Homotopy groups of $Sp(n)/Sp(n-2)$

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§ 1. Main results.

For a couple of positive integers (n, k) such that $n \geq k$, the space of k-frames in the quarternionic n-space is known as the quarternionic Stiefel manifold. Let us denote it by $X_{n,k}$. Thus, $X_{n,n}$ is regarded as the symplectic group $Sp(n)$, while $X_{n,1}$ is the $(4n-1)$-sphere S^{4n-1}. $X_{n,k}$ is also interpreted as the quotient space $Sp(n, n-k) = Sp(n)/Sp(n-k)$. Let $X_{n,0} = Sp(n, n)$ be a point space.

Denote by (E, B, F) the fibering whose total space is E, the base space B and the fiber F. There are two fiberings associated to $X_{n,k}$:

(A) $\quad (X_{n,k}, S^{4n-1}, X_{n-1,k-1}) = (Sp(n, n-k), S^{4n-1}, Sp(n-1, n-k))$;

(B) $\quad (Sp(n), X_{n,k}, Sp(n-k))$.

The associated principal bundle of the fibering (A) is

(C) $\quad (Sp(n), S^{4n-1}, Sp(n-1))$

The bundle projection $p_B : Sp(n) \to X_{n,k}$ of the fibering (B) induces a homomorphism of the homotopy exact sequence of (C) into that of (A):

$$\begin{array}{ccccccc}
\cdots \longrightarrow \pi_{i+1}(S^{4n-1}) & \xrightarrow{\Delta} & \pi_i(Sp(n-1)) & \xrightarrow{i_*} & \pi_i(Sp(n)) & \xrightarrow{p_*} & \pi_i(S^{4n-1}) \\
\| & & \downarrow p_{B*} & & \downarrow p_{B*} & & \| \\
\cdots \longrightarrow \pi_{i+1}(S^{4n-1}) & \xrightarrow{\Delta} & \pi_i(Sp(n-1, k)) & \xrightarrow{i_*} & \pi_i(Sp(n, k)) & \xrightarrow{p_*} & \pi_i(S^{4n-1})\ .
\end{array}$$

In this paper, we compute $\pi_i(X_{n,2})$. Since $X_{n,2}$ is the total space of a sphere bundle over a sphere, its homotopy groups are obtained from the exact sequence of the fibering (A) together with the explicit description of their generators in terms of composition, coextension and secondary composition of those of homotopy groups of spheres. The results are shown in the tables below.

Denote by ∞ the infinite cyclic group Z, by m the cyclic group of order m, by $m+n$ the direct sum $Z_m \oplus Z_n$, and by $(m)^n$ the n copies of Z_m. For example, $\infty + (2)^3$ denotes $Z \oplus Z_2 \oplus Z_2 \oplus Z_2$.

(i) $\pi_i(X_{n,2})$ for $i \leq 4n+1$, $n>2$

i	$i<4n-5$	$4n-5$	$4n-4$	$4n-3$	$4n-2$	$4n-1$	$4n$	$4n+1$ (n odd)	$4n+1$ (n even)
π_i	0	∞	2	2	$d(24, n)$	∞	2	2	$(2)^2$

where $d(m, n)$ denotes the greatest common divisor of m and n.

(ii) $\pi_i(X_{n,2})$ for $i>4n+1$, $n>2$

Denote by $\pi_i(Y; p)$ the p-primary component of $\pi_i(Y)$.

(1) $\pi_i(X_{n,2}; p) \cong \pi_i(S^{4n-5}; p) \oplus \pi_i(S^{4n-1}; p)$ for prime $p>3$, or $p=3$, $n \equiv 0 \bmod 3$, or $p=2$, $n \equiv 0 \bmod 8$.

(2) $\pi_i(X_{n,2}; 3)$ for $n \not\equiv 0 \bmod 3$

r	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
π_{4n+r}	9	0	0	3	27	0	0	0	27	0	3	0	9	3	0

r	17	18	19	20	21	22	23	24	25	26	27
π_{4n+r}	0	27	0	0	3	3+27	0	0	3	9	0

where this table is valid for $n>6$, and it is valid up to $r \leqq 24$ if $n=5$, up to $r \leqq 16$ if $n=4$.

(3) $\pi_i(X_{n,2}; 2)$ for $n \not\equiv 0 \bmod 8$

	r	2	3	4	5	6	7	8	9	10
	n odd	4+16	$(2)^2$	$(2)^2$	2	128	$(2)^2$	$(2)^3$	$(2)^3$	2+4+64
π_{4n+r}	$n \equiv 2 \bmod 4$	8+16	$(2)^2$	$(2)^3$	$(2)^2$	2+64	$(2)^2$	$(2)^3$	$(2)^3$	2+8+32
	$n \equiv 4 \bmod 8$	8+16	$(2)^2$	$(2)^3$	$(2)^2$	4+32	$(2)^2$	$(2)^3$	$(2)^3$	2+8+32

	r	11	12	13	14	15	16
	n odd	$(2)^2$	$(2)^3$	2+16	$(2)^2+256$	$(2)^2+4$	$(2)^4$
π_{4n+r}	$n \equiv 2 \bmod 4$	$(2)^2$	$(2)^4$	$(2)^3+8$	$(2)^3+128$	$(2)^2+8$	$(2)^6$
	$n \equiv 4 \bmod 8$	$(2)^2$	$(2)^4$	$(2)^3+8$	$(2)^2+4+64$	$(2)^2+8$	$(2)^6$

where this table is valid for $n \geqq 7$, and it is valid up to $r \leqq 12$ if $n=6$, up to $r \leqq 8$ if $n=5$, and up to $r \leqq 4$ if $n=4$.

(4) $\pi_{24+r}(X_{6,2}; 2)$

r	13	14	15	16
π_{24+r}	$(2)^4+8$	$(2)^3+128$	$(2)^3+8$	$(2)^8$

(5) $\pi_{20+r}(X_{5,2})$

r	9	10	11	12	13	14	15	16
π_{20+r}	$(2)^2+4$	$2+4+64$	$(2)^3$	$(2)^4$	$2+8+16$	$(2)^3+256$	$(2)^2+4$	$(2)^4$

(6) $\pi_{16+r}(X_{4,2}; 2)$

r	5	6	7	8	9	10	11	12	13	14	15	16
π_{16+r}	$(2)^3$	$4+32$	$(2)^3$	$(2)^5$	$(2)^2+16$	$2+8+16$	2	$(2)^3$	$(2)^2+(4)^2+8$	$(2)^4+4+64$	$(2)^5+8$	$(2)^8$

(7) $\pi_{12+r}(X_{3,2}; 2)$

r	2	3	4	5	6	7	8	9	10	11	12	13
π_{12+r}	$4+8$	$(2)^3$	$(2)^3$	2	$2+16$	$(2)^2$	$(2)^3$	$2+(4)^2$	$(2)^2+4+8$	$(2)^3$	$(2)^4$	$2+t$

where $t=32$ or 64 or 128.

§ 2. $\pi_i(X_{n,2})$ for $i \leq 4n+1$, $n>2$.

Throughout this paper, we use the same notations as in [6], [8] and [9], e.g.
π_m^k for $\pi_m(S^k)$,
G_q for the q-th stable homotopy group of spheres,
$\iota_m \in \pi_m^m$, $\eta_m \in \pi_{m+1}^m$, $\nu_m \in \pi_{m+3}^m$, etc.,
$E^n: \pi_m^k \to \pi_{m+n}^{k+n}$ for the n-fold suspension homomorphism, with $E^1=E$.
We sometimes omit the subscript m in the notations $\iota_m, \eta_m, \cdots$.

Denote by $\chi(n) \in \pi_{4n-2}(Sp(n-1))$ the homotopy class of the characteristic map $S^{4n-2} \to Sp(n-1)$ of the fibering $(Sp(n), S^{4n-1}, Sp(n-1))$ which is defined in [4]. Let Δ_C be the homomorphism $\pi_i^{4n-1} \to \pi_{i-1}(Sp(n-1))$ which appears in the exact sequence of the fibering (C) in § 1, then

$$\Delta_C(E\alpha)=\chi(n)\circ\alpha \qquad \text{for any} \quad \alpha \in \pi_{i-1}^{4n-2}\,.$$

I. M. James has proved in [3] that

$$p_{B*}\chi(n)=n\nu_{4n-5}\,,$$

where $p_B: Sp(n-1) \to S^{4n-5}$ denotes the bundle projection. From the commutative diagram in § 1 we have

PROPOSITION 2.1. *$\Delta_A(E\alpha)=n\nu_{4n-5}\circ\alpha$ for $\alpha \in \pi_{i-1}^{4n-2}$, where Δ_A: $\pi_i^{4n-1} \to \pi_i^{4n-5}$ is analogous to Δ_C.*

Consider the exact sequence:

$$\cdots \longrightarrow \pi_{i+1}^{4n-1} \xrightarrow{\Delta} \pi_i^{4n-5} \xrightarrow{i_*} \pi_i(X_{n,2}) \xrightarrow{p_*} \pi_i^{4n-1} \longrightarrow \cdots .$$

For $4n-5 \leqq i \leqq 4n+1$, the groups π_i^{4n-5}, π_i^{4n-1} and their generators are listed below:

i	$4n-5$	$4n-4$	$4n-3$	$4n-2$	$4n-1$	$4n$	$4n+1$	$4n+2$
π_i^{4n-5}	∞	2	2	24	0	0	2	
generator	ι	η	η^2	ν			ν^2	
π_i^{4n-1}	0	0	0	0	∞	2	2	24
generator					ι	η	η^2	ν

We identify S^{4n-5} with the fiber over the reference point of $X_{n,2}$, and the image of $\alpha \in \pi_i^{4n-5}$ with α itself via $i_*: \pi_i^{4n-5} \to \pi_i(X_{n,2})$ when no confusion occurs. Let us denote by $[\beta] \in \pi_i(X_{n,2})$ the element such that $p_*[\beta]=\beta$ in π_i^{4n-1}. Denote by $q(n)=24/d(24, n)$. Now, we have

THEOREM 2.2. $\pi_i(X_{n,2})=0$ *for* $i<4n-5$

$\pi_{4n-5}(X_{n,2}) \cong Z=\{\iota\}$ $\qquad \pi_{4n-4}(X_{n,2}) \cong Z_2=\{\eta\}$

$\pi_{4n-3}(X_{n,2}) \cong Z=\{\eta^2\}$ $\qquad \pi_{4n-2}(X_{n,2}) \cong Z_{d(24,n)}=\{\nu\}$

$\pi_{4n-1}(X_{n,2}) \cong Z=\{[q(n)\iota]\}$ $\qquad \pi_{4n}(X_{n,2}) \cong Z_2=\{[\eta]\}$

$$\pi_{4n+1}(X_{n,2}) \cong \begin{cases} Z_2=\{[\eta]\circ\eta\} & \textit{if } n \textit{ is odd} \\ Z_2 \oplus Z_2=\{\nu^2\} \oplus \{[\eta]\circ\eta\} . \end{cases}$$

§ 3. p-primary component.

Denote by $\pi_i(X; p)$ the p-primary component of $\pi_i(X)$. Since the order of $p_{B*}\chi(n)$ is prime to p for any prime >3, we have the direct sum decomposition below:

PROPOSITION 3.1. $\pi_i(X_{n,2}; p) \cong \pi_i(S^{4n-5}; p) \oplus \pi_i(S^{4n-1}; p)$ *for* $p>3$.

The same argument applies to the cases when $p=3$, $n \equiv 0 \bmod 3$, and $p=2$, $n \equiv 0 \bmod 8$. In the remainder, we will exclude those cases.

§ 4. Homotopy periodicity.

I. M. James has defined in [1] the intrinsic join operation

$$\pi_i(X_{n,k}) \otimes \pi_j(X_{m,k}) \to \pi_{i+j+1}(X_{n+m,k})$$

in which the image of $\alpha \otimes \beta$ is denoted by $\alpha * \beta$. For a fixed element $\beta \in \pi_j(X_{m,k})$, $\alpha \to \alpha * \beta$ defines a homomorphism

$$*\beta : \pi_i(X_{n,k}) \to \pi_{i+j+1}(X_{n+m,k}) .$$

The element $\theta=[q\iota] \in \pi_{4m-1}(X_{m,k})$ is called a q-section according to James [2], and he has proved that if $\theta \in \pi_{4m-1}(X_{m,k})$ is a q-section, with q relatively prime to p,

$$*\theta : \pi_i(X_{n,k}; p) \to \pi_{4m+i}(X_{m+n,k}; p)$$

is an isomorphism for $i \leqq 4p(n-k+1)-4$.

