

A note on pseudo groups

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(Comm. by N. Iwahori)

Introduction

A study of the character table of a finite group led Brauer in [1] to define and study the notion of a pseudo-group. In this note we continue this investigation.

The motivation for our study will, of course, be derived from the case of the character table of a finite group. Since the motivation for and interpretation of our results will usually be apparent from the case of the character table of a finite group, we will rarely interpret our results in terms of finite group theory.

In §1, we define the scalars of multiplication of a pseudo-group and show how a knowledge of the scalars of multiplication is equivalent to a knowledge of the character table of a pseudo-group. In §2, we define and study Abelian pseudo-groups. These pseudo-groups are seen to be the pseudo-groups of finite Abelian groups. In §3, we define and study the center of a pseudo-group. The concept of normal subgroups of a finite group motivates our study in §4 of normal sets of pseudo-groups. Finally, in §5, we present examples, including two infinite classes, of pseudo-groups which are not derived from finite groups.

As is concluded from [1, §1], a pseudo-group $G = \{\mathfrak{R}, \mathfrak{E}, v, E\}$ consists of:

- a) a non-empty finite set $\mathfrak{R} = \{K_i\}$ of objects K_1, \dots, K_k (called the classes of G),
- b) a non-empty finite set $\mathfrak{E} = \{\chi_j\}$ of functions (called the irreducible characters of G) from \mathfrak{R} into \mathcal{C} (the complex numbers),
- c) a function v from \mathfrak{R} into \mathcal{C} ,
- d) a set $E = \{e_n | n \in \mathbb{Z}\}$ (where \mathbb{Z} denotes the rational integers) of functions $e_n: \mathfrak{R} \rightarrow \mathfrak{R}$ for each $n \in \mathbb{Z}$ (called the exponentiation functions).

We shall often write $K^{[n]}$ for $e_n(K)$ for $n \in \mathbb{Z}$ and $K \in \mathfrak{R}$.

Each finite group H gives rise to such a system (denoted by $G(H)$) where: \mathfrak{R} is the set of conjugacy classes of H , \mathfrak{E} is the set of irreducible characters of H , if $h = \text{card}(H)$ (the cardinality of H) then $v(K) = \frac{1}{h} \text{card}(K)$ for each

* This article contains a part of the doctoral dissertation of the author at Harvard University under the guidance of Professor Richard Brauer. The author wishes to express his gratitude to Professor Brauer for his advice and suggestions.

conjugacy class K of H , and where if $\sigma \in K$ then $e_n(K)$ is the well defined conjugacy class of H containing σ^n for any conjugacy class K of H .

By a character χ of a system G is meant a linear combination of the irreducible characters of G , $\chi = \sum a_j \chi_j$, where $a_j \in \mathbf{Z}$ and $a_j \geq 0$ for all $1 \leq j \leq \text{card}(\mathcal{E})$.

If $\tilde{G} = \{\tilde{\mathfrak{K}}, \tilde{\mathcal{E}}, \tilde{v}, \tilde{E}\}$ is a second such system, a mapping ζ of $\tilde{\mathfrak{K}}$ into \mathfrak{K} will be called an embedding of \tilde{G} into G if:

- (α) $\chi \in \mathcal{E}$ implies $\chi \circ \zeta$ is a character of \tilde{G} ,
- (β) $\zeta \circ \tilde{e}_n = e_n \circ \zeta$ for all $n \in \mathbf{Z}$,
- (γ) $\zeta(\tilde{K}^{[n]}) = \zeta(\tilde{K}^{[0]})$ for some $\tilde{K} \in \tilde{\mathfrak{K}}$ implies $\tilde{K}^{[n]} = \tilde{K}^{[0]}$ for any $n \in \mathbf{Z}$.

If \tilde{H} is a subgroup of H then there is an obvious embedding of the system $G(\tilde{H})$ into the system $G(H)$.

A system $G = \{\mathfrak{K}, \mathcal{E}, v, E\}$ is a pseudo-group if the following axioms are satisfied:

- (I) $\text{card}(\mathfrak{K}) = \text{card}(\mathcal{E})$ (this positive integer is called the class number of G and will be denoted by k) and

$$(1.1) \quad \sum_{K \in \mathfrak{K}} v(K) \chi_i(K) \overline{\chi_j(K)} = \delta_{ij} \quad \text{for all } 1 \leq i, j \leq k,$$

where δ_{ij} denotes the Kronecker delta and where the bar denotes complex conjugation,

- (II) the product of two irreducible characters of G is a character of G ,
- (III) the constant 1 function is an irreducible character of G ,
- (IV) there is a fixed class denoted by 1_G such that $K^{[0]} = 1_G$ for all $K \in \mathfrak{K}$,
- (V) for each class $K \in \mathfrak{K}$, there exists a positive integer m (depending on K) such that if $Z_m = \langle \sigma \rangle$ is a cyclic group of order m then there is an embedding ζ of $G(Z_m)$ into G such that $\zeta(\sigma) = K$.

Clearly m is uniquely determined by K . We call m the order of K .

- (VI) if $K \in \mathfrak{K}$ has order m and if $n \in \mathbf{Z}$ is such that $(m, n) = 1$ then $v(K^{[n]}) = v(K)$.

Clearly if H is a group then $G(H)$ is a pseudo-group.

As in [1, § 1], define the function $c: \mathfrak{K} \rightarrow \mathbf{C}$ by:

$$c(K) = \frac{1}{v(K)} \quad \text{for all } K \in \mathfrak{K}.$$

It is demonstrated in [1, § 1] that $c(K) \in \mathbf{Z}$ and $c(K) > 0$ for all $K \in \mathfrak{K}$.

The order g of the pseudo-group G is defined to be $g = c(1_G)$. If χ is a character of the pseudo-group G then the degree of χ is defined to be $\text{degree } \chi = \chi(1_G)$. As shown in [1, § 1], $\text{degree } \chi$ is a positive rational integer.

The exponent e of the pseudo-group G is defined to be the l.c.m. of the

orders of the classes of G . In what follows, Q will denote the field of rational numbers, Ω will denote the subfield of C obtained by adjoining a primitive e -th root of unity to Q and $\text{Gal}(\Omega/Q)$ will denote the Galois group of Ω/Q . It is clear from [1, § 1] that the values of the irreducible characters of G are actually in Ω .

By the character table of a pseudo-group G is meant the $k \times k$ matrix $(\chi_i(K_\alpha))$ with $1 \leq i, \alpha \leq k$. Two pseudo-groups $G = \{\mathfrak{K}, \Xi, v, E\}$ and $\tilde{G} = \{\tilde{\mathfrak{K}}, \tilde{\Xi}, \tilde{v}, \tilde{E}\}$ are said to have the same character table if there exist bijections $\rho: \tilde{\mathfrak{K}} \rightarrow \mathfrak{K}$ and $t: \tilde{\Xi} \rightarrow \Xi$ such that $t(\tilde{\chi})(\rho(\tilde{K})) = \chi(\tilde{K})$ for all $\tilde{\chi} \in \tilde{\Xi}$ and all $\tilde{K} \in \tilde{\mathfrak{K}}$. It is clear from [1, (1.8)] that if $e_n \circ \rho = \rho \circ \tilde{e}_n$ for all $n \in \mathbf{Z}$ (i.e., if ρ is an embedding) then (ρ, t) is a pseudo-group isomorphism.

As is well known, for any prime p , the two non-isomorphic non-Abelian groups of order p^3 give rise to pseudo-groups which have the same character table. Moreover, in [3], Dade has shown that for any prime $p \geq 5$ there are two non-isomorphic groups (of exponent p) whose pseudo-groups are isomorphic.

Finally, there is a character-class duality in a pseudo-group. That is, in the pseudo-group $G = \{\mathfrak{K}, \Xi, v, E\}$, we may also view the classes as complex valued functions on the irreducible characters by defining $K^*(\chi) = \chi(K)$ for each $K \in \mathfrak{K}$ and $\chi \in \Xi$. This duality will be amplified by means of [6] as we shall soon see.

