

# Regular mappings associated with elliptic differential operators of second order in a manifold

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## Introduction.

In their recent book [1], L. Ahlfors and L. Sario introduced the notion of *normal operators* in open Riemann surfaces and showed the existence theorem for 'principal functions'. Using the existence theorem, they obtained some remarkable results, including elegant proofs of some classical theorems, in the theory of open Riemann surfaces. More recently, H. Yamaguchi [6] has introduced the notion of *regular operators*, a modification of the notion of normal operators in [1], and discussed the correspondence between regular operators and spaces of harmonic functions with finite Dirichlet integral.

In the present paper, we shall define the *regular mapping*  $L$  associated with the elliptic operator  $A^*$  of the form  $A^*u = \text{div}(\nabla u - bu)$  under a certain assumption for  $b$  (§1); the mapping  $L$  is a normal operator in [1] and also a regular operator in [6] in case  $A$  is Laplacian. Using the regular mapping, we shall define a kernel function which plays a role of 'Green function' for the elliptic boundary value problem with vanishing normal flux at the 'point at infinity'. We shall also prove a theorem analogous to the existence theorem for principal functions.

The regular mapping and the kernel function play important roles in the construction of the ideal boundary of Neumann type associated with the operator  $A^*$  which is a generalization of Kuramochi boundary [5]. The construction of such ideal boundary will be discussed elsewhere [4].

## §1. Preliminaries.

Let  $R$  be an orientable  $C^\infty$ -manifold of dimension  $m \geq 2$ , and  $A$  be an elliptic differential operator of the form:

$$\begin{aligned} Au(x) &= \text{div} [\nabla u(x)] + (b(x) \cdot \nabla u(x)) \\ &= \sum_{i,j} \frac{1}{\sqrt{a(x)}} \frac{\partial}{\partial x^i} \left\{ \sqrt{a(x)} a^{ij}(x) \frac{\partial u(x)}{\partial x^j} \right\} + \sum_i b^i(x) \frac{\partial u(x)}{\partial x^i} \end{aligned}$$

where  $\|a^{ij}(x)\|$  and  $\|b^i(x)\|$  are contravariant tensors of class  $C^2$  in  $R$ ,  $\|a^{ij}(x)\|$  is symmetric and strictly positive-definite for each  $x \in R$  and  $a(x) = \det \|a_{ij}(x)\| =$

$\det \|a^{ij}(x)\|^{-1}$ . Throughout this paper, we are concerned with the formally adjoint operator  $A^*$  of  $A$ :

$$A^*u = \operatorname{div}(\nabla u - bu).$$

We shall denote by  $dx$  and  $dS(x)$  respectively the volume element and the  $m-1$  dimensional hypersurface element with respect to the 'Riemann metric' defined by  $\|a_{ij}(x)\|$ . Given positive-valued and continuous function  $\omega(x)$  on a subdomain  $\Omega$  of  $R$ , we define the measure  $d_{\omega}x = \omega(x)dx$  and put

$$(\nabla u, \nabla v)_{\Omega, \omega} = \int_{\Omega} (\nabla u(x) \cdot \nabla v(x)) d_{\omega}x$$

(where  $(\nabla u \cdot \nabla v) = \sum_{i,j} a^{ij} \frac{\partial u}{\partial x^i} \cdot \frac{\partial v}{\partial x^j}$ ) and

$$\|\nabla u\|_{\Omega, \omega} = (\nabla u, \nabla u)_{\Omega, \omega}^{1/2}$$

whenever the right-hand side of each formula makes sense. We denote by  $L_{\omega}^2(\Omega)$  the completion of the space of all  $m$ -vector field  $\phi$  in  $\Omega$  whose covariant components  $\phi_1, \dots, \phi_m$  satisfy

$$\|\phi\|_{\Omega, \omega}^2 = \int_{\Omega} \sum_{i,j} a^{ij} \phi_i \phi_j d_{\omega}x < \infty,$$

and by  $P_{\omega}(\Omega)$  the totality of functions  $\phi \in C^1(\Omega)$  such that  $\nabla \phi \in L_{\omega}^2(\Omega)$ . Given compact set  $K \subset \Omega$ , we denote by  $P_{\omega}(\Omega; K)$  the totality of functions  $\phi \in C^0(\bar{\Omega} - K^{\circ}) \cap C^1(\Omega - K)$  such that  $\phi|_{\partial K} = 0$  and  $\nabla \phi \in L_{\omega}^2(\Omega - K)$  ( $\bar{\Omega}$  and  $K^{\circ}$  respectively denote the closure of  $\Omega$  and the interior of  $K$ ).

A function  $u$  is said to be *harmonic* in a domain  $\Omega \subset R$  if it satisfies  $A^*u = 0$  in  $\Omega$ . A subset  $E$  of  $R$  is said to be *regular* if the boundary of  $E$  consists of a finite number of simple hypersurfaces of class  $C^3$  ( $E$  is not necessarily relatively compact).

We fix a point  $x_0 \in R$ , which we call *normalizing point*. For every relatively compact regular domain  $D \ni x_0$ , let  $w^D$  be the solution of the following elliptic boundary value problem (1.1) satisfying the *normalizing condition*  $w^D(x_0) = 1$ :

$$(1.1) \quad A^*w = 0 \text{ in } D, \quad \left( \frac{\partial w}{\partial \mathbf{n}_D} - \beta_D w \right) \Big|_{\partial D} = 0$$

where  $\frac{\partial w}{\partial \mathbf{n}_D}$  and  $\beta_D$  respectively denote the *outer* normal derivative of  $w$  and the *outer* normal component of  $b$  on  $\partial D$ ; as is shown in [3]<sup>1)</sup>, the solution  $w$  of

<sup>1)</sup> Differential operators  $A$  and  $A^*$  in the present paper respectively correspond to  $A^*$  and  $A$  in [3] (also those in [2] cited in §2).

(1.1) uniquely exists up to a multiplicative constant and does not change sign on  $\bar{D}$ , and accordingly, by means of the normalizing condition,  $\omega^D$  is uniquely determined and  $\omega^D > 0$  on  $\bar{D}$ . We put  $p^D = \log \omega^D$ . Then we have

$$(1.2) \quad \begin{cases} b - \nabla p^D \in L_w^2(D) \text{ and} \\ (b - \nabla p^D, \nabla \phi)_{D, \omega^D} = 0 \text{ for any } \phi \in P_{\omega^D}(D). \end{cases}$$

Throughout this paper, we set the following

ASSUMPTION (A): *There exist functions  $q \in C^1(R)$  and  $w > 0$  on  $R$  such that*

$$(1.3) \quad b - \nabla q \in L_w^2(R) \text{ and}$$

$$(1.4) \quad \limsup_{D \uparrow R} \sup_{x \in D} \left| \log \frac{\omega^D(x)}{w(x)} \right| < \infty.$$

It may easily be seen that the existence of such functions  $q$  and  $w$  does not depend on the choice of the normalizing point  $x_0$ . The condition (1.4) is equivalent to the following one: there exists a monotone increasing sequence  $\{D_n\}$  of relatively compact regular domains such that

$$(1.5) \quad \lim_{n \rightarrow \infty} D_n = R \text{ and } \sup_n \sup_{x \in D_n} \left| \log \frac{\omega^{D_n}(x)}{w(x)} \right| < \infty.$$

Hereafter  $\{D_n\}$  always denotes a sequence of domains with this property. All results in this paper are independent of the special choice of the normalizing point  $x_0$  and the sequence  $\{D_n\}$ .

## § 2. Some properties of solutions of boundary value problems in compact subdomains.

We first mention some properties of Green functions of boundary value problems implied by the results of [2].<sup>2)</sup>

Let  $K$  be a regular compact set and  $D$  be a relatively compact regular domain containing  $K$ . Let  $f$ ,  $\varphi$  and  $\varphi_1$  be functions Hölder-continuous on  $\bar{D}$ ,  $\partial K$  and  $\partial D$  respectively.

Denote by  $G^{D-K}(x, y)$  the Green function of the elliptic boundary value problem:

$$(2.1) \quad Au = -f \text{ in } D - K, \quad u|_{\partial K} = \varphi, \quad u|_{\partial D} = \varphi_1,$$

and by  $N^{D-K}(x, y)$  the kernel function of the elliptic boundary value problem:

<sup>2)</sup> See the foot-note 1).

$$(2.2) \quad Av = -f \text{ in } D-K, \quad v|_{\partial K} = \varphi, \quad \frac{\partial v}{\partial \mathbf{n}_D} \Big|_{\partial D} = \varphi_1.$$

$G^{D-K}(x, y)$  and  $N^{D-K}(x, y)$  respectively are also the Green function of the adjoint boundary value problem to (2.1):

$$(2.1^*) \quad A^*u = -f \text{ in } D-K, \quad u|_{\partial K} = \varphi, \quad u|_{\partial D} = \varphi_1,$$

and the kernel function of the adjoint boundary value problem to (2.2):

$$(2.2^*) \quad A^*v = -f \text{ in } D-K, \quad v|_{\partial K} = \varphi, \quad \left( \frac{\partial v}{\partial \mathbf{n}_D} - \beta_D v \right) \Big|_{\partial D} = \varphi_1;$$

this statement means that, for instance, the unique solution  $v$  of (2.2\*) is given by the formula

$$(2.3) \quad v(y) = \int_D f(x) N^{D-K}(x, y) dx + \int_{\partial K} \varphi(x) \frac{\partial N^{D-K}(x, y)}{\partial \mathbf{n}_K(x)} dS(x) \\ + \int_{\partial D} \varphi_1(x) N^{D-K}(x, y) dS(x).$$

The following three lemmas will be proved in Appendix.

