hence we have

$$|C_P(y_4)| = 6q^4\ .$$

Next suppose y_5 is centralized by an element $uh\omega_a x_a(t)$, $u \in \mathfrak{U}$, $h \in \mathfrak{H}$, $t \in K$. Then we have $t^2+t+\eta=0$ as above, but this contradicts the choice of η. Thus we obtain

$$|C_P(y_5)| = |C_B(y_5)| = 2q^4\ ,$$

since

$$C_B(y_5) = \mathfrak{X}_{3a+b} \mathfrak{X}_{3a+2b} \cdot \{x_a(\delta)x_b(t)x_{a+b}(u)x_{2a+b}(\delta+\eta t+u) \mid \delta = 0 \text{ or } 1,\ t, u \in K\}\ .$$

Finally let

$$y = x_a(t_0)x_b(t_1)x_{a+b}(t_2)x_{2a+b}(t_3)x_{3a+b}(t_4)x_{3a+2b}(t_5)h(\chi)\omega_a x_a(t),\quad t_i,\ t \in K,\ h(\chi) \in \mathfrak{H}\ ,$$

be an element of $B\omega_a \mathfrak{X}_a$. Then direct computations show that y_6 and y commute if and only if the following equations are satisfied:

$$1+\chi(b)(t^3+t+\zeta)=0\ , \tag{2.5.1}$$

$$t_0+\chi(a+b)(t^2+1)=0\ , \tag{2.5.2}$$

$$1+t_0^2+\chi(2a+b)t=0\ , \tag{2.5.3}$$

$$\zeta+\chi(2a+b)t_0 t+\chi(3a+b)=0\ , \tag{2.5.4}$$

(2.5.5) $$t_0+\chi(2a+b)t_2t+t_3t_0+t_4+\chi(3a+b)(t_1+1)+\chi(3a+2b)t^3=0\ .$$

Then we have

(2.5.2)′ $$\chi(a)(t^2+1)=t_0(t^3+t+\zeta)\ ,$$

(2.5.3)′ $$\chi(a)^2t=(t_0^2+1)(t^3+t+\zeta)\ ,$$

(2.5.4)′ $$\chi(a)^3=(t_0^3+t_0+\zeta)(t^3+t+\zeta)\ .$$

(Note that $t^3+t+\zeta\neq 0$ for any $t\in K$, from the choice of ζ.)
From (2.5.2)′ and (2.5.3)′ we have

$$\begin{aligned}\chi(a)^2t(t^2+1)^2&=t_0^2(t^3+t+\zeta)^2t\\&=(t_0^2+1)(t^3+t+\zeta)(t^2+1)^2\ ,\end{aligned}$$

then

$$t_0^2(t^3+t+\zeta)t=(t_0^2+1)(t^2+1)^2\ .$$

Therefore we have

(2.5.6) $$t_0^2(t^2+t\zeta+1)=t^4+1\ .$$

From (2.5.2)′, (2.5.3)′ and (2.5.4)′ we have

$$\begin{aligned}\chi(a)^3t(t^2+1)&=t_0(t_0^2+1)(t^3+t+\zeta)^2\\&=t(t^2+1)(t_0^3+t_0+\zeta)(t^3+t+\zeta)\ ,\end{aligned}$$

then

$$t_0(t_0^2+1)(t^3+t+\zeta)=t(t^2+1)(t_0^3+t_0+\zeta)\ .$$

Therefore we have

(2.5.7) $$(t+t_0)(t^2+tt_0+t_0^2+1)=0\ .$$

Suppose $t+t_0=0$. Then we have

(2.5.6)′ $$t^3\zeta+t^2+1=0$$

from (2.5.6), but this contradicts Lemma 2.2 (i), therefore

(2.5.7)′ $$t^2+tt_0+t_0^2+1=0\ .$$

Then

$$\begin{aligned}(t^2+tt_0+t_0^2+1)(t^2+t\zeta+1)&=(t^2+tt_0+1)(t^2+t\zeta+1)+t^4+1\\&=t(t^2\zeta+\zeta+t_0(t^2+t\zeta+1))=0\ .\end{aligned}$$

Therefore $t=0$ or $(t^2+1)\zeta+t_0(t^2+t\zeta+1)=0$. If $t\neq 0$, we have

$$(t^2+1)^2\zeta^2+t_0^2(t^2+t\zeta+1)^2=(t^4+1)\zeta^2+(t^4+1)(t^2+t\zeta+1)=0\ .$$

This implies $t=1$ or $t^2+t\zeta+\zeta^2+1=0$.

