Hauptvermutung for $\pi_1 = Z$

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§1. Introduction and statement of results.

In [6, 7], Sullivan has studied the homotopy smoothing (resp. homotopy triangulation) of simply-connected manifolds of dimension greater than four, and has succeeded to prove Hauptvermutung for such manifolds under some reasonable conditions. The purpose of this paper is to extend his result to the case of $\pi_1 = Z$. A manifold with $\pi_1 = Z$ has an interesting property, that is, roughly speaking, by cutting along a suitable simply-connected submanifold of codimension 1, we get a simply-connected manifold with boundary (Browder [1]). Our method is to reduce a surgery problem for $\pi_1 = Z$ to the simply-connected surgery, using the above "splitting property".

Before describing our results, we shall give some definitions (cf. Sullivan [7]).

Let M be a connected smooth (resp. PL) manifold. A pair (L, f) is called a homotopy smoothing (resp. homotopy triangulation) of M, if L is a smooth (resp. PL) manifold and $f: (L, \partial L) \to (M, \partial M)$ is a homotopy equivalence. For simplicity, the terminology h-smoothing (resp. h-triangulation) is also used. Two such pairs (L_1, f_1) and (L_2, f_2) are said to be concordant if there exists a diffeomorphism (resp. PL homeomorphism) $g: (L_1, \partial L_1) \to (L_2, \partial L_2)$ such that the following diagram is homotopically commutative.

$$\begin{array}{ccc} (L_1, \partial L_1) & \xrightarrow{f_1} & \\ g \downarrow \simeq & & (M, \partial M) \\ (L_2, \partial L_2) & \xrightarrow{f_2} & \end{array}$$

The concordance relation is proved to be an equivalence relation. Let $hS(M)$ (resp. $hT(M)$) denote the set of concordance classes of h-smoothings (resp. h-triangulations) of M. Note that $hS(M)$ (resp. $hT(M)$) has a base element (M, id).

A triple (E, ϖ, φ) is called an F/O-bundle (resp. F/PL-bundle) over M if $\varpi: E \to M$ is a vector bundle (resp. PL euclidean bundle) over M of fibre dimension s and $\varphi: E \to R^s$ is a map which restricted to each fibre of E gives a proper homotopy equivalence. $(E_1, \varpi_1, \varphi_1)$ and $(E_2, \varpi_2, \varphi_2)$ are said to be equivalent if there exists a bundle equivalence $b: E_1 \to E_2$ such that the following diagram is properly homotopically commutative.

$$\begin{array}{ccc} E_1 & \xrightarrow{\varphi_1} & \\ b\downarrow & \cong & R^s \\ E_2 & \xrightarrow{\varphi_2} & \end{array}$$

We can define the Whitney sum of F/O (resp. F/PL) bundles and stabilize a bundle by adding a large dimensional trivial bundle.

Let F/O (resp. F/PL) denote the classifying space of such stable bundles. (For the existence of the classifying space, see Sullivan, [6].)

Our result is the following

THEOREM 1.1. *Let M be a connected closed smooth (resp. PL) manifold with $\pi_1 \cong Z$ of dimension $n \geqq 6$ such that $\pi_i(M)$ are finitely-generated groups for $i \leqq 4$. Then we have the following exact sequences (of based sets).*

i) *If M is orientable,*

$$\theta_n(\partial\pi)\oplus\theta_{n-1}(\partial\pi)\longrightarrow hS(M)\longrightarrow[M, F/O]\xrightarrow{\mathscr{S}_Q}P_n\oplus P_{n-1}$$

$$0\longrightarrow hT(M)\longrightarrow[M, F/PL]\xrightarrow{\mathscr{S}_Q}P_n\oplus P_{n-1}\,.$$

ii) *If M is non-orientable,*

$n\equiv 0 \pmod 4$ $\quad 0\longrightarrow hS(M)\longrightarrow[M, F/O]\xrightarrow{\mathscr{S}_Q}Z$

$\quad 0\longrightarrow hT(M)\longrightarrow[M, F/PL]\xrightarrow{\mathscr{S}_Q}Z$

$n\equiv 1 \pmod 4$ $\quad \theta_n(\partial\pi)\longrightarrow hS(M)\longrightarrow[M, F/O]\longrightarrow 0$

$\quad 0\longrightarrow hT(M)\longrightarrow[M, F/PL]\longrightarrow 0$

$n\equiv 2 \pmod 4$ $\quad Z_2\longrightarrow hS(M)\longrightarrow[M, F/O]\xrightarrow{\mathscr{S}_Q}Z_2$

$\quad Z_2\longrightarrow hT(M)\longrightarrow[M, F/PL]\xrightarrow{\mathscr{S}_Q}Z_2$

$n\equiv 3 \pmod 4$ $\quad \theta_n(\partial\pi)\longrightarrow hS(M)\longrightarrow[M, F/O]\xrightarrow{\mathscr{S}_Q}Z_2$

$\quad 0\longrightarrow hT(M)\longrightarrow[M, F/PL]\xrightarrow{\mathscr{S}_Q}Z_2\,,$

where P_n stand for the simply-connected surgery obstruction groups which are given as follows (Kervaire and Milnor [3]):

$$P_n\cong\begin{cases} Z & (n\equiv 0 \bmod 4)\\ Z_2 & (n\equiv 2 \bmod 4)\\ 0 & (n \textit{ is odd})\,. \end{cases}$$

REMARK. In the above, the sequence $Z_2\to hS(M)\to[M, F/O]$ only means that the latter map has at most two elements as its "kernel". We do not define map $Z_2\to hS(M)$. Similarly for the PL case.