LEMMA 2.1. *Let  $\Omega$  and  $\Omega_1$  be relatively compact regular domains such that  $\Omega \supset \bar{\Omega}_1 \supset \Omega_1 \supset K$ . Then, for any  $D \supset \bar{\Omega}$ , it holds that*

$$(2.4) \quad N^{D-K}(x, y) \\ = G^{D-K}(x, y) + \int_{\partial D} \int_{\partial \Omega_1} \frac{\partial G^{D-K}(x, z)}{\partial \mathbf{n}_D(z)} N^{D-K}(z, z_1) \frac{\partial G^{\Omega_1-K}(z_1, y)}{\partial \mathbf{n}_{\Omega_1}(z_1)} dS(z_1) dS(z) \\ \text{for } x, y \in \Omega_1 - K^\circ.$$

LEMMA 2.2. *Let  $K$  be a fixed regular compact set, and let  $E$  and  $F$  be arbitrarily given mutually disjoint compact subsets of  $R-K^\circ$ . Then, for any relatively compact domain  $\Omega$  containing  $K \cup E \cup F$ , the system of functions*

$$\left\{ \begin{array}{l} N^{D-K}(x, y), \nabla_x N^{D-K}(x, y), \nabla_y N^{D-K}(x, y), \nabla_x \nabla_y N^{D-K}(x, y); \\ D \text{ running over all relatively compact regular domains} \\ \text{containing } \bar{\Omega} \end{array} \right\}$$

*is uniformly bounded and equi-continuous on  $E \times F$ .*

LEMMA 2.3. *Let  $K$  be the same as in Lemma 2.2, and  $F$  be arbitrarily given compact subset of  $R-K^\circ$ . Let  $v^D$  be the solutions of (2.2\*) where we assume  $f=0$ ,  $\varphi_1=0$  and  $\varphi$  is any fixed function  $\in C^1(\partial K)$ . Then, for any relatively compact domain  $\Omega$  containing  $K \cup F$ , the system of functions*

$$\left\{ \begin{array}{l} v^D, \nabla v^D; \\ D \text{ running over all relatively compact regular domains} \\ \text{containing } \bar{\Omega} \end{array} \right\}$$

*is uniformly bounded and equi-continuous on  $F$ .*

### §3. Regular mapping

Let  $\{\omega^n\}$  be the system of functions defined in §1 and  $\{D_n\}$  be the subsequence of  $\{D\}$  mentioned at the end of §1.

In this §, we shall prove the following two theorems and define a mapping of  $C^1(\partial K)$  into the set of harmonic functions on  $R-K$ , which we shall call *regular mapping*.

**THEOREM 3.1.** *There exists a function  $\omega$  on  $R$  satisfying that*

$$(B) \quad \begin{cases} \omega > 0 \text{ on } R, \quad b - \nabla p \in L_\omega^2(R) \text{ and} \\ (b - \nabla p, \nabla \phi)_{R, \omega} = 0 \text{ for any } \phi \in P_\omega(R) \end{cases}$$

where  $p = \log \omega$ ; such  $\omega$  is unique up to a multiplicative constant. The function  $\omega$  is harmonic in  $R$  and, if we normalize  $\omega$  by  $\omega(x_0) = 1$ , we have  $\omega = \lim_{n \rightarrow \infty} \omega^n$  and  $\nabla \omega = \lim_{n \rightarrow \infty} \nabla \omega^n$  uniformly on every compact subset of  $R$  for any sequence  $\{D_n\}$  satisfying (1.5).

**THEOREM 3.2.** *For any regular compact set  $K$  and any function  $\varphi \in C^1(\partial K)$ , there exists unique function  $u$  on  $R-K^\circ$  satisfying that*

$$(C) \quad \begin{cases} u|_{\partial K} = \varphi, \quad \left\| \nabla \frac{u}{\omega} \right\|_{R-K, \omega} < \infty, \quad \sup_{R-K} \left| \frac{u}{\omega} \right| < \infty \text{ and} \\ \left( \nabla \frac{u}{\omega} - [b - \nabla p] \frac{u}{\omega}, \nabla \phi \right)_{R-K, \omega} = 0 \text{ for any } \phi \in P_\omega(R; K). \end{cases}$$

The function  $u$  is harmonic in  $R-K$  and satisfies  $\sup_{R-K} \left| \frac{u}{\omega} \right| \leq \max_{\partial K} \left| \frac{\varphi}{\omega} \right|$ . If we denote by  $v^n$  the solution of the boundary value problem:  $\Delta v = 0$  in  $D-K$ ,  $v|_{\partial K} = \varphi$ ,  $\left( \frac{\partial v}{\partial \mathbf{n}_D} - \beta_D v \right) \Big|_{\partial D} = 0$ , then we have  $u = \lim_{n \rightarrow \infty} v^n$  and  $\nabla u = \lim_{n \rightarrow \infty} \nabla v^n$  uniformly on every compact subset of  $R-K^\circ$  for any sequence  $\{D_n\}$  satisfying (1.5).

To prove these theorems, we first mention the following three lemmas whose proofs will be given in Appendix.

**LEMMA 3.1.** *Let  $\{D_n\}$  be as in Theorems 3.1 and 3.2, and put  $\omega_n = \omega^{D_n}$  ( $n = 1, 2, \dots$ ). Let  $K$  be a regular compact set. Assume that  $\phi_n \in L_{\omega_n}^2(D_n - K)$  and  $(\phi_n, \nabla \phi)_{D_n - K, \omega_n} = 0$  for any  $\phi \in P_{\omega_n}(D_n; K)$  for every  $n$  and that  $\sup_n \|\phi_n\|_{D_n - K, \omega_n} < \infty$ . Assume further that  $\lim_{n \rightarrow \infty} \omega_n = \omega$  and  $\lim_{n \rightarrow \infty} \phi_n = \phi$  uniformly on every compact subset of  $R-K^\circ$ . Then*

$$(3.1) \quad \phi \in L_\omega^2(R-K) \text{ and } (\phi, \nabla \phi)_{R-K, \omega} = 0 \text{ for any } \phi \in P_\omega(R; K).$$

This proposition holds even when  $K$  is empty if we read  $P_{\omega_n}(D_n)$  and  $P_\omega(R)$

for  $P_{\omega_n}(D_n; K)$  and  $P_\omega(R; K)$  respectively.

LEMMA 3.2. If  $\phi$  satisfies (3.1), then  $(\phi\phi, \nabla\phi)_{R-K, \omega} = 0$  for any  $\phi \in P_\omega(R; K)$  (for any  $\phi \in P_\omega(R)$  if  $K$  is empty) which is bounded on  $R-K$ .

LEMMA 3.3. Assume that  $\omega_1$  and  $\omega_2$  satisfy the condition (B) in Theorem 3.1. Then  $\omega = \omega_1 + \omega_2$  also satisfies (B), and we have, for each  $\omega_\nu$  ( $\nu=1, 2$ ),  $\nabla \frac{\omega_\nu}{\omega} \in L^2_\omega(R)$  and

$$(3.2) \quad \left( \nabla \frac{\omega_\nu}{\omega} - [b - \nabla p] \frac{\omega_\nu}{\omega}, \nabla \phi \right)_{R, \omega} = 0 \text{ for any } \phi \in P_\omega(R) \quad (p = \log \omega).$$

PROOF OF THEOREM 3.1. Let  $\{D_n\}$  be a sequence of domains satisfying (1.5). Then there exists a constant  $M > 0$  such that

$$(3.3) \quad M^{-1}\omega \leq \omega^{D_n} \leq M\omega \quad \text{on } D_n \quad (n=1, 2, \dots).$$

For any relatively compact regular domain  $\Omega$ , denote by  $G^0(x, y)$  the Green function of the boundary value problem:  $Au = -f$  in  $\Omega$ ,  $u|_{\partial\Omega} = \varphi$ , which is also the Green function of the adjoint boundary value problem:  $A^*v = -f$  in  $\Omega$ ,  $v|_{\partial\Omega} = \varphi$ .<sup>3)</sup> Then we have

$$(3.4) \quad \omega^{D_n}(y) = - \int_{\partial\Omega} \omega^{D_n}(x) \frac{\partial G^0(x, y)}{\partial \mathbf{n}_\Omega(x)} dS(x) \quad \text{for any } y \in \Omega$$

whenever  $D_n \supset \Omega$ . It follows from (3.3) and (3.4) that the system of functions  $\{\omega^{D_n}, \nabla \omega^{D_n}; D_n \supset \Omega\}$  is uniformly bounded and equi-continuous on every compact subset of  $\Omega$ . Since  $\Omega$  is arbitrary,  $\{D_n\}$  contains a subsequence  $\{D_{(n)}\}$  for which  $\{\omega^{D_{(n)}}\}$  and  $\{\nabla \omega^{D_{(n)}}\}$  converge uniformly on every compact subset of  $R$ ; for the sake of simplicity, we temporarily denote the subsequence by the same notation  $\{D_n\}$  as the original one. The function  $\omega = \lim_{n \rightarrow \infty} \omega^{D_n}$  is positive and harmonic in  $R$  and satisfies  $\nabla \omega = \lim_{n \rightarrow \infty} \nabla \omega^{D_n}$  and  $\omega(x_0) = 1$ . We put  $\omega_n = \omega^{D_n}$ ,  $p_n = \log \omega_n$  and  $p = \log \omega$ . Then

$$(3.5) \quad \lim_{n \rightarrow \infty} p_n = p \quad \text{and} \quad \lim_{n \rightarrow \infty} \nabla p_n = \nabla p$$

uniformly on every compact subset of  $R$ . It follows from (1.3) and (3.3) that  $b - \nabla q \in L^2_{\omega_n}(D_n)$ ; this fact and (1.2) imply that  $\nabla(p_n - q) \in L^2_{\omega_n}(D_n)$ . Hence

$$(b - \nabla p_n, \nabla(p_n - q))_{D_n, \omega_n} = 0$$

by means of (1.2). Therefore

$$(3.6) \quad \|b - \nabla p_n\|_{D_n, \omega_n} \leq \|b - \nabla q\|_{D_n, \omega_n} \leq M^{1/2} \|b - \nabla q\|_{R, \omega} < \infty$$

<sup>3)</sup> This situation is the same as that of  $G^{D-K}(x, y)$  in § 2.

by virtue of (3.3) and (1.3). Hence we may apply Lemma 3.1 (where  $K$  is empty) to  $\phi_n = \mathbf{b} - \nabla p_n$  ( $n=1, 2, \dots$ ) and  $\phi = \mathbf{b} - \nabla p$ , and obtain (B) in Theorem 3.1.

Now let  $\{D_n\}$  be the original sequence. Then, from the above argument and the uniqueness of  $\omega$  to be proved below, we may see that any subsequence of  $\{D_n\}$  contains a subsequence  $\{D_{\nu}\}$  for which  $\{\omega^{D_{\nu}}\}$  and  $\{\nabla \omega^{D_{\nu}}\}$  respectively converge to  $\omega$  and  $\nabla \omega$  which are independent of the subsequence and each convergence holds uniformly on every compact subset of  $R$ . Hence the original sequences  $\{\omega^{D_n}\}$  and  $\{\nabla \omega^{D_n}\}$  converge in the same way.