If $t=0$, we have $t_0=1$. Therefore $\chi(a)=\zeta$ and $\chi(a)^3=\zeta^2$, hence $\zeta=1$. If $t=1$, we have $t_0=0$. Therefore $\chi(a)^2=\zeta$ and $\chi(a)^3=\zeta^2$, hence $\zeta=1$. Thus we have $t^2+t\zeta+\zeta^2+1=0$ in any case and this equation has two distinct roots from Lemma 2.2 (ii). For each solution t, we have $t^3+t+\zeta=\zeta^3\neq 0$, $t^2+t\zeta+1=\zeta^2\neq 0$, hence we have $\chi(b)=\zeta^{-3}$ and $t_0=t+\zeta$ from (2.5.1) and (2.5.6). Then we have $\chi(a)=\zeta^2$ from (2.5.2)′ or (2.5.3)′. These satisfy (2.5.1) through (2.5.4). We may assign any value to t_1, t_2, t_3, t_5, then t_4 is uniquely determined by (2.5.5). Therefore we have

$$|C_{B\omega_a\mathfrak{X}_a}(y_6)|=2q^4.$$

It is easily checked that

$$C_B(y_6)=\{x_b(t_1)x_{a+b}(t_2)x_{2a+b}(t_3)x_{3a+b}(t_1\zeta+t_2)x_{3a+2b}(t_4)\mid t_i\in K\}\ ,$$

hence we have

$$|C_B(y_6)|=q^4\ .$$

Therefore we have

$$|C_P(y_6)|=3q^4\ .$$

Now we know that no two of y_i's are conjugate in P, since they have different orders of the centralizers respectively.[1] The number of elements in the conjugacy classes of P containing y_i is equal to $|P|/|C_P(y_i)|$, and we have

$$\sum_{i=1}^{6}|P|/|C_P(y_i)|=q^5-1=|S_3-\{1\}|\ .$$

This means that every element of $S_3-\{1\}$ is conjugate in P to one and only one y_i, and the proposition is proved.

Now we will determine the orders of the centralizers of y_i in G. Since $\omega_b^{-1}x_{3a+2b}(1)\omega_b=x_{3a+b}(1)$, y_1 and y_2 are conjugate in G and we have

$$|C_G(y_2)|=|C_G(y_1)|=|C_P(y_1)|=q^6(q^2-1)\ ,$$

for if y_1 is centralized by $uh\omega(w)v$, $u, v\in\mathfrak{U}$, $h\in\mathfrak{H}$, $w\in W$, then we have

$$\omega(w)x_{3a+2b}(1)\omega(w)^{-1}\in\mathfrak{X}_{3a+2b}$$

and so $w(3a+2b)=3a+2b$, i.e. $w=1$ or w_a. Similarly we have

$$|C_G(y_4)|=|C_P(y_4)|=6q^4\ ,$$
$$|C_G(y_5)|=|C_P(y_5)|=2q^4\ ,$$
$$|C_G(y_6)|=|C_P(y_6)|=3q^4\ .$$

It is easily checked that

[1] The centralizers of y_1 and y_4 (resp. y_3 and y_6) are of the same order when $q=2$ (resp. $q=4$). But they are not conjugate even in G, since y_1, y_2 and y_3 are of order 2, and y_4, y_5 and y_6 are of order 4.

$$C_G(y_3)=C_B(y_3)\cup C_B(y_3)\omega_b\mathfrak{X}_b\ ,$$

hence we have

$$|C_G(y_3)|=(q+1)\cdot|C_B(y_3)|=q^4(q^2-1)\ .$$

Now we know that no two of y_1, y_3, y_4, y_5 and y_6 are conjugate in G, since they have different orders of the centralizers respectively.[1]

Summarizing the above, we have

PROPOSITION 2.6. *Including the identity class, the group G contains 8 conjugacy classes of 2-elements. The representatives and the orders of the centralizers in G are as follows:*

$1,$	$\lvert C_G(1)\rvert=q^6(q^2-1)(q^6-1)\ ,$
$x_1=x_a(1)x_b(1)\ ,$	$\lvert C_G(x_1)\rvert=2q^2\ ,$
$x_2=x_a(1)x_b(1)x_{2a+b}(\eta)\ ,$	$\lvert C_G(x_2)\rvert=2q^2\ ,$
$x_3=x_{3a+2b}(1)\ ,$	$\lvert C_G(x_3)\rvert=q^6(q^2-1)\ ,$
$x_4=x_{2a+b}(1)\ ,$	$\lvert C_G(x_4)\rvert=q^4(q^2-1)\ ,$
$x_5=x_{a+b}(1)x_{2a+b}(1)\ ,$	$\lvert C_G(x_5)\rvert=6q^4\ ,$
$x_6=x_{a+b}(1)x_{2a+b}(1)x_{3a+b}(\eta)\ ,$	$\lvert C_G(x_6)\rvert=2q^4\ ,$
$x_7=x_b(1)x_{2a+b}(1)x_{3a+b}(\zeta)\ ,$	$\lvert C_G(x_7)\rvert=3q^4\ .$