§ 1. The scalars of multiplication

As suggested by the finite group case:

DEFINITION 1.1. The k^3 complex numbers defined by:

$$a(K_\alpha, K_\beta, K_\gamma) = \frac{g}{c(K_\alpha)c(K_\beta)} \sum_{i=1}^k \frac{\chi_i(K_\alpha)\chi_i(K_\beta)\overline{\chi_i(K_\gamma)}}{\chi_i(1_G)}$$

where $1 \leq \alpha, \beta, \gamma \leq k$ are called the scalars of multiplication of the pseudo-group $G = \{\mathfrak{K}, \Xi, v, E\}$.

LEMMA 1.1. If χ is an irreducible character of G , then so is $\bar{\chi}$ (the complex conjugate of χ).

PROOF. If $\chi_i \in \Xi$, then $\chi_i \bar{\chi} = \sum_{j=1}^k a_j \chi_j$ where $a_j = (\chi_i \bar{\chi}, \chi_j)_G = (\chi_i, \chi \chi_j)_G$. Hence a_j is a non-negative integer. Thus $1 \bar{\chi} = \bar{\chi}$ is a character of G and since $(\bar{\chi}, \bar{\chi})_G = (\chi, \chi)_G = 1$, $\bar{\chi}$ is irreducible.

COROLLARY 1.2. The scalars of multiplication of G are real numbers.

Let S be a ring such that $Z[\{a(K_\alpha, K_\beta, K_\gamma) | 1 \leq \alpha, \beta, \gamma \leq k\}] \subseteq S \subseteq C$. Then we can define an associative and commutative free S -algebra $\mathfrak{U}(S)$ with an identity whose elements are formal linear combinations $\sum_{\alpha=1}^k b_\alpha K_\alpha$ with coefficients $b_\alpha \in S$ and where multiplication in $\mathfrak{U}(S)$ is defined by means of: $K_\alpha K_\beta = \sum_{\gamma=1}^k a(K_\alpha, K_\beta, K_\gamma) K_\gamma$.

For $1 \leq i, \alpha \leq k$, set $\omega_i(K_\alpha) = \frac{gv(K_\alpha)\chi_i(K_\alpha)}{\chi_i(1_G)}$ and extend ω_i by linearity to all of $\mathfrak{U}(S)$. Then $\{\omega_i | 1 \leq i \leq k\}$ is a set of k distinct S -algebra homomorphisms of $\mathfrak{U}(S)$ into C which includes every such non-zero S -algebra homomorphism. Also if $A \in \mathfrak{U}(S)$ is such that $\omega_i(A) = 0$ for all $1 \leq i \leq k$ then $A = 0$. If S is a field then $\mathfrak{U}(S)$ is semi-simple. If $\left\{ \frac{\chi_i(1)}{g} \overline{\chi_i(K_\alpha)} | 1 \leq i, \alpha \leq k \right\} \subseteq S$, then the k -elements $e_i = \sum_{\alpha=1}^k K_\alpha \frac{\chi_i(1_g)}{g} \overline{\chi_i(K_\alpha)}$ are distinct primitive orthogonal idempotents of $\mathfrak{U}(S)$ and $1 = \sum_{i=1}^k e_i$. If the scalars of multiplication of G are algebraic integers and if S is a ring of algebraic integers then the elements $\{K_\alpha | 1 \leq \alpha \leq k\}$ of $\mathfrak{U}(S)$ can be obtained from $\mathfrak{U}(S)$ up to multiples by roots of unity in S (cf. [7, § 2, Theorem C]).

Clearly, if we are given the $k \times k$ character table $(\chi_i(K_\alpha))$ of a pseudo-group G then $\{c(K_\alpha) | 1 \leq \alpha \leq k\}$ is determined and hence the scalars of multiplication of G are determined. Conversely, given the indexed set of the k^3 scalars of multiplication of a pseudo-group G we can determine the $k \times k$ character table of G . To see this, note that $K_\alpha = 1_G$ if and only if $a(K_\alpha, K_\beta, K_\gamma) = \delta_{\beta\gamma}$ and hence the index corresponding to the one class, 1_G , is determined. But

$$a(K_\alpha, K_\beta, 1_G) = \begin{cases} 0 & \text{if } K_\alpha \neq K_\beta^{[-1]} \\ gv(K_\alpha) & \text{if } K_\alpha = K_\beta^{[-1]}. \end{cases}$$

Thus $gv(K_\alpha) = \sum_{\beta=1}^k a(K_\alpha, K_\beta, 1_G)$ and $g = \sum_{\alpha=1}^k gv(K_\alpha)$ are determined. Now the $k \times k$ matrix $(\chi_i(K_\alpha))$ can be determined by the method of [2, § 223].

COROLLARY 1.3. *The following three conditions on a pseudo-group G are equivalent:*

- The scalars of multiplication of G are rational numbers.*
- For any $\sigma \in \text{Gal}(\Omega/K)$, χ^σ is a character of G if χ is a character of G .*
- For any $\sigma \in \text{Gal}(\Omega/K)$, χ^σ is an irreducible character of G if χ is an irreducible character of G .*

Note that no example of a pseudo-group having scalars of multiplication which are not non-negative rational integers seems to be known.

This intimate relationship between a pseudo-group and its scalars of multiplication will be frequently exploited as we proceed. Moreover, we now have reason to suspect the importance of an axiom such as Axiom B of [1, § 5] for pseudo-groups.

Note that $e_{-1}: \mathfrak{R} \rightarrow \mathfrak{R}$ is an involutory permutation of \mathfrak{R} and that the k^3 complex numbers $a(K_\alpha, K_\beta, K_\gamma) = a(e_{-1}(K_\alpha), e_{-1}(K_\beta), e_{-1}(K_\gamma))$ for all $1 \leq \alpha, \beta, \gamma \leq k$. Moreover, as we have seen, $a(K_\alpha, K_\beta, 1_G) = 0$ if $K_\alpha \neq e_{-1}(K_\beta)$ and $a(K_\alpha, K_\beta, 1_G) =$

$gv(k_\alpha) > 0$ if $K_\alpha = e_{-1}(K_\beta)$; also $gv(1_G) = 1$. Thus the k^3 scalars of multiplication of the pseudo-group $G = \{\mathfrak{R}, \varepsilon, v, E\}$ are real numbers which satisfy conditions (a), (a₁), (b), (c) and (c₁) of [5] and, using pseudo-group axiom (III), it is easy to verify that G satisfies conditions (d) and (d₁) of [5]. However, as is demonstrated in [5], if a set of k^3 real numbers $\{a_{\alpha, \beta, \gamma} | 1 \leq \alpha, \beta, \gamma \leq k\}$ is given which satisfies conditions (a), (a₁), (b), (c), (c₁), (d) and (d₁) of [5] then one obtains a $k \times k$ "character table" which satisfies pseudo-group axioms (I) and (III). It would be of interest to determine a set of conditions on a set of k^3 real numbers as above which are satisfied by the scalars of multiplication of a pseudo-group and which imply that the $k \times k$ "character table" determined by these k^3 real numbers is the "character table" of some pseudo-group. One could not expect to have determined a unique pseudo-group because of the existence of non-isomorphic pseudo-groups with the same character table and hence the same scalars of multiplication.

We also can conclude that $\mathfrak{U}(C)$ satisfies the axioms of [6] for a C -algebra. Hence Theorems 1 and 2 of [6] hold for $\mathfrak{U}(C)$ and the character-class duality in a pseudo-group is further emphasized.

§ 2. Abelian pseudo-groups

DEFINITION 2.1. A pseudo-group G is said to be Abelian if all of the irreducible characters of G have degree 1.

LEMMA 2.1. A pseudo-group G is Abelian if and only if $v(K) = \frac{1}{g}$ for all classes K of G .