The uniqueness of  $\omega$  is proved as follows. Suppose that  $\omega_1$  and  $\omega_2$  satisfy (B), and put  $\omega = \omega_1 + \omega_2$  and  $p = \log \omega$ . Then, by Lemma 3.3, we have  $\nabla \frac{\omega_\nu}{\omega} \in L_{\omega^2}(R)$  and accordingly

$$\left( \nabla \frac{\omega_\nu}{\omega} - [\mathbf{b} - \nabla p] \frac{\omega_\nu}{\omega}, \nabla \frac{\omega_1 - \omega_2}{\omega} \right)_{R, \omega} = 0 \quad (\text{by (3.2)})$$

for  $\nu=1$  and 2. Hence we get

$$\left( \nabla \frac{\omega_1 - \omega_2}{\omega} - [\mathbf{b} - \nabla p] \frac{\omega_1 - \omega_2}{\omega}, \nabla \frac{\omega_1 - \omega_2}{\omega} \right)_{R, \omega} = 0.$$

On the other hand, by virtue of Lemma 3.3, we may apply Lemma 3.2 (where  $K$  is empty) to  $\phi = \mathbf{b} - \nabla p$  and  $\psi = \frac{\omega_1 - \omega_2}{\omega}$  to obtain that

$$\left( [\mathbf{b} - \nabla p] \frac{\omega_1 - \omega_2}{\omega}, \nabla \frac{\omega_1 - \omega_2}{\omega} \right)_{R, \omega} = 0.$$

Hence we get  $\left\| \nabla \frac{\omega_1 - \omega_2}{\omega} \right\|_{R, \omega}^2 = 0$ . Therefore  $\frac{\omega_1 - \omega_2}{\omega_1 + \omega_2} = c$  on  $R$  where  $c$  is a constant and  $0 \leq c < 1$ ; this result implies that  $\omega_1/\omega_2$  is constant on  $R$ . In particular, if  $\omega_1(x_0) = \omega_2(x_0) = 1$ , then we have  $\omega_1 = \omega_2$  on  $R$ .

PROOF OF THEOREM 3.2. Let  $D_0$  be a fixed relatively compact domain containing  $K$ , and  $u_0$  be a function of class  $C^1$  on  $R$  satisfying  $u_0|_{\partial K} = \varphi$  and whose support is contained in  $D_0$ . Since  $\operatorname{div} \{\omega^D [\mathbf{b} - \nabla p^D]\} = 0$  in  $D - K$  and  $\beta_D - \partial p^D / \partial \mathbf{n}_D = 0$  on  $\partial D$ , we may show by means of integration by part that

$$\left( [\mathbf{b} - \nabla p^D] \frac{v^D - u_0}{\omega^D}, \nabla \frac{v^D - u_0}{\omega^D} \right)_{D-K, \omega^D} = 0$$

and

$$\left( \nabla \frac{v^D}{\omega^D} - [\mathbf{b} - \nabla p^D] \frac{v^D}{\omega^D}, \nabla \frac{v^D - u_0}{\omega^D} \right)_{D-K, \omega^D} = 0$$

whenever  $D \supset D_0$ . Hence we have

$$\begin{aligned} \left\| \nabla \frac{v^D - u_0}{\omega^D} \right\|_{D-K, \omega^D}^2 &= \left( \nabla \frac{v^D - u_0}{\omega^D} - [\mathbf{b} - \nabla p^D] \frac{v^D - u_0}{\omega^D}, \nabla \frac{v^D - u_0}{\omega^D} \right)_{D-K, \omega^D} \\ &= - \left( \nabla \frac{u_0}{\omega^D} - [\mathbf{b} - \nabla p^D] \frac{u_0}{\omega^D}, \nabla \frac{v^D - u_0}{\omega^D} \right)_{D-K, \omega^D}, \end{aligned}$$

which implies

$$(3.7) \quad \left\| \nabla \frac{v^D - u_0}{\omega^D} \right\|_{D-K, \omega^D} \leq \left\| \nabla \frac{u_0}{\omega^D} - [\mathbf{b} - \nabla p^D] \frac{u_0}{\omega^D} \right\|_{D-K, \omega^D};$$

accordingly

$$(3.8) \quad \left\| \nabla \frac{v^D}{\omega^D} \right\|_{D-K, \omega^D} \leq 2 \left\| \nabla \frac{u_0}{\omega^D} \right\|_{D-K, \omega^D} + \left\| [\mathbf{b} - \nabla p^D] \frac{u_0}{\omega^D} \right\|_{D-K, \omega^D}.$$

Since  $\frac{v^D}{\omega^D}$  satisfies  $\operatorname{div} \left\{ \omega^D \left[ \nabla \frac{v^D}{\omega^D} \right] \right\} - ([\mathbf{b} - \nabla p^D] \cdot \nabla \frac{v^D}{\omega^D}) = 0$  in  $D-K$  and  $\frac{\partial}{\partial \mathbf{n}_D} \left( \frac{v^D}{\omega^D} \right) = 0$  on  $\partial D$ , we may see that

$$(3.9) \quad \sup_{D-K} \left| \frac{v^D}{\omega^D} \right| \leq \max_{\partial K} \left| \frac{\varphi}{\omega^D} \right|.$$

Now let  $\{D_n\}$  be a sequence of domains satisfying (1.5). By virtue of Lemma 2.3,  $\{D_n\}$  contains a subsequence  $\{D_{(v)}\}$  for which  $\{v^{D_{(v)}}\}$  and  $\{\nabla v^{D_{(v)}}\}$  converge uniformly on every compact subset of  $R-K^\circ$ . It is sufficient to prove Theorem 3.2 where the sequence  $\{D_n\}$  in the last assertion is replaced by the subsequence  $\{D_{(v)}\}$ ; the convergence of the original sequence  $\{\omega^{D_n}\}$  may be shown by the same argument as in the proof of Theorem 3.1. So we denote the subsequence by  $\{D_n\}$  again, and put  $\omega_n = \omega^{D_n}$ ,  $v_n = v^{D_n}$  and  $u = \lim_{n \rightarrow \infty} v_n$ . Then  $u$  is harmonic in  $R-K$  and  $u|_{\partial K} = \varphi$ . Since  $\omega = \lim_{n \rightarrow \infty} \omega_n$  by Theorem 3.1 and since the support of  $u_0$  is compact, we obtain from (3.9), (3.8) and by the Lebesgue-Fatou lemma that  $\sup_{R-K} \left| \frac{u}{\omega} \right| \leq \max_{\partial K} \left| \frac{\varphi}{\omega} \right| < \infty$  and  $\left\| \nabla \frac{u}{\omega} \right\|_{R-K, \omega} \leq \sup_n \left\| \nabla \frac{v_n}{\omega_n} \right\|_{D_n-K, \omega_n} < \infty$ . Furthermore we may see from the definition of  $v_n = v^{D_n}$ , that  $\left( \nabla \frac{v_n}{\omega_n} - [\mathbf{b} - \nabla p_n] \frac{v_n}{\omega_n}, \nabla \phi \right)_{D_n-K, \omega_n} = 0$  for any  $\phi \in P_{\omega_n}(D_n; K)$ . Hence we may apply Lemma 3.1 to  $\phi_n = \nabla \frac{v_n}{\omega_n} - [\mathbf{b} - \nabla p_n] \frac{v_n}{\omega_n}$  ( $n=1, 2, \dots$ ) and  $\phi = \nabla \frac{u}{\omega} - [\mathbf{b} - \nabla p] \frac{u}{\omega}$  to obtain that  $(\phi, \nabla \phi)_{R-K, \omega} = 0$  for any  $\phi \in P_\omega(R; K)$ . Thus we have proved that  $u$  satisfies the condition (C).

The uniqueness of  $u$  is proved as follows. Suppose that  $u$  and  $v$  satisfy (C) for given  $\varphi$ . Then we may apply Lemma 3.2 to  $\phi = \mathbf{b} - \nabla p$  and  $\psi = \frac{u-v}{\omega}$  to obtain that



$$\left( [b - \nabla p] \frac{u-v}{\omega}, \nabla \frac{u-v}{\omega} \right)_{R-K, \omega} = 0.$$

Hence

$$\left( \nabla \frac{u-v}{\omega}, \nabla \frac{u-v}{\omega} \right)_{R-K, \omega} = \left( \nabla \frac{u-v}{\omega} - [b - \nabla p] \frac{u-v}{\omega}, \nabla \frac{u-v}{\omega} \right)_{R-K, \omega} = 0,$$

which implies  $u=v$  in  $R-K$  since  $u=v=\varphi$  on  $\partial K$ .

By virtue of Theorems 3.1 and 3.2, we can define a mapping  $L=L_K$  of  $C^1(\partial K)$  into the space of harmonic functions in  $R-K$  with the boundary value  $\varphi$  on  $\partial K$  in such a way that  $u=L_K\varphi$  satisfies the condition (C). The mapping  $L$  is called a *regular mapping*. We may easily see from Theorem 3.2 that

$$(3.10) \quad L\omega = \omega,$$

$$(3.11) \quad L(c_1\varphi_1 + c_2\varphi_2) = c_1L\varphi_1 + c_2L\varphi_2 \quad (c_1, c_2: \text{constant}),$$

$$(3.12) \quad L\varphi \geq 0 \quad \text{if} \quad \varphi \geq 0$$

and

$$(3.13) \quad \begin{cases} \text{if } u=L\varphi \text{ and if } \phi \text{ is a function } \in C^0(R-K^\circ) \cap C^1(R-K) \\ \text{such that } \nabla\phi \in L_\omega^2(R-K), \text{ then} \\ \left( \nabla \frac{u}{\omega} - (b - \nabla p) \frac{u}{\omega}, \nabla\phi \right)_{R-K, \omega} = - \int_{\partial K} \left( \frac{\partial u}{\partial \mathbf{n}_K} - \beta_K \varphi \right) \phi dS. \end{cases}^{(4)}$$

In case  $b=0$  (whence  $A=A^*$ ), we have  $\omega=1$  by means of the uniqueness of  $\omega$  in Theorem 3.1, and accordingly  $p=0$ . Therefore the equality in (3.13) becomes:

$$(3.14) \quad (\nabla u, \nabla\phi)_{R-K} = - \int_{\partial K} \frac{\partial u}{\partial \mathbf{n}_K} \phi dS.$$

Hence we may say that the mapping  $L$  is a normal operator in [1] and also a regular operator in [6] if  $A$  is Laplacian in a Riemann surface. Since (3.14) implies  $(\nabla u, \nabla v)_{R-K} = 0$  for any  $v \in L^2(R-K)$  satisfying  $v|_{\partial K} = 0$ ,  $u=L\varphi$  is the unique function with the minimum Dirichlet integral over  $R-K$  among the functions with the boundary value  $\varphi$  on  $\partial K$ .