3. Conjugacy classes of semi-simple elements.

In our case every element of odd order in G is semi-simple and is conjugate in $\bar{G}$ to an element of $\bar{\mathfrak{H}}$. We put $\mathfrak{H}(\omega)=\Gamma(\omega)\cap\bar{\mathfrak{H}}$ for $\omega\in\mathfrak{W}$. Then from Proposition 1.3 we know that if $h\in\bar{\mathfrak{H}}$ is conjugate to an element $x\in G$, then $h\in\mathfrak{H}(\omega(w))$ for some $w\in W$ and $C_G(x)\simeq C_{\Gamma(\omega(w))}(h)$.

The next proposition is proved in the same way as (4.1) in [1]. (See also [2] and [4].)

PROPOSITION 3.1. $C_{\bar{G}}(h(\chi))=\langle\bar{\mathfrak{H}}, \bar{\mathfrak{X}}_r \mid r\in\Sigma, \chi(r)=1\rangle$.

This proposition means that the centralizer of any semi-simple element in $\bar{G}$ is a connected algebraic group. Suppose $h_1, h_2\in\bar{\mathfrak{H}}$ and h_i is conjugate to $x_i\in G$, $i=1,2$. By applying Proposition 1.2 we know that if h_1 and h_2 are conjugate in $\bar{G}$, then x_1 and x_2 are conjugate in G. Therefore there is a one-to-one correspondence between the conjugacy classes of semi-simple elements in G and the classes of conjugate elements of $\bigcup_{w\in W}\mathfrak{H}(\omega(w))$ (conjugation being taken under W).

The cojugacy classes of W are represented by $1, w_a, w_b, w_6=w_bw_a, w_3=w_6^2$ and $w_2=w_6^3$. Now we have

$$\begin{aligned}\mathfrak{H}(\omega(vwv^{-1}))&=\mathfrak{H}(\omega(v)\omega(w)\omega(v)^{-1})\\&=\omega(v)\mathfrak{H}(\omega(w))\omega(v)^{-1}\end{aligned}$$

for $w, v\in W$. Hence we need only to consider

$$\begin{aligned}\mathfrak{H}(1)&=\{h(z_1, z_2, z_3)\mid z_i^{q-1}=1\}\ ,\\ \mathfrak{H}(\omega_a)&=\{h(z, z^{q-1}, z^{-q})\mid z^{q^2-1}=1\}\ ,\\ \mathfrak{H}(\omega_b)&=\{h(z, z^{q}, z^{-q-1})\mid z^{q^2-1}=1\}\ ,\\ \mathfrak{H}(\omega_6)&=\{h(z, z^{-q}, z^{q^2})\mid z^{q^2-q+1}=1\}\ ,\\ \mathfrak{H}(\omega_3)&=\{h(z, z^{q}, z^{q^2})\mid z^{q^2+q+1}=1\}\ ,\\ \mathfrak{H}(\omega_2)&=\{h(z_1, z_2, z_3)\mid z_i^{q+1}=1\}\ ,\end{aligned}$$

where $\omega_i=\omega(w_i)$.

Let ω be a fixed element of K_2 whose order is 3.

First, we shall consider the centralizer of $h(\omega, \omega, \omega)$. In case $q\equiv 1$ (mod. 3), ω is an element of K and $h(\omega, \omega, \omega)\in G$. From Proposition 3.1 we have

$$\begin{aligned}C_G(h(\omega, \omega, \omega))&=\langle\mathfrak{H}, \mathfrak{X}_{\pm b}, \mathfrak{X}_{\pm(3a+b)}, \mathfrak{X}_{\pm(3a+2b)}\rangle\\&\cong SL(3, K)\end{aligned}$$

and

$$|C_G(h(\omega, \omega, \omega))|=q^3(q^2-1)(q^3-1)\ .$$

In case $q\equiv -1$ (mod. 3), $\omega\in K_2-K$ and

$$\begin{aligned}C_{\Gamma(\omega_a)}(h(\omega, \omega, \omega))&=\langle\mathfrak{H}(\omega_a), \mathfrak{U}_0, \omega_{3a+2b}\rangle\\&\cong SU(3, K_2)\ ,\end{aligned}$$