Combining the above result and the homotopy theory of F/PL due to Sullivan [7], we can easily deduce

COROLLARY 1.2 (Hauptvermutung for $\pi_1=Z$). *Let M be a connected closed PL manifold with $\pi_1=Z$ of dimension $n\geqq 6$ such that $\pi_i(M)$ are finitely generated groups for $i\leqq 4$. Assume M is either orientable or $n\not\equiv 2$ (mod 4). If $H^4(M;Z)$ has no two torsion, Hauptvermutung holds for M, that is, if $f: L\to M$ is a topological homeomorphism from a PL manifold L to M, then f is homotopic to a PL homeomorphism.*

Throughout this paper, only the simply-connected surgery is used. In §2 we shall prove a fundamental lemma, which is important to reduce a $\pi_1=Z$ case to the simply-connected surgery. §3 is devoted to the proof of the main theorem.

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Having prepared this paper, the author was noticed that J. L. Shaneson had studied the homotopy smoothing of 5-dimensional closed manifold with $\pi_1=Z$ as an application of his more general surgery theorems and proved Hauptvermutung for such 5-manifolds (Shaneson [5]).

§2. Fundamental lemma.

Let M be a connected closed smooth (resp. PL) manifold with $\pi_1=Z$ of dimension $n\geqq 6$, and let Q be a connected simply-connected bi-collared submanifold of codimension 1. We can find an embedding $e: Q\times(-1, 1)\to M$ such that the image of $Q\times 0$ is identified with the submanifold Q.

Let $M_Q=M-e\left(Q\times\left(-\frac{1}{2}, \frac{1}{2}\right)\right)$. Then, the diffeomorphism class (resp. PL homeomorphism class) of M_Q is independent of the choice of the embedding e. The above process is called "cutting M along Q".

If M_Q is connected and simply-connected, Q is called "a splitting of M". The boundary ∂M_Q consists of two copies Q_0, Q_1 of Q. If the following additional condition (1) is satisfied, we shall call Q "a nice-splitting of M";

$$(1)\qquad \pi_i(M_Q, Q_0)\cong\pi_i(M_Q, Q_1)\cong 0 \quad \text{for} \quad i\leqq 3.$$

In §3, it will be proved that, if $\pi_i(M)$ are finitely generated groups for $i\leqq 4$, then M has a nice-splitting. Let $\varpi: E\to M$ be a vector (resp. PL euclidean) bundle over M of large fibre dimension, and let L be a closed submanifold of the total space E such that ϖ restricted to L gives a homotopy equivalence from L to M. Let Q be a nice-splitting of M. We shall denote by $E|Q$ the bundle restricted over Q and at the same time the total space of the bundle by an abuse of language. $E|Q$ is a submanifold of the total space of codimension 1.

FUNDAMENTAL LEMMA 2.1. *Let M, L, E, $E|Q$ be as above. Then, we can find a submanifold L' of E which satisfies the following conditions* (i)~(iii).

(i) *L is isotopic to L' by a smooth (resp. PL) isotopy in E.*

(ii) *L' intersects with $E|Q$ transversally.*

(iii) *The intersection manifold $L'\cap E|Q$ is homotopically equivalent to Q by ϖ restricted to $L'\cap E|Q$.*

The rest of this section is devoted to the proof of the fundamental lemma. By the general position argument, we may assume that L intersects with $E|Q$ transversally and $L\cap E|Q$ is a manifold. We shall denote the intersection by N. N may not be connected. But by cutting L along N, we get L_N. This is possible because the normal line bundle of N in L is trivial. The boundary ∂L_N consists of two copies N_0, N_1 of N. The projection $\varpi: E\to M$ induces naturally a projection $L_N\to M_Q$, which we also denote by the same letter ϖ.

Our aim is to kill the "kernels" of $\varpi_{\sharp}{}^0: \pi_i(L_N, N_0)\to\pi_i(M_Q, Q_0)$ and $\varpi_{\sharp}{}^1: \pi_i(L_N, N_1)\to\pi_i(M_Q, Q_1)$ by a sequence of local isotopies constructed below. (Here, $\pi_i(L_N, N_0)$ etc. are considered to be homotopy sets.)

2.2. *Construction of local isotopy.*

Let i be any integer such that $1\leqq i\leqq n-3$, and let d be an i-dimensional disk embedded in L_N such that d intersects with N_0 transversally and $d\cap N_0=\partial d$. The embedding $(d, \partial d)\subset(L_N, N_0)$ represents an element δ of $\pi_i(L_N, N_0)$. If δ is mapped to the zero element by the map

$$\varpi_{\sharp}: \pi_i(L_N, N_0)\to\pi_i(M_Q, Q_0)$$

the local isotopy of L on d is constructed as follows.