#### § 4. Extension of the regular mapping and some properties.

Let  $K$  be a regular compact set and  $L=L_K$  be the regular mapping defined in the preceding §. Then, for any fixed  $y \in R-K^\circ$ , we have

<sup>4)</sup> Note that  $\frac{\partial u}{\partial \mathbf{n}_K}$  and  $\beta_K$  are respectively the *outer* normal derivative of  $u$  and the *outer* normal component of  $b$  on  $\partial K$  as the boundary of  $K$  (not of  $R-K$ ).

$$(4.1) \quad \left| \frac{(L_K \varphi)(y)}{\omega(y)} \right| \leq \max_{\partial K} \left| \frac{\varphi}{\omega} \right|$$

for any  $\varphi \in C^1(\partial K)$ . Hence the mapping  $\frac{\varphi}{\omega} \rightarrow \frac{(L_K \varphi)(y)}{\omega(y)}$  is uniquely extended to a bounded positive linear functional on  $C(\partial K)$ , and accordingly there exists a Borel measure  $\nu_K^\omega$  on  $\partial K$  such that  $\nu_K^\omega(\partial K) \leq 1$  and

$$(4.2) \quad (L_K \varphi)(y) = \omega(y) \int_{\partial K} \frac{\varphi(x)}{\omega(x)} d\nu_K^\omega(x)$$

for any  $\varphi \in C^1(\partial K)$ .

For any lower semi-continuous function  $\varphi$  on  $\partial K$ , we define  $(L_K \varphi)(y)$  by the formula (4.2). Thus the regular mapping  $L_K$  is extended to a mapping defined on the space of all lower semi-continuous functions on  $\partial K$ . Since the limit of a monotone increasing sequence of harmonic functions in a domain is harmonic whenever the limit is not identically equal to  $\infty$ , we may see that:

**THEOREM 4.1.** *For any lower semi-continuous function  $\varphi$  on  $\partial K$ ,  $L_K \varphi$  is harmonic in any connected component of  $R-K$  in which  $L_K \varphi$  is not identically equal to  $\infty$ .*

**THEOREM 4.2.** *Let  $K_1$  and  $K_2$  be regular compact sets such that  $K_1 \subset K_2$ , and  $\varphi$  be a lower semi-continuous function on  $\partial K_1$ . Then  $L_{K_2}(L_{K_1} \varphi) = L_{K_1} \varphi$  in  $R-(K_2)^\circ$ .*

**PROOF.** By means of the definition of the extension of  $L$ , it is sufficient to prove this theorem under the assumption  $\varphi \in C^1(\partial K_1)$ . For such  $\varphi$ , the function  $u = L_{K_1} \varphi$  satisfies (C) with  $K = K_1$  and we have  $u|_{\partial K_2} \in C^1(\partial K_2)$ , and accordingly the function  $v = L_{K_2} u$  satisfies (C) with  $K = K_2$  and  $\varphi = u|_{\partial K_2}$ . For any  $\psi \in P_\omega(R; K_2)$ , the function  $\tilde{\psi}$  on  $R-(K_1)^\circ$  defined by

$$\tilde{\psi} = \psi \text{ on } R-K_2 \text{ and } = 0 \text{ on } K_2-(K_1)^\circ$$

is continuous on  $R-(K_1)^\circ$  and satisfies  $\|\nabla \tilde{\psi}\|_{R-K_1, \omega} < \infty$  ( $\nabla \tilde{\psi}$  is defined in  $R-K_1-\partial K_2$ ). Hence there exists a sequence  $\{\phi_n\} \subset P_\omega(R; K_1)$  such that  $\lim_{n \rightarrow \infty} \|\nabla \phi_n - \nabla \tilde{\psi}\|_{R-K_1, \omega} = 0$ . Since

$$\left( \nabla \frac{\omega}{u} - [b - \nabla p] \frac{u}{\omega}, \nabla \phi_n \right)_{R-K_1, \omega} = 0 \quad (n=1, 2, \dots),$$

we get

$$\left( \nabla \frac{u}{\omega} - [b - \nabla p] \frac{u}{\omega}, \nabla \psi \right)_{R-K_1, \omega} = 0.$$

Hence  $u$  satisfies (C) with  $K = K_2$  and  $\varphi = u|_{\partial K_2}$ . Consequently we get  $u = v$  in  $R-(K_2)^\circ$  by Theorem 3.2.

We shall mention a theorem similar to the *main existence theorem* in [1; Chap. III, 3A].

As we have shown, the regular mapping  $L$  has the following properties:

- (L.1)  $L\varphi = \varphi$  on  $\partial K$ ,  
 (L.2)  $L(c_1\varphi_1 + c_2\varphi_2) = c_1L\varphi_1 + c_2L\varphi_2$ ,  
 (L.3)  $L\omega = \omega$ ,  
 (L.4)  $L\varphi \geq 0$  in  $R-K$  if  $\varphi \geq 0$  on  $\partial K$ ,  
 (L.5)  $\int_{\partial K} \left( \frac{\partial u}{\partial \mathbf{n}_K} - \beta_K u \right) dS = 0$  for  $u = L\varphi$ .

In Theorem 4.3 below, we assume that  $L$  is just a mapping of  $C^1(\partial K)$  into the set of functions continuous on  $R-K^\circ$  and harmonic in  $R-K$  satisfying (L.1–5);  $L$  is not necessarily the regular mapping defined in §3. We may easily derive from (L.2–4) that

$$(L.6) \quad \sup_{R-K} \left| \frac{L\varphi}{\omega} \right| \leq \max_{\partial K} \left| \frac{\varphi}{\omega} \right|.$$

**THEOREM 4.3.** *Given function  $u$  continuous on  $R-K^\circ$  and harmonic in  $R-K$ , a necessary and sufficient condition that there exists a harmonic function  $w$  on  $R$  satisfying*

$$(4.3) \quad w - u = L(w - u) \quad \text{in } R - K$$

*is that*

$$(4.4) \quad \int_{\partial K} \left( \frac{\partial u}{\partial \mathbf{n}_K} - \beta_K u \right) dS = 0.$$

*The function  $w$  is unique up to an additional term of a constant multiple of the function  $\omega$ , and  $w = c\omega$  for some constant  $c$  if and only if  $u = Lu$ .*

We may prove this theorem by way of the entirely same arguments as those in [1; Chap. III, 3B–3E] using the following three lemmas; proof of Lemma 4.1 also is essentially same as that of corresponding lemma in the book cited above.

**LEMMA 4.1.** *Let  $F$  be a compact set in a subdomain  $\Omega$  of  $R$ . Then there exists a positive constant  $k < 1$ , depending only on  $F$  and  $\Omega$ , such that*

$$\max_F \left| \frac{v}{\omega} \right| \leq k \sup_\Omega \left| \frac{v}{\omega} \right|$$

*for all functions  $v$  harmonic in  $\Omega$  and not of constant sign on  $F$ .*

**LEMMA 4.2.** *Let  $D$  be a relatively compact regular domain containing  $K$ ,*

and  $\phi$  be the solution of the elliptic boundary value problem:

$$(4.5) \quad A\phi = 0 \text{ in } D-K, \quad \phi|_{\partial K} = 0, \quad \phi|_{\partial D} = 1.$$

Let  $v$  be a function continuous on  $\overline{D-K}$  and harmonic in  $D-K$ .

i) If  $v$  satisfies

$$(4.6) \quad \int_{\partial K} \left( \frac{\partial v}{\partial \mathbf{n}_K} - \beta_K v \right) dS = 0 \quad \text{and} \quad \int_{\partial K} v \frac{\partial \phi}{\partial \mathbf{n}_K} dS = 0,$$

then we have

$$(4.7) \quad \int_{\partial D} \left( \frac{\partial v}{\partial \mathbf{n}_D} - \beta_D v \right) dS = 0 \quad \text{and} \quad \int_{\partial D} v \frac{\partial \phi}{\partial \mathbf{n}_D} dS = 0,$$

and accordingly  $v$  changes sign on  $\partial D$  unless  $v = 0$  on  $\partial D$ .

ii) Conversely, if  $v$  satisfies (4.7), then we have (4.6) and accordingly  $v$  changes sign on  $\partial K$  unless  $v = 0$  on  $\partial K$ .

PROOF. By means of Green's formula, we have

$$\int_{\partial K} \left( \frac{\partial v}{\partial \mathbf{n}_K} - \beta_K v \right) dS = \int_{\partial D} \left( \frac{\partial v}{\partial \mathbf{n}_D} - \beta_D v \right) dS$$

and

$$\int_{\partial K} \frac{\partial \phi}{\partial \mathbf{n}_K} v dS = \int_{\partial D} \left\{ \frac{\partial \phi}{\partial \mathbf{n}_D} v - \left( \frac{\partial v}{\partial \mathbf{n}_D} - \beta_D v \right) \right\} dS.$$

Using these relations, we may see that (4.6) implies (4.7). Since the solution  $\phi$  of (4.5) satisfies  $\frac{\partial \phi}{\partial \mathbf{n}_D} > 0$  on  $\partial D$ , it follows from the second equality in (4.7) that  $v$  changes sign on  $\partial D$  unless  $v = 0$  on  $\partial D$ . Part i) is thus proved. Part ii) may be proved similarly.

LEMMA 4.3. Let  $w$  be a harmonic function on  $R$ . Then  $w = Lw$  if and only if  $w = c\omega$  for some constant  $c$ .

PROOF. 'If' part is clear from (L.3). 'Only if' part is proved as follows. Since  $\operatorname{div} \{ \omega [\mathbf{b} - \nabla p] \} = 0$ , the function  $\frac{w}{\omega}$  satisfies that

$$\operatorname{div} \left\{ \omega \left( \nabla \frac{w}{\omega} \right) \right\} = \omega \left( [\mathbf{b} - \nabla p] \cdot \nabla \frac{w}{\omega} \right) = \operatorname{div} \{ \nabla w - \mathbf{b} w \} = A^* w = 0.$$

On the other hand, the assumption  $w = Lw$  implies that  $\sup_{R-K} \left| \frac{w}{\omega} \right| \leq \max_{\partial K} \left| \frac{w}{\omega} \right|$  by (L.6), and accordingly  $\frac{w}{\omega}$  takes its maximum at a certain point in  $K$ . Hence  $\frac{w}{\omega}$  must be constant.

### §5. The kernel function $N(x, y)$ .

In this §, we fixed a regular compact set  $K_0 \subset R$ , and denote by  $N^D(x, y)$  the function  $N^{D-K_0}(x, y)$  defined in §2 for any relatively compact regular domain  $D \supset K_0$ . For any  $x \in R - K_0$ , we denote by  $K(x)$  the totality of regular compact subsets  $K$  of  $R - K_0$  such that  $x \in K^\circ$ .