where

$$\begin{aligned}\mathfrak{U}_0&=\{x_b(t)x_{3a+b}(s)x_{3a+2b}(u)\mid t, s, u\in K_2\}\cap\Gamma(\omega_a)\\&=\{x_b(t)x_{3a+b}(t^q)x_{3a+2b}(u)\mid t, u\in K_2,\ t^{q+1}=u+u^q\}\ ,\end{aligned}$$

and therefore

$$|C_{\Gamma(\omega_a)}(h(\omega, \omega, \omega))|=q^3(q^2-1)(q^3+1)\ .$$

For the other types, the centralizers are easily calculated as follows:

$$\begin{aligned}&C_G(h(z, z^{-1}, 1))=\langle\mathfrak{H}, \mathfrak{X}_{\pm(2a+b)}\rangle && (z^{q-1}=1,\ z\neq 1)\ ,\\ &C_G(h(z^{-1}, z^2, z^{-1}))=\langle\mathfrak{H}, \mathfrak{X}_{\pm(3a+2b)}\rangle && (z^{q-1}=1,\ z^3\neq 1)\ ,\\ &C_G(h(z_1, z_2, z_3))=\mathfrak{H} && (z_i^{q-1}=1,\ z_iz_j^{\pm1}\neq 1)\ ,\\ &C_{\Gamma(\omega_b)}(h(z, z^{-1}, 1))=\langle\mathfrak{H}(\omega_b), \mathfrak{X}_{\pm(2a+b)}\rangle && (z^{q+1}=1,\ z\neq 1)\ ,\\ &C_{\Gamma(\omega_a)}(h(z^{-1}, z^2, z^{-1}))=\langle\mathfrak{H}(\omega_a), \mathfrak{X}_{\pm(3a+2b)}\rangle && (z^{q+1}=1,\ z^3\neq 1)\ ,\end{aligned}$$

$$C_{\Gamma(\omega_2)}(h(z_1, z_2, z_3)) = \mathfrak{H}(\omega_2) \qquad (z_i^{q+1}=1,\ z_i z_j^{\pm 1} \neq 1)\,,$$
$$C_{\Gamma(\omega_b)}(h(z, z^q, z^{-q-1})) = \mathfrak{H}(\omega_b) \qquad (z^{q^2-1}=1,\ z^{q\pm 1} \neq 1)\,,$$
$$C_{\Gamma(\omega_a)}(h(z, z^{q-1}, z^{-q})) = \mathfrak{H}(\omega_a) \qquad (z^{q^2-1}=1,\ z^{q\pm 1} \neq 1)\,,$$
$$C_{\Gamma(\omega_3)}(h(z, z^q, z^{q^2})) = \mathfrak{H}(\omega_3) \qquad (z^{q^2+q+1}=1,\ z^3 \neq 1)\,,$$
$$C_{\Gamma(\omega_6)}(h(z, z^{-q}, z^{q^2})) = \mathfrak{H}(\omega_6) \qquad (z^{q^2-q+1}=1,\ z^3 \neq 1)\,.$$

4. Conjugacy classes of 2-singular elements.

Let the order of an element $x \in G$ be $2^r \cdot m$, where $r \geqq 1, m > 1$ and $2 \nmid m$. Then we can write uniquely as $x = yz = zy$, where the orders of y and z are 2^r and m respectively. We call y (resp. z) the 2-part (resp. the odd part) of x. If two elements are conjugate, their 2-parts and odd parts must be conjugate. Let y be a semi-simple element and z_1, z_2 be 2-elements in the centralizer of y. Suppose we have $w^{-1} y z_1 w = y z_2$, $w \in G$, then we must have $w \in C_G(y)$ and $w^{-1} z_1 w = z_2$, i.e. z_1 and z_2 are conjugate in $C_G(y)$. Therefore we need only to determine the conjugacy classes of 2-elements in the centralizer of semi-simple elements.