In particular, set $k=2$, $p=3$, $n \not\equiv 0 \bmod 3$, then the element $[8\iota] \in \pi_{11}(X_{3,2})$ is an 8-section. Hence we have

COROLLARY 4.1. $\pi_{4n+r}(X_{n,2}; 3) \cong \pi_{4(n+3)+r}(X_{n+3,2}; 3)$ *for* $r \leqq 8(n-2)$.

Hence, if $n=4$ this isomorphism is valid for $r \leqq 16$, and if $n=5$ it is valid for $r \leqq 24$, and so on. In the same way we have

COROLLARY 4.2. $\pi_{4n+r}(X_{n,2}; 2) \cong \pi_{4(n+8)+r}(X_{n+8,2}; 2)$ *for* $n \not\equiv 0 \bmod 8$, $r \leqq 4(n-3)$.

§ 5. Quasi-projective space Q_n.

Cellular decomposition of the Stiefel manifolds is given in [5]. In the first step of construction, there appears a *CW*-complex Q_n which is called by I. M. James a quasi-projective n-space [3]. We restrict our attention to the quaternionic case.

Q_n is a *CW*-complex with a 0-cell Q_0 and with a (4n−1)-cell for each integer m such that $1 \leqq m \leqq n$. It is naturally embedded in $Sp(n)$ as a subcomplex, which means that the inclusion map $Sp(m) \to Sp(n)$ induces a commutative diagram below:

$$\begin{array}{ccc} Q_m & \longrightarrow & Q_n \\ \downarrow & & \downarrow \\ Sp(m) & \longrightarrow & Sp(n) \end{array} \qquad (m \leqq n) .$$

Let us define the stunted quasi-projective space $Q_{n,k}$ to be the complex obtained from Q_n by identifying Q_{n-k} with Q_0 (see [3]). Then, it has the cell structure

$$Q_{n,k} = Q_0 \smile e_{n-k+1} \smile e_{n-k+2} \smile \cdots \smile e_n ,$$

where e_m denotes $(4m-1)$-cell. The projection $p_B : Sp(n) \to Sp(n, n-k)$ induces a map $Q_n \to Q_{n,k}$ and $Q_{n,k}$ is embedded in $X_{n,k}$ as a subcomplex.

According to [5], $H^*(X_{n,k}; Z)$ is an exterior algebra over Z generated by x_{4m-1} $(n-k<m \leqq n)$, where x_{4m-1} denotes the cohomology class dual to the homology class $\{e_m\}$. Hence the difference between cohomologies of $X_{n,k}$ and $Q_{n,k}$ first appears in the dimension

$$\{4(n-k+1)-1\} + \{4(n-k+2)-1\} = 8(n-k)+10 .$$

Thus we have the following

PROPOSITION 5.1. *The inclusion map $Q_{n,k} \to X_{n,k}$ induces an isomorphism*

$$\pi_{4n+r}(Q_{n,k}) \cong \pi_{4n+r}(X_{n,k}) \quad \textit{for} \quad r \leqq 4(n-2k)+8 .$$

§ 6. $\pi_i(X_{n,2}; 3)$.

Throughout this section, we assume that $n \not\equiv 0 \mod 3$.

Consider the homomorphism $\pi_{4n+r}(Q_{n,2}) \to \pi_{4n+r}(X_{n,2})$ induced by the inclusion map $Q_{n,2} \to X_{n,2}$. Take the range of r to be $2 \leqq r \leqq r_0$ for some r_0, and take n so large that $r_0 \leqq 4n-8$. It follows from Proposition 5.1 that the above homomorphism is an isomorphism for such n. Using the homotopy periodicity in § 4, we shall compute $\pi_{4n+r}(Q_{n,2})$.

The cell-structure of $Q_{n,2}$ is given by $S^{4n-5} \cup_{n\nu} e^{4n-1}$, where $n\nu$ generates the 3-primary component of G_3, and we denote it by α_1. In the remainder of this section, all the homotopy groups are taken to be 3-primary components of them.

Let $\psi: (E^{4n-1}, S^{4n-2}) \to (Q_{n,2}, S^{4n-5})$ be the characteristic map for the cell e^{4n-1}. Consider the commutative diagram below:

$$\begin{array}{ccc} \pi_i(E^{4n-1}, S^{4n-2}) & \xrightarrow{\psi_*} & \pi_i(Q_{n,2}, S^{4n-5}) \\ \partial_* \downarrow & & \downarrow p'_* \\ \pi_{i-1}(S^{4n-2}) & \xrightarrow{E} & \pi_i(S^{4n-1}) \end{array}$$

where p'_* is induced by the map collapsing S^{4n-5} into a point. If the suspension homomorphism E is an isomorphism, then ψ_* is a monomorphism, and p'_* an epimorphism, and besides $\pi_i(Q_{n,2}, S^{4n-5})$ is the direct sum $\operatorname{im} \psi_* \oplus \ker p'_*$. Since S^{4n-5} is $(4n-6)$-connected, $\ker p'_* = 0$ whenever $i \leqq (4n-2)+(4n-6) = 8n-8$ ([10] p. 324). Comparing with the inequality $r \leqq 4n-8$, we conclude that for the values of n and i in question p'_* is an isomorphism. Substituting $\pi_i(Q_{n,2}, S^{4n-5})$ with $\pi_i(S^{4n-1})$ in the homotopy exact sequence of the pair $(Q_{n,2}, S^{4n-5})$, we have an exact sequence

$$\cdots \longrightarrow \pi_i^{4n-5} \xrightarrow{i_*} \pi_i(Q_{n,2}) \xrightarrow{p_*} \pi_i^{4n-1} \xrightarrow{\Delta} \pi_i^{4n-5} \longrightarrow \cdots$$

where Δ is given by

$$\Delta(\gamma) = \alpha_{1*} E^{-1}(\gamma) \qquad \text{for } \gamma \in \pi_i^{4n-1}.$$

Note that π_i^{4n-5}, π_i^{4n-1} are in the stable range. According to [7], 3-primary components of stable homotopy groups of spheres up to 32-stem are generated by the following elements and their compositions:

(i) $G_3 \cong Z_3 = \{\alpha_1\}$. Choose an element α_{i+1} from the secondary composition

$$\alpha_{i+1} \in \langle \alpha_i, 3\iota, \alpha_1 \rangle = \langle \alpha_1, 3\iota, \alpha_i \rangle,$$

for each i such that $1 \leqq i \leqq 7$,

(ii) $\beta_1 \in G_{10}$, $\beta_2 \in G_{26}$,

(iii) $\alpha_3' \in G_{11}$, $\alpha_6' \in G_{23}$ such that $3\alpha_3' = \alpha_3$, $3\alpha_6' = \alpha_6$.

Thus, π_{4n+r}^{4n-5} and π_{4n+r}^{4n-1} are given in the table below:

r	1	2	3	4	5	6	7	8	9	10	11	12	13	14
π_{4n+r}^{4n-5}	0	3	0	0	3	9	0	3	0	3	0	0	0	3
generator		α_2			β_1	α_3'		$\alpha_1 \circ \beta_1$		α_4				α_5
π_{4n+r}^{4n-5}	0	3	0	0	0	3	0	0	3	9	0	3	0	3
generator		α_1				α_2			β_1	α_3'		$\alpha_1 \circ \beta_1$		α_4

r	15	16	17	18	19	20	21	22	23	24	25	26	27
π_{4n+r}^{4n-5}	3	0	0	9+3	0	0	3	3	0	3	3	3	0
generator	β_1^2			$\alpha_6', \alpha_1 \circ \beta_1^2$			β_2	α_7		$\alpha_1 \circ \beta_2$	β_1^3	α_8	
π_{4n+r}^{4n-1}	0	0	0	3	3	0	0	9+3	0	0	3	3	0
generator				α_5	β_1^2			$\alpha_6', \alpha_1 \circ \beta_1^2$			β_2	α_7	

In [6] (or [7]), H. Toda has given the relations:

$$\alpha_1 \circ \alpha_i = 0 \qquad \text{for } i = 1, 2, 4, 5, 7 \text{ and } 8\,,$$

$$\alpha_1 \circ \alpha_i' = 0 \qquad \text{for } i = 3 \text{ and } 6\,.$$

Now, let us construct some elements of $\pi_i(Q_{n,2})$. Since $\alpha_1 \circ \alpha_1 = 0$, there exists an element $[\alpha_1] \in \pi_{4n+2}(Q_{n,2})$ chosen from $\text{Coext}(\alpha_1, \alpha_1)$ (see [6] Ch. I). We have $p_*[\alpha_1] = \alpha_1$. The following commutative diagram is well known:

$$\begin{array}{ccccc}
S^n & \xleftarrow{f_1} & S^m \underset{\beta}{\cup} e^{q+1} & \xleftarrow{f_2} & S^{r+1} \\
\downarrow i & & \downarrow p & & \| \\
S^n \underset{\alpha}{\cup} e^{m+1} & \xleftarrow{-f_3} & S^{q+1} & \xleftarrow{f_4} & S^{r+1}
\end{array}$$

where $f_1 \in \text{Ext}(\alpha, \beta)$ (see [6], Ch. I), $f_2 \in \text{Coext}(\beta, \gamma)$, $f_3 \in \text{Coext}(\alpha, \beta)$ and $f_4 \in E\gamma$. Applying this diagram to the case when $n = m-3$, $q = m+3 = r$, $\alpha = \beta = \alpha_1$, $\gamma = 3\iota$, we have

$$\text{Coext}(\alpha_1, \alpha_1) \circ (3\iota) = -i_* \langle \alpha_1, \alpha_1, 3\iota \rangle \qquad \text{mod } 3G_7 = 0\,.$$

Hence the right hand side consists of a single element $\pm i_* \alpha_2$ because of the relation given in [7]:

$$\langle \alpha_1, \alpha_1, 3\iota \rangle = \pm \langle \alpha_1, 3\iota, \alpha_1 \rangle\,.$$

Thus we have proved that

$$3[\alpha_1]=\pm i_*\alpha_2 .$$

Similarly, let us choose an element $[\alpha_i]$ from Coext(α_1, α_i) for each i such that $i=2, 4, 5, 7$ and 8. We prove that

for $i=2, 5$ Coext$(\alpha_1, \alpha_i)\circ(3\iota)=-i_*\langle\alpha_1, \alpha_i, 3\iota\rangle \ni \pm i_*\alpha'_{i+1}$ mod $\{\alpha_{i+1}\}$,

for $i=3, 6$ Coext$(\alpha_1, \alpha_i')\circ(9\iota)=-i_*\langle\alpha_1, \alpha_i, 3\iota\rangle \ni \pm i_*\alpha_{i+1}$ mod 0 ,

for $i=4, 7$ Coext$(\alpha_1, \alpha_i)\circ(3\iota)=-i_*\langle\alpha_1, \alpha_i, 3\iota\rangle \ni \pm i_*\alpha_{i+1}$ mod 0 .

(See [7] Proposition 4.17.) Thus, we have proved that

LEMMA 6.1. $3[\alpha_i]=\pm i_*\alpha_{i+1}$ *for* $i=1, 4, 7$,

$3[\alpha_i]=x i_*\alpha'_{i+1}$ *for* $i=2, 5$ *with some integer* x *relatively prime to* 3

$9[\alpha_i']=\pm i_*\alpha_{i+1}$ *for* $i=3, 6$.