Let  $K$  be a regular compact subset of  $R - K_0$  and, for any relatively compact regular domain  $D \supset K_0 \cup K$ , denote by  $L_{K+K_0}^D$  the regular mapping of  $C^1(\partial K + \partial K_0)$  into the space of harmonic functions in  $D - (K + K_0)$ . For any  $\varphi \in C^1(\partial K)$ , we put

$$(5.1) \quad L_{K+K_0}^D \varphi = L_{K+K_0}^D \tilde{\varphi} \quad \text{where} \quad \tilde{\varphi} = \begin{cases} \varphi & \text{on } \partial K \\ 0 & \text{on } \partial K_0. \end{cases}$$

Then the function  $v = L_{K+K_0}^D \varphi$  is the solution of the boundary value problem:

$$(5.2) \quad \begin{cases} A^*v = 0 \text{ in } D - K, \quad v|_{\partial K_0} = 0, \quad v|_{\partial K} = \varphi, \\ \left( \frac{\partial v}{\partial \mathbf{n}_D} - \beta_D v \right)_{\partial D} = 0, \end{cases}$$

and accordingly,

$$(5.3) \quad \sup_{D-K-K_0} \left| \frac{L_{K+K_0}^D \varphi}{\omega^D} \right| \leq \max_{\partial K} \left| \frac{\varphi}{\omega^D} \right|$$

and

$$(5.4) \quad L_{K+K_0} \tilde{\varphi} = \lim_{n \rightarrow \infty} L_{K_n+K_0}^{D_n} \varphi \quad \begin{array}{l} \text{uniformly on every compact} \\ \text{subset of } R - (K + K_0)^\circ \end{array}$$

where  $L_{K+K_0}$  is the regular mapping defined in §3 and  $\{D_n\}$  is a sequence of relatively compact regular domains satisfying (1.5) and  $D_n \supset K + K_0$  ( $n=1, 2, \dots$ ). These facts are evident from Theorem 3.2.

Hereafter  $L_{K+K_0}^D N^D(x, \cdot)$  denotes the image of  $N^D(x, y)$  as a function of  $y \in \partial K$  through the mapping  $L_{K+K_0}^D$  for any fixed  $x$  ( $L_{K+K_0} N(x, \cdot)$  should be understood analogously for the function  $N(x, y)$  mentioned below).

We remark, among others, the following properties of the kernel function  $N^D(x, y)$ : a) For any Hölder-continuous function  $f(x)$  whose support is a compact subset of  $D - K_0$ , the function  $v(y) = \int_D f(x) N^D(x, y) dx$  satisfies that  $A^*v = -f$  in  $D - K_0$  and  $v|_{\partial K_0} = 0$ . b) For any fixed  $x \in D - K_0$ , it holds that  $L_{K+K_0}^D N^D(x, \cdot) = N^D(x, \cdot)$  in  $D - K_0 - K$  for any  $K \in K(x)$  since  $v(y) = N(x, y)$  is the solution of (5.2) with  $\varphi(y) = N^D(x, y)$  (for  $y \in \partial K$ ). Hence it is natural to consider the function  $N(x, y)$  in the following theorem to be a generalization of  $N^D(x, y)$  to the case:  $D = R$ .

THEOREM 5.1. *There exists unique function  $N(x, y)$  continuous on*

$$(5.5) \quad \overline{(R-K_0)} \times \overline{(R-K_0)} - \{(z, z); z \in \overline{R-K_0}\}$$

*with the following properties i) and ii):*

i) *For any Hölder-continuous function  $f(x)$  whose support is a compact subset of  $R-K_0$ , the function*

$$(5.6) \quad v(y) = \int_{R-K_0} f(x) N(x, y) dx$$

*satisfies that*

$$(5.7) \quad A^*v = -f \text{ in } R-K_0 \text{ and } v|_{\partial K_0} = 0.$$

ii) *For any fixed  $x \in R-K_0$ , it holds that*

$$(5.8) \quad L_{K+K_0} N(x, \cdot) = N(x, \cdot) \text{ in } R-K_0-K$$

*for any  $K \in \mathbf{K}(x)$ .*

*Further we have*

$$(5.9) \quad \lim_{n \rightarrow \infty} N^{D_n}(x, y) = N(x, y) \quad \text{uniformly on every compact subset of the set (5.5)}$$

*for any sequence  $\{D_n\}$  of relatively compact regular domains satisfying (1.5).*

PROOF. It follows from Lemma 2.2 that the sequence  $\{D_n\}$  contains a subsequence  $\{D_{(\nu)}\}$  for which  $\{N^{D_{(\nu)}}(x, y)\}$  converges to a function  $N(x, y)$  continuous on the set (5.5) and the convergence is uniform on every compact subset of the set (5.5). We denote the subsequence  $\{D_{(\nu)}\}$  simply by  $\{D_\nu\}$ , and the corresponding  $\{\omega^{D_{(\nu)}}\}$  (defined in §1) and  $\{L_{K,0}^{D_{(\nu)}}\}$  (defined by (5.1))—by  $\{\omega_\nu\}$  and  $\{L_{K,0}^\nu\}$  respectively. Put  $K = K_0$  and  $D = D_\nu$  in (2.4) and let  $\nu \rightarrow \infty$ . Then we obtain

$$(5.10) \quad N(x, y) = G^{D-K_0}(x, y) + \int_{\partial\Omega} \int_{\partial\Omega_1} \frac{\partial G^{D-K_0}(x, z)}{\partial \mathbf{n}_\Omega(z)} N(z, z_1) \frac{\partial G^{D_1-K_0}(z_1, y)}{\partial \mathbf{n}_{\Omega_1}(z_1)} dS(z_1) dS(z) \quad \text{for } x, y \in \Omega_1 - (K_0)^\circ$$

whenever  $\Omega \supset \bar{\Omega}_1 \supset \Omega_1 \supset K_0$ . Hence we get (5.7) from properties of Green functions  $G^{D-K_0}(x, y)$  and  $G^{D_1-K_0}(x, y)$ . To prove (5.8), we fix  $x \in R-K_0$  and  $K \in \mathbf{K}(x)$ , and denote  $N^{D_\nu}(x, y)$  and  $N(x, y)$  simply by  $N_\nu$  and  $N$  respectively. Then we have

$$\begin{aligned} \left| \frac{L_{K+K_0} N - N}{\omega} \right| &\leq \left| \frac{L_{K+K_0} N}{\omega} - \frac{L_{K,0}^\nu N}{\omega_\nu} \right| + \left| \frac{L_{K,0}^\nu (N - N_\nu)}{\omega_\nu} \right| + \left| \frac{L_{K,0}^\nu N}{\omega_\nu} - \frac{N}{\omega} \right| \\ &\leq \left| \frac{L_{K+K_0} N}{\omega} - \frac{L_{K,0}^\nu N}{\omega_\nu} \right| + \max_{\partial K} \left| \frac{N - N_\nu}{\omega_\nu} \right| + \left| \frac{N_\nu}{\omega_\nu} - \frac{N}{\omega} \right| \end{aligned}$$

since the regular mapping  $L_{K,0}^\nu (=L_{K,0}^{D_\nu})$  satisfies (5.3) and since  $N_\nu (=N^{D_\nu})$  has the property b) mentioned above. Letting  $\nu \rightarrow \infty$ , we obtain  $L_{K+K_0} N = N$  by virtue of (5.4) and the uniform convergence of  $\{N_\nu(x, y)\}$  as a function of  $y$  on every compact subset of  $R - K_0 - K$ . (5.8) is thus proved.

From the above argument and the uniqueness of  $N$  to be proved below, we may see that any subsequence of  $\{N^{D_n}\}$  contains a subsequence  $\{N_\nu\}$  which converges to the unique function  $N$  uniformly on every compact subset of the set (5.5). Hence the convergence (5.9) holds for the original sequence  $\{N^{D_n}\}$ .

In order to show the uniqueness of  $N(x, y)$ , we first verify the following fact from the properties i) and ii). For any Hölder-continuous function  $f(x)$  with compact support contained in  $R - K_0$ , the function  $v(y)$  defined by (5.6) satisfies that

$$(5.11) \quad L_{K+K_0} v = v \quad \text{in } R - K - K_0$$

for any regular compact subset  $K$  of  $R - K_0$  containing the support of  $f$  in its interior  $K^\circ$ , and that

$$(5.12) \quad \left( \nabla \frac{v}{\omega} - [\mathbf{b} - \nabla p] \frac{v}{\omega}, \nabla \phi \right)_{R-K_0, \omega} = (f, \phi)_{R-K_0, 1}$$

for any  $\phi \in P_\omega(R; K_0)$ .

(5.11) is verified from (4.2), (5.6) and (5.8) in the following way:

$$\begin{aligned} (L_{K+K_0} v)(y) &= \omega(y) \int_{\partial K} \omega(z)^{-1} d\mu_{K+K_0}^\nu \int_{R-K_0} f(x) N(x, z) dx \\ &= \int_{R-K_0} f(x) dx \int_{\partial K} \frac{\omega(y)}{\omega(z)} N(x, z) d\mu_{K+K_0}^\nu(z) \\ &= \int_{R-K_0} f(x) N(x, y) dx = v(y) \quad \text{for any } y \in R - K - K_0. \end{aligned}$$

To prove (5.12), we take a  $C^1$ -function  $h$  on  $R$  which equals 1 on  $K$  and has a compact support, and put

$$\phi = \phi_1 + \phi_2 \quad \text{where } \phi_1 = h\phi \quad \text{and } \phi_2 = (1-h)\phi.$$

Then we have

$$\begin{aligned} (5.13) \quad \left( \nabla \frac{v}{\omega} - [\mathbf{b} - \nabla p] \frac{v}{\omega}, \nabla \phi_1 \right)_{R-K_0, \omega} &= -(A^* v, \phi_1)_{R-K_0, 1} \\ &= (f, \phi_1)_{R-K_0, 1} = (f, \phi)_{R-K_0, 1} \end{aligned}$$

on account of (5.7), while it may be seen from (5.11) that

$$(5.14) \quad \left( \nabla \frac{v}{\omega} - [\mathbf{b} - \nabla p] \frac{v}{\omega}, \nabla \phi_2 \right)_{R-K_0, \omega} = 0$$

since  $\phi_2 \in P_\omega(R; K + K_0)$ . (5.12) follows immediately from (5.13) and (5.14).