Let E_Q denote the manifold obtained by cutting the total space E along the submanifold $E|Q$. The projection $E_Q\to M_Q$ is naturally defined and is denoted by the same letter ϖ. $\varpi: (E_Q, E|Q_0)\to(M_Q, Q_0)$ is a homotopy equivalence, where $E|Q_0$ is one of the two copies of $E|Q$ which is over Q_0. Hence, by hypothesis, δ equals to zero in $\pi_i(E_Q, E|Q_0)$. Note $(L_N, N_0)\subset(E_Q, E|Q_0)$. Embed an $i+1$-disk Δ in E_Q in such a way that Δ intersects with $E|Q_0$ and L_N transversally along the northern and the southern hemispheres of i-sphere $\partial\Delta$, respectively, and that $\Delta\cap L_N=$(the southern hemisphere of $\partial\Delta$)$=d$. This is possible, for the fibre dimension of E is sufficiently large and δ equals zero in

$\pi_i(E_Q, E|Q_0)$. Of course, Δ has the angle along the equator of $\partial\Delta$. By pasting E_Q up along $E|Q$, we may consider that Δ is embedded in E. Note that N has a collar embedding $e: N\times(-1, 1)\to L$. By sliding Δ along this collar, we may assume that Δ intersects with $E|Q$ transversally and $\Delta\cap E|Q=d'$, an i-disk attached to N along the boundary $\partial d'$. Since $1\leqq i\leqq n-3$, the i-disk d ($=$the southern hemisphere of $\partial\Delta$) has a product neighbourhood in L, which is extended to the whole of $i+1$-disk Δ. After all, we have found $n+1$-disk $D_0^{n+1}=\Delta^{i+1}\times D^{n-i}$ embedded in E and has the following properties (i)$\sim$(iii):

(i) $D_0^{n+1}\cap L=d\times D^{n-i}$ ($=$the thickening of d in L):

(ii) $D_0^{n+1}\cap E|Q=d'\times D^{n-i}$ ($=$the thickening of d' in $E|Q$):

(iii) D_0^{n+1} has the angle along $\partial\Delta^{i+1}\times\partial D^{n-i}\cup S^{i-1}\times D^{n-i}$, where S^{i-1} denotes the equator of $\partial\Delta$.

Consider the embedding $\mathscr{E}: L\times[0, 1]\to E$ such that the embedded image of $L\times 0$ is identified with L and that $\mathscr{E}(L\times[0, 1])\cap D_0^{n+1}=\mathscr{E}(L\times 0)\cap D_0^{n+1}=L\cap D_0^{n+1}$. Moreover, assume $\mathscr{E}(L\times[0, 1])\cap E|Q=\mathscr{E}(N\times[0, 1])$.

Let $W=\mathscr{E}(L\times[0, 1])\cup D_0^{n+1}$. In the smooth case, straighten the angle of W by sufficiently small movement. D_0^{n+1} is a knob attached to $\mathscr{E}(L\times[0, 1])$ and so W is diffeomorphic (resp. *PL* homeomophic) to $L\times[0, 1]$. The boundary ∂W consists of the embedded image of $L\times 1$ and another manifold L_1. L_1 is isotopic to the image of $L\times 1$ by the smooth (resp. *PL*) isotopy $\mathscr{I}_W$ which has W as the isotopy trace. Obviously, the image of $L\times 1$ is isotopic to that of $L\times 0$ by the level changing isotopy $\mathscr{L}$.

The composite isotopy

$$L=\mathscr{E}(L\times 0)\overset{\mathscr{L}}{\rightsquigarrow}\mathscr{E}(L\times 1)\overset{\mathscr{I}_W}{\rightsquigarrow}L_1\,,$$

is called *the local isotopy of L on d.*

The new intersection $N_1=L_1\cap E|Q$ is obtained from $N\times 1$ by a surgery on $\partial d\times 1\subset N\times 1$, and the surgery trace is just $W\cap E|Q$.

If N_0 and N_1 (or Q_0 and Q_1) are interchanged in the construction, the argument goes similarly.

LEMMA 2.3. *By a sequence of local isotopies L can be moved to* L_1 *such that* $L_1\cap E|Q$ *is connected and simply-connected.*

PROOF. If $N=L\cap E|Q$ is not connected, let $C_1, C_2, \cdots, C_r$ be the connected components of N. First assume that the copies C_1^0, C_2^0 of C_1, C_2 in $N_0\subset L_N$ are joined by an arc $\alpha\subset L_N$. α may be assumed to have no self-intersection and to intersect N_0 transversally at exactly its two terminal points. α represents an element of $\pi_1(L_N, N_0)$ and map $\varpi_\#: \pi_1(L_N, N_0)\to\pi_1(M_Q, Q_0)\cong 0$ is obviously a zero map. Therefore we can construct the local isotopy of L on α and move L to L'. The components of the new intersection $N'=L'\cap E|Q$ consists of $C_1\sharp C_2$, C_3, $\cdots$, C_r, where $\sharp$ denotes the connected sum operation. If the copies C_1^1, C_2^1 of C_1, C_2 in $N_1\subset L_N$ are joined by an arc in L_N, the argument is similar. It is not difficult to prove that if N is not connected there exist at least two components of N C_i, C_j whose copies C_i^0, C_j^0 (or C_i^1, C_j^1) are joined by an arc in L_N, so the proof is left to the reader.