Hence, we have the results below:

THEOREM 6.2. *For* $n\not\equiv 0$ mod 3, $\pi_{4n+r}(Q_{n,2}; 3)$ *and their generators are given in the following table if* n *is large enough comparing with* r:

r	2	3	4	5	6	7	8	9	10	11	12	13	14	15
π_{4n+r}	9	0	0	3	27	0	0	0	27	0	3	0	9	3
generator	$[\alpha_1]$			β_1	$[\alpha_2]$				$[\alpha_3']$		$[\alpha_1]\circ\beta_1$		$[\alpha_4]$	β_1^2

r	16	17	18	19	20	21	22	23	24	25	26	27
π_{4n+r}	0	0	27	0	0	3	27+3	0	0	3	9	0
generator			$[\alpha_5]$			β_2	$[\alpha_6'], [\alpha_1]\circ\beta_1^2$			β_1^3	$[\alpha_1]$	

REMARK. Combined with Corollary 4.1 and Proposition 5.1, this table gives the values of $\pi_{4n+r}(X_{n,2}; 3)$ for $n>6$, and it is valid up to $r\leqq 24$ if $n=5$, and up to $r\leqq 16$ if $n=4$.

For $n\equiv 0$ mod 3, $\pi_{4n+r}(X_{n,2}; 3)$ is obtained as the direct sum of the groups in the same column of the preceding table.

§ 7. $\pi_i(X_{n,2}; 2)$ for $n\geqq 7$.

Throughout this section, we assume that $n\not\equiv 0$ mod 8. Since we have the informations about 2-primary components of homotopy groups of spheres up to 22-stem, our calculation of $\pi_{4n+r}(X_{n,2}; 2)$ is restricted to $r\leqq 16$. For these values of r, the homotopy periodicity $\pi_{4n+r}(X_{n,2}; 2)\cong\pi_{4m+r}(X_{m,2}; 2)$ with $n\leqq m$, $n\equiv m$ mod 8 holds for $n\geqq 7$ (see § 4). Take n large enough so that both of the inclusion map $Q_{n,2}\to X_{n,2}$ and the pinching map $p': (Q_{n,2}, S^{4n-5})\to(S^{4n-1}, *)$ induce isomorphisms $\pi_{4n+r}(Q_{n,2}; 2)\cong\pi_{4n+r}(X_{n,2}; 2)$ and $\pi_{4n+r}(Q_{n,2}, S^{4n-5}; 2)$.

For the remainder of this section, $\pi_i(X)$, π_i^m denote the 2-primary components of them. We have an exact sequence:

$$\cdots \longrightarrow \pi_{4n+r+1}^{4n-1} \xrightarrow{\Delta} \pi_{4n+r}^{4n-5} \xrightarrow{i_*} \pi_{4n+r}(Q_{n,2}) \xrightarrow{p_*} \pi_{4n+r}^{4n-1} \xrightarrow{\Delta} \pi_{4n+r-1}^{4n-5} \longrightarrow \cdots$$

where Δ is defined by

$$\Delta(\gamma)=n\nu_* E^{-1}(\gamma) \qquad \text{for} \quad \gamma \in \pi_{4n+r}^{4n-1} .$$

Note that π_{4n+r}^{4n-5} and π_{4n+r}^{4n-1} are in the stable range, and they are known as follows ([6], [8] and [9]):

r	2	3	4	5	6	7	8	9	10
π_{4n+r}^{4n-5}	16	2+2	2+2+2	2	8	0	0	2+2	32+2
generator	σ	$\eta\circ\sigma$, ε	ν^3, $\eta\circ\varepsilon$, μ	$\eta\circ\mu$	ζ			σ^2, κ	ρ, $\eta\circ\kappa$
π_{4n+r}^{4n-1}	8	0	0	2	16	2+2	2+2+2	2	8
generator x	ν			ν^2	σ	$\eta\circ\sigma$, ε	ν^3, $\eta\circ\varepsilon$, μ	$\eta\circ\mu$	ζ
$\nu\circ x$	ν^2			ν^3	0	0	0	0	0

r	11	12	13	14	15	16	17
π_{4n+r}^{4n-5}	2+2	2+2+2+2	8+2	8+2	8	2+2	
generator	$\eta\circ\rho$, η^*	$\bar{\mu}$, $\nu\circ\kappa$, $\eta^2\circ\rho$, $\eta\circ\eta^*$	ν^*, $\eta\circ\bar{\mu}$	$\bar{\zeta}$, $\bar{\sigma}$	$\bar{\kappa}$	$\eta\circ\bar{\kappa}$, σ^3	
π_{4n+r}^{4n-1}	0	0	2+2	32+2	2+2	2+2+2+2	8+2
generator x			σ^2, κ	ρ, $\eta\circ\kappa$	$\eta\circ\rho$, η^*	$\bar{\mu}$, $\nu\circ\kappa$, $\eta^2\circ\rho$, $\eta\circ\eta^*$	ν^3, $\eta\circ\bar{\mu}$
$\nu\circ x$			0 $\nu\circ\kappa$	0	0	0, $4\bar{\kappa}$, 0, 0	σ^3, 0

The last row of the table gives the values of $\nu\circ\pi_{4n+r}^{4n-1}$ depending on [6], [8] and [9] except the following:

LEMMA 7.1. $\langle\nu, \sigma, \nu\rangle=\sigma^2$, $\nu\circ\nu^*=\sigma^3$.

PROOF. In [6], we find the facts below:

(a) $\nu\circ G_{11}=0$, hence $\langle\nu, \sigma, \nu\rangle$ consists of a single element.

(b) $\langle\nu, \sigma, \nu\rangle$ is essential.

(c) $\langle\nu, \sigma, \nu\rangle$ and $-\langle\sigma, \nu, 2\nu\rangle$ have a common element.

(d) $2\nu\in\langle\eta, 2\iota, \eta\rangle$, $\langle\nu, \eta, 2\iota\rangle=0$. $\langle\sigma, \nu, \eta\rangle=0$.

Now since

$$\langle\sigma, \nu, \langle\eta, 2\iota, \eta\rangle\rangle-\langle\sigma, \langle\nu, \eta, 2\iota\rangle, \eta\rangle+\langle\langle\sigma, \nu, \eta\rangle, 2\iota, \eta\rangle\equiv 0 ,$$

mod $\sigma\circ G_7+(2\nu)\circ G_{11}+\eta\circ G_{13}=\{\sigma^2\}$, we conclude that $\langle\nu, \sigma, \nu\rangle=\sigma^2$. Hence we have

$$\nu\circ\nu^*=\nu\circ\xi=\nu\circ\langle\sigma, \nu, \sigma\rangle=\langle\nu, \sigma, \nu\rangle\circ\sigma=\sigma^3 .$$

To construct some elements of $\pi_i(Q_{n,2};2)$, we need some lemmas.

LEMMA 7.2. $\langle\nu,\zeta,8\iota\rangle=4\rho+8G_{15}$.

This depends on several sublemmas.

SUBLEMMA 1. $\langle\nu,\zeta,8\iota\rangle\subset 2G_{15}$.

PROOF. Consider the following formula:

$$\langle\langle 2\iota,\varepsilon,\eta^2\rangle,\nu,8\iota\rangle+\langle 2\iota,\langle\varepsilon,\eta^2,\nu\rangle,8\iota\rangle+\langle 2\iota,\varepsilon,\langle\eta^2,\nu,8\iota\rangle\rangle\equiv 0\ .$$

Note that $\langle 2\iota,\varepsilon,\eta^2\rangle=\zeta+2G_{11}$, $\langle\varepsilon,\eta^2,\nu\rangle$ is of order 2, and $\langle\eta^2,\nu,8\iota\rangle=\eta\circ G_5=0$. Hence $\langle\zeta,\nu,8\iota\rangle$ is divisible by 2. Next, consider

$$\langle\langle\nu,2\sigma,8\iota\rangle,\nu,8\iota\rangle-\langle\nu,\langle 2\sigma,8\iota,\nu\rangle,8\iota\rangle+\langle\nu,2\sigma,\langle 8\iota,\nu,8\iota\rangle\rangle\equiv 0 \mod 8G_{15}\ .$$

Note that $\langle 2\sigma,8\iota,\nu\rangle=\zeta$ $\langle\nu,2\sigma,8\iota\rangle=x\zeta$ and $\langle 8\iota,\nu,8\iota\rangle=0$. Hence $\langle\zeta,\nu,8\iota\rangle=x\langle\nu,\zeta,8\iota\rangle$ for an odd integer x (see [6]). Thus we have proved that $\langle\nu,\zeta,8\iota\rangle\subset 2G_{15}$.

SUBLEMMA 2. $\langle\nu,8\iota,\zeta\rangle=16\rho$.

PROOF. Observe that $\nu\circ G_{12}=G_4\circ\zeta=0$, and hence $\langle\nu,8\iota,\zeta\rangle$ consists of a single element. Consider the following formula:

$$-\langle\nu,8\iota,\langle\nu,8\iota,2\sigma\rangle\rangle-\langle\nu,\langle 8\iota,\nu,8\iota\rangle,2\sigma\rangle+\langle\langle\nu,8\iota,\nu\rangle,8\iota,2\sigma\rangle\equiv 0 \mod 0\ .$$

Note that $\langle\nu,8\iota,2\sigma\rangle=\zeta$, $\langle 8\iota,\nu,8\iota\rangle=0$, $\langle\nu,8\iota,\nu\rangle=8\sigma$, and $\langle 8\sigma,8\iota,2\sigma\rangle=4\langle 2\sigma,8\iota,2\sigma\rangle=4\langle 8\iota,2\sigma,4\sigma\rangle=16\langle 8\iota,2\sigma,\sigma\rangle=16\rho$. Hence we have $\langle\nu,8\iota,\zeta\rangle=16\rho$.

PROOF OF LEMMA 7.2. Consider the following formula:

$$\langle\langle\nu,8\iota,\nu\rangle,\sigma,16\iota\rangle-\langle\nu,\langle 8\iota,\nu,\sigma\rangle,16\iota\rangle-\langle\nu,8\iota,\langle\nu,\sigma,16\iota\rangle\rangle\equiv 0 \mod 16G_{15}\ .$$

Note that

(a) $\langle\nu,8\iota,\nu\rangle=8\sigma$, and $\langle 8\sigma,\sigma,16\iota\rangle\ni 4\langle 2\sigma,\sigma,16\iota\rangle=4\langle 2\sigma,2\sigma,16\iota\rangle=8\langle\sigma,2\sigma,8\iota\rangle=8\rho$.

(b) $\langle 8\iota,\nu,\sigma\rangle=x\zeta$ for an odd integer x ([6] p. 94), and $\langle\nu,x\zeta,16\iota\rangle=(2x)\langle\nu,\zeta,8\iota\rangle$.

(c) $\langle\nu,\sigma,16\iota\rangle=y\zeta$ for an odd integer y, and $\langle\nu,8\iota,y\zeta\rangle=(16y)\rho$ by Sublemma 2.

Hence we conclude that $2\langle\nu,\zeta,8\iota\rangle\equiv 8\rho \mod 16G_{15}$. While $\langle\nu,\zeta,8\iota\rangle\subset 2G_{15}$, we have $\langle\nu,\zeta,8\iota\rangle=4\rho+8G_{15}$.

LEMMA 7.3. *$\langle\nu,4\rho,8\iota\rangle$ consists of a single element $\bar{\zeta}$.*

PROOF. Note the followings:

(a) $\nu\circ G_{16}=0$, $8G_{19}=0$, hence $\langle\nu,4\rho,8\iota\rangle$ consists of a single element.

(b) $\langle\nu,4\rho,8\iota\rangle-\langle 4\rho,8\iota,\nu\rangle+\langle 8\iota,\nu,4\rho\rangle=0$.

(c) $\langle 8\iota,\nu,4\rho\rangle=\langle 8\iota,4\nu,\rho\rangle=\langle 8\iota,\eta^3,\rho\rangle=(4\iota)\circ\langle 2\iota,\eta,\varepsilon\rangle\circ\mu=0$ ([6]).

Hence we conclude that $\langle\nu,4\rho,8\iota\rangle=\langle 4\rho,8\iota,\nu\rangle=\langle\nu,8\iota,4\rho\rangle$. Consider the following formula:

$$\langle\langle\nu,8\iota,2\sigma\rangle,8\iota,2\sigma\rangle-\langle\nu,\langle 8\iota,2\sigma,8\iota\rangle,2\sigma\rangle-\langle\nu,8\iota,\langle 2\sigma,8\iota,2\sigma\rangle\rangle\equiv 0\ .$$

Then Lemma 7.3 follows from the following facts:

(d) $\langle \nu, 8\iota, 2\sigma\rangle=\zeta$, $\langle 8\iota, 2\sigma, 8\iota\rangle=0$, $\langle 2\sigma, 8\iota, 2\sigma\rangle=4\zeta$.