Now suppose that  $N_1(x, y)$  and  $N_2(x, y)$  be continuous functions on the set (5.5) with properties i) and ii), and put

$$(5.6') \quad v_\nu(y) = \int_{R-K_0} f(x) N_\nu(x, y) dx, \quad \nu=1, 2,$$

where  $f$  is an arbitrary Hölder-continuous function with compact support contained in  $R-K_0$ . Then it follows from (5.7) and (5.11) that  $\frac{v_\nu}{\omega}$  belongs to  $P_\omega(R; K_0)$  and is bounded on  $R-K_0$  for each  $\nu$ , and accordingly  $\phi = \frac{v_1 - v_2}{\omega}$  has the same property. Hence

$$(5.15) \quad \left( [b - \nabla p] \frac{v_1 - v_2}{\omega}, \nabla \frac{v_1 - v_2}{\omega} \right)_{R-K_0, \omega} = 0$$

by Lemma 3.2, while

$$\left( \nabla \frac{v_\nu}{\omega} - [b - \nabla p] \frac{v_\nu}{\omega}, \nabla \frac{v_1 - v_2}{\omega} \right)_{R-K_0, \omega} = \left( f, \frac{v_1 - v_2}{\omega} \right)_{R-K_0, 1} \quad (\nu=1, 2)$$

on account of (5.12), and accordingly

$$(5.16) \quad \left( \nabla \frac{v_1 - v_2}{\omega} - [b - \nabla p] \frac{v_1 - v_2}{\omega}, \nabla \frac{v_1 - v_2}{\omega} \right)_{R-K_0, \omega} = 0.$$

(5.15) and (5.16) imply  $\left\| \nabla \frac{v_1 - v_2}{\omega} \right\|_{R-K_0, \omega} = 0$ . Hence we get  $v_1 = v_2$  in  $R-K_0$  since  $v_1 = v_2 = 0$  on  $\partial K_0$ . Therefore  $N_1 = N_2$  on the set (5.5) by means of the arbitrariness of  $f$  and the continuity of  $N_1$  and  $N_2$ . The uniqueness of  $N(x, y)$  is thus proved.

COROLLARY 5.1.1. i) For any fixed  $y \in R - (K_0)^\circ$ ,  $N(x, y)$  satisfies that

$$(5.17) \quad A_x N(x, y) = 0 \text{ in } x \in R - K_0 - \{y\} \text{ and } N(x, y) = 0 \text{ for } x \in \partial K_0 - \{y\}$$

and that

$$(5.18) \quad \int_{R-K_0} A^* f(x) \cdot N(x, y) dx = -f(y)$$

for any  $f \in C^2(R-K_0)$  with compact support  $\subset R-K_0$ .

ii) For any fixed  $x \in R - (K_0)$ ,  $N(x, y)$  satisfies that

$$(5.17^*) \quad A_y^* N(x, y) = 0 \text{ in } y \in R - K_0 - \{x\} \text{ and } N(x, y) = 0 \text{ for } y \in \partial K_0 - \{x\}$$

and that

$$(5.18^*) \quad \int_{R-K_0} N(x, y) \cdot A f(y) dy = -f(x)$$



for any  $f \in C^2(R-K_0)$  with compact support  $\subset R-K_0$ .

These properties of  $N(x, y)$  may be seen from (5.10) and properties of Green functions  $G^{q-K_0}(x, y)$  and  $G^{q_1-K_0}(x, y)$  in (5.10).

**COROLLARY 5.1.2.** *Let  $E$  be a compact subset of  $R-K_0$  and  $\Omega$  be a relatively compact regular subdomain of  $R$  such that  $\Omega \supset E$  and that  $\partial\Omega$  does not intersect  $K_0$ . Then*

$$(5.19) \quad \sup_{x \in E, y \in R-K_0 \cup \Omega} \frac{N(x, y)}{\omega(y)} < \infty \quad \text{and}$$

$$(5.20) \quad \sup_{x \in E} \int_{R-K_0 \cup \Omega} \left| \nabla_y \frac{N(x, y)}{\omega(y)} \right|^2 d_\omega y < \infty.$$

**PROOF.** We fix a regular compact set  $K$  such that  $E \subset K^\circ \subset K \subset \Omega - K_0$  and a function  $f$  of class  $C^2$  on  $R$  such that  $f(y) = 1$  on  $K$  and  $f(y) = 0$  on  $R - \Omega$ , and put  $N_0(x, y) = N(x, y)f(y)$ . Let  $x$  be any fixed point in  $E$  and consider  $N(x, y)$  and  $N_0(x, y)$  as functions of  $y$ . Then, by the same argument as the proof of Theorem 3.2, we may show that

$$(5.21) \quad \left\| \nabla \frac{L_{K+K_0} N}{\omega} \right\|_{R-K-K_0, \omega} \leq 2 \left\| \nabla \frac{N_0}{\omega} \right\|_{\Omega-K-K_0, \omega} + \left\| [b - \nabla p] \frac{N_0}{\omega} \right\|_{\Omega-K-K_0, \omega}$$

(consider the inequality obtained by putting  $D = D_n$  in (3.8) and let  $n \rightarrow \infty$ ). The right-hand side of (5.21) is bounded by a constant independent of  $x \in E$  since  $E$  and  $\overline{\Omega - K}$  are mutually disjoint and  $L_{K+K_0} N = N$  by Theorem 5.1. Hence (5.21) implies (5.20). Similarly (5.19) is proved as follows:

$$\begin{aligned} \sup_{x \in E, y \in R-K_0 \cup \Omega} \frac{N(x, y)}{\omega(y)} &\leq \sup_{x \in E} \left\{ \sup_{y \in R-K-K_0} \frac{L_{K+K_0} N(x, y)}{\omega(y)} \right\} \\ &\leq \sup_{x \in E} \left\{ \max_{y \in \partial K} \frac{N(x, y)}{\omega(y)} \right\} < \infty. \end{aligned}$$

The following theorem is a generalization of ii) in Theorem 5.1.

**THEOREM 5.2.** *For any fixed  $x \in R - (K_0)^\circ$  and any regular compact set  $K \subset R - K_0$ , it holds that  $L_{K+K_0} N(x, \cdot) \leq N(x, \cdot)$  in  $R - K - K_0$ ; the equality holds if  $x \in K^\circ$ .*

**PROOF.** If  $x \in K^\circ$ , this assertion is implied by Theorem 5.1.

We consider the case:  $x \in R - K$ . Since  $\sup_{y \in R-K-K_0} \frac{L_{K+K_0} N}{\omega} \leq \max_{y \in \partial K} \frac{N}{\omega}$ ,  $L_{K+K_0} N$  is bounded in any compact neighborhood of  $x$ . Therefore  $L_{K+K_0} N \leq N$  on sufficiently small  $K_1 \in K(x)$  as  $\dim R \geq 2$ . Hence, on account of Theorem 4.2 and Theorem 5.1, we get

$$L_{K+K_0} N = L_{K_1+K+K_0}(L_{K+K_0} N) \leq N_{K_1+K+K_0} N = N \text{ in } R - K_0 - K - K_1.$$

Since  $K_1$  can be chosen arbitrarily small and since  $N(x, x) = \infty$ , we obtain  $L_{K+K_0} N \leq N$  in  $R - K_0 - K$ .

Finally we consider the case:  $x \in \partial K$ . We approximate  $N(x, \cdot)$  by an increasing sequence  $\{\varphi_n\}$  of non-negative and continuous functions on  $\partial K + \partial K_0$ . Then  $\lim_{n \rightarrow \infty} L_{K+K_0} \varphi_n = L_{K+K_0} N$  by the definition of the extended regular mapping in § 4. Since  $L_{K+K_0} \varphi_n$  is continuous on  $R - K^\circ - (K_0)^\circ$  and  $L_{K+K_0} \varphi_n = \varphi_n \leq N$  on  $\partial K + \partial K_0$ , there exists a sequence  $\{F_n\}$  of regular compact sets such that  $(F_n)^\circ \supset K + K_0$  and  $L_{K+K_0} \varphi_n \leq N + \frac{1}{n} \omega$  on  $\partial F_n$  for every  $n$  and that  $\bigcap_{n=1}^{\infty} F_n = K + K_0$ . Hence by Theorems 4.2 and 5.1,

$$L_{K+K_0} \varphi_n = L_{F_n}(L_{K+K_0} \varphi_n) \leq L_{F_n} N + \frac{1}{n} L_{F_n} \omega = N + \frac{1}{n} \omega \text{ in } R - F_n$$

since  $x \in \partial K \subset (F_n)^\circ$ . Letting  $n \rightarrow \infty$ , we obtain  $L_K N \leq N$  in  $R - K - K_0$ .

## § 6. A boundary value problem.

In this §, we consider the following boundary value problem with vanishing *normal flux*<sup>5)</sup> on 'the point at infinity'. Given regular compact set  $K_0$ , Hölder-continuous function  $f(x)$  on  $R - (K_0)^\circ$  whose support is a compact subset of  $R - (K_0)^\circ$ , and Hölder continuous function  $\varphi(x)$  on  $\partial K_0$ , find a function  $v(x)$  satisfying:

$$(6.1) \quad A^*v = -f \text{ in } R - K_0,$$

$$(6.2) \quad v|_{\partial K_0} = \varphi \text{ and}$$

$$(6.3) \quad \lim_{D \uparrow R} \int_{\partial D} \left( \frac{\partial v}{\partial \mathbf{n}_D} - \beta_D v \right) \phi dS \text{ for any } \phi \in P_\omega(R; K_0)$$

( $D$  being relatively compact regular domains in  $R$ ).

If we consider a similar boundary value problem in a fixed  $D$ , the condition

$$(6.3') \quad \int_{\partial D} \left( \frac{\partial v}{\partial \mathbf{n}_D} - \beta_D v \right) \phi dS = 0 \text{ for any } \phi \in P_\omega(D; K_0)$$

is equivalent to

$$(6.4) \quad \frac{\partial v}{\partial \mathbf{n}_D} - \beta_D v = 0 \text{ on } \partial D.$$

<sup>5)</sup> The vector field  $\nabla u - bu$  is called *flux* as a terminology in diffusion theory. So we call *normal flux* the normal component  $\frac{\partial u}{\partial \mathbf{n}_D} - \beta_D u$  on  $\partial D$  of the vector field.

Hence it is natural to consider (6.3) to be the condition of vanishing normal flux on the point at infinity.

There is nothing essentially new in this § concerning boundary value problems. The following theorems are only to show the relation between the function  $N(x, y)$  and the boundary value problem (6.1–3).