PROOF. If G is Abelian then $c(K) = \sum_{\alpha=1}^k |\chi_\alpha(K)|^2 = k = \sum_{\alpha=1}^k \chi_\alpha(1_G)^2 = g$ and hence $v(K) = \frac{1}{g}$ for all classes K of G . Conversely if $v(K) = \frac{1}{g}$ for all classes K of G then $1 = \sum_{\alpha=1}^k v(K_\alpha) = k/g$. But $g = \sum_{\alpha=1}^k \chi_\alpha(1_G)^2$ and hence $\chi_\alpha(1_G) = 1$ for all $1 \leq \alpha \leq k$.

THEOREM 2.2. An Abelian pseudo-group G is the pseudo-group of an Abelian group.

PROOF. The irreducible characters of G form an Abelian group H under multiplication. Each class K of G gives rise to a representation K^* of H in C defined by: $K^*(\chi) = \chi(K)$. Thus $\{K^* | K \in \mathfrak{R}\}$ is the set of irreducible representations of H in C . Also $(K_\alpha^* \cdot K_\beta^*)(\chi) = \chi(K_\alpha)\chi(K_\beta) = \chi(K_\alpha \cdot K_\beta) = \sum_{\gamma=1}^k a(K_\alpha, K_\beta, K_\gamma)\chi(K_\gamma)$ and hence if $K_\alpha^* \cdot K_\beta^* = K_\rho^*$ then $a(K_\alpha, K_\beta, K_\gamma) = \delta_{\gamma\rho}$ which implies that $K_\alpha \cdot K_\beta = K_\rho$. Thus the set of classes of G is closed under the multiplication in $\mathfrak{U}(S)$ and since $K_\alpha \cdot K_\alpha^{[-1]} = 1_G$, the theorem follows.

COROLLARY 2.3. A pseudo-group G is Abelian if and only if $a(K_\alpha, K_\beta, K_\gamma)$

is either 0 or 1 for all $1 \leq \alpha, \beta, \gamma \leq k$.

PROOF. If G is Abelian then we see, as above, that $a(K_\alpha, K_\beta, K_\gamma) = \delta_{\gamma\rho}$. Conversely, since

$$a(K_\alpha, K_\beta, 1_G) = \begin{cases} 0 & \text{if } K_\beta \neq K_\alpha^{[-1]} \\ gv(K_\alpha) & \text{if } K_\beta = K_\alpha^{[-1]} \end{cases}$$

we have $v(K_\alpha) = \frac{1}{g}$ for all classes K_α and hence G is Abelian.

LEMMA 2.4. In the pseudo-group G , if χ is an irreducible character of G of degree greater than 1, then there exists a class K of G such that $\chi(K) = 0$.

PROOF. Assume that the lemma is false. On the set \mathfrak{R} of classes of G define an equivalence relation \sim by: $K_\alpha \sim K_\beta$ if there exists an $n \in \mathbb{Z}$ with $(n, e) = 1$ such that $K_\beta = K_\alpha^{[n]}$. Pseudo-group axiom VI implies that v is constant on the \sim equivalence classes $\{A_1, \dots, A_s\}$ where we assume that $A_1 = \{1_G\}$. Also $\sum_{K \in A_i} |\chi(K)|^2$ is a positive real, an algebraic integer and a rational number since it is invariant under $\text{Gal}(\Omega/K)$. Hence $\sum_{K \in A_i} |\chi(K)|^2$ is a positive rational integer. But if $\alpha_i = \text{card } A_i$, then $\frac{\sum_{K \in A_i} |\chi(K)|^2}{\alpha_i} \geq (\prod_{K \in A_i} |\chi(K)|^2)^{\frac{1}{\alpha_i}}$. However $\prod_{K \in A_i} |\chi(K)|^2$ is also a positive rational integer. Thus $\sum_{K \in A_i} |\chi(K)|^2 \geq \alpha_i$. Hence $1 = \frac{1}{g} \chi(1_G)^2 + \sum_{i=2}^s \sum_{K \in A_i} v(K) |\chi(K)|^2 \geq \frac{1}{g} \chi(1_G)^2 + \sum_{i=2}^s \sum_{K \in A_i} v(K) = \frac{1}{g} \chi(1_G)^2 + 1 - \frac{1}{g}$ since v is constant on each A_i . Thus $1 \geq \chi(1_G)^2 - \frac{1}{g}$ contradiction.

COROLLARY 2.5. Every pseudo-group G of prime order $g = p$ comes from a cyclic group (of order p).

PROOF. If χ is an irreducible character of G , then $\chi(1_G) \leq p-1$. If, moreover, $\chi(1_G) > 1$, then $\chi(K) = 0$ for some class K of G . But $\chi(K)$ is a sum of $\chi(1_G)$ p -th roots of unity and this is impossible since $\chi(1_G) \leq p-1$.

§ 3. The center

DEFINITION 3.1. $Z(G) = \{K \in \mathfrak{R} | c(K) = g\}$ is called the center of the pseudo-group G .

Clearly $1_G \in Z(G)$ and Lemma 2.1 states that G is Abelian if and only if $Z(G) = \mathfrak{R}$.

LEMMA 3.1. If $K \in \mathfrak{R}$ then the following three conditions are equivalent:

- $K \in Z(G)$.
- $|\chi(K)| = \chi(1_G)$ for all $\chi \in \Xi$.
- If K has order r and if $\chi \in \Xi$ then $\chi(K) = \chi(1_G)\eta$ where η is an r -th root of

unity.

PROOF. Clearly b) and c) are equivalent. Also $c(K) = \sum_{j=1}^k |\chi_j(K)|^2 \leq \sum_{j=1}^k \chi_j(1_G)^2 = g$. Hence a) and b) are equivalent.

COROLLARY 3.2. 1) If $K \in Z(G)$, $\chi \in \Xi$ and $\chi(K) = \chi(1_G)\eta$ where η is a root of unity then $\chi(K^{[i]}) = \chi(1_G)\eta^i$ and hence $K^{[i]} \in Z(G)$ for all $i \in Z$.

2) If $K \in Z(G)$ and $r, s \in Z$ then $K_a^{[r]} \cdot K_a^{[s]} = K_a^{[r+s]}$ where the multiplication takes place in $\mathfrak{U}(S)$.

The center of a pseudo-group is "well behaved" with regard to subpseudo-groups as can be seen in the following two theorems.

THEOREM 3.3. If $G = \{\mathfrak{K} = \{K_a\}, \Xi = \{\chi_i\}, v, E = \{e_n\}\}$ is a pseudo-group with subpseudo-group $\{\tilde{G}, \zeta\}$ where $\tilde{G} = \{\tilde{\mathfrak{K}} = \{\tilde{K}_a\}, \tilde{\Xi} = \{\tilde{\chi}_i\}, \tilde{v}, \tilde{E} = \{\tilde{e}_n\}\}$ and ζ is an embedding of \tilde{G} into G and if $K \in (\text{Image } \zeta) \cap Z(G)$ then there exists a unique $\tilde{K} \in Z(\tilde{G})$ such that $\zeta(\tilde{K}) = K$.

PROOF. $\zeta(\tilde{K}) = K$, let $\tilde{\chi}$ be an arbitrary irreducible character of \tilde{G} and let χ be an irreducible character of G such that $(\tilde{\chi}, \chi)_G = a \neq 0$ (cf. [1, § 2] for the definition of $\tilde{\chi}$). Hence $\chi \circ \zeta = a\tilde{\chi} + \tilde{X}$ where \tilde{X} is a character of \tilde{G} (possibly zero) and $\chi(K) = a\tilde{\chi}(\tilde{K}) + \tilde{X}(\tilde{K}) = \chi(1_G)\varepsilon$ where ε is a root of unity. Thus $|\chi(K)| = \chi(1_G) \leq a|\tilde{\chi}(\tilde{K})| + |\tilde{X}(\tilde{K})| \leq a|\tilde{\chi}(1_{\tilde{G}})| + |\tilde{X}(1_{\tilde{G}})| = \chi(1_G)$. This implies that $\tilde{\chi}(\tilde{K}) = \tilde{\chi}(1_{\tilde{G}})\varepsilon$ and hence $\tilde{K} \in Z(\tilde{G})$ and \tilde{K} is the unique class of \tilde{G} such that $\zeta(\tilde{K}) = K$.