Now, we may assume that $N=L\cap E|Q$ is already connected. Let $i: N\to L$ be the inclusion. By the diagram;

$$\begin{array}{ccc} \pi_1(N) & \xrightarrow{\ i_\sharp\ } & \pi_1(L) \\ \downarrow & & \downarrow \cong \\ 0=\pi_1(Q) & \longrightarrow & \pi_1(M) \end{array}$$

$i_\sharp$: $\pi_1(N)\to\pi_1(L)$ is a zero map. For any element $x\in\pi_1(N)$, we can find an embedded 2-disk $D^2\subset L$ such that $\partial D^2\subset N$ represents x. Suppose D^2 intersects with N transversally and $\mathring{D}^2\cap N=\gamma_1\cup\cdots\cup\gamma_s$ (disjoint circles). Let γ_i be one of the innermost circles. Denote by d^2 the closed region of D^2 bounded by γ_i. Since Q is a nice splitting of M, $\pi_2(M_Q, Q_0)=\pi_2(M_Q, Q_1)=0$, and we can construct the local isotopy of L on d^2. Proceeding inductively, N is improved to be simply-connected. The lemma follows.

By Lemma 2.3, we may suppose N is already connected and simply-connected. Let X and Y be the universal coverings of M and L, respectively. There is induced a natural projection $Y\to X$ denoted by $\tilde{\varpi}$. $t: Y\to Y$ and $t': X\to X$ are the covering transformations corresponding fixed generators of $\pi_1(L)$ and $\pi_1(M)$ respectively, and $\tilde{\varpi}$ is assumed to be equivariant with respect to t and t'.

Identify $N_0\subset\partial L_N$ with a fixed lift of N into Y, N_1 with $tN_0\subset Y$, and L_N with the closed region of Y between N_0 and N_1. For convenience, let N_m denote $t^m N_0$ for any integer $m\in Z$. Similarly for Q, M, X, t'. We may suppose $\tilde{\varpi}(N_m)=Q_m$ for any $m\in Z$. Let V_m denote the closed region of Y between N_0 and N_m, and let U_m be the closed region of X between Q_0 and Q_m. Define $A=\bigcup_{m<0} U_m$, $B=\bigcup_{m\geq 0} U_m$, $C=\bigcup_{m<0} V_m$, $D=\bigcup_{m\geq 0} V_m$. $\tilde{\varpi}|V_m: (V_m, \partial V_m)\to(U_m, \partial U_m)$ is proved to have degree 1 for any $m\in Z$. Hence, the groups $K_i(V_m)$ (resp. $K^i(V_m)$) can be defined as the kernel (resp. the cokernel) of the surjective (resp. injective) homomorphism $\tilde{\varpi}_*: H_i(V_m; Z)\to H_i(U_m; Z)$ (resp. $\tilde{\varpi}^*: H^i(U_m; Z)\to H^i(V_m; Z)$).

Groups $K_i(N_m)$, $K^i(N_m)$, $K_i(V_m, N_0)$ and $K^i(V_m, N_0)$ are similarly defined. N_m, Q_m, V_m, U_m are assumed to be simply-connected below. The following lemma is well-known (Wall [8]).

LEMMA 2.4. *K_* and K^* have the same properties as the ordinary homology or cohomology theory, for example* (i) *the exact sequences,* (ii) *the Poincaré-Lefschetz duality,* (iii) *the universal coefficient theorems,* (iv) *the excision theorems. These properties are formally deduced from the corresponding properties of the ordinary theory.*

The surjective homomorphism $\tilde{\varpi}_*: H_i(V_m) \to H_i(U_m)$ (integral coefficients are assumed) induces a surjective homomorphism $\tilde{\varpi}_*: H_i(D) \to H_i(B)$ as the inductive limit. Define $K_i(D) = \mathrm{Ker}(\tilde{\varpi}_*: H_i(D) \to H_i(B))$. Similarly for the definition of $K_i(C)$. Obviously

$$K_i(D) = \lim_{m \to +\infty}.\mathrm{ind}\, K_i(V_m),$$

$$K_i(C) = \lim_{m \to -\infty}.\mathrm{ind}\, K_i(V_m).$$

The analogous equations hold for $K_i(D, N_0)$ and $K_i(C, N_0)$.

LEMMA 2.5. *If there exists an integer r $(0 \leqq r \leqq n)$ such that*

$$K_i(C, N_0) = 0 \text{ for } i \leqq n-r,$$

$$K_i(D, N_0) = 0 \text{ for } i \leqq r,$$

then $K_i(N_0) = 0$ for all i, therefore, $\varpi | N: N \to Q$ is a homotopy equivalence.