In case $q \equiv 1$ (mod. 3), we have shown

$$C_G(h(\omega, \omega, \omega)) = \langle \mathfrak{H}, \mathfrak{X}_{\pm b}, \mathfrak{X}_{\pm(3a+b)}, \mathfrak{X}_{\pm(3a+2b)} \rangle$$
$$\cong SL(3, K)\,.$$

The conjugacy classes of 2-elements in $C_G(h(\omega, \omega, \omega))$ are represented by 1, $x_{3a+2b}(1)$ and $x_b(1) x_{3a+b}(t)$, $t = 1, \tau, \tau^2$, where τ is a fixed generator of the multiplicative group K^*, and

$$C_{C(h(\omega,\omega,\omega))}(x_{3a+2b}(1)) = \langle \{h(z^{-1}, z^2, z^{-1}) \mid z \in K^*\}, \mathfrak{X}_b \mathfrak{X}_{3a+b} \mathfrak{X}_{3a+2b} \rangle\,,$$
$$C_{C(h(\omega,\omega,\omega))}(x_b(1) x_{3a+b}(t)) = \langle h(\omega, \omega, \omega), \{x_b(u) x_{3a+b}(tu) \mid u \in K\}, \mathfrak{X}_{3a+2b} \rangle\,.$$

Hence we have

$$|C_G(h(\omega, \omega, \omega) x_{3a+2b}(1))| = q^3(q-1)\,,$$
$$|C_G(h(\omega, \omega, \omega) x_b(1) x_{3a+b}(t))| = 3q^2\,.$$

In case $q \equiv -1$ (mod. 3), we have shown

$$C_{\Gamma(\omega_a)}(h(\omega, \omega, \omega)) = \langle \{h(z, z^{q-1}, z^{-q}) \mid z \in K_2^*\}, \mathfrak{U}_0, \omega_{3a+2b} \rangle$$
$$\cong SU(3, K_2)\,.$$

The conjugacy classes of 2-elements in $C_{\Gamma(\omega_a)}(h(\omega, \omega, \omega))$ are represented by 1, $x_{3a+2b}(1)$ and $x_b(t) x_{3a+b}(t^{-1}) x_{3a+2b}(\omega)$, $t = 1, \sigma, \sigma^2$, where $\sigma = \rho^{q-1}$ and ρ is a fixed generator of the multiplicative group K_2^*, and we have

$$C_{C(h(\omega,\omega,\omega))}(x_{3a+2b}(1))=\langle\{h(z^{-1}, z^2, z^{-1}) \mid z^{q+1}=1\}, \mathfrak{U}_0\rangle ,$$

$$C_{C(h(\omega,\omega,\omega))}(x_b(t)x_{3a+b}(t^{-1})x_{3a+2b}(\omega))$$
$$=\langle h(\omega, \omega, \omega), \{x_b(u)x_{3a+b}(u^q)x_{3a+2b}(v) \mid u, v \in K_2, u^{q+1}=v+v^q, t^{-1}u+tu^q=0\}\rangle .$$

Hence we have

$$|C_{\Gamma(\omega_a)}(h(\omega, \omega, \omega)x_{3a+2b}(1))| = q^3(q+1) ,$$
$$|C_{\Gamma(\omega_a)}(h(\omega, \omega, \omega)x_b(t)x_{3a+b}(t^{-1})x_{3a+2b}(\omega))| = 3q^2 .$$

In $C_G(h(z, z^{-1}, 1))=\langle\mathfrak{H}, \mathfrak{X}_{\pm(2a+b)}\rangle$ $(z^{q-1}=1, z\neq 1)$, every non-identity 2-element is conjugate to $x_{2a+b}(1)$, and

$$C_{C(h(z,z^{-1},1))}(x_{2a+b}(1))=\langle\{h(v, v^{-1}, 1) \mid v \in K^*\}, \mathfrak{X}_{2a+b}\rangle .$$

Hence we have

$$|C_G(h(z, z^{-1}, 1)x_{2a+b}(1))| = q(q-1) \quad (z^{q-1}=1, z\neq 1) .$$

Similarly we have

$$|C_G(h(z^{-1}, z^2, z^{-1})x_{3a+2b}(1))| = q(q-1) \qquad (z^{q-1}=1, z^3\neq 1) ,$$
$$|C_{\Gamma(\omega_b)}(h(z, z^{-1}, 1)x_{2a+b}(1))| = q(q+1) \qquad (z^{q+1}=1, z\neq 1) ,$$
$$|C_{\Gamma(\omega_a)}(h(z^{-1}, z^2, z^{-1})x_{3a+2b}(1)| = q(q+1) \qquad (z^{q+1}=1, z^3\neq 1) .$$

5. Summarizing the above, we have the main theorem.

THEOREM 1. *In case $p=2$, the group $G=G_2(q)$ has q^2+q+8 conjugacy classes, including 8 classes of 2-elements. The representatives and the orders of the centralizers in G are given by Table 1.*

III. The Case ch$(K)=3$

We shall only give a sketch for this case, because the proof is quite similar to the case $p=2$. In fact, we do not need a lemma corresponding to Lemma 2.2 and the calculation is a little easier.