(e) $\langle \zeta, 8\iota, 2\sigma\rangle \ni \bar{\zeta}$.

(f) $\zeta\circ G_8=0$, $G_{12}\circ(2\sigma)=0$, $\nu\circ G_{16}=0$.

Now, let us construct some elements of $\pi_i(Q_{n,2}; 2)$ as follows:

(i) There is an element $[\eta]\in \mathrm{Coext}(n\nu, \eta)$ such that $p_*[\eta]=\eta$, $2[\eta]=0$.

Recall that p is the pinching map $S^{4n-5}\underset{n\nu}{\cup} e^{4n-1}\to S^{4n-1}$ of S^{4n-5}. Hence $p_*[\eta]=\eta$ by the definition of coextension. Now,

$$\mathrm{Coext}(n\nu, \eta)\circ(2\iota)=-i_*\langle n\nu, \eta, 2\iota\rangle\subset i_*G_5=0 ,$$

where $i: S^{4n-5}\to S^{4n-5}\underset{n\nu}{\cup} e^{4n-1}$ is the inclusion map.

(ii) For even n, there is an element $[\nu]\in \mathrm{Coext}(n\nu, \nu)$ such that $p_*[\nu]=\nu$, $8[\nu]=0$.

Indeed, $\mathrm{Coext}(n\nu, \nu)\circ(8\iota)=-i_*\langle n\nu, \nu, 8\iota\rangle$. Note that $\langle 2\nu, \nu, 8\iota\rangle$ and $\langle \nu, 8\iota, \nu\rangle=8\sigma$ have a common element. Hence $\langle n\nu, \nu, 8\iota\rangle$ is divisible by 8. We choose an element $[\nu]$ such that $8[\nu]=0$.

In the remainder of this section, we put $\pi_{4n+r}=\pi_{4n+r}(Q_{n,2}; 2)$. We have from the preceding table that

$$\pi_{4n+2}\cong Z_{16}\oplus Z_8=\{i_*\sigma\}\oplus\{[\nu]\} \qquad \text{for } n \text{ even} .$$

(ii′) For odd n, there is an element $[2\nu]\in\{[\eta], 2\iota, \eta\}$ such that $p_*[2\nu]=2\nu$, $4[2\nu]=0$.

Indeed, $p_*\{[\eta], 2\iota, \eta\}\subset\langle p_*[\eta], 2\iota, \eta\rangle=\langle\eta, 2\iota, \eta\rangle=2\nu+4G_3$.

Note that the modulus of $\{[\eta], 2\iota, \eta\}$ is generated by $[\eta]\circ\eta^2$. Hence we choose $[2\nu]$ so that $p_*[2\nu]=2\nu$. Now,

$$\{[\eta], 2\iota, \eta\}\circ(4\iota)=\pm[\eta]\circ\langle 2\iota, \eta, 4\iota\rangle=\pm[\eta]\circ\eta^2\circ(2\iota)=0 .$$

Thus we have

$$\pi_{4n+2}\cong Z_{16}\oplus Z_4=\{i_*\sigma\}\oplus\{[2\nu]\} \qquad \text{for } n \text{ odd}.$$

(iii) It follows from the preceding table that

$$\pi_{4n+3}\cong Z_2\oplus Z_2=\{i_*\eta\circ\sigma\}\oplus\{i_*\varepsilon\} ,$$

$$\pi_{4n+4}\cong Z_2\oplus Z_2\oplus Z_2=\{i_*\nu^3\}\oplus\{i_*\eta\circ\varepsilon\}\oplus\{i_*\mu\} \qquad \text{for } n \text{ even} ,$$

$$\pi_{4n+4}\cong Z_2\oplus Z_2=\{i_*\mu\}\oplus\{i_*\eta\circ\varepsilon\} \qquad \text{for } n \text{ odd} ,$$

$$\pi_{4n+5}\cong Z_2=\{i_*\eta\circ\mu\} \qquad \text{for } n \text{ odd} .$$

Note that for even n

$$\mathrm{Coext}(n\nu, \nu)\circ(2\nu)=-\langle n\nu, \nu, 2\nu\rangle=(-n/2)i_*\langle\nu, 2\nu, 4\nu\rangle=0,$$

since $\langle\nu, 2\nu, 4\nu\rangle\subset 4G_{10}=0$. Thus we have $2([\nu]\circ\nu)=0$, and that

$$\pi_{4n+5}\cong Z_2\oplus Z_2=\{i_*\eta\circ\mu\}\oplus\{[\nu]\circ\nu\} .$$

(iv) There is an element $[\sigma]\in \mathrm{Coext}(n\nu, \sigma)$ such that $p_*[\sigma]=\sigma$, $16[\sigma]=nxi_*\zeta$ for an odd integer x.

Note that $\langle\nu, \sigma, 16\iota\rangle=x'\zeta$ for an odd integer x'. Hence

$$\mathrm{Coext}(n\nu, \sigma)\circ(16\iota)=-i_*\langle n\nu, \sigma, 16\iota\rangle=nxi_*\zeta \text{ with } x\equiv -x'.$$

We have the following:

$$\pi_{4n+6}\cong Z_{128}=\{[\sigma]\} \quad \text{for } n \text{ odd};$$
$$\pi_{4n+6}\cong Z_2\oplus Z_{64}=\{i_*\zeta, [\sigma]\} \quad \text{for } n\equiv 2 \bmod 4;$$
$$\pi_{4n+6}\cong Z_4\oplus Z_{32}=\{i_*\zeta, [\sigma]\} \quad \text{for } n\equiv 4 \bmod 8.$$

(v) There are elements $[\varepsilon]\in\{[\eta], 2\iota, \nu^2\}$, $[\mu]\in\{[\eta], 2\iota, 8\sigma\}$ such that $p_*[\varepsilon]=\varepsilon$, $p_*[\mu]=\mu$, $2[\varepsilon]=0$, $2[\mu]=0$.

The proofs are quite similar as before, and we have

$$\pi_{4n+7}\cong Z_2\oplus Z_2=\{[\eta]\circ\sigma\}\oplus\{[\varepsilon]\},$$
$$\pi_{4n+8}\cong Z_2\oplus Z_2\oplus Z_2=\{[\eta]\circ\eta\circ\sigma\}\oplus\{[\eta]\circ\varepsilon\}\oplus\{[\mu]\},$$
$$\pi_{4n+9}\cong Z_2\oplus Z_2\oplus Z_2=\{i_*\sigma^2\}\oplus\{i_*\kappa\}\oplus\{[\eta]\circ\mu\}.$$

(vi) There is an element $[\zeta]\in \mathrm{Coext}(n\nu, \zeta)$ such that $p_*[\zeta]=\zeta$, $8[\zeta]=4ni_*\rho$.

Indeed, $\mathrm{Coext}(n\nu, \zeta)\circ(8\iota)=-ni_*\langle\nu, \zeta, 8\iota\rangle=4ni_*\rho+8i_*G_{15}$ by Lemma 7.2. Hence we choose an element $[\zeta]$ so that $8[\zeta]=4ni_*\rho$. We have the following:

$$\pi_{4n+10}\cong Z_2\oplus Z_{32}\oplus Z_8=\{i_*\eta\circ\kappa\}\oplus\{i_*\rho\}\oplus\{[\zeta]\} \quad \text{for } n \text{ even};$$
$$\pi_{4n+10}\cong Z_2\oplus Z_{64}\oplus Z_4=\{i_*\eta\circ\kappa\}\oplus\{i_*\rho, [\zeta]\} \quad \text{for } n \text{ odd};$$
$$\pi_{4n+11}\cong Z_2\oplus Z_2=\{i_*\eta\circ\rho\}\oplus\{i_*\eta^*\};$$
$$\pi_{4n+12}\cong Z_2\oplus Z_2\oplus Z_2=\{i_*\eta^2\circ\rho\}\oplus\{i_*\eta\circ\eta^*\}\oplus\{i_*\bar{\mu}\} \quad \text{for } n \text{ odd};$$
$$\pi_{4n+12}\cong Z_2\oplus Z_2\oplus Z_2\oplus Z_2=\{i_*\eta^2\circ\rho\}\oplus\{i_*\eta\circ\eta^*\}\oplus\{i_*\bar{\mu}\}\oplus\{i_*\nu\circ\kappa\} \quad \text{for } n \text{ even}.$$

(vii) There is an element $[\kappa]\in \mathrm{Coext}(n\nu, \kappa)$ such that $p_*[\kappa]=\kappa$, $2[\kappa]=0$ for n even.

Indeed, $\mathrm{Coext}(n\nu, \kappa)\circ(2\iota)=-i_*\langle n\nu, \kappa, 2\iota\rangle \bmod 2i_*G_{18}$. Note that $\langle 2\nu, \kappa, 2\iota\rangle\supset \nu\circ\langle 2\iota, \kappa, 2\iota\rangle=\nu\circ\kappa\circ\eta=0$. Hence $\langle n\nu, \kappa, 2\iota\rangle\subset 2G_{18}$. We choose an element $[\kappa]$ so that $2[\kappa]=0$.

(viii) $2([\sigma]\circ\sigma)=-ni_*\nu^*$.

Indeed, $\mathrm{Coext}(n\nu, \sigma)\circ(2\sigma)=-i_*\langle n\nu, \sigma, 2\sigma\rangle$, which consists of a single elment because $\nu\circ G_{15}=0$, $G_{11}\circ\sigma=0$. Now,

$$\langle n\nu, \sigma, 2\sigma\rangle=n\langle\nu, \sigma, 2\sigma\rangle=n\nu^*.$$

Thus we have

$$\pi_{4n+13}\cong Z_2\oplus Z_{16}=\{i_*\eta\circ\bar{\mu}\}\oplus\{[\sigma]\circ\sigma\} \quad \text{for } n \text{ odd},$$

$\pi_{4n+13}\cong Z_2\oplus Z_8\oplus Z_2\oplus Z_2=\{i_*\eta\circ\bar{\mu}\}\oplus\{i_*\nu^*, [\sigma]\circ\sigma\}\oplus\{[\kappa]\}$ for n even.

(ix) There is an element $[\rho]\in\mathrm{Coext}(n\nu, \rho)$ such that $p_*[\rho]=\rho$, $32[\rho]=-ni_*\bar{\zeta}$.

Indeed, $\mathrm{Coext}(n\nu, \rho)\circ(32\iota)=-i_*\langle n\nu, \rho, 32\iota\rangle=-ni_*\langle\nu, 4\rho, 8\iota\rangle=-ni_*\bar{\zeta}$ by Lemma 7.3. We have

$$\pi_{4n+14}\cong Z_2\oplus Z_2\oplus A=\{i_*\bar{\sigma}\}\oplus\{[\eta]\circ\kappa\}\oplus A ,$$

where

$$A\cong\begin{cases} Z_{256}=\{[\rho]\} & \text{for } n \text{ odd},\\ Z_2\oplus Z_{128}=\{i_*\bar{\zeta}, [\rho]\} & \text{for } n\equiv 2 \bmod 4,\\ Z_2\oplus Z_{64}=\{i_*\bar{\zeta}, [\rho]\} & \text{for } n\equiv 4 \bmod 8.\end{cases}$$

(x) There are elements $[\eta^*]\in\{[\eta], 2\sigma, \sigma\}$, $[\bar{\mu}]\in\{[\mu], 2\iota, 8\sigma\}$ such that $p_*[\eta^*]=\eta^*$, $p_*[\bar{\mu}]=\bar{\mu}$, $2[\eta^*]=0$, $2[\bar{\mu}]=0$.