PROOF. The exact sequence

$$\begin{array}{l} K_{i+1}(D, V_m) \to K_i(V_m, N_0) \to K_i(D, N_0) \quad (m > 0) \\ \quad \wr\!\| \; t_*^{-m} \\ K_{i+1}(D, N_0) \end{array}$$

implies $K_i(V_m, N_0) = 0$ for $i \leqq r-1$. Note $\partial V_m = N_0 \cup N_m$. $K_i(V_m, N_m) = 0$ for $i \leqq n-r-1$ is similarly shown. On the other hand,

$$K_i(V_m, N_0) \cong K^{n-i}(V_m, N_m) \cong \mathrm{Hom}(K_{n-i}(V_m, N_m), Z) \oplus \mathrm{Ext}(K_{n-i-1}(V_m, N_m), Z).$$

Thus $K_i(V_m, N_0) = 0$ if $n-i \leqq n-r-1$. Hence $K_i(V_m, N_0) = 0$ for $i \neq r$. Hence,

$$K_i(D, N_0) = \lim_{m \to +\infty}.\mathrm{ind}\, K_i(V_m, N_0) = 0 \quad i \neq r.$$

On the other hand, $K_r(D, N_0) = 0$ by the hypothesis. In consequence, $K_i(D, N_0) = 0$ for all i. Similarly, $K_i(C, N_0) = 0$ for all i. By considering the diagram

$$\begin{array}{ccccc} & & & & K_i(Y) \cong 0 \\ & & & & \uparrow \\ K_{i+1}(D, N_0) & \longrightarrow & K_i(N_0) & \longrightarrow & K_i(D) \\ \wr\!\| & & & & \uparrow \\ 0 & & & & K_{i+1}(Y, D) \cong K_{i+1}(C, N_0) \cong 0, \end{array}$$

the lemma is proved.

The effect on C of local isotopy on $(d, \partial d)\subset(L_N, N_0)$ (resp. $(d, \partial d)\subset(L_N, N_1)$) is to exchange an i-handle from D to C (resp. from C to D). By Lemma 2.5, the proof of our fundamental lemma is reduced to the fibering problem of Browder and Levine [2] except that K_* homology is used in place of H_*. However, we need a few more words to show that K_i's are finitely generated and that elements of K_i to be killed are spherical.

LEMMA 2.6. *$K_i(C, N_0)$ and $K_i(D, N_0)$ are finitely generated for all i.*

PROOF. *Let $\rho: M\to L$ be a homotopy inverse of $\varpi: L\to M$.*

Let H_s ($s\in[0, 1]$) be a homotopy $\rho\circ\varpi\simeq 1_L$. Lifting to the universal coverings, we obtain homotopy equivalances $\tilde\varpi: Y\to X$, $\tilde\rho: X\to Y$ and a homotopy $\tilde H_s: \tilde\rho\circ\tilde\varpi\simeq 1_Y$. Let $D_m=t^mD$ and $B_m=t'^mB$. If m is sufficiently large, $\tilde H_s(D_m)\subset D$ for all $s\in[0, 1]$. For such m, $\tilde\rho\circ(\tilde\varpi|D_m): D_m\to D$ is homotopic to the inclusion $D_m\subset D$.

Consider the commutative diagram

$$\begin{array}{ccc} K_i(D_m) & & \\ \downarrow & & \\ H_i(D_m) & \xrightarrow{\ \mathrm{inc}_*\ } & H_i(D) \; . \\ \downarrow{\scriptstyle (\tilde\varpi|D_m)_*} & \nearrow{\scriptstyle \tilde\rho_*} & \\ H_i(B_m) & & \end{array}$$

By the diagram, $K_i(D_m)\to K_i(D)$ is zero. Thus K_* exact sequence: $K_i(D_m)\to K_i(D)\to K_i(D, D_m)$ shows that $K_i(D)$ is a subgroup of $K_i(D, D_m)\cong K_i(V_m, N_m)$. Hence $K_i(D)$ is finitely generated. The lemma follows easily.

LEMMA 2.7. *If $K_i(C, N_0)\cong K_i(D, N_0)\cong 0$ for $i<k$, where $k\geqq 4$, then the elements of $K_k(V_1, N_0)$ and $K_k(V_1, N_1)$ are spherical, i.e. each element has a representative map $f: (d^k, \partial d^k)\to(V_1, N_0)$ (resp. (V_1, N_1)) which belongs to the kernel of $\pi_k(V_1, N_0)\to\pi_k(U_1, Q_0)$ (resp. $\pi_k(V_1, N_1)\to\pi_k(U_1, Q_1)$).*

PROOF. Consider the quadruple $\Phi=\begin{pmatrix} N_0 \longrightarrow V_1 \\ \downarrow\omega_0 \quad \omega_1\downarrow \\ Q_0 \longrightarrow U_1 \end{pmatrix}$. (For the homotopy theory of quadruples, see Namioka [4].) Assume that $\pi_i(V_1, N_0)=0$ for $i<u$, $\pi_i(\omega_1)=0$ for $i<v$ and $\pi_i(U_1, Q_0)=0$ for $i<w$. Namioka's theorem (Namioka [4]) asserts that if N_0, Q_0, V_1, U_1 are all simply-connected (this is our case) and if $1<r\leqq u+v-2$, then $H_q(\Phi)=0$ for $q\leqq r$ is equivalent to $\pi_q(\Phi)=0$ for $0<q\leqq r$. Moreover if $2<r<v+w-2$, and if $\pi_q(\Phi)=0$ for $1<q\leqq r$, then the Hurewicz map $h: \pi_q(\Phi)\to H_q(\Phi)$ is injective for $q\leqq r+1$ and surjective for $q\leqq r+2$. The homology exact

sequence of the quadruple implies that $H_i(\Phi)\cong K_{i-1}(V_1, N_0)$.