6. Conjugacy classes of 3-elements.

In this case the commutator relations (2.1) in [1] are reduced to:

$$[x_a(t), x_b(u)]=x_{a+b}(-tu)x_{2a+b}(-t^2u)x_{3a+b}(t^3u)x_{3a+2b}(t^3u^2) ,$$
$$[x_a(t), x_{a+b}(u)]=x_{2a+b}(tu) ,$$
$$[x_b(t), x_{3a+b}(u)]=x_{3a+2b}(tu) ,$$
$$[x_r(t), x_s(u)]=1 , \quad \text{for all other pairs of } r, s \in \Sigma^+.$$

PROPOSITION 6.1. (i) *Any element of* $S_1=\{x_a(z_1)x_b(z_2)u|z_1, z_2\in K^*, u\in \mathfrak{U}_2=[\mathfrak{U}, \mathfrak{U}]\}$ *is conjugate to one of* $x_1=x_a(1)x_b(1)$, $x_2=x_a(1)x_b(1)x_{3a+b}(\zeta)$ *or* $x_3=x_a(1)x_b(1)x_{3a+b}(-\zeta)$, *where* ζ *is a fixed element of* K *such that* $X^3-X-\zeta$ *is an irreducible polynomial of* $K[X]$.

(ii) *Let* $g^{-1}xg=y, x\in S_1, y\in \mathfrak{U}, g\in G$. *Then one has* $g\in \mathfrak{U}\mathfrak{H}=B$ *and* $y\in S_1$.

(iii) $|G(x_i)|=3q^2$, $i=1, 2, 3$.

(iv) *No two of* x_i*'s are conjugate in* G.

PROPOSITION 6.2. *Any element of* $S_2=\{x_a(z)u\mid z\in K^*, u\in \mathfrak{U}_2\}$ *is conjugate in* G *to an element of* $S_3=\{x_b(t)u|t\in K, u\in \mathfrak{U}_2\}$.

PROPOSITION 6.3. *Any element of* $S_3-\{1\}$ *is conjugate in* $P=\langle B, \omega_a\rangle$ *to one and only one of the following seven elements:*

$$\begin{array}{ll}
y_1=x_{a+b}(1)x_{3a+b}(1)\ , & |C_P(y_1)|=2q^4\ , \\
y_2=x_{a+b}(1)x_{3a+b}(\mu)\ , & |C_P(y_2)|=2q^4\ , \\
y_3=x_{2a+b}(1)\ , & |C_P(y_3)|=q^6(q-1)\ , \\
y_4=x_{2a+b}(1)x_{3a+b}(1)\ , & |C_P(y_4)|=q^5\ , \\
y_5=x_{2a+b}(1)x_{3a+2b}(1)\ , & |C_P(y_5)|=q^6\ , \\
y_6=x_{3a+b}(1)\ , & |C_P(y_6)|=q^5(q-1)\ , \\
y_7=x_{3a+2b}(1)\ , & |C_P(y_7)|=q^6(q+1)(q-1)\ ,
\end{array}$$

where μ *is a fixed non-square element of* K.

PROPOSITION 6.4. *Including the identity class, the group* G *contains* 9 *conjugacy classes of* 3*-elements. The representatives and the orders of the centralizers in* G *are as follows:*

$$\begin{array}{ll}
1, & |C_G(1)|=q^6(q^2-1)(q^6-1)\ , \\
x_1=x_a(1)x_b(1)\ , & |C_G(x_1)|=3q^2\ , \\
x_2=x_a(1)x_b(1)x_{3a+b}(\zeta)\ , & |C_G(x_2)|=3q^2\ , \\
x_3=x_a(1)x_b(1)x_{3a+b}(-\zeta)\ , & |C_G(x_3)|=3q^2\ , \\
x_4=x_{a+b}(1)x_{3a+b}(1)\ , & |C_G(x_4)|=2q^4\ , \\
x_5=x_{a+b}(1)x_{3a+b}(\mu)\ , & |C_G(x_5)|=2q^4\ , \\
x_6=x_{2a+b}(1)\ , & |C_G(x_6)|=q^6(q^2-1)\ , \\
x_7=x_{2a+b}(1)x_{3a+2b}(1)\ , & |C_G(x_7)|=q^6\ , \\
x_8=x_{3a+2b}(1)\ , & |C_G(x_8)|=q^6(q^2-1)\ .
\end{array}$$

7. The conjugacy classes of semi-simple elements and 3-singular elements are determined by the same way as in the case $p=2$.

THEOREM 2. *In case* $p=3$, *the group* $G=G_2(q)$ *has* q^2+q+8 *conjugacy classes,* *including* 9 *cojugacy classes of* 3-*elements. The representatives and the orders of the centralizers in* G *are given by Table* 2.