Denote by  $\mathcal{D}$  the totality of functions  $v \in C^1(R - K_0)$  satisfying that  $\left\| \nabla \frac{v}{\omega} \right\|_{R - K_0, \omega} < \infty$  and  $\sup_{R - K_0} \left| \frac{v}{\omega} \right| < \infty$ . Then we have the following

**THEOREM 6.1.** *Let  $f(x)$  and  $\varphi(x)$  be as stated above, and  $N(x, y)$  be the kernel function defined in §5. Then the function  $v$  defined by*

$$(6.5) \quad v(y) = \int_{R - K_0} f(x) N(x, y) dx + \int_{\partial K_0} \varphi(x) \frac{\partial N(x, y)}{\partial \mathbf{n}_{K_0}(x)} dS(x)$$

*belongs to  $\mathcal{D}$  and satisfies (6.1–3). The function  $v \in \mathcal{D}$  satisfying (6.1–3) is unique.*

Proof of this theorem is essentially contained in the proof of Theorem 5.1. So we only sketch the proof of Theorem 6.1.

Let  $v$  be defined by (6.5). Then it follows from (5.10) and properties of Green functions  $G^{2-K_0}(x, y)$  and  $G^{2_1-K_0}(x, y)$  that  $v$  satisfies (6.1) and (6.2) and that

$$(6.6) \quad v \text{ and } |\nabla v| \text{ are bounded on } K - K_0$$

for any compact subset  $K$  of  $R$ . Furthermore we have

$$(6.7) \quad L_K v = v \text{ in } R - K$$

for any regular compact set  $K$  containing  $K_0$  and the support of  $f$  in its interior  $K^\circ$ , and

$$(6.8) \quad \left( \nabla \frac{v}{\omega} - [\mathbf{b} - \nabla p] \frac{v}{\omega}, \nabla \phi \right)_{R - K_0, \omega} = (f, \phi)_{R - K_0, 1}$$

for any  $\phi \in P_\omega(R; K_0)$ ; proofs of these facts are essentially same as those of (5.11) and (5.12). On account of Theorem 3.2, (6.7) implies that

$$(6.9) \quad \sup_{R - K} \left| \frac{v}{\omega} \right| < \infty \text{ and } \left\| \nabla \frac{v}{\omega} \right\|_{R - K, \omega} < \infty.$$

From (6.6) and (6.9), it follows that  $v \in \mathcal{D}$  and accordingly

$$(6.10) \quad \begin{aligned} \left( \nabla \frac{v}{\omega} - [\mathbf{b} - \nabla p] \frac{v}{\omega}, \nabla \phi \right)_{R - K_0, \omega} &= \lim_{D \uparrow R} \left( \nabla \frac{v}{\omega} - [\mathbf{b} - \nabla p] \frac{v}{\omega}, \nabla \phi \right)_{D - K_0, \omega} \\ &= \lim_{D \uparrow R} \int_{\partial D} \left( \frac{\partial v}{\partial \mathbf{n}_D} - \beta_D v \right) \phi dS + (f, \phi)_{R - K_0, 1} \end{aligned}$$

for any  $\phi \in P_\omega(R; K_0)$ . (6.8) and (6.10) imply (6.3).

Uniqueness of  $v$  is proved as follows. If  $u$  and  $v$  belong to  $\mathcal{D}$  and satisfy (6.1–3), then, applying (6.10) to  $u-v$ , we have

$$\left( \nabla \cdot \frac{u-v}{\omega} - [b \cdot \nabla p] \frac{u-v}{\omega}, \nabla \phi \right)_{R-K_0, \omega} = 0 \quad \text{for any } \phi \in P_\omega(R; K_0).$$

Since  $u-v=\varphi$  on  $\partial K_0$ , we get  $u=v$  in  $R-K_0$  by Theorem 3.2.

From the ‘uniqueness’ part of the above theorem, we obtain the following uniqueness theorem for kernel function of the boundary value problem (6.1–3).

**THEOREM 6.2.** *If  $\tilde{N}(x, y)$  is continuous on*

$$[R-(K_0)^\circ] \times [R-(K_0)^\circ] - \{(z, z); z \in R-(K_0)^\circ\},$$

*and if  $v(y) = \int_{R-K_0} f(x) \tilde{N}(x, y) dx$  satisfies  $A^*v = -f$  in  $R-K_0$ ,  $v|_{\partial K_0} = 0$  and (6.3) for any Hölder-continuous function  $f$  whose support is a compact subset of  $R-(K_0)^\circ$ , then  $\tilde{N}(x, y)$  is identical with  $N(x, y)$  defined in §5.*

### Appendix. Proofs of Lemmas stated in §§ 2 and 3.

In what follows, notations are referred to §2.

**PROOF OF LEMMA 2.1.** By means of Green’s formula, we may show that

$$(1) \quad N^{D-K}(x, y) = G^{D-K}(x, y) - \int_{\partial\Omega} \frac{\partial G^{D-K}(x, z)}{\partial \mathbf{n}_\Omega(z)} N^{D-K}(z, y) dS(z) \\ \text{for } x, y \in \Omega - K^\circ,$$

and that

$$(3) \quad N^{D-K}(z, y) = - \int_{\partial\Omega_1} N^{D-K}(z, z_1) \frac{\partial G^{D_1-K}(z_1, y)}{\partial \mathbf{n}_{\Omega_1}(z_1)} dS(z_1) \\ \text{for } z \in D - \bar{\Omega}_1 \text{ and } y \in \Omega_1 - K^\circ.$$

Substituting the right-hand side of (2) for  $N^{D-K}(z, y)$  in the right-hand side of (1), we obtain (2.4).

To prove Lemmas 2.2 and 2.3, we first show the following

**LEMMA A.** *Let  $\Omega$  be a relatively compact regular domain containing  $K$ . Then*

$$\sup_{D \supset \Omega} \left\{ \sup_{x \in D - \bar{\Omega}} \int_{\partial\Omega} N^{D-K}(x, y) dS(y) \right\} < \infty.$$

**PROOF.** Let  $u$  be the solution of the boundary value problem:  $Au=0$  in  $\Omega-K$ ,  $u|_{\partial K}=1$ ,  $u|_{\partial\Omega}=0$ . Then  $\frac{\partial u}{\partial \mathbf{n}_\Omega} < 0$  on  $\partial\Omega$ ; accordingly

$$(3) \quad \min_{y \in \partial \Omega} \left\{ -\frac{\partial u(y)}{\partial \mathbf{n}_\Omega} \right\} > 0.$$

Since  $A_y^* N^{D-K}(x, y) = 0$  in  $\Omega - K$  for any fixed  $x \in D - \bar{\Omega}$ , we have, by Green's formula

$$(4) \quad \int_{\partial \Omega} N^{D-K}(x, y) \frac{\partial u(y)}{\partial \mathbf{n}_\Omega} dS - \int_{\partial K} \frac{\partial N^{D-K}(x, y)}{\partial \mathbf{n}_K(y)} dS(y) = 0.$$

On the other hand, since the function  $w(x) \equiv 1$  is the solution of the boundary value problem:  $Aw = 0$  in  $D - K$ ,  $w|_{\partial K} = 1$ ,  $\frac{\partial w}{\partial \mathbf{n}_D} \Big|_{\partial D} = 0$ , we have

$$(5) \quad 1 = \int_{\partial K} \frac{\partial N^{D-K}(x, y)}{\partial \mathbf{n}_K(y)} dS(y) \quad \text{for any } x \in D - K.$$

It follows from (3), (4) and (5) that

$$\int_{\partial \Omega} N^{D-K}(x, y) dS(y) \leq \left[ \min_{y \in \partial \Omega} \left\{ -\frac{\partial u(y)}{\partial \mathbf{n}_\Omega} \right\} \right]^{-1} < \infty$$

for any  $D \supset \Omega$  and any  $x \in D - \bar{\Omega}$ .

**PROOF OF LEMMA 2.2.** Let  $\Omega_1$  be a regular domain such that  $K \cup E \cup F \subset \Omega_1 \subset \bar{\Omega}_1 \subset \Omega$ . Then we have (2.4) in Lemma 2.1 for any  $D \supset \bar{\Omega}$ . Since  $N^{D-K}(z, z_1)$  in (2.4) satisfies

$$\sup_{D \supset \bar{\Omega}} \sup_{z \in \partial \Omega} \int_{\partial \Omega_1} N^{D-K}(z, z_1) dS(z_1) < \infty$$

by Lemma A, we may easily derive the conclusion of Lemma 2.2 from (2.4).

**PROOF OF LEMMA 2.3.** We fix two domains  $\Omega_1$  and  $\Omega_2$  such that  $K \cup F \subset \Omega_1 \subset \bar{\Omega}_1 \subset \Omega_2 \subset \bar{\Omega}_2 \subset \Omega$ , and a function  $h$  of class  $C^3$  on  $R$  satisfying that  $h(y) \equiv 1$  for  $y \in \Omega_1$  and  $h(y) = 0$  for  $y \in R - \Omega_2$ . Denote by  $u$  the solution of the boundary value problem:  $A^*u = 0$  in  $\Omega - K$ ,  $u|_{\partial K} = \varphi$ ,  $u|_{\partial \Omega} = 0$ , and put

$$w(y) = \begin{cases} h(y)u(y) & \text{for } y \in \Omega - K \\ 0 & \text{for } y \in R - \Omega. \end{cases}$$

Then  $A^*w$  is of class  $C^1$  on  $R - K$  and the support of  $A^*w$  is contained in  $\Omega - \Omega_1$ . For any  $D \supset \bar{\Omega}$ , the function  $v^D - w$  satisfies:

$$\begin{cases} A^*(v^D - w) = -A^*w \text{ in } D - K, (v^D - w)|_{\partial K} = 0 \text{ and} \\ \left[ \frac{\partial(v^D - w)}{\partial \mathbf{n}_D} - \beta_D(v^D - w) \right] \Big|_{\partial D} = 0. \end{cases}$$

Therefore, by the formula (2.3), we obtain

$$v^p(y) - w(y) = \int_{y - \varrho_1} A^* w(x) \cdot N^{p-K}(x, y) dx \quad \text{for any } y \in \Omega - K.$$

Hence, on account of Lemma 2.1 and Lemma A, we may get the conclusion of Lemma 2.3.

We shall prove Lemmas 3.1 and 3.2 in the case where  $K$  is not empty. The modification of these proofs to the case where  $K$  is empty is quite easy.