Now suppose inductively $K_i(C, N_0)\cong K_i(D, N_0)\cong 0$ for $i<k$. This implies $K_i(V_1, N_0)\cong K_i(V_1, N_1)=0$ for $i<k-1$ (see the proof of Lemma 2.5). Thus $H_i(\Phi)=0$ for $i<k$. By the above Namioka's results, $h: \pi_{k+1}(\Phi)\to H_{k+1}(\Phi)\cong K_k(V_1,N_0)$ is surjective. Hence the elements of $K_k(V_1, N_0)$ are spherical. The argument is similar for (V_1, N_1). The restriction on r in Namioka's theorem requires $k\geqq 4$.

2.8. *Completion of the proof of the fundamental lemma.*

Suppose both N_0 and V_1 are connected and simply-connected. Since Q is a nice-splitting of M, $\pi_i(M_Q, Q_0)=\pi_i(M_Q, Q_1)=0$ for $i\leqq 3$. Thus, $K_i(V_1, N_0)=H_i(V_1, N_0)$ and $K_i(V_1, N_1)=H_i(V_1, N_1)$ for $i\leqq 3$. By the argument of Browder-Levine [2] all the groups $H_i(C, N_0)$ and $H_i(D, N_0)$ are killed for $i\leqq 3$ by exchanging handles. And the assumption $\pi_i(M_Q, Q_0)=\pi_i(M_Q, Q_1)=0$ for $i\leqq 3$ guarantees that the exchanging handles is realized by a sequence of local isotopies. It is proved that if $K_i(V_1, N_0)\to K_i(D, N_0)$ is a zero map, then $K_i(D, N_0)=0$ (Browder-Levine [2]). Hence, if $K_i(D, N_0)\neq 0$, we can find an embedded disk $(d^i, \partial d^i)\subset (V_1, N_0)$ representing a non-zero element of $K_i(D, N_0)$ and construct a local isotopy on d, which kills the element. Lemmas 2.6 and 2.7 imply that after finite processes all the groups K_i vanish below the middle dimension.

The middle dimension. Let $n=2k$ or $2k-1$. We may suppose

$$K_i(D, N_0)=0 \text{ for } i<k\,,$$
$$K_i(C, N_0)=0 \text{ for } i<n-k+1\,.$$

This implies

$$K_i(V_1, N_0)=0 \text{ for } i\neq k-1,\, k\,,$$
$$K_i(V_1, N_1)=0 \text{ for } i\neq n-k,\; n-k+1\,,$$

and moreover $K_k(V_1, N_0)$ and $K_{n-k+1}(V_1, N_1)$ are free. $K_k(V_1, N_0)$ is proved to be a direct summand of the free group $K_k(D, N_0)$. If $K_k(D, N_0)\neq 0$, a non-zero free generator y of $K_k(V_1, N_0)$ can be found. By Lemma 2.7, y is spherical. y is represented by an embedded disk $(d^k, \partial d^k)\subset (V_1, N_0)$ which belongs to the kernel of $\varpi_\sharp: \pi_k(V_1, N_0)\to\pi_k(U_1, Q_0)$. In the odd case, the condition $\pi_2(M_Q, Q_0)=0$ guarantees the embedding. The local isotopy on d^k kills y. This isotopy does not affect $K_i(D, N_0)=0$ for $i>k+1$. $K_{k+1}(D, N_0)=0$ is deduced from the choice of y. By induction, we have succeeded to obtain

$$K_i(D, N_0)=0 \text{ for all } i\,.$$

The fundamental lemma follows by Lemma 2.5.

§ 3. The proof of the main theorem.

3.1. *Definition of the map* $hS(M)\to[M, F/O]$ *(resp.* $hT(M)\to[M, F/PL]$*).*

Let $f: L\to M$ be a h-smoothing (resp. h-triangulation) of M. Let $\bar{f}: M\to L$ be a homotopy inverse of f. Consider an embedding $M\to L\times R^s$ ($s\gg n$) approximating the composite map $M\xrightarrow{\bar{f}}L\xrightarrow{\times 0}L\times R^s$. Let $E\xrightarrow{\varpi}M$ be the open tubular neighbourhood of the embedding. It is proved that E has a homotopy trivialization $\varphi: E\to R^s$. Correspondence from (L, f) to (E, ϖ, φ) defines the map $hS(M)\to [M, F/O]$ (resp. $hT(M)\to[M, F/PL]$). A geometric argument shows that we may assume $\varphi^{-1}(0)\xrightarrow{\varpi}M$ is concordant to $L\xrightarrow{f}M$. For the details, see Sullivan [7].