Table 1

The Representatives and the Orders of the Centralizers in Case $p=2$.

1		1	$q^6(q^2-1)(q^6-1)$
$x_1=x_a(1)x_b(1)$		1	$2q^2$
$x_2=x_a(1)x_b(1)x_{2a+b}(\eta)$		1	$2q^2$
$x_3=x_{3a+2b}(1)$		1	$q^6(q^2-1)$
$x_4=x_{2a+b}(1)$		1	$q^4(q^2-1)$
$x_5=x_{a+b}(1)x_{2a+b}(1)$		1	$6q^4$
$x_6=x_{a+b}(1)x_{2a+b}(1)x_{3a+b}(\eta)$		1	$2q^4$
$x_7=x_b(1)x_{2a+b}(1)x_{3a+b}(\zeta)$		1	$3q^4$
$h(\omega, \omega, \omega)$		1	$q^3(q^2-1)(q^3-\varepsilon)$
$h(z, z^{-1}, 1)$	$z^{q-1}=1, z\neq 1$	$(q-2)/2$	$q(q-1)^2(q+1)$
$h(z^{-1}, z^2, z^{-1})$	$z^{q-1}=1, z^3\neq 1$	$(q-3-\varepsilon)/2$	$q(q-1)^2(q+1)$
$h(z_1, z_2, z_3)$	$z_i^{q-1}=1, z_iz_j^{\pm 1}\neq 1$	$(q^2-8q+14+2\varepsilon)/12$	$(q-1)^2$
$h(z, z^{-1}, 1)$	$z^{q+1}=1, z\neq 1$	$q/2$	$q(q-1)(q+1)^2$
$h(z^{-1}, z^2, z^{-1})$	$z^{q+1}=1, z^3\neq 1$	$(q-1+\varepsilon)/2$	$q(q-1)(q+1)^2$
$h(z_1, z_2, z_3)$	$z_i^{q+1}=1, z_iz_j^{\pm 1}\neq 1$	$(q^2-4q+2-2\varepsilon)/12$	$(q+1)^2$
$h(z, z^q, z^{-q-1})$	$z^{q^2-1}=1, z^{q\pm 1}\neq 1$	$q(q-2)/4$	q^2-1
$h(z, z^{q-1}, z^{-q})$	$z^{q^2-1}=1, z^{q\pm 1}\neq 1$	$q(q-2)/4$	q^2-1
$h(z, z^q, z^{q^2})$	$z^{q^2+q+1}=1, z^3\neq 1$	$(q^2+q-1-\varepsilon)/6$	q^2+q+1
$h(z, z^{-q}, z^{q^2})$	$z^{q^2-q+1}=1, z^3\neq 1$	$(q^2-q-1+\varepsilon)/6$	q^2-q+1
$h(\omega, \omega, \omega)x_{3a+2b}(1)$		1	$q^3(q-\varepsilon)$
$h(\omega, \omega, \omega)y_i$	$i=0, 1, 2$	3	$3q^2$
$h(z, z^{-1}, 1)x_{2a+b}(1)$	$z^{q-1}=1, z\neq 1$	$(q-2)/2$	$q(q-1)$
$h(z^{-1}, z^2, z^{-1})x_{3a+2b}(1)$	$z^{q-1}=1, z^3\neq 1$	$(q-3-\varepsilon)/2$	$q(q-1)$
$h(z, z^{-1}, 1)x_{2a+b}(1)$	$z^{q+1}=1, z\neq 1$	$q/2$	$q(q+1)$
$h(z^{-1}, z^2, z^{-1})x_{3a+2b}(1)$	$z^{q+1}=1, z^3\neq 1$	$(q-1+\varepsilon)/2$	$q(q+1)$

Each entry S of the first column denotes a set of elements in $\Gamma(\omega)$ for some $\omega\in\mathfrak{W}$. Every element of G is conjugate to an element in one and only one of the sets $\bar{\eta}(S)$, where $\bar{\eta}$ is an isomorphism of $\Gamma(\omega)$ onto G given in the remark after Proposition 1.3 such that $\bar{\eta}(S)\subseteq G$. The second column denotes the number of conjugacy classes of G into which the elements of the set $\bar{\eta}(S)$ are divided. The third column denotes the order of the centralizer of an element in $\bar{\eta}(S)$. $\varepsilon=1$ or -1 according as $q\equiv 1$ or -1 (mod. 3). ω is a fixed element of K_2 whose order is 3. η and ζ are fixed elements of K such that $X^2+X+\eta$ and $X^3+X+\zeta$ are irreducible polynomials of $K[X]$. $y_i=x_b(1)x_{3a+b}(\tau^i)$ in case $q\equiv 1$ (mod. 3), where τ is a fixed generator of K^*, and $y_i=x_b(\sigma^i)x_{3a+b}(\sigma^{-i})x_{3a+2b}(\omega)$ in case $q\equiv -1$ (mod. 3), where $\sigma=\rho^{q-1}$ and ρ is a fixed generator of K_2^*.