The proofs are quite similar as before. We have the following

$\pi_{4n+15}\cong Z_8\oplus Z_2\oplus Z_2=\{i_*\bar{\kappa}\}\oplus\{[\eta]\circ\rho\}\oplus\{[\eta^*]\}$ for n even,

$\pi_{4n+15}\cong Z_4\oplus Z_2\oplus Z_2=\{i_*\bar{\kappa}\}\oplus\{[\eta]\circ\rho\}\oplus\{[\eta^*]\}$ for n odd,

$\pi_{4n+16}\cong Z_2\oplus\cdots\oplus Z_2$ (rank 6) $=\{i_*\eta\circ\bar{\kappa}\}\oplus\{i_*\sigma^3\}\oplus\{[\eta]\circ\eta\circ\rho\}\oplus\{[\eta]\circ\eta^*\}\oplus\{[\bar{\mu}]\}\oplus\{[\nu]\circ\kappa\}$ for n even,

$\pi_{4n+16}\cong Z_2\oplus\cdots\oplus Z_2$ (rank 4) $=\{i_*\eta\circ\bar{\kappa}\}\oplus\{[\eta]\circ\eta\circ\rho\}\oplus\{[\eta]\circ\eta^*\}\oplus\{[\bar{\mu}]\}$ for n odd.

§8. $\pi_{24+r}(X_{6,2}; 2)$.

It follows from Corollary 4.2 that for $r\leqq 12$ $\pi_{24+r}(X_{6,2}; 2)$ is obtained from the results in §7. Throughout this section, $\pi_i(X)$, π_i^m denote $\pi_i(X; 2)$, $\pi_i(S^m; 2)$, respectively.

Note that $\pi_{24+r}(Q_{6,2})\to\pi_{24+r}(X_{6,2})$ and $\pi_{24+r}(X_{6,2}; S^{19})\to\pi_{24+r}(S^{23})$ are isomorphisms for $r\leqq 16$, and hence we compute $\pi_{24+r}=\pi_{24+r}(Q_{6,2})$ from the exact sequence

$$\cdots\longrightarrow\pi^{23}_{24+r+1}\xrightarrow{\Delta}\pi^{19}_{24+r}\xrightarrow{i_*}\pi_{24+r}\xrightarrow{p_*}\pi^{23}_{24+r}\longrightarrow\cdots,$$

where Δ is given by the formula:

$$\Delta(\gamma)=(6\nu)_*E^{-1}(\gamma)\qquad\text{for } \gamma\in\pi_i^{23}.$$

The groups π^{19}_{24+r}, π^{23}_{24+r} for $13\leqq r\leqq 17$ are konwn as follows:

r	13	14	15	16	17
π^{19}_{24+r} generator	$8+2+2$ $\nu^*,\ \nu^*+\xi,\ \eta\circ\bar{\mu}$	$2+8$ $\bar{\sigma},\ \bar{\zeta}$	$8+2$ $\bar{\kappa},\ \bar{\beta}$	$2+2+2+2$ $\eta\circ\bar{\kappa},\ \sigma^3,\ \nu^*\circ\nu,\ \bar{\beta}\circ\eta$	
π^{23}_{24+r} generator	$2+2$ $\sigma^2,\ \kappa$	$32+2$ $\rho,\ \eta\circ\kappa$	$2+2$ $\eta\circ\rho,\ \eta^*$	$2+2+2+2$ $\bar{\mu},\ \nu\circ\kappa,\ \eta^2\circ\rho,\ \eta\circ\eta^*$	$8+2$ $\nu^*,\ \eta\circ\bar{\nu}$

Observe that $(6\nu)\circ\pi^{23}_{24+r}=0$ for $13\leqq r\leqq 17$. $[\kappa]\in \mathrm{Coext}(6\nu,\kappa)$, $[\rho]\in(6\nu,\rho)$, $[\eta^*]\in\{[\eta], 2\sigma, \sigma\}$, $[\bar{\mu}]\in\{[\mu], 2\iota, 8\sigma\}$ are chosen so that $p_*[\kappa]=\kappa$, $p_*[\rho]=\rho$, $p_*[\eta^*]=\eta^*$, $p_*[\bar{\mu}]=\bar{\mu}$. It is obvious that the following elements are of order 2:

$$[\kappa],\ [\eta]\circ\kappa,\ [\eta]\circ\rho,\ [\eta^*],\ [\bar{\mu}],\ [\nu]\circ\kappa,\ [\eta]\circ\eta\circ\rho,\ [\eta]\circ\eta^*\,.$$

We need the relations below:

LEMMA 8.1. $32[\rho]=2i_*\bar{\zeta}$.

PROOF. This follows from the commutative diagram below (see Appendix):

$$\begin{array}{ccc} \pi_{38}(Q_{6,2}) & \xrightarrow{p_*} & \pi_{38}(S^{23}) \\ \downarrow{\scriptstyle *\theta} & & \downarrow{\scriptstyle E^{32}} \\ \pi_{70}(Q_{14,2}) & \xrightarrow{p_*} & \pi_{70}(S^{55}) \end{array}$$

Note that $E^\infty:\pi_{38}(S^{23})\to G_{15}$ is an isomorphism, and $32[\rho]=-14i_*\bar{\zeta}=2i_*\bar{\zeta}$ in $\pi_{70}(Q_{14,2})$.

LEMMA 8.2. $2([\sigma]\circ\sigma)=2i_*\nu^*$.

PROOF. $\alpha_3\in\pi^3_6$, $\sigma_8\in\pi^8_{15}$, then $\alpha_3\times\sigma_8=\pm 2\nu_{11}\circ\sigma_{14}=\pm 4\sigma_{11}\circ\nu_{18}=0$, where $\alpha\times\beta$ denotes the reduced join of α and β. While $(2\sigma_8)\times\sigma_8=2\sigma_{16}\circ\sigma_{23}=0$. Applying the Proposition 3.4 of [6], we have

$$\{2\nu_{19},\ \sigma_{22},\ 2\sigma_{29}\}=\{2\sigma_{19},\ \sigma_{26},\ 2\nu_{33}\}=2\{\sigma_{19},\ 2\sigma_{26},\ \nu_{33}\}\ni 2\nu_{19}^*\ ,$$

which consists of a single element because $2\sigma_{19}\circ\pi^{26}_{37}+\pi^{19}_{34}\circ(2\nu_{34})=0$. Hence we have

$$\mathrm{Coext}(2\nu,\sigma)\circ(2\sigma)=-i_*\{2\nu_{19},\ \sigma_{22},\ 2\sigma_{29}\}=-2i_*\nu_{19}^*\ .$$

Since $[\sigma]\in\mathrm{Coext}(6\nu,\sigma)$, it follows that $[\sigma]\circ(2\sigma)=2i_*\nu_{19}^*$.

Denote by $\pi_i=\pi_i(Q_{6,2};2)$, then it follows that

$$\pi_{37}\cong Z_8\oplus Z_2\oplus Z_2\oplus Z_2\oplus Z_2=\{i_*\nu^*\}\oplus\{i_*\nu^*+i_*\xi\}\oplus\{i_*\eta\circ\bar{\mu}\}\oplus\{[\kappa]\}\oplus\{i_*\nu^*-[\sigma]\circ\sigma\}\ ,$$

$$\pi_{38}\cong Z_2\oplus Z_2\oplus Z_2\oplus Z_{128}=\{i_*\bar{\sigma}\}\oplus\{[\eta]\circ\kappa\}\oplus\{i_*\bar{\zeta}-16[\rho]\}\oplus\{[\rho]\}\ ,$$

$$\pi_{39}\cong Z_8\oplus Z_2\oplus Z_2\oplus Z_2=\{i_*\bar{\kappa}\}\oplus\{i_*\bar{\beta}\}\oplus\{[\eta]\circ\rho\}\oplus\{[\eta^*]\}\ ,$$

$$\begin{aligned}\pi_{40}&\cong Z_2\oplus\cdots\oplus Z_2\quad(\text{rank } 8)\\ &=\{i_*\eta\circ\bar{\kappa}\}\oplus\{i_*\sigma^3\}\oplus\{i_*\nu^*\circ\nu\}\oplus\{i_*\bar{\beta}\circ\eta\}\oplus\{[\bar{\mu}]\}\oplus\{[\nu]\circ\kappa\}\oplus\{[\eta]\circ\eta\circ\rho\}\oplus\{[\eta]\circ\eta^*\}\ .\end{aligned}$$

§ 9. $\pi_{20+r}(X_{5,2}\,;2)$.

It follows from Corollary 4.2 that for $r\leqq 8$ $\pi_{20+r}(X_{5,2};2)$ is obtained from the results in § 7. We use the same notations as in § 8.

Note that $\pi_{20+r}(Q_{5,2})\to\pi_{20+r}(X_{5,2})$ and $\pi_{20+r}(Q_{5,2};S^{15})\to\pi_{20+r}(S^{19})$ are isomorphisms for $r\leqq 12$, and hence we compute $\pi_{20+r}=\pi_{20+r}(Q_{5,2})$ from the sequence

$$\cdots\longrightarrow\pi^{19}_{20+r+1}\xrightarrow{\Delta}\pi^{15}_{20+r}\xrightarrow{i_*}\pi_{20+r}\xrightarrow{p_*}\pi^{19}_{20+r}\xrightarrow{\Delta}\cdots,$$

where $\Delta(\gamma)=(5\nu_{15})\circ E^{-1}(\gamma)$ for $\gamma\in\pi^{19}_{20+r}$. The groups π^{15}_{20+r} and π^{19}_{20+r} for $9\leqq r\leqq 13$ are known as follows:

r	9	10	11	12	13
π^{15}_{20+r} generator	$2+4$ $\kappa,\ \sigma^2$	$32+2$ $\rho,\ \eta\circ\kappa$	$2+2+2$ $\eta^{*\prime},\ \omega,\ \sigma\circ\mu$	$2+2+2+2+2$ $\eta^{*\prime}\circ\eta,\ \varepsilon^*,\ \sigma\circ\eta\circ\mu,\ \bar{\mu},\ \nu\circ\kappa$	
π^{19}_{20+r} generator	2 $\eta\circ\mu$	8 ζ	0	0	$2+2$ $\kappa,\ \sigma^2$

Note that $\nu_k\circ\zeta_{k+3}=0$ for $k\geqq 14$, $\nu_k\circ\sigma_{k+3}=0$ for $k\geqq 11$, and that the element $[\eta]\circ\mu$ is of order 2.

LEMMA 9.1. *There is an element* $[\zeta]\in\mathrm{Coext}\,(5\nu,\zeta)$ *such that* $p_*[\zeta]=\zeta$, *and* $8[\zeta]=4i_*\rho$.

PROOF. The proof is similar to that of Lemma 8.1. Note that $E^\infty:\pi^{15}_{30}\to G_{15}$ is an isomorphism.

Consequently, we have the following:

$$\pi_{29}\cong Z_2\oplus Z_4\oplus Z_2=\{i_*\kappa\}\oplus\{i_*\sigma^2\}\oplus\{[\eta]\circ\mu\},$$

$$\pi_{30}\cong Z_2\oplus Z_4\oplus Z_{64}=\{i_*\eta\circ\kappa\}\oplus\{i_*\rho,\ [\zeta]\},$$

$$\pi_{31}\cong Z_2\oplus Z_2\oplus Z_2=\{i_*\eta^{*\prime}\}\oplus\{i_*\omega\}\oplus\{i_*\sigma\circ\mu\},$$

$$\pi_{32}\cong Z_2\oplus Z_2\oplus Z_2\oplus Z_2=\{i_*\eta^{*\prime}\circ\eta\}\oplus\{i_*\varepsilon^*\}\oplus\{i_*\sigma\circ\eta\circ\mu\}\oplus\{i_*\bar{\mu}\}.$$

For the values of r such that $13\leqq r\leqq 16$, we have to study more about the inclusion map $k':(Q_{n,2},S^{4n-5})\to(X_{n,2},S^{4n-5})$, which induces a homomorphism of the homotopy exact sequence of $(Q_{n,2},S^{4n-5})$ into that of $(X_{n,2},S^{4n-5})$:

$$\begin{array}{ccccccccc}
\to\pi_i^{4n-5} & \xrightarrow{i'_*} & \pi_i(Q_{n,2}) & \xrightarrow{j'_*} & \pi_i(Q_{n,2},S^{4n-5}) & \xrightarrow{\partial'_*} & \pi_{i-1}^{4n-5} & \to \\
\| & & \downarrow k_* & & \downarrow k'_* & & \| & \\
\to\pi_i^{4n-5} & \xrightarrow{i_*} & \pi_i(X_{n,2}) & \xrightarrow{j_*} & \pi_i(X_{n,2},S^{4n-5}) & \xrightarrow{\partial_*} & \pi_{i-1}^{4n-5} & \to
\end{array}$$