THEOREM 3.4. Let G, \tilde{G}, ζ be as in the theorem above. If for each irreducible character χ of G and for each irreducible character $\tilde{\chi}$ of \tilde{G} we have:

$$(*) \quad (\tilde{\chi}, \chi)_G = (\tilde{\chi}, \chi \circ \zeta)_{\tilde{G}} = 0 \quad \text{or} \quad \chi(1_G),$$

then $(\text{Image } \zeta) \subseteq Z(G)$ and \tilde{G} is Abelian. Conversely, if $(\text{Image } \zeta) \subseteq Z(G)$, then (*) holds for all irreducible characters χ of G and all irreducible characters $\tilde{\chi}$ of \tilde{G} . In either case, ζ is one to one.

PROOF. Let $\tilde{\chi}$ be an arbitrary irreducible character of \tilde{G} and choose an irreducible character χ of G such that $(\tilde{\chi}, \chi)_G = (\chi, \chi \circ \zeta)_{\tilde{G}} \neq 0$. Assuming (*), we have $\chi \circ \zeta = \chi(1_G)\tilde{\chi}$, which implies that $\tilde{\chi}(1_{\tilde{G}}) = 1$ and thus \tilde{G} is Abelian. Moreover, for any class \tilde{K} of \tilde{G} , we have $|\chi(\zeta(\tilde{K}))| = \chi(1_G)$ for any irreducible character χ of G . Hence $(\text{Image } \zeta) \subseteq Z(G)$. Conversely, assume that $(\text{Image } \zeta) \subseteq Z(G)$. Thus \tilde{G} is Abelian by Theorem 3.3. Also $|\chi(\zeta(\tilde{K}))| = \chi(1_G)$ for all $\tilde{K} \in \tilde{\mathfrak{K}}$ and all $\chi \in \Xi$. If $\chi \circ \zeta = \sum_{j=1}^k a_j \tilde{\chi}_j$ where $a_j = (\chi \circ \zeta, \tilde{\chi}_j)_G$ is a non-negative rational integer then $\chi(1_G) = |\chi(\zeta(\tilde{K}))| = |\sum_{j=1}^k a_j \tilde{\chi}_j(\tilde{K})| \leq \sum_{j=1}^k a_j |\tilde{\chi}_j(\tilde{K})| \leq \sum_{j=1}^k a_j \tilde{\chi}_j(1_{\tilde{G}}) = \chi(1_G)$ for any $\tilde{K} \in \tilde{\mathfrak{K}}$. Hence only one $a_j \neq 0$, say a_{j_0} , and then $\chi(1_G) = a_{j_0} \tilde{\chi}_{j_0}(1_{\tilde{G}}) = a_{j_0}$ — completing the proof.

We conclude this section with a reply to the question: when does a pseudo-

group G have an Abelian subpseudo-group $\{\tilde{G}, \zeta\}$ such that $(\text{Image } \zeta) = Z(G)$?

LEMMA 3.5. *Let the scalars of multiplication of G be such that $a(K_\alpha, K_\beta, K_\tau) \geq 0$ whenever $K_\beta \in Z(G)$. Then: for any $K_\beta \in Z(G)$ and any $K_\alpha \in \mathfrak{K}$, we have $K_\alpha \cdot K_\beta$ (multiplication in $\mathfrak{U}(S)$) is a positive real multiple of a single class of G .*

PROOF. Since $K_\alpha K_\beta = \sum_{\tau=1}^k a(K_\alpha, K_\beta, K_\tau) K_\tau$, we have $K_\alpha = K_\alpha (K_\beta K_\beta^{[-1]}) = (K_\alpha K_\beta) K_\beta^{[-1]}$. Hence $K_\beta^{[-1]} = \sum_{\tau=1}^k a(K_\alpha, K_\beta, K_\tau) K_\tau K_\beta^{[-1]} = \sum_{\tau=1}^k \sum_{\rho=1}^k a(K_\alpha, K_\beta, K_\tau) a(K_\tau, K_\beta^{[-1]}, K_\rho) K_\rho$. Hence $\sum_{\tau=1}^k a(K_\alpha, K_\beta, K_\tau) a(K_\tau, K_\beta^{[-1]}, K_\rho) = \delta_{\alpha\rho}$. Choose τ_1 such that $a(K_\alpha, K_\beta, K_{\tau_1}) > 0$ then $a(K_{\tau_1}, K_\beta^{[-1]}, K_\rho) = 0$ for all $\rho \neq \alpha$. Hence $K_{\tau_1} K_\beta^{[-1]} = a(K_{\tau_1}, K_\beta^{[-1]}, K_\alpha) K_\alpha$ and $K_{\tau_1} = a(K_{\tau_1}, K_\beta^{[-1]}, K_\alpha) K_\alpha K_\beta$ since $K_\beta \cdot K_\beta^{[-1]} = 1_G$. Consequently $a(K_\alpha, K_\beta, K_\tau) \neq 0$ if and only if $\tau = \tau_1$.

Here we again point out that all known pseudo-groups have non-negative rational integer scalars of multiplication.

THEOREM 3.6. *If $K_\alpha \in Z(G)$ and $K_\beta \in \mathfrak{K}$ imply that $K_\alpha \cdot K_\beta$ (multiplication in $\mathfrak{U}(S)$) is a real multiple of a single class then:*

- Under the multiplication in $\mathfrak{U}(S)$, $Z(G)$ is an Abelian group (denoted by $A(Z(G))$).*
- The restriction of each ω_i to $Z(G)$ is an irreducible character of $A(Z(G))$.*
- Each irreducible character of $A(Z(G))$ is thereby obtained.*
- If $G(A(Z(G)))$ denotes the pseudo-group of $Z(G)$ then $\{G(A(Z(G))), \zeta\}$ is a subpseudo-group of G where ζ is the obvious imbedding.*

PROOF. If $K_\alpha, K_\beta \in Z(G)$ and $K_\alpha K_\beta = a(K_\alpha, K_\beta, K_\tau) K_\tau$ then $g^2 v(K_\alpha) v(K_\beta) = a(K_\alpha, K_\beta, K_\tau) g v(K_\tau) = 1$. Also $K_\alpha = (K_\alpha K_\beta) K_\beta^{[-1]} = a(K_\alpha, K_\beta, K_\tau) a(K_\tau, K_\beta^{[-1]}, K_\rho) K_\rho$ for some ρ ; hence $\rho = \alpha$ and $a(K_\alpha, K_\beta, K_\tau) a(K_\tau, K_\beta^{[-1]}, K_\alpha) = 1$. Thus $g v(K_\tau) = a(K_\tau, K_\beta^{[-1]}, K_\alpha)$; but $a(K_\tau, K_\beta^{[-1]}, K_\alpha) = a(K_\alpha, K_\beta, K_\tau)$ by Lemma 1.1 and hence $g^2 v(K_\tau)^2 = 1$. Therefore $K_\tau \in Z(G)$ and $K_\alpha K_\beta = K_\tau$. Now a), b) and d) readily follow. To prove c), let ψ be any irreducible character of $A(Z(G))$ and let $\chi_i \in \mathfrak{E}$ be such that $(\psi^i, \chi_i)_G \neq 0$. Then $(\psi^i, \chi_i \circ \zeta)_{G(A(Z(G)))} = \chi_i(1_G)$ which implies that $\psi = \omega_i$, completing the proof.