The meaning of the following table is the same as that of Table 1. ζ is a fixed element of K such that $X^3-X-\zeta$ is an irreducible polynomial of $K[X]$. μ is a fixed non-square element of K.

Table 2

The Representatives and the Orders of the Centralizers in Case $p=3$.

1		1	$q^6(q^2-1)(q^6-1)$
$x_1=x_a(1)x_b(1)$		1	$3q^2$
$x_2=x_a(1)x_b(1)x_{3a+b}(\zeta)$		1	$3q^2$
$x_3=x_a(1)x_b(1)x_{3a+b}(-\zeta)$		1	$3q^2$
$x_4=x_{a+b}(1)x_{3a+b}(1)$		1	$2q^4$
$x_5=x_{a+b}(1)x_{3a+b}(\mu)$		1	$2q^4$
$x_6=x_{2a+b}(1)$		1	$q^6(q^2-1)$
$x_7=x_{2a+b}(1)x_{3a+2b}(1)$		1	q^6
$x_8=x_{3a+2b}(1)$		1	$q^6(q^2-1)$
$h(-1, -1, 1)$		1	$q^2(q^2-1)^2$
$h(z, z^{-1}, 1)$	$z^{q-1}=1,\ z^2\neq 1$	$(q-3)/2$	$q(q-1)^2(q+1)$
$h(z^{-1}, z^2, z^{-1})$	$z^{q-1}=1,\ z^2\neq 1$	$(q-3)/2$	$q(q-1)^2(q+1)$
$h(z_1, z_2, z_3)$	$z_i^{q-1}=1,\ z_iz_j^{\pm 1}\neq 1$	$(q^2-8q+15)/12$	$(q-1)^2$
$h(z, z^{-1}, 1)$	$z^{q+1}=1,\ z^2\neq 1$	$(q-1)/2$	$q(q-1)(q+1)^2$
$h(z^{-1}, z^2, z^{-1})$	$z^{q+1}=1,\ z^2\neq 1$	$(q-1)/2$	$q(q-1)(q+1)^2$
$h(z_1, z_2, z_3)$	$z_i^{q+1},\ z_iz_j^{\pm 1}\neq 1$	$(q^2-4q+3)/12$	$(q+1)^2$
$h(z, z^q, z^{-q-1})$	$z^{q^2-1}=1,\ z^{q\pm 1}\neq 1$	$(q^2-2q+1)/4$	q^2-1
$h(z, z^{q-1}, z^{-q})$	$z^{q^2-1}=1,\ z^{q\pm 1}\neq 1$	$(q^2-2q+1)/4$	q^2-1
$h(z, z^q, z^{q^2})$	$z^{q^2+q+1}=1,\ z\neq 1$	$(q^2+q)/6$	q^2+q+1
$h(z, z^{-q}, z^{q^2})$	$z^{q^2-q+1}=1,\ z\neq 1$	$(q^2-q)/6$	q^2-q+1
$h(-1, -1, 1)x_b(1)$		1	$q^2(q^2-1)$
$h(-1, -1, 1)x_{2a+b}(1)$		1	$q^2(q^2-1)$
$h(-1, -1, 1)x_b(1)x_{2a+b}(1)$		1	$2q^2$
$h(-1, -1, 1)x_b(1)x_{2a+b}(\mu)$		1	$2q^2$
$h(z, z^{-1}, 1)x_{2a+b}(1)$	$z^{q-1}=1,\ z^2\neq 1$	$(q-3)/2$	$q(q-1)$
$h(z^{-1}, z^2, z^{-1})x_{3a+2b}(1)$	$z^{q-1}=1,\ z^2\neq 1$	$(q-3)/2$	$q(q-1)$
$h(z, z^{-1}, 1)x_{2a+b}(1)$	$z^{q+1}=1,\ z^2\neq 1$	$(q-1)/2$	$q(q+1)$
$h(z^{-1}, z^2, z^{-1})x_{3a+2b}(1)$	$z^{q+1}=1,\ z^2\neq 1$	$(q-1)/2$	$q(q+1)$

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