Let $p':(Q_{n,2};S^{4n-5})\to(S^{4n-1},*)$ be the pinching map. It follows from the last line of p. 40 of [5] that there is a commutative diagram below:

$$\begin{array}{ccc} \pi_i(Q_{n,2}, S^{4n-5}) & \xrightarrow{k'_*} & \pi_i(X_{n,2}, S^{4n-5}) \\ \downarrow p'_* & & \downarrow p_{A*} \\ \pi_i^{4n-2} & = & \pi_i^{4n-1} \end{array} .$$

Thus, we have a commutative diagram:

$$\begin{array}{ccccccccc} \to \pi_{i+1}(Q_{n,2}, S^{4n-5}) & \xrightarrow{\partial'_*} & \pi_i^{4n-5} & \xrightarrow{i'_*} & \pi_i(Q_{n,2}) & \xrightarrow{j'_*} & \pi_i(Q_{n,2}, S^{4n-5}) & \to \\ \downarrow p'_* & & \| & & \downarrow k_* & \searrow p_* & \downarrow p'_* & \\ \to \pi_{i+1}^{4n-5} & \xrightarrow{\Delta} & \pi_i^{4n-5} & \xrightarrow{i_*} & \pi_i(X_{n,2}) & \xrightarrow{p_{A*}} & \pi_i^{4n-5} & \to \end{array} .$$

A direct consequence of the diagram is:

LEMMA 9.2. *If there is an element* $[\alpha]' \in \pi_i(Q_{n,2})$ *snch that* $p_*[\alpha]' = \alpha \neq 0$ *and* $m[\alpha]' = i'_*\beta$ *for some* $\beta \in \pi_i^{4n-5}$, *then* $k_*[\alpha]'$ *which we denote by* $[\alpha]$ *satisfies* $p_{A*}[\alpha] = \alpha$, $m[\alpha] = i_*\beta$.

Now, set $n=5$ and consider the 2-primary components of the groups for $i=20+r$, $13 \leqq r \leqq 16$. π_{20+r}^{15} and π_{20+r}^{19} are known as follows:

r	13	14	15	16	17
π_{20+r}^{15}	2+8+8	8+2+2	8	2+2+8	
generator	$\eta\circ\bar{\mu}$, ξ, $E^2\lambda$	$\bar{\zeta}$, $\bar{\sigma}$, $\omega\circ\nu$	$\bar{\kappa}$	$\eta\circ\bar{\kappa}$, σ^3, $E^2\lambda\circ\nu$	
π_{20+r}^{19}	2+2	2+32	2+2	2+2+2+2	2+2+8
generator	κ, σ^2	$\eta\circ\kappa$, ρ	$\eta\circ\rho$, η^*	$\bar{\mu}$, $\nu\circ\kappa$, $\eta^2\circ\rho$, $\eta\circ\eta^*$	$\eta\circ\bar{\mu}$, $\nu^*+\xi$, ν^*

Note that $\nu_k\circ\rho_{k+3}=0$ for $k \geqq 11$, $\nu_k\circ\sigma_{k+3}\circ\mu_{k+10}=0$ for $k \geqq 7$, $\nu_k\circ\eta^*_{k+3}=0$ for $k \geqq 14$, $\nu_k\circ\bar{\mu}_{k+3}=0$ for $k \geqq 7$, and $\nu_k^2\circ\kappa_{k+6}=4\bar{\kappa}_k$ for $k \geqq 7$ (see [6], [8], [9]).

LEMMA 9.3. (i) $\nu_{15}\circ\xi_{18}=\sigma_{15}^3$, (ii) $\nu_{15}\circ(\xi_{18}+\nu_{18}^*)=E^2\lambda\circ\nu_{33}$.

PROOF. (i) Since $E: \pi_{36}^{15} \to \pi_{37}^{16}$ is a monomorphism, it suffices to show that $\nu_{16}\circ\xi_{19}=\sigma_{16}^3$. Note that $\xi_{16}\circ\nu_{34} \in \{\sigma_{16}, \nu, \sigma\}\circ\nu = \sigma_{16}\circ\{\nu, \sigma, \nu\} \bmod \sigma_{16}\circ\pi_{34}^{23}\circ\nu_{34}=0$. Since $\{\nu_{23}, \sigma, \nu\}=\sigma^2$ (Lemma 7.1), we have $\xi_{16}\circ\nu_{34}=\sigma_{16}^3$. While, $\nu_{16}\circ\xi_{19}=\nu_4\times\xi_{12}=\xi_{16}\circ\nu_{34}$. (ii) $\nu_{16}\circ(\xi_{19}+\nu_{19}^*)=\nu_{16}\circ[\iota_{19}, \iota_{19}]=[\nu_{16}, \nu_{16}]=[\iota_{16}, \nu_{16}]\circ\nu=\pm(E^3\lambda-2\nu_{16}^*)\circ\nu=(E^3\lambda)\circ\nu$ (see [6]). Hence we have $\nu_{15}\circ(\xi_{18}+\nu_{8}^*)=(E^2\lambda)\circ\nu_{33}$.

LEMMA 9.4. (i) Denote by $[\sigma]'$ an element in Coext $(5\nu, \sigma)$ such that $p_*[\sigma]'=\sigma$. Then $([\sigma]'\circ\sigma)=3i'_*\xi$ or $i'_*(3\xi+4E^2\lambda)$.

(ii) There is an element $[\rho]' \in$ Coext $(5\nu, \rho)$ in $\pi_{34}(Q_{5,2})$ such that $p_*[\rho]'=\rho$, $32[\rho]'=3i_*\bar{\zeta}$ or $i_*(3\bar{\zeta}+\omega\circ\nu)$.

(iii) There is an element $[\eta^*]' \in \{[\eta], 2\iota, \sigma^2\}$ in $\pi_{35}(Q_{5,2})$ such that $p_*[\eta^*]'=\eta^*$,

$2[\eta^*]'=0$.

PROOF. (i) $\mathrm{Coext}(5\nu,\sigma)\circ(2\sigma)=-i_*\{5\nu_{15},\sigma_{18},2\sigma_{25}\}$ mod $i'_*[\nu_{15}\circ\pi^{18}_{33}+\pi^{15}_{26}\circ(2\sigma_{26})]=0$ (see [6]). While in the stable range, $\langle\nu,\sigma,2\sigma\rangle=-\langle\sigma,\nu,\sigma\rangle=-\xi$ and $\ker(\pi^{15}_{33}\to G_{18})$ is generated by $4E^2\lambda$. Hence we have (i).

(ii) The proof is similar to that of Lemma 8.1 (i). Note that $\ker(\pi^{15}_{34}\to G_{19})$ is generated by $\omega\circ\nu$.

(iii) $\{[\eta],2\iota,\sigma^2\}\circ(2\iota)=[\eta]\circ\{2\iota,\sigma^2,2\iota\}=[\eta]\circ\sigma^2\circ\eta=0$.

Applying Lemma 9.2 to those results, we have the following:

$$\pi_{33}\cong Z_2\oplus Z_8\oplus Z_{16}=\{i_*\eta\circ\bar{\mu}\}\oplus\{i_*E^2\lambda\}\oplus\{[\sigma]\circ\sigma\},$$
$$\pi_{34}\cong Z_2\oplus Z_2\oplus Z_{256}\oplus Z_2=\{i_*\bar{\sigma}\}\oplus\{i_*\omega\circ\nu\}\oplus\{[\rho]\}\oplus\{[\eta]\circ\kappa\},$$
$$\pi_{35}\cong Z_4\oplus Z_2\oplus Z_2=\{i_*\bar{\kappa}\}\oplus\{[\eta]\circ\rho\}\oplus\{[\eta^*]\},$$
$$\pi_{36}\cong Z_2\oplus Z_2\oplus Z_2\oplus Z_2=\{i_*\eta\circ\bar{\kappa}\}\oplus\{[\bar{\mu}]\}\oplus\{[\eta]\circ\eta\circ\rho\}\oplus\{[\eta]\circ\eta^*\},$$

where π_i denotes $\pi_i(X_{n,2};2)$.

§ 10. $\pi_{16+r}(X_{4,2};2)$.

It follows from Corollary 4.2 that $\pi_{16+r}(X_{4,2};2)$ is obtained from the results in § 7 if $r\leqq 4$. We use the same notations as before.

(i) $5\leqq r\leqq 8$.

Since the inclusion map $Q_{4,2}\to X_{4,2}$ induces an isomorphism $\pi_{16+r}(Q_{4,2})\cong\pi_{16+r}(X_{4,2})$ for $r\leqq 8$, we compute $\pi_{16+r}(X_{4,2})$ from the exact sequence

$$\longrightarrow\pi^{15}_{16+r+1}\xrightarrow{\Delta}\pi^{11}_{16+r}\xrightarrow{i_*}\pi_{16+r}(Q_{4,2})\xrightarrow{p_*}\pi^{15}_{16+r}\longrightarrow.$$

Recall that $\Delta(\gamma)=(4\nu)_*E^{-1}(\gamma)$ for $\gamma\in\pi^{15}_i$. Table of π^{11}_{16+r}, π^{15}_{16+r} is given below:

r	5	6	7	8	9
π^{11}_{16+r} generator	2+2 $\eta\circ\mu$, $\sigma\circ\nu$	8 ζ	2 θ'	2+2 $\theta'\circ\eta$, $\sigma\circ\nu^2$	
π^{15}_{16+r} generator	2 ν^2	16 σ	2+2 ε, $\eta\circ\sigma$	2+2+2 $\eta\circ\varepsilon$, μ, ν^3	2+2 $\eta\circ\mu$, $\sigma\circ\nu$

Observe that Δ is trivial. There exist elements

$$[\eta]\in\mathrm{Coext}(4\nu,\eta),\ [\nu]\in\mathrm{Coext}(4\nu,\nu),\ [\sigma]\in\mathrm{Coext}(4\nu,\sigma),$$
$$[\varepsilon]\in\{[\eta],2\iota,\nu^2\},\ [\mu]\in\{[\eta],2\iota,8\sigma\}$$

such that

$$p_*[\eta]=\eta,\ p_*[\nu]=\nu,\ p_*[\sigma]=\sigma,\ p_*[\varepsilon]=\varepsilon,\ p_*[\mu]=\mu;$$

We prove similarly as before that the elements $[\nu]\circ\nu$, $[\varepsilon]$, $[\eta]\circ\sigma$, $[\eta]\circ\varepsilon$, $[\mu]$, $[\nu]\circ\nu^2$ are of order 2, and that $16[\sigma]=4i_*\zeta$. Thus we have

$$\pi_{21}\cong Z_2\oplus Z_2\oplus Z_2=\{i_*\eta\circ\mu\}\oplus\{i_*\sigma\circ\nu\}\oplus\{[\nu]\circ\nu\}\ ,$$

$$\pi_{22}\cong Z_4\oplus Z_{32}=\{i_*\zeta,\ [\sigma]\}\ ,$$

$$\pi_{23}\cong Z_2\oplus Z_2\oplus Z_2=\{i_*\theta'\}\oplus\{[\varepsilon]\}\oplus\{[\eta]\circ\sigma\}\ ,$$

$$\pi_{24}\cong Z_2\oplus\cdots\oplus Z_2\,(\text{rank }5)=\{i_*\theta'\circ\eta\}\oplus\{i_*\sigma\circ\nu^2\}\oplus\{[\eta]\circ\varepsilon\}\oplus\{[\mu]\}\oplus\{[\nu]\circ\nu^2\}\ ,$$

where π_i denotes $\pi_i(Q_{4,2};2)\cong\pi_i(X_{4,2};2)$.

(ii) $9\leqq r\leqq 12$.