COROLLARY 3.7. *If two groups give rise to pseudo-groups which have the same character table then the centers of the two groups are isomorphic groups.*

As the final result of this section we give:

THEOREM 3.8. *If G, \tilde{G} are two pseudo-groups which satisfy the hypotheses of the last theorem and if $\{\tilde{G}, \zeta\}$ is a subpseudo-group of G then if $K_\alpha, K_\beta \in (\text{Image } \zeta) \cap Z(G)$ and if $\zeta(\tilde{K}_\alpha) = K_\alpha$, $\zeta(\tilde{K}_\beta) = K_\beta$ for (unique) classes $\tilde{K}_\alpha, \tilde{K}_\beta$ of \tilde{G} then $\tilde{K}_\alpha, \tilde{K}_\beta \in Z(\tilde{G})$, $\tilde{K}_\alpha \cdot \tilde{K}_\beta$ is a class in $Z(\tilde{G})$ and $\zeta(\tilde{K}_\alpha \cdot \tilde{K}_\beta) = K_\alpha \cdot K_\beta$. Moreover, $\zeta^{-1}(\zeta(\tilde{K}) \cap Z(G))$ forms a subgroup of $A(Z(G))$ which is isomorphic under ζ to the*

subgroup $(\text{Image } \zeta) \cap Z(G)$ of $Z(G)$.

PROOF. By Theorem 3.3, $\tilde{K}_\alpha, \tilde{K}_\beta \in Z(G)$ and are unique and hence $\tilde{K}_\alpha \cdot \tilde{K}_\beta$ is a class in $Z(\tilde{G})$. If $\chi \in \Xi$ and $\tilde{\chi} \in \tilde{\Xi}$ are such that $(\chi, \tilde{\chi})_G \neq 0$ and if $\chi(\zeta(\tilde{K}_\alpha)) = \chi(K_\alpha) = \chi(1_G)\varepsilon_\alpha$ and $\chi(\zeta(\tilde{K}_\beta)) = \chi(K_\beta) = \chi(1_G)\varepsilon_\beta$, then $\tilde{\chi}(\tilde{K}_\alpha) = \tilde{\chi}(1_G)\varepsilon_\alpha$ and $\tilde{\chi}(\tilde{K}_\beta) = \tilde{\chi}(1_G)\varepsilon_\beta$. Hence $\chi(\tilde{K}_\alpha \cdot \tilde{K}_\beta) = \tilde{\chi}(1_G)\varepsilon_\alpha\varepsilon_\beta$ which implies that $\chi(\zeta(\tilde{K}_\alpha \cdot \tilde{K}_\beta)) = \chi(1_G)\varepsilon_\alpha\varepsilon_\beta = \chi(K_\alpha \cdot K_\beta)$. Since $\chi \in \Xi$ is arbitrary, $\zeta(\tilde{K}_\alpha \cdot \tilde{K}_\beta) = K_\alpha \cdot K_\beta$ and the rest follows immediately.

§ 4. Normal sets

Various definitions for "normality" in pseudo-groups are suggested by the finite group case. In this section we investigate some of these possibilities.

DEFINITION 4.1. If $\mathfrak{M} \subseteq \mathfrak{K}$, set $\mathfrak{M}^\perp = \{\chi \in \Xi \mid \chi(K) = \chi(1_G) \text{ for all } K \in \mathfrak{M}\}$ and if $\Phi \subseteq \Xi$ set $\Phi^\perp = \{K \in \mathfrak{K} \mid \chi(K) = \chi(1_G) \text{ for all } \chi \in \Phi\}$.

Clearly $\Phi \subseteq \Phi^{\perp\perp}$, $\mathfrak{M} \subseteq \mathfrak{M}^{\perp\perp}$, $\mathfrak{M}^\perp = \mathfrak{M}^{\perp\perp\perp}$ and $\Phi^\perp = \Phi^{\perp\perp\perp}$.

THEOREM 4.1. If $\Phi \subseteq \Xi$ then:

- If ψ is a finite product of characters in Φ and if $\chi \in \Xi$ is such that $(\psi, \chi)_G \neq 0$ then $\chi \in \Phi^{\perp\perp}$.
- If $\chi \in \Phi^{\perp\perp}$ then there is a character ψ which is a finite product of irreducible characters in Φ such that $(\psi, \chi)_G \neq 0$.

PROOF. If ψ is a product of characters in Φ then ψ is a character and $\psi = \sum_{i=1}^k a_i \chi_i$ where all $a_i \geq 0$. If $K \in \Phi^\perp$ then $\psi(K) = \psi(1_G) = \sum_{i=1}^k a_i \chi_i(K) = \sum_{i=1}^k a_i \chi_i(1_G)$.

Hence $\psi(1_G) = |\sum_{i=1}^k a_i \chi_i(K)| \leq \sum_{i=1}^k a_i |\chi_i(K)| \leq \sum_{i=1}^k a_i \chi_i(1_G) = \psi(1_G)$; this implies that if $a_i \neq 0$ then $\chi_i(K) = \chi_i(1_G)$, whence $\chi_i \in \Phi^{\perp\perp}$. To prove b), let $\chi \in \Phi^{\perp\perp}$ and set $\psi = \sum_{\chi_i \in \Phi} \chi_i$. It suffices to show that there is a non-negative rational integer n such that $(\psi^n, \chi)_G \neq 0$. Thus we assume that $(\psi^n, \chi)_G = 0$ for $n = 0, 1, 2, \dots$; that is, $\sum_{\alpha=1}^k v(K_\alpha) (\psi(K_\alpha))^n \overline{\chi(K_\alpha)} = 0$ for $n = 0, 1, 2, \dots$. Define an equivalence relation \sim on \mathfrak{K} by: $K_\alpha \sim K_\beta$ if and only if $\psi(K_\alpha) = \psi(K_\beta)$. If \tilde{K} denotes the \sim equivalence class containing K and if $\{\tilde{K}_1, \dots, \tilde{K}_r\}$ denotes the set of \sim equivalence classes then $\sum_{j=1}^k (\psi(K_{\alpha_j}))^n (\sum_{K \in \tilde{K}_{\alpha_j}} v(K) \overline{\chi(K)}) = 0$ for $n = 0, 1, 2, \dots$. View the first r of these equations as a set of linear homogeneous equations with coefficient matrix $(\psi(K_{\alpha_j}))^n$ where $0 \leq n \leq r-1$, $1 \leq j \leq r$. Since the determinant of this matrix is a Vandermonde determinant and so is non-zero, we have $\sum_{K \in \tilde{K}_{\alpha_j}} v(K) \overline{\chi(K)} = 0$ for $1 \leq j \leq r$.

Since $K \in \Phi^\perp$ if and only if $K \sim 1_G$, $\sum_{K \in 1_G} v(K) \chi(1_G) = 0$, which is a contradiction.

COROLLARY 4.2. If Φ is a non-empty subset of Ξ containing the one character, then the following two conditions are equivalent:

a) $\phi^{\perp\perp} = \phi$.

b) If $\chi_\alpha, \chi_\beta \in \phi$ and if $\chi \in \Xi$ is such that $(\chi_\alpha \chi_\beta, \chi) \neq 0$ then $\chi \in \phi$.

The fact that condition b) implies condition a) is related to the second conclusion of [6, Theorem 3]. Note that this theorem requires the hypotheses that ϕ generates a characteristic subalgebra of the conjugate algebra $\overline{\mathfrak{U}(C)}$ of $\mathfrak{U}(C)$ (the conjugate algebra of a C -algebra is defined in § 2) and that the scalars of multiplication of both the algebras $\mathfrak{U}(C)$ and $\overline{\mathfrak{U}(C)}$ are non-negative. (We are however certain that the scalars of multiplication of $\overline{\mathfrak{U}(C)}$ are non-negative by pseudo-group Axiom II).

DEFINITION 4.2. A non-empty subset of Ξ satisfying one and hence both conditions of Corollary 4.2 is called a normal set of irreducible characters.

Obviously, the set of all irreducible characters of degree one of a pseudo-group form a normal set.

Now we proceed to investigate the character-class duals of these results.

THEOREM 4.3. Assume that \mathfrak{M} is a non-empty set of classes of G . If $K_\alpha \in \mathfrak{M}^{\perp\perp}$ then there is in $\mathfrak{U}(S)$ a finite product of classes in \mathfrak{M} in which K_α appears with a non-zero coefficient. If the scalars of multiplication of G are non-negative then the converse also holds.