PROOF OF LEMMA 3.1. We put  $\omega_n = 0$  and  $\phi_n = 0$  in  $R - \bar{D}_n$ . Then, from (1.5) and the assumption of this lemma, it follows that  $\sup_n \left\| \frac{\omega_n}{\omega} \phi_n \right\|_{R-K, \omega} < \infty$  and accordingly that  $\|\phi\|_{R-K, \omega} < \infty$  by the Lebesgue-Fatou lemma. For any  $\psi \in P_\omega(R; K)$ , it is clear that  $\phi|_{D_n-K} \in P_\omega(D_n; K)$  and also that, for any  $\varepsilon > 0$ , there exists a relatively compact domain  $D \supset K$  such that

$$\left( \sup_n \left\| \frac{\omega_n}{\omega} \phi_n \right\|_{R-D, \omega} + \|\phi\|_{R-D, \omega} \right) \|\nabla \psi\|_{R-D, \omega} < \varepsilon.$$

Hence, from the assumption, we get

$$\begin{aligned} |(\phi, \nabla \psi)_{R-K, \omega}| &= |(\phi, \nabla \psi)_{R-K, \omega} - (\phi_n, \nabla \psi)_{D_n-K, \omega_n}| \\ &\leq \left| \left( \phi - \frac{\omega_n}{\omega} \phi_n, \nabla \psi \right)_{D-K, \omega} \right| + \left( \|\phi\|_{R-D, \omega} + \left\| \frac{\omega_n}{\omega} \phi_n \right\|_{R-D, \omega} \right) \|\nabla \psi\|_{R-D, \omega} \\ &\leq \left\| \phi - \frac{\omega_n}{\omega} \phi_n \right\|_{D-K, \omega} \cdot \|\nabla \psi\|_{R-K, \omega} + \varepsilon \quad \text{whenever } D_n \supset D. \end{aligned}$$

Letting  $n \rightarrow \infty$ , we obtain  $|(\phi, \nabla \psi)_{R-K, \omega}| \leq \varepsilon$ . Hence we get  $(\phi, \nabla \psi)_{R-K, \omega} = 0$  as  $\varepsilon$  is arbitrary.

To prove Lemma 3.2, we shall show that

LEMMA B. Let  $\Psi \in L_\omega^2(R-K)$  and assume that  $\Psi$  is continuous on  $R-K^\circ$  and satisfies  $(\phi, \Psi)_\omega = 0$  for any  $m$ -vector function  $\phi$  of class  $C^1$  on  $R-K^\circ$  such that

(7)  $\operatorname{div}(\omega \phi) = 0$  in  $R-K$  and

(8) the support of  $\phi$  is a compact subset of  $R-K^\circ$ .

Then there exists a function  $\psi \in C^1(R-K^\circ)$  such that  $\phi|_{\partial K} = 0$  and  $\Psi = \nabla \psi$  in  $R-K$ .

PROOF. Let  $C$  be a simple closed oriented curve of class  $C^1$  in  $R-K$  and, for every point  $x \in C$ , let  $t = t(x)$  be the unit tangent vector to  $C$  at  $x$  with the same direction as the orientation of  $C$ . We first show that

$$(9) \quad \int_C (\Psi \cdot t) ds = 0 \quad (ds \text{ denotes the line element on } C)$$

for any such  $C$ . It is sufficient to prove (9) under the assumption that  $C$  is contained in a coordinate neighborhood  $U$  relatively compact in  $R-K$ .

We fix a local coordinate system in  $U$ , and define

$$\Phi_\varepsilon(y) = \int_C \frac{1}{\omega(y)} \rho_\varepsilon(x-y) t(x) ds(x) \quad (\varepsilon > 0)$$

where  $\{\rho_\varepsilon, \varepsilon > 0\}$  is a system of mollifiers which 'tends to Dirac  $\delta$ -function' as  $\varepsilon \rightarrow 0$ . Then

$$\operatorname{div} \{\omega(y) \Phi_\varepsilon(y)\} = - \int_C (\nabla_x \rho_\varepsilon(x-y) \cdot t(x)) ds(x) = - \int_C \frac{\partial \rho_\varepsilon(x-y)}{\partial t(x)} ds(x) = 0.$$

Hence, by the assumption of this lemma, we have

$$\int_U (\Psi(y) \cdot \Phi_\varepsilon(y)) \omega(y) dy = 0,$$

namely

$$\int_C \left( \left[ \int_U \rho_\varepsilon(x-y) \Psi(y) dy \right] \cdot t(x) \right) ds(x) = 0.$$

Letting  $\varepsilon \rightarrow 0$  in this equality, we obtain (9).

We fix a point  $x_0 \in R-K$ . For any point  $x \in R-K^\circ$ , we define

$$(10) \quad \phi(x) = \int_{C_x} (\Psi \cdot t) ds$$

where  $C_x$  is a curve of class  $C^1$  which starts at  $x_0$ , ends at  $x$  and is contained in  $R-K$  except at most the end point  $x$ . On account of (9), the value of  $\phi(x)$  is uniquely determined by the point  $x$  and is independent of such path  $C_x$ . Accordingly  $\phi \in C^1(R-K)$  and  $\Psi = \nabla \phi$  by means of (10). Since  $\phi$  can be replaced by  $\phi - c$ ,  $c$  being any constant, it remains only to prove  $\phi|_{\partial K} = c$  instead of  $\phi|_{\partial K} = 0$ .

Let  $z_1$  and  $z_2$  be arbitrary points on  $\partial K$ , and let  $C$  be a curve of class  $C^1$  which starts at  $z_1$ , ends at  $z_2$  and is contained in  $R-K$  except both terminals  $z_1$  and  $z_2$ . Define  $t$  and  $\Phi_\varepsilon$  similarly to those in the above argument. Then we have  $\int_{R-K^\circ} (\Phi_\varepsilon \cdot \nabla \phi) \omega dy = 0$  and, letting  $\varepsilon \rightarrow 0$ , we get

$$0 = \int_C (t \cdot \nabla \phi) ds = \int_C \frac{\partial \phi}{\partial t} ds = \phi(z_2) - \phi(z_1).^{61}$$

<sup>61</sup> We should not assume that both  $z_1$  and  $z_2$  belong to a common coordinate neighborhood for  $K$  is not necessarily connected. However, we may perform similar arguments to those in preceding paragraphs by using the partition of unity.

Thus we may see that  $\phi$  is constant on  $\partial K$ .

PROOF OF LEMMA 3.2. Let  $H_0$  be the closed linear subspace of the Hilbert space  $L_{\omega^2}(R-K)$  spanned by  $\{\nabla\phi; \phi \in P_{\omega}(R; K)\}$ , and  $H_1$  be the orthogonal complement of  $H_0$ . Then any  $\phi$  satisfying (3.1) clearly belongs to  $H_1$ . If  $\phi$  satisfies (7) and (8) and if  $\phi \in P_{\omega}(R; K)$ , then we have by Green's formula

$$(\phi\phi, \nabla\phi)_{R-K, \omega} = ((\nabla\phi \cdot \phi) + \phi \operatorname{div} [\omega\phi], \phi)_{R-K, \omega} = -(\phi\phi, \nabla\phi)_{R-K, \omega};$$

accordingly  $(\phi\phi, \nabla\phi)_{R-K, \omega} = 0$ . The totality of  $\phi \in H_1$  satisfying (7) and (8) is dense in  $H_1$  by Lemma B. Hence we have  $(\phi\phi, \nabla\phi)_{R-K, \omega} = 0$  for any  $\phi \in H_1$  and any  $\phi \in P_{\omega}(R; K)$  bounded in  $R-K$ .

PROOF OF LEMMA 3.3. We may show by simple computation that

$$\frac{|\nabla\omega_1|^2}{\omega_1} + \frac{|\nabla\omega_2|^2}{\omega_2} - \frac{|\nabla\omega|^2}{\omega} \geq 0 \quad \text{in } R$$

and accordingly

$$\begin{aligned} |\mathbf{b} - \nabla p|^2 \omega &= |\mathbf{b}|^2 \omega - 2(\mathbf{b} \cdot \nabla \omega) + \frac{|\nabla \omega|^2}{\omega} \\ &\leq |\mathbf{b} - \nabla p_1|^2 \omega_1 + |\mathbf{b} - \nabla p_2|^2 \omega_2 \quad \text{in } R \quad (p = \log \omega). \end{aligned}$$

Integrating both sides of this inequality over  $R$ , we get

$$\|\mathbf{b} - \nabla p\|_{R, \omega}^2 \leq \|\mathbf{b} - \nabla p_1\|_{R, \omega_1}^2 + \|\mathbf{b} - \nabla p_2\|_{R, \omega_2}^2 < \infty.$$

For any  $\phi \in P_{\omega}(R)$ , we may show that

$$\begin{aligned} ([\mathbf{b} - \nabla p] \cdot \nabla \phi) \omega &= ([\omega \mathbf{b} - \nabla \omega] \cdot \nabla \phi) \\ &= ([\mathbf{b} - \nabla p_1] \cdot \nabla \phi) \omega_1 + ([\mathbf{b} - \nabla p_2] \cdot \nabla \phi) \omega_2 \quad \text{in } R. \end{aligned}$$

Integrating both sides of this equality over  $R$ , we obtain  $(\mathbf{b} - \nabla p, \nabla \phi)_{R, \omega} = 0$  since  $P_{\omega}(R) \subset P_{\omega_1}(R) \cap P_{\omega_2}(R)$ . Thus we see that  $\omega$  satisfies (B). Next we have, for  $\nu = 1$  and  $2$ ,

$$(11) \quad \left\{ \nabla \frac{\omega_{\nu}}{\omega} - [\mathbf{b} - \nabla p] \frac{\omega_{\nu}}{\omega} \right\} \omega = -[\mathbf{b} - \nabla p_{\nu}] \omega_{\nu} \quad \text{on } R,$$

and accordingly

$$\left| \nabla \frac{\omega_{\nu}}{\omega} \right| \omega^{1/2} \leq |\mathbf{b} - \nabla p| \omega^{1/2} + |\mathbf{b} - \nabla p_{\nu}| \omega_{\nu}^{1/2} \quad \text{on } R.$$

Integrating the squares of both sides over  $R$ , we obtain

$$\left\| \nabla \frac{\omega_{\nu}}{\omega} \right\|_{R, \omega}^2 \leq 2(\|\mathbf{b} - \nabla p\|_{R, \omega}^2 + \|\mathbf{b} - \nabla p_{\nu}\|_{R, \omega_{\nu}}^2) < \infty$$



since  $\omega$  and  $\omega_\nu$  satisfy (B). Thus we get  $\nabla \frac{\omega_\nu}{\omega} \in L_{\omega^2}(R)$ , and accordingly  $\nabla \frac{\omega_\nu}{\omega} - [b - \nabla p] \frac{\omega_\nu}{\omega} \in L_{\omega^2}(R)$ . Hence it follows from (11) that

$$\left( \nabla \frac{\omega_\nu}{\omega} - [b - \nabla p] \frac{\omega_\nu}{\omega}, \nabla \psi \right)_{R, \omega} = -(b - \nabla p, \nabla \psi)_{R, \omega_\nu} = 0$$

for any  $\psi \in P_\omega(R) \subset P_{\omega_\nu}(R)$ . Lemma 3.3 is thus proved.

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(Received August 21, 1969)