In the exact sequence

$$\longrightarrow\pi^{15}_{16+r+1}\xrightarrow{\ \Delta\ }\pi^{11}_{16+r}\xrightarrow{\ i_*\ }\pi_{16+r}(X_{4,2})\xrightarrow{\ p_{A*}\ }\pi^{15}_{16+r}\longrightarrow\ ,$$

the homomorphism Δ is easily known because $E:\pi^{14}_{16+r}\to\pi^{15}_{16+r+1}$ is an epimorphism for $r\leqq 13$. Table of π^{11}_{16+r}, π^{15}_{16+r} is given below:

r	9	10	11	12	13
π^{11}_{16+r} generator	$2+16$ $\kappa,\ \sigma^2$	$2+16$ $\eta\circ\kappa,\ \rho'$	2 $\eta\circ\rho$	$2+2+2$ $\bar{\mu},\ \nu\circ\kappa,\ \eta^2\circ\rho$	
π^{15}_{16+r} generator	2 $\eta\circ\mu$	8 ζ	0	0	$2+4$ $\kappa,\ \sigma^2$

Note that $\nu_{11}\circ\zeta_{14}=8\sigma^2_{11}$, and hence Δ is trivial for $r\leqq 13$. Now, we apply Lemma 9.2, and we use the same notation as in § 9. Then we see that there is an element $[\zeta]'\in\text{Coext}\,(4\nu,\zeta)$ such that $p_*[\zeta]'=\zeta$, $8[\zeta]'=0$, since $E^\infty:\pi^{11}_{26}\cong G_{15}$. Thus we have

$$\pi_{25}\cong Z_2\oplus Z_{16}\oplus Z_2=\{i_*\kappa\}\oplus\{i_*\sigma^2\}\oplus\{[\eta]\circ\mu\}\ ,$$

$$\pi_{26}\cong Z_2\oplus Z_{16}\oplus Z_8=\{i_*\eta\circ\kappa\}\oplus\{i_*\rho'\}\oplus\{[\zeta]\}\ ,$$

$$\pi_{27}\cong Z_2=\{i_*\eta\circ\rho\}\ ,$$

$$\pi_{28}\cong Z_2\oplus Z_2\oplus Z_2=\{i_*\bar{\mu}\}\oplus\{i_*\nu\circ\kappa\}\oplus\{i_*\eta^2\circ\rho\}\ .$$

(iii) $13\leqq r\leqq 16$.

The table of π^{11}_{16+r}, π^{15}_{16+r} is given as follows:

r	13	14	15	16
π^{11}_{16+r} generator	$2+4+8$ $\eta\circ\bar{\mu},\ \{\xi',\ \lambda'\}$	$8+2+2+2$ $\bar{\zeta},\ \bar{\sigma},\ \xi'\circ\eta,\ \lambda'\circ\eta$	$8+2+2$ $\bar{\kappa},\ \theta'\circ\varepsilon,\ \beta''$	$2+2+2+2$ $\eta\circ\bar{\kappa},\ \sigma\circ\kappa,\ \sigma^3,\ \theta'\circ\mu$
π^{15}_{16+r} generator	$2+4$ $\kappa,\ \sigma^2$	$2+32$ $\eta\circ\kappa,\ \rho$	$2+2+2$ $\eta\circ\rho,\ \omega,\ \eta^{*\prime}$	$2+2+2+2+2$ $\bar{\mu},\ \nu\circ\kappa,\ \sigma\circ\eta\circ\mu,\ \eta\circ\omega,\ \eta^{*\prime}\circ\eta$

The elements in the last row are suspension elements except $\eta^{*\prime}$ and $\eta^{*\prime}\circ\eta$. Note that $(4\nu_{11})\circ\rho_{14}=0$, and $\pi_{33}^{15}=E\pi_{32}^{14}$, $4\pi_{32}^{11}=0$. Hence $\varDelta$ is trivial except possibly $\varDelta(\eta^{*\prime})$ and $\varDelta(\eta^{*\prime}\circ\eta)$.

We use the same notations as in Lemma 9.2.

LEMMA 10.1. *There are elements* $[\kappa]'\in\mathrm{Coext}(4\nu,\kappa)$, $[\bar{\mu}]'\in\{[\mu]'\,2\iota, 8\sigma\}$ *such that* $p_*[\kappa]'=\kappa$, $p_*[\bar{\mu}]'=\bar{\mu}$, $2[\kappa]'=0$, $2[\bar{\mu}]'=0$.

The proof is quite similar as before.

The proofs of the following two lemmas will be given later.

LEMMA 10.2. *There is an element* $[\sigma^2]'\in\mathrm{Coext}(4\nu,\sigma^2)$ *such that* $p_*[\sigma^2]'=\sigma^2$, $4[\sigma^2]'=0$.

LEMMA 10.3. *There is an element* $[\omega]'\in\mathrm{Coext}(4\nu,\omega)$ *such that* $p_*[\omega]'=\omega$, $2k_*[\omega]'=0$.

Now, we prove the followings:

LEMMA 10.4. *There is an element* $[\eta^{*\prime}]'\in\{[\sigma^2]', 4\iota, \eta\}$ *such that* $p_*[\eta^{*\prime}]'=\eta^{*\prime}$, $2[\eta^{*\prime}]'=0$.

PROOF. $p_*[\eta^{*\prime}]'\in\{\sigma^2, 4\iota, \eta\}=\eta^{*\prime}+\sigma_{15}^2\circ\pi_{31}^{29}+\pi_{30}^{15}\circ\eta_{30}$. Since $\sigma_{15}^2\circ\eta^2=0$, we can choose $[\eta^{*\prime}]'$ such that $p_*[\eta^{*\prime}]'=\eta^{*\prime}$. Now, $\{\sigma^2, 4\iota, \eta\}\circ(2\iota)=\sigma^2\circ\{4\iota, \eta, 2\iota\}=\sigma^2\circ(2\iota)\circ\eta^2=0$.

LEMMA 10.5. *There is an element* $[\rho]'\in\mathrm{Coext}(4\nu,\rho)$ *such that* $p_*[\rho]'=\rho$ *and* $32[\rho]'=4i'_*\bar{\zeta}$.

PROOF. $\ker(\pi_{30}^{11}\to G_{19})$ is generated by $\xi'\circ\eta$ and $\lambda'\circ\eta$. Hence it follows from the results in §7 that $32[\rho]'=i'_*(4\bar{\zeta}+x\xi'\circ\eta+y\lambda'\circ\eta)$ with $x, y=0$ or 1.

On the other hand, $\mathrm{Coext}(4\nu_{11},\rho_{14})\circ(32\iota_{30})=-i'_*\{4\nu_{11},\rho_{14},32\iota_{29}\}$, hence $32[\rho]'$ is divisible by 4. Thus, we conclude that $x=y=0$.

Now we have the following results:

$$\pi_{29}\cong Z_2\oplus Z_4\oplus Z_8\oplus Z_2\oplus Z_4=\{i_*\eta\circ\bar{\mu}\}\oplus\{i_*\xi', i_*\lambda'\}\oplus\{[\kappa]\}\oplus\{[\sigma^2]\},$$

$$\pi_{30}\cong Z_2\oplus Z_2\oplus Z_2\oplus Z_2\oplus Z_4\oplus Z_{64}=\{i_*\bar{\sigma}\}\oplus\{i_*\xi'\circ\eta\}\oplus\{i_*\lambda'\circ\eta\}\oplus\{[\eta]\circ\kappa\}\oplus\{i_*\bar{\zeta}, [\rho]\},$$

$$\pi_{31}\cong Z_8\oplus Z_2\oplus Z_2\oplus Z_2\oplus Z_2\oplus Z_2=\{i_*\bar{\kappa}\}\oplus\{i_*\theta'\circ\varepsilon\}\oplus\{i_*\beta''\}\oplus\{[\eta]\circ\rho\}\oplus\{[\omega]\}\oplus\{[\eta^{*\prime}]\},$$

$$\pi_{32}\cong Z_2\oplus\cdots\oplus Z_2\ (\text{rank }9)=\{i_*\eta\circ\bar{\kappa}\}\oplus\{i_*\sigma\circ\kappa\}\oplus\{i_*\sigma^3\}\oplus\{i_*\theta'\circ\mu\}\oplus\{[\bar{\mu}]\}\oplus\{[\nu]\circ\kappa\}$$
$$\oplus\{[\sigma]\circ\eta\circ\mu\}\oplus\{[\eta]\circ\omega\}\oplus\{[\eta^{*\prime}]\circ\eta\}.$$

PROOF OF LEMMA 10.2. Consider the commutative diagram below:

$$\begin{array}{ccccc}
 & & \pi_{29}(S^{11}) & \xrightarrow{E} & \pi_{30}(S^{12}) \\
 & \swarrow^{i_*} & \downarrow{i'_*} & & \downarrow{i'_*} \\
\pi_{39}(X) & \xleftarrow{k_*} & \pi_{29}(Q) & \xrightarrow{E} & \pi_{30}(EQ) \\
 & {}_{p_{\varDelta*}}\searrow & \swarrow{p_*} & & \\
 & & \pi_{29}(S^{15}) & &
\end{array}$$

where $X=X_{4,2}$, $Q=Q_{4,2}$. Note that:

(a) k_* is a monomorphism, because $\pi_{30}(X, Q)\cong\pi_{30}^{26}=0$,

(b) the top row in the square is a monomorphism ([6]).

Now, there is an element $[\sigma^2]'\in \mathrm{Coext}(4\nu, \sigma^2)$ such that $p_*[\sigma^2]'=\sigma^2$, which is an element of order 4. It follows from (a) that $4[\sigma^2]'\in \mathrm{im}\, i'_*$. Let us consider $E(4[\sigma^2]')$. Since $\mathrm{Coext}(4\nu_{12}, \sigma_{15}^2)\circ(4\iota_{30})=-i'_*\{4\nu_{12}, \sigma_{15}^2, 4\iota_{29}\}\subset 4i'_*\pi_{30}^{12}$, and since $E\pi_{29}^{11}\cap 4\pi_{30}^{12}=0$, it follows from (b) that $4[\sigma^2]'=0$.

PROOF OF LEMMA 10.3. Note that $\nu_{12}\circ\omega_{15}$ is an element of order 2, hence $E[(4\nu_{11})\circ\omega_{14}]=4(\nu_{12}\circ\omega_{15})=0$. Since $E:\pi_{30}^{11}\to\pi_{31}^{12}$ is a monomorphism, we have $(4\nu_{11})\circ\omega_{14}=0$. Applying the similar discussion as in the previous proof, we conclude that $2[\omega]'\in 2i'_*\pi_{31}^{11}+\ker k_*$. So, we choose an element $[\omega]'$ such that $2[\omega]'\in \ker k_*$.

§ 11. $\pi_{12+r}(X_{3,2}\,;2)$.

Throughout this section, we use the same notations as in § 10, except for $X=X_{3,2}$, $Q=Q_{3,2}$. Consider the exact sequence

$$\longrightarrow \pi_{12+r+1}^{11} \xrightarrow{\Delta} \pi_{12+r}^{7} \xrightarrow{i_*} \pi_{12+r}(X) \xrightarrow{p_{A*}} \pi_{12+r}^{11} \longrightarrow .$$

(i) $2\leqq r\leqq 5$.

π_{12+r}^{7}, π_{12+r}^{11} are given in the table below:

r	2	3	4	5	6
π_{12+r}^{7} generator	8 σ'	2+2+2 $\varepsilon, \bar{\nu}, \sigma'\circ\eta$	2+2+2+2 $\eta\circ\varepsilon, \mu, \nu^3, \sigma'\circ\eta^2$	2+8 $\eta\circ\mu, \nu\circ\sigma$	
π_{12+r}^{11} generator x	8 ν	0	0	2 ν^2	16 σ
$\Delta(x)$	$\nu^2\neq 0$			ν^3	$\nu\circ\sigma$

There is an element $[2\nu]'\in\{[\eta]', 2\iota, \eta\}\subset\pi_{14}(Q)$ such that $p_*[2\nu]'=2\nu$, $4[2\nu]'=0$. Applying Lemma 9.2, we have an element $[2\nu]=k_*[2\nu]'$ such that $p_{A*}[2\nu]=2\nu$, $4[2\nu]=0$. Denoting $\pi_i=\pi_i(X_{3,2}\,;2)$, we have

$$\pi_{14}\cong Z_8\oplus Z_4=\{i_*\sigma'\}\oplus\{[2\nu]\},$$

$$\pi_{15}\cong Z_2\oplus Z_2\oplus Z_2=\{i_*\varepsilon\}\oplus\{i_*\bar{\nu}\}\oplus\{i_*\sigma'\circ\eta\},$$

$$\pi_{16}\cong Z_2\oplus Z_2\oplus Z_2=\{i_*\eta\circ\varepsilon\}\oplus\{i_*\mu\}\oplus\{i_*\sigma'\circ\eta^2\},$$

$$\pi_{17}\cong Z_2=\{i_*\eta\circ\mu\}.$$

(ii) $6 \leqq r \leqq 9$.