PROOF. Let $\mathfrak{M} = \{K_{\alpha_1}, \dots, K_{\alpha_r}\}$ and distribute $\{\omega_1, \dots, \omega_k\}$ into subsets X_1, \dots, X_s such that two ω 's belong to the same subset if and only if they take the same value on each element of \mathfrak{M} . Choose a representative $\omega_i \in X_i$ for $1 \leq i \leq s$. By renumbering if necessary, we may assume that $\mathfrak{M}^\perp = X_1$ and $\omega_1 = gv$. Choose positive integers u_1, \dots, u_r such that $\sum_{i=1}^r u_i (\omega_\alpha(K_i) - \omega_\beta(K_i)) \neq 0$ for all $\alpha \neq \beta$, $1 \leq \alpha, \beta \leq s$. Thus if $h = \sum_{i=1}^r u_i K_i$ then $\omega_\alpha(h) = \omega_\beta(h)$ if and only if ω_α and ω_β belong to the same X_i . Suppose that K_j does not appear in h^0, h^1, h^2, \dots with non-zero coefficient and let $h^n = \sum_{j \neq j'} b_{nj} K_j$ for $n = 0, 1, 2, \dots$. Since $\omega_i(h)^n = \omega_i(h^n) = \sum_{j \neq j'} b_{nj} g v(K_j) \frac{\chi_i(K_j)}{\chi_i(1_G)}$, we have $\sum_{i=1}^k \omega_i(h^n) \overline{\chi_i(K_j)} \chi_i(1_G) = 0$. Then as in the proof of Theorem 4.1 part b, we obtain $\sum_{\chi \in X_j} \overline{\chi(K_j)} \chi(1_G) = 0$ for $1 \leq j \leq s$. Since $X_1 = \mathfrak{M}^\perp$ we have $K_j \in \mathfrak{M}^{\perp\perp}$, a contradiction. For the second part of the theorem, assume that the scalars of multiplication of G are non-negative and observe that it suffices to consider a product of just two classes in \mathfrak{M} . So, let $K_\alpha, K_\beta \in \mathfrak{M}$ and $\chi_i \in \mathfrak{M}^\perp$ and apply ω_i to $K_\alpha K_\beta = \sum_{\tau=1}^k a(K_\alpha, K_\beta, K_\tau) K_\tau$ to obtain $g^2 v(K_\alpha) v(K_\beta) = \sum_{\tau=1}^k a(K_\alpha, K_\beta, K_\tau) g v(K_\tau) \frac{\chi_i(K_\tau)}{\chi_i(1_G)}$. Hence $g^2 v(K_\alpha) v(K_\beta) = \left| \sum_{\tau=1}^k a(K_\alpha, K_\beta, K_\tau) g v(K_\tau) \frac{\chi_i(K_\tau)}{\chi_i(1_G)} \right| \leq \sum_{\tau=1}^k a(K_\alpha, K_\beta, K_\tau) g v(K_\tau) \left| \frac{\chi_i(K_\tau)}{\chi_i(1_G)} \right| \leq \sum_{\tau=1}^k a(K_\alpha, K_\beta, K_\tau) g v(K_\tau) = g^2 v(K_\alpha) v(K_\beta)$. Thus

$a(K_\alpha, K_\beta, K_\gamma) \neq 0$ implies $\frac{\chi_i(K_\alpha)}{\chi_i(1_G)} = 1$ and so $K_\gamma \in \mathcal{M}^\perp$.

DEFINITION 4.3. A non-empty subset $\mathcal{M} \subseteq \mathcal{R}$ is said to be *R-normal* if $\mathcal{M}^\perp = \mathcal{M}$ and is said to be *S-normal* if $K_\alpha, K_\beta \in \mathcal{M}$ and $a(K_\alpha, K_\beta, K_\gamma) \neq 0$ imply $K_\gamma \in \mathcal{M}$.

COROLLARY 4.4. Any *S-normal* set of classes is *R-normal*. If the scalars of multiplication of G are non-negative then any *R-normal* set of classes of G is *S-normal*.

The dual of the remark following Corollary 4.2 can be applied to the first conclusion of Corollary 4.4.

Lemma 3.5 and Theorem 3.6 imply that if the scalars of multiplication of a pseudo-group G are non-negative then $Z(G)$ is *R-normal*.

Forming a quotient group with regard to a normal subgroup suggests:

DEFINITION 4.4. An *R-normal* subset \mathcal{M} of \mathcal{R} is said to be *T-normal* if there is a pseudo-group $\hat{G} = \{\hat{\mathcal{R}}, \hat{\mathcal{E}}, \hat{\nu}, \hat{E}\}$ and a mapping λ of \mathcal{R} onto $\hat{\mathcal{R}}$ such that:

- a) if $\chi \in \mathcal{M}^\perp$ then $\chi \cdot \lambda^{-1}$ is well defined on $\hat{\mathcal{R}}$ and is an irreducible character of \hat{G} .
- b) $\{\chi \cdot \lambda^{-1} | \chi \in \mathcal{M}^\perp\}$ is the set $\hat{\mathcal{E}}$ of the irreducible characters of G .

In this case, $\{\hat{G}, \lambda\}$ is called a *T-normal quotient* of G modulo \mathcal{M} . If, moreover, λ is such that $e_n \cdot \lambda = \lambda \cdot e_n$ for all $n \in Z$ then \mathcal{M} is said to be a *U-normal* subset of \mathcal{R} and $\{\hat{G}, \lambda\}$ is called a *U-normal quotient* of G modulo \mathcal{M} .

In §5, we shall exhibit a pseudo-group with an *S* and *T-normal* subset which is not *U-normal*.

It is straightforward to prove:

THEOREM 4.5. If $\{\hat{G}, \lambda\}$ is a *T-normal quotient* of G modulo \mathcal{M} and if $\{\hat{G}, \rho\}$ is a *T-normal quotient* of \hat{G} modulo $\hat{\mathcal{M}}$ then $\{\hat{G}, \rho \circ \lambda\}$ is a *T-normal quotient* of G modulo $\{K \in \mathcal{R} | \lambda(K) \in \hat{\mathcal{M}}\}$. Similarly for *U-normal* quotients.

Assume that $\{\hat{G}, \lambda\}$ is a *T-normal quotient* of G modulo \mathcal{M} and define an equivalence relation \sim on \mathcal{R} by: $K_\alpha \sim K_\beta$ if $\chi(K_\alpha) = \chi(K_\beta)$ for all $\chi \in \mathcal{M}^\perp$. Clearly $\lambda(K_\alpha) = \lambda(K_\beta)$ if and only if $K_\alpha \sim K_\beta$. Let \tilde{K} denote the \sim equivalence class containing K and let \mathcal{B} denote the set of \sim equivalence classes. For $\tilde{K} \in \mathcal{B}$, set $\lambda(\tilde{K}) = \lambda(K)$ and if $\chi \in \mathcal{M}^\perp$, set $\chi(\tilde{K}) = \chi(K)$. Clearly $\gamma: \mathcal{B} \rightarrow \hat{\mathcal{R}}$ defined by: $\gamma(\tilde{K}) = \lambda(\tilde{K})$ and $\rho: \mathcal{M}^\perp \rightarrow \hat{\mathcal{E}}$ defined by $\rho(\chi) = \chi \circ \lambda^{-1}$ are both bijections and are such that $\chi(\tilde{K}) = \chi(\lambda^{-1}(\lambda(K))) = \rho(\chi)(\gamma(\tilde{K}))$ for all $\chi \in \mathcal{M}^\perp$ and $\tilde{K} \in \mathcal{B}$. Set:

$$(**) \quad t(\tilde{K}) = \sum_{K \in \tilde{K}} v(K) \quad \text{for } \tilde{K} \in \mathcal{B}.$$