LEMMA 3.2. *Let* M^n *be a connected closed smooth (resp. PL) manifold with* $\pi_1=Z$ *of dimension* $n\geqq 6$ *such that* $\pi_i(M)$ *are finitely generated abelian groups for* $i\leqq 4$. *Then* M *has a nice-splitting.*

PROOF. Let $f: M\to S^1$ be a map corresponding to a generator of $[M, S^1]=H^1(M; Z)\simeq Z$. Make f t-regular at a point $p\in S^1$. Let $Q'=f^{-1}(p)$. Q' is a closed submanifold of codimension 1 of M and has the trivial normal bundle. By exchanging handles, we can make Q' connected. The homology class $[Q']$ is not zero in $H_{n-1}(M; Z_2)$. Thus $M_{Q'}$ is connected. Let X be the universal covering of M, and let Q_0' be a lifting of Q' which divides X into two parts A', B'. By the handle exchanging argument of Browder-Levine [2], we can change Q' to Q such that $H_i(A, Q_0)\cong H_i(B, Q_0)\cong 0$ for $i\leqq 4$, where Q_0 is a lifting of Q into X. This is possible, because $\pi_i(M)$ is finitely generated for $i\leqq 4$. Note that $H_i(A, Q_0)\simeq H_i(B, Q_0)\cong 0$ for $i<k$ implies that $H_i(M_Q, Q_0)\cong H_i(M_Q, Q_1)=0$ for $i<k-1$ (see the proof of Lemma 2.5). By the Hurewicz theorem, the lemma follows.

Throughout this section, a nice-splitting Q is fixed.

3.3. *The definition of* $\mathscr{S}_Q$: $[M, F/O]$ (resp. $[M, F/PL]$)$\to P_n\oplus P_{n-1}$.

(i) The orientable case.

Let $\varpi: E\to M$ be a F/O (resp. F/PL) bundle over M of large fibre dimension s. Let $\varphi: E\to R^s$ be the homotopy trivialization of E. Make φ t-regular at the origin of the s-dimensional euclidean space R^s.

Let $K^n=\varphi^{-1}(0)$. K is a framed n-submanifold of E. Let T^{n-1} be the intersection manifold $T=K\cap E|Q$[1]. ϖ restricted to T has degree 1. If n is odd (thus $\dim T=n-1$ is even), define $\mathscr{S}_Q$ as the surgery obstruction $\sigma(T)$ of the map $\varpi|T: T\to Q$ (Kervaire and Milnor [3], Sullivan [6]). $\sigma(T)\in P_{n-1}$. If n is even, $\sigma(T)=0$, for dim T is odd, and we can do surgery on T to make $\varpi|T: T\to Q$

1) Without loss of generality, T may be assumed to be connected.

be a homotopy equivalence ($n-1\geq 5$). This surgery can be extended to the surgery of K. In this section, "surgery" means the framed surgery, that is, there exists a homotopy trivialization $\tilde{\varphi}$: $E\times I\to R^s$ such that $\tilde{\varphi}^{-1}(0)$ (0 is the origin of R^s) is the surgery trace. Now, in the case n=even, we may assume that $\varpi|T$: $T\to Q$ is already a homotopy equivalence and by cutting K along T we obtain K_T. ϖ: $K_T\to M_Q$, restricted to the boundary, gives a homotopy equivalence. Define $\mathscr{S}_Q$ as the interior surgery obstruction $\sigma(K_T)$ of ϖ: $K_T\to M_Q$. $\sigma(K_T)\in P_n$. In the above, if we choose an orientation of M and a generator of $\pi_1(M)=Z$, the orientations of Q and M_Q are canonically determined. The well-definedness of $\mathscr{S}_Q$ is easily verified by framed cobordism invariance of surgery obstructions.

(ii) The non-orientable case.

If $n\not\equiv 1$ (mod 4), the argument is the same as the orientable case. If $n\equiv 1$ (mod 4), the following simple observation tells us that there is no surgery obstruction of $K\to M$.

LEMMA 3.4. *Let* W^{4m+1} *be a compact connected closed* $4m+1$*-manifold and* V^{4m} *be a connected closed submanifold of codimension* 1 *with the trivial normal bundle. Suppose* V *is orientable and* W_V *(the manifold obtained from* W *by cutting along* V*) is connected and orientable.*

If W *is non-orientable, the oriented cobordism class of* V *is two torsion. And thus, the index of* V *equals to zero.*

3.5 *The exactness at* $[M, F/O]$ *(resp.* $[M, F/PL]$*).*

By the definition of $\mathscr{S}_Q$, the kernel of $\mathscr{S}_Q$ is contained in the image of $hS(M)\to[M, F/O]$ (resp. $hT(M)\to[M, F/PL]$). Conversely, if (E, ϖ, φ) is in the image, where ϖ: $E\to M$ is an F/O (resp. F/PL) bundle over M, then $K^n=\varphi^{-1}(0)$ may be assumed to be homotopically equivalent to M, so by virtue of our fundamental lemma, K is moved in the total space E until $T=K\cap E|Q$ is homotopically equivalent to Q. Thus, $\sigma(T)$ and $\sigma(K_T)$ vanish.