The following table depends on [6].

r	6	7	8	9	10
π^7_{12+r} generator	$2+8$ $\bar{\nu}\circ\nu,\ \zeta$	0	2 $\nu\circ\sigma\circ\nu$	$8+4$ $\sigma'\circ\sigma,\ \kappa$	
π^{11}_{12+r} generator x	16 σ	$2+2$ $\varepsilon,\ \eta\circ\sigma$	$2+2+2$ $\eta\circ\varepsilon,\ \mu,\ \eta\circ\bar{\nu}$	$2+2$ $\eta\circ\mu,\ \sigma\circ\nu$	8 ζ
$\Delta(x)$	$\nu\circ\sigma$	0	0	$0,\ \nu\circ\sigma\circ\nu$	$4\sigma'\circ\sigma$

We shall construct some elements in $\pi_i(Q)$ to apply Lemma 9.2.

LEMMA 11.1. *Let* $[8\iota]' \in \mathrm{Coext}\,(3\nu, 8\iota) \subset \pi_{11}(Q)$. *Then* $2([8\iota]'\circ\sigma) = x i_* \zeta_7$ *for an odd integer* x.

PROOF. $\mathrm{Coext}\,(3\nu, 8\iota)\circ(2\sigma) = -i'_*\{3\nu, 8\iota, 2\sigma\}$ which contains $x\zeta_7$ for an odd integer x. Note that $\nu_7\circ\pi^{10}_{18} + \pi^7_{11}\circ 2\sigma_{11} = 0$.

It follows from the similar arguments as before that there are elements $[\varepsilon]' \in \{[\eta]', 2\iota, \nu^2\}$, $[\mu]' = \{[\eta]', 2\iota, 8\sigma\}$ such that $2[\varepsilon]' = 0$, $2[\mu]' = 0$, $p_*[\varepsilon]' = \varepsilon$, $p_*[\mu]' = 0$. Denote by $[\alpha] = k_*[\alpha]'$, where k is the inclusion map $Q \to X$. Now we have

$$\pi_{18} \cong Z_2 \oplus Z_{16} = \{i_* \bar{\nu}\circ\nu\} \oplus \{[8\iota]\circ\sigma\}\ ,$$

$$\pi_{19} \cong Z_2 \oplus Z_2 = \{[\varepsilon]\} \oplus \{[\eta]\circ\sigma\}\ ,$$

$$\pi_{20} \cong Z_2 \oplus Z_2 \oplus Z_2 = \{[\eta]\circ\varepsilon\} \oplus \{[\mu]\} \oplus \{[\eta]\circ\bar{\nu}\}\ ,$$

$$\pi_{21} \cong Z_4 \oplus Z_4 \oplus Z_2 = \{i_* \sigma'\circ\sigma\} \oplus \{i_* \kappa\} \oplus \{[\eta]\circ\mu\}\ .$$

(iii) $10 \leqq r \leqq 13$.

Due to [6], we have the table below:

r	10	11	12	13	14
π^7_{12+r} generator	$2+8+2+2$ $\bar{\varepsilon},\ \rho'',\ \sigma'\circ\varepsilon,\ \sigma'\circ\bar{\nu}$	$2+2+2+2$ $\mu\circ\bar{\varepsilon},\ \mu\circ\sigma,\ \sigma'\circ\varepsilon\circ\eta,\ \sigma'\circ\mu$	$2+2+2+2$ $\eta\circ\mu\circ\sigma,\ \bar{\mu},\ \nu\circ\kappa,\ \sigma'\circ\eta\circ\mu$	$2+8$ $\eta\circ\bar{\mu},\ \zeta\circ\sigma$	
π^{11}_{12+r} generator x	8 ζ	2 θ'	$2+2$ $\theta'\circ\eta,\ \sigma\circ\nu^2$	$2+16$ $\kappa,\ \sigma^2$	$2+16$ $\eta\circ\kappa,\ E^2\rho'$
(Δx)	$4\sigma'\circ\sigma$		$\eta\circ\bar{\varepsilon}$	$\nu\circ\kappa,\ 0$	0

LEMMA 11.2. $\Delta(\theta') = \sigma'\circ\bar{\nu}$, $\Delta(\theta'\circ\eta) = \eta\circ\bar{\varepsilon}$.

PROOF. Consider the commutative diagram below:

$$
\begin{array}{ccccc}
\pi_{23}(Q. S^7) & \xrightarrow{k_*} & \pi_{23}(X, S^7) & \xrightarrow{l_*} & \pi_{23}(X, Q) \\
\downarrow \partial_* & \searrow p'_* & \| & & \| \\
\pi^7_{22} & \xleftarrow{\Delta} & \pi^{11}_{23} & & \pi^{18}_{23} = 0 .
\end{array}
$$

Since k_* is an epimorphism, there is an element $\alpha \in \pi_{23}(Q, S^7)$ such that $p'_*\alpha = \theta'$. We want to compute $\partial_*\alpha$. Consider the commutative diagram below:

$$
\begin{array}{ccc}
S^{11}_{23} & \xrightarrow{E} & S^{12}_{24} \\
\uparrow p'_* & & \uparrow p'_* \\
\pi_{23}(Q, S^7) & \xrightarrow{E} & \pi_{24}(EQ, S^8) \\
\downarrow \partial_* & & \downarrow \partial_* \\
\pi^7_{22} & \xrightarrow{E} & \pi^8_{23}
\end{array} .
$$

Since $\nu_9 \circ E\theta' = \nu_9 \circ [\iota_{12}, \eta_{12}] = [\nu_9, \nu_9 \circ \eta_{12}] = 0$, and $\ker(\pi^8_{23} \to \pi^9_{24})$ is generated by $\sigma' \circ \varepsilon$ and $\sigma' \circ \varepsilon$ and $\sigma' \circ \bar{\nu}$, we see that $\nu_8 \circ \theta' = xE\sigma' \circ \varepsilon + yE\sigma' \circ \bar{\nu}$ with $x, y = 0$ or 1. Hence it follows from the diagram that

$$\Delta(\theta) = \partial_*\alpha = x\sigma' \circ \varepsilon + y\sigma' \circ \bar{\nu} \qquad \text{with } x, y = 0 \text{ or } 1 .$$

Note that $\Delta(\sigma_{11} \circ \nu^2_{18}) = \nu_7 \circ \sigma_{10} \circ \nu^2_{17} = \eta_7 \circ \bar{\varepsilon}_8$, and $\eta \circ \theta = \theta' \circ \eta + \sigma \circ \nu^2$. Since $[\eta]' \circ \theta \in \pi_{24}(Q)$ and $p_*([\eta]' \circ \theta) = \eta \circ \theta$, we have $\Delta(\eta \circ \theta) = 0$. Hence $\Delta(\theta' \circ \eta) = \Delta(\sigma \circ \nu^2) = \eta \circ \bar{\varepsilon}$. Since $\sigma' \circ \varepsilon \circ \eta = \zeta'$ and $\sigma' \circ \bar{\nu} \circ \eta = \sigma' \circ \nu^3 = \nu \circ \sigma \circ \nu^2 = \eta \circ \bar{\varepsilon}$, we conclude that $x = 0$, $y = 1$.

LEMMA 11.3. *There is an element* $[\sigma^2]' \in \mathrm{Coext}\,(3\nu_7, \sigma^2_{10})$ *such that* $p_*[\sigma^2]' = \sigma^2_{11}$, $16[\sigma^2]' = x\zeta_7 \circ \sigma_{18}$ *for an odd integer* x.

PROOF. It follows from Proposition 2.6 of [6] that

$$H\{\nu_5, 2\sigma^2_8, 8\iota_{22}\}_1 = (\sigma^2_9 + 2\kappa_9) \circ (8\iota_{23}) = 8\sigma^2_9 ,$$

where H denotes the generalized Hopf homomorphism (see [6]). On the other hand, $H(\zeta_5 \circ \sigma_{16}) = 8\sigma^2_9$. Hence $\{\nu_5, 2\sigma^2_8, 8\iota_{22}\}_1 \ni \zeta_5 \circ \sigma_{16} \mod \nu_5 \circ E\pi^7_{22} = 4\pi^5_{33}$. Now, $\mathrm{Coext}\,(3\nu_7, \sigma^2_{10}) \circ (16\iota) = \pm 3i'_*\{\nu_7, \sigma^2_{10}, 16\iota_{24}\}_2 \mod \nu_7 \circ E^2\pi^8_{23} = 4\pi^7_{35}$. Hence, we conclude that $16[\sigma^2]' = x\zeta_7 \circ \sigma_{18}$ for an odd integer x.

It is easy to see that there is an element $[2\zeta]' \in \{[2\nu]', 4\iota, 4\sigma\}$ such that $p_*[2\zeta]' = 2\zeta$, $4[2\zeta]' = 0$. It is known by [6] that $\nu_7 \circ E\rho' \equiv 0 \mod 2\pi^7_{25}$. However, I can not decide exact value of $\nu_7 \circ E\rho'$. We conclude that

$$\pi_{22} \cong Z_2 \oplus Z_8 \oplus Z_2 \oplus Z_4 = \{i_*\bar{\varepsilon}\} \oplus \{i_*\rho''\} + \{i_*\sigma' \circ \varepsilon\} \oplus \{[2\zeta]\} ,$$

$$\pi_{23} \cong Z_2 \oplus Z_2 \oplus Z_2 = \{i_*\mu \circ \sigma\} \oplus \{i_*\sigma' \circ \varepsilon \circ \eta\} \oplus \{i_*\sigma' \circ \mu\} ,$$

$$\pi_{24} \cong Z_2 \oplus Z_2 \oplus Z_2 \oplus Z_2 = \{i_*\eta \circ \mu \circ \sigma\} \oplus \{i_*\bar{\mu}\} \oplus \{i_*\sigma' \circ \eta \circ \mu\} \oplus \{[\eta] \circ \theta\} ,$$

$$\pi_{25} \cong Z_2 \oplus Z_t = \{i_*\eta \circ \bar{\mu}\} \oplus \{[\sigma^2]\}, \text{ where t} = 32 \text{ or } 64 \text{ or } 128 .$$

Appendix

Suppose that $X_{m,k}$ admits a cross-section. In [3], I. M. James has proved that there is a homotopy equivalence

$$G: E^{4m}Q_{n,k} \to Q_{n+m,k},$$

which admits a commutative diagram

$$\begin{array}{ccccc} \pi_i(Q_{n-l,k-l}) & \xrightarrow{i_*} & \pi_i(Q_{n,k}) & \xrightarrow{p_*} & \pi_i(Q_{n,l}) \\ \downarrow G_* & & \downarrow G_* & & \downarrow G_* \\ \pi_{i+4m}(E^{4m}Q_{n-l,k-l}) & \xrightarrow{i_*} & \pi_{i+4m}(E^{4m}Q_{n,k}) & \xrightarrow{p_*} & \pi_{i+4m}(E^{4m}Q_{n,l}) . \end{array}$$

This proposition holds when we speak of p-primary components of the homotopy groups by replacing the cross-section with "q-section" such that q is relatively prime to p.

In particular, since $X_{3,2}$ has a 3-section, we have a commutative diagram below: Let $m-n=8k$, $m>n$, and E^* denote E^{32k},

$$\begin{array}{ccccc} \pi_{4n+r}(S^{4n-5};2) & \xrightarrow{i_*} & \pi_{4n+r}(Q_{n,2};2) & \xrightarrow{p_*} & \pi_{4n+r}(S^{4n-1};2) \\ \downarrow E^* & & \downarrow G_* & & \downarrow E^* \\ \pi_{4m+r}(S^{4m-5};2) & \xrightarrow{i_*} & \pi_{4m+r}(Q_{m,2};2) & \xrightarrow{p_*} & \pi_{4m+r}(S^{4m-1};2) . \end{array}$$

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