Then if $\chi_i, \chi_j \in \mathcal{M}^\perp$ we have

$$\sum_{\tilde{K} \in \mathcal{B}} t(\tilde{K}) \chi_i(\tilde{K}) \overline{\chi_j(\tilde{K})} = \delta_{ij}.$$

Since $\text{card}(\mathfrak{B}) = \text{card}(\mathfrak{M}^\perp)$, we may apply the method of [1] used to obtain (1.8) of [1] from (1.5) of [1]; this yields:

$$\sum_{\chi \in \mathfrak{M}^\perp} \chi(\tilde{K}_\alpha) \chi(\tilde{K}_\beta) = \begin{cases} 0 & \text{if } \tilde{K}_\alpha \neq \tilde{K}_\beta \\ \frac{1}{t(\tilde{K}_\alpha)} & \text{if } \tilde{K}_\alpha = \tilde{K}_\beta. \end{cases}$$

Thus $\hat{c}(\gamma(\tilde{K}_\alpha)) = \frac{1}{t(\tilde{K}_\alpha)}$ and $\hat{v}(\gamma(\tilde{K}_\alpha)) = t(\tilde{K}_\alpha)$ for all $\tilde{K}_\alpha \in \mathfrak{B}$ and $\tilde{g} = \frac{1}{t(1_G)}$ since $\tilde{1}_G \in \mathfrak{M}$. Consequently the function \hat{v} of \tilde{G} has been obtained from G . Setting $h_n(\tilde{K}) = \lambda^{-1}(\hat{e}_n(\lambda(\tilde{K})))$ for all $n \in Z$ and all $\tilde{K} \in \mathfrak{B}$ then $\{\mathfrak{B}, \mathfrak{M}^\perp, t, \{h_n\}\} = H$ is a pseudo-group isomorphic to \tilde{G} via γ and ρ and if $\tilde{\lambda}: \mathfrak{R} \rightarrow \mathfrak{B}$ is defined by $\tilde{\lambda}(K) = \tilde{K}$ then $\{H, \tilde{\lambda}\}$ is a T -normal quotient of G modulo \mathfrak{M} . Hence the pseudo-groups of any two T -normal quotients modulo \mathfrak{M} have the same character table.

If $\{\tilde{G}, \lambda\}$ is a U -normal quotient of G modulo \mathfrak{M} then $h_n(\tilde{K}) = \widehat{e}_n(\tilde{K})$ for $n \in Z$ and the exponentiation in H is also obtained from that of G and therefore the pseudo-groups of any two U -normal quotients of G modulo \mathfrak{M} are isomorphic.

On the other hand, the question arises: when does a given R -normal subset \mathfrak{M} of a pseudo-group G have a T - or U -normal quotient modulo \mathfrak{M} ?

LEMMA 4.6. *If \mathfrak{M} is any subset of the set \mathfrak{R} of classes of the pseudo-group G and if the equivalence relation \sim on \mathfrak{R} is defined by: $K_i \sim K_j$ if $\chi(K_i) = \chi(K_j)$ for all $\chi \in \mathfrak{M}^\perp$ then the number of \sim equivalence classes equals $\text{card}(\mathfrak{M}^\perp)$.*

PROOF. Let $r = \text{card}(\mathfrak{M}^\perp)$, let \mathfrak{B} denote the set of \sim equivalence classes and let $s = \text{card}(\mathfrak{B})$. Since \mathfrak{M}^\perp is a set of linearly independent functions on \mathfrak{B} , $r \leq s$. Let $\mathfrak{M}^\perp = \{\chi_1, \dots, \chi_r\}$ and choose positive integer u_1, \dots, u_r such that $f = \sum_{j=1}^r u_j \chi_j$ takes distinct values on \mathfrak{B} . Now $\mathfrak{M}^{\perp\perp\perp} = \mathfrak{M}^\perp$ so that Theorem 4.1 implies that f^n is a linear combination with non-negative integral coefficients of the characters of \mathfrak{M}^\perp for any non-negative integer n . Hence $\{1, f, f^2, \dots, f^r\}$ is a linearly dependent set of functions on \mathfrak{B} . Thus for some integer $1 \leq m \leq r$ and some set $\{c_0, c_1, \dots, c_{m-1}\}$ of complex numbers we have $f^m + c_{m-1}f^{m-1} + \dots + c_0 1_G = 0$. If $\{K_1, \dots, K_s\}$ are representatives from the s distinct \sim equivalence classes then $\{f(K_i) | 1 \leq i \leq s\}$ is a set of s distinct roots of the polynomial $X^m + c_{m-1}X^{m-1} + \dots + c_0$ and hence $s \leq r$. Thus $s = r$.

Note that we could have used the proof of [6, Theorem 3] to conclude Lemma 4.6 for pseudo-groups G having non-negative scalars of multiplication. For, since $\mathfrak{M}^\perp = (\mathfrak{M}^{\perp\perp})^\perp$, we may assume that \mathfrak{M} is R -normal. But then \mathfrak{M} is S -normal by Corollary 4.4 and hence \mathfrak{M} generates a characteristic subalgebra of $\mathfrak{U}(C)$ and the conclusion of Lemma 4.6 follows from the proof of [6, Theorem 3].

Consequently if an R -normal subset \mathfrak{M} of \mathfrak{K} is given, if the equivalence relation \sim on \mathfrak{K} is defined as above, if \mathfrak{B} denotes the \sim equivalence classes and if t is defined in \mathfrak{B} by (**) then $\{\mathfrak{B}, \mathfrak{M}^\perp, t\}$ satisfies pseudo-group Axioms I-III. Thus the R -normal subset \mathfrak{M} of \mathfrak{K} is T -normal if and only if there exist exponentiation functions $\{h_n | n \in \mathbb{Z}\}$ defined on \mathfrak{B} such that the system $\{\mathfrak{B}, \mathfrak{M}^\perp, t, \{h_n\}\}$ satisfies pseudo-group Axioms IV-VI. Also the R -normal subset \mathfrak{M} of \mathfrak{K} is U -normal if and only if $K_i \sim K_j$ implies $K_i^{[n]} \sim K_j^{[n]}$ for all $n \in \mathbb{Z}$ and all $K_i, K_j \in \mathfrak{K}$.

These methods used to construct $\{\mathfrak{B}, \mathfrak{M}^\perp, t\}$ satisfying pseudo-group Axioms I-III out of a given R -normal subset \mathfrak{M} of \mathfrak{K} are closely related to the methods used at the end of § 3 of [6] to construct a quotient C -algebra out of a given characteristic subalgebra.

COROLLARY 4.7. *If \mathfrak{M} is an R -normal subset of the pseudo-group G such that \mathfrak{M}^\perp consists only of characters of degree one then \mathfrak{M} is a U -normal subset of \mathfrak{K} . In that case, \mathfrak{M}^\perp forms an Abelian group under multiplication and the U -normal quotient of G -modulo \mathfrak{M} is the pseudo-group of the group dual to \mathfrak{M}^\perp .*

COROLLARY 4.8. *If finite groups G and H give rise to pseudo-groups which have the same character table then $G/G' \cong H/H'$ and the Abelian factor groups of the upper central series of G and H are isomorphic in ascending order.*

It is easy to see that in any pseudo-group the irreducible characters of degree one under multiplication form an Abelian group.

COROLLARY 4.9. *If L is any subgroup of the group of irreducible characters of degree one of the pseudo-group G then:*

- a) *For each $K \in \mathfrak{K}$ the map $K^*: L \rightarrow C$ defined by $K^*(\chi) = \chi(K)$ for $\chi \in L$ is an irreducible representation of L in C .*
- b) *Every irreducible representation of L in C is obtained in this way.*

PROOF. Only b) is not obvious. To prove b), apply Lemma 4.6 to L^\perp and use the fact that $L^{\perp\perp} = L$.

§ 5. Examples

From the pseudo-group point of view, it is of interest to give examples of pseudo-groups which do not come from groups. In this section, we give various examples, including two infinite classes, of pseudo-groups not coming from groups. We also give an example of a pseudo-group having a T -normal set of classes (which is also R -normal) but which is not U -normal. Clearly this pseudo-group does not come from a group.