3.6. *The exactness at* $hS(M)$ *(resp.* $hT(M)$*).*

If two homotopy smoothings (resp. homotopy triangulations) $L\to M$ and $L'\to M$ have equivalent characteristic bundles, then we get the following situation. $\tilde{E}\to M^n\times I$ is an s-dimensional vector (resp. PL euclidean) bundle over $M^n\times I$, where $I=[0, 1]$ ($s\gg n$). $\tilde{E}$ is equipped with a homotopy trivialization $\tilde{\varphi}$: $\tilde{E}\to R^s$. $U^{n+1}=\tilde{\varphi}^{-1}(0)$ is a framed submanifold of E. We may suppose $L=U\cap\tilde{E}|M\times 0$, $L'=U\cap\tilde{E}|M\times 1$. Our fundamental lemma admits us to move L (resp. L') in $\tilde{E}|M\times 0$ (resp. in $\tilde{E}|M\times 1$) and to make $N=L\cap\tilde{E}|Q\times 0$ (resp. $N'=L'\cap\tilde{E}|Q\times 1$)

homotopically equivalent to $Q\times 0$ (resp. $Q\times 1$). Let $W=U^{n+1}\cap \tilde{E}|Q\times I$. $\varpi:\partial W\to\partial(Q\times I)$ is a homotopy equivalence ($\partial W=N\cup N'$).

First, assume n is odd, we can perform surgery on Int$W$ to make $W\to Q\times I$ be a homotopy equivalence. And then cut U along W. $\partial U_W\to\partial(M_Q\times I)$ is a homotopy equivalence, and the surgery obstruction $\sigma(U_W)\in P_{n+1}$ is defined. $\sigma(U_W)$ is the obstruction to do surgery on IntU_W to make U_W homotopically equivalent to $M_Q\times I$. But the obstruction $\sigma(U_W)$ can be resolved by the following process.

Resolving obstruction. Any element of P_n ($n\geq 5$) has the following standard model. Let D^n be a standard n-disk. A suitable framed smooth n-submanifold V^n can be found in $D^n\times R^s$ ($s\gg n$) such that the boundary ∂V^n is PL homeomorphic to the standard sphere and that the obstruction to do surgery on Int V^n to make V^n contractible represents the element of P_n (Kervaire and Milnor [3], Sullivan [6]). Let V^{n+1} be a standard model of $-\sigma(U_W)\in P_{n+1}$. Attach V^{n+1} to U_W along L_N by the boundary connected sum. Then $\sigma(U_W\natural V^{n+1})=\sigma(U_W)-\sigma(U_W)=0$, and the obstruction is resolved.

In the above process, L is changed to $L''=L\#\Sigma^n$, where $\Sigma^n\in\theta_n(\partial\pi)$. Since the obstruction $\sigma(U_W\natural V^{n+1})=0$, we can perform surgery on Int$(U_W\natural V^{n+1})$ to obtain a manifold U'_W such that $U'_W\to M_Q\times I$ is a homotopy equivalence. Paste up U'_W along W, then we shall recover manifolds L', L'' and an h-cobordism between them. Note that $Wh(Z)=0$. By the s-cobordism theorem, U' is diffeomorphic (resp. PL homeomorphic) to $L'\times I$. This product structure gives a concordance between L' and $L''=L\#\Sigma^n$. In the PL case, we can construct an isotopy from L to L''. For this purpose, we have only to take an embedded $n+1$-disk $\Delta^{n+1}\subset D^{n+1}\times R^s$ such that $\partial\Delta^{n+1}=\partial V^{n+1}$. Thus L' and L are concordant. Hence, the exactness $0\to hT(M)\to[M, F/PL]$ is verified. The odd dimensional case is completed.

Secondly, consider the case where n is even and M^n is orientable. The surgery obstruction $\sigma(W^n)$ appears. But this is also resolved as follows. Embed $S^1\times D^{n-1}$ in L^n in such a way that $S^1\times D^{n-1}\cap N=\{p\}\times D^{n-1}$ ($p\in S^1$) and that the $S^1\times D^{n-1}$ represents a generator of $\pi_1(L)=Z$. This is possible, for M is orientable and thus L, which is homotopically equivalent to M, is also orientable. Let V^n be a standard model of $-\sigma(W^n)$. Make the "boundary connected sum" of $S^1\times V^n$ and U^{n+1} along the above embedded $S^1\times D^{n-1}$. The resulting framed submanifold U' in E has the intersection $W'=U'\cap\tilde{E}|Q\times I=W^n\natural V^n$. The obstruction $\sigma(W')=0$. After making $W'\to Q\times I$ be a homotopy equivalence, cut U' along W'. dim $U'_{W'}=n+1$ is odd, and no obstruction appears.

The argument for the *PL* case is similar to the odd case and we can conclude that L and L' are concordant. For the smooth case $L\tilde{\#}(S^1\times\Sigma^{n-1})$ and L' are concordant, ($\Sigma^{n-1}\in\theta_{n-1}(\partial\pi)$), where $\tilde{\#}$ denotes "the connected sum along a generator of $\pi_1(L)=Z$".

Finally, we consider the even dimensional non-orientable case. In this case the above resolving process is not available, for no embedded $S^1\times D^{n-1}$ can represent a generator of $\pi_1(L)=Z$. But if $n\equiv 0 \pmod 4$, $\sigma(W)=0$ by the observation of Lemma 3.4. If $n\equiv 2 \pmod 4$, $\sigma(W)\in P_n=Z_2$. This is the only possible kernel of $hT(M)\to[M, F/PL]$.

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