The two non-Abelian groups of order 8 (the quaternion and dihedral groups) both have the same character table:

(1)

	K_1	K_2	K_3	K_4	K_5
χ_1	1	1	1	1	1
χ_2	1	1	-1	1	-1
χ_3	1	1	1	-1	-1
χ_4	1	1	-1	-1	1
χ_5	2	-2	0	0	0

The exponentiation for the quaternion group is determined by: $K_2^{[2]}=K_1$, $K_3^{[2]}=K_4^{[2]}=K_5^{[2]}=K_2$ and $K_j^{[3]}=K_j$ for $j=3, 4, 5$ and the exponentiation for the dihedral group is determined by: $K_2^{[2]}=K_3^{[2]}=K_4^{[2]}=K_1$ and $K_5^{[2]}=K_2$, $K_5^{[3]}=K_5$.

From (1) we can define two other non-isomorphic pseudo-groups which have (1) as their character table and which obviously do not come from groups. Since (1) is the character table of a pseudo-group we only have to define the exponentiation and check pseudo-group Axioms IV-VI. To facilitate the verification of Axiom V, let $Z_2=\langle t \rangle$ be the cyclic group of order two with irreducible characters $\{1_{Z_2}, \lambda\}$ where $1_{Z_2}(t)=1$ and $\lambda(t)=-1$ and let $Z_4=\langle \sigma \rangle$ be the cyclic group of order 4 with irreducible characters $\{1_{Z_4}, \omega, \omega^2, \omega^3\}$ where $1_{Z_4}(\sigma)=1$ and $\omega(\sigma)=i$. We now define two sets of exponentiation which turn (1) into two non-isomorphic pseudo-groups:

- $K_2^{[2]}=K_3^{[2]}=K_4^{[2]}=K_5^{[2]}=K_1$. Here if $3 \leq j \leq 5$ and if $\zeta(t^i)=K_j^{[i]}$ for all $i \in Z$, then $\chi_5 \circ \zeta = 1_{Z_2} + \lambda$. The verification of Axioms IV-VI is now trivial.
- $K_3^{[2]}=K_1$, $K_4^{[2]}=K_5^{[2]}=K_2$ and $K_j^{[3]}=K_j$ for $j=4, 5$. If $j=4$ or 5 and if $\zeta(\sigma^i)=K_j^{[i]}$ for all $i \in Z$, then $\chi_5 \circ \zeta = \omega + \omega^3$ and again the verification of Axioms IV-VI is trivial.

A similar method can be applied to the two non-Abelian groups of order p^3 where p is any odd prime. Both non-Abelian groups of order p^3 have the same character table:

	K_1	K_2	\dots	K_p	K_{p+1}	\dots	K_{p^2+p-1}
χ_1	1	1	\dots	1	B		
χ_2	1	1	\dots	1			
\vdots	\vdots	\vdots		\vdots			
χ_{p^2}	1	1	\dots	1			
χ_{p^2+1}	p	$p\varepsilon$	\dots	$p\varepsilon^{p-1}$	0	\dots	0
χ_{p^2+2}	\vdots	$p\varepsilon^2$	\dots	$p(\varepsilon^2)^{p-1}$	0	\dots	0
\vdots	\vdots	\vdots		\vdots	\vdots		\vdots
χ_{p^2+p+1}	p	$p\varepsilon^{p-1}$	\dots	$p(\varepsilon^{p-1})^{p-1}$	0	\dots	0

where ε is a primitive p -th root of unity and where B is a $p^2 \times (p^2 - 1)$ matrix whose columns give all of the $p^2 - 1$ irreducible characters not the 1 character of the Abelian group of type (p, p) .

Let $\mathfrak{M} = \{K_{p+1}, \dots, K_{p^2+p-1}\}$ and define for each integer r with $(r, p) = 1$ the exponentiation function $e_r: \mathfrak{M} \rightarrow \mathfrak{M}$ such that $e_r(K_\alpha) = K_\beta$ if when every element in the column of B under K_α is raised to the r -th power we get the column of B under K_β . Define an equivalence relation \sim on \mathfrak{M} by: $K_\alpha \sim K_\beta$ if there exists an integer r with $(r, p) = 1$ such that $e_r(K_\alpha) = K_\beta$. By renumbering the classes of \mathfrak{M} (if necessary) we may assume that the \sim equivalence classes are:

$$T_1 = \{e_j(K_{p+1}) = K_{p+j} | 1 \leq j \leq p-1\}$$

$$T_2 = \{e_j(K_{2p}) = K_{2p+j-1} | 1 \leq j \leq p-1\}$$

.....

$$T_{p+1} = \{e_j(K_{p^2+1}) = K_{p^2+j} | 1 \leq j \leq p-1\}.$$

If we are to have $e_p(K_\alpha) = K_1$ then $e_p(e_s(K_\alpha)) = e_s(e_p(K_\alpha)) = e_s(K_1) = K_1$. Hence e_p must be defined so that it maps all or none of each \sim equivalence class into K_1 .

Let $\{T_{\alpha_1}, \dots, T_{\alpha_r}\}$ be any r of the \sim equivalence classes ($0 \leq r \leq p+1$) and for each $K \in \bigcup_{i=1}^r T_{\alpha_i}$ set $h_p(K) = K_1$. If $T_i = \{K_{i p - i + 1 + j} | 1 \leq j \leq p-1\}$ is one of the remaining $(p+1-r) \sim$ equivalence classes, set $e_p(K_{i p - i + 2}) = K_2$ and $e_p(K_{i p - i + 1 + j}) = e_p(e_j(K_{i p - i + 2})) = e_j(K_2) = K_{j+1}$ for $1 \leq j \leq p-1$. Thus exponentiation functions $\{e_n | n \in \mathbb{Z}\}$ are uniquely determined and it is straightforward to verify that the

pseudo-group axioms are satisfied.

For the two non-Abelian groups of order p^3 we have $r=1$ and $r=p+1$. Hence for a choice of $r \neq 1, p+1$ we obtain a pseudo-group not coming from a group.

This method of altering the exponentiation function can be used to construct a pseudo-group with an S and T -normal subset of the set of classes which is not U -normal. To see this, consider the group of order 16 defined by:

$P^4=Q^2=R^2=1, R^{-1}PR=PQ, Q^{-1}PQ=P, R^{-1}QR=Q$; the classes of this group are: $K_1=\{1\}, K_2=\{Q\}, K_3=\{P^2\}, K_4=\{QP^2\}, K_5=\{P, PQ\}, K_6=\{R, RQ\}, K_7=\{PR, PRQ\}, K_8=\{P^3, P^3Q\}, K_9=\{RP^2, RP^2Q\}$ and $K_{10}=\{P^3R, P^3RQ\}$. The exponentiation for this group is determined by:

$$\begin{aligned} K_2^{[2]} &= K_3^{[2]} = K_4^{[2]} = K_6^{[2]} = K_9^{[2]} = K_1 \\ K_5^{[2]} &= K_8^{[2]} = K_3, K_6^{[3]} = K_5, K_5^{[3]} = K_8 \\ K_7^{[2]} &= K_{10}^{[2]} = K_4, K_7^{[3]} = K_{10}, K_{10}^{[3]} = K_7 \end{aligned}$$

and its character table is:

	K_1	K_2	K_3	K_4	K_5	K_6	K_7	K_8	K_9	K_{10}
χ_1	1	1	1	1	1	1	1	1	1	1
χ_2	1	1	-1	-1	i	1	i	- i	-1	- i
χ_3	1	1	1	1	-1	1	-1	-1	1	-1
χ_4	1	1	-1	-1	- i	1	- i	i	-1	i
χ_5	1	1	1	1	1	-1	-1	1	-1	-1
χ_6	1	1	-1	-1	i	-1	- i	- i	1	i
χ_7	1	1	1	1	-1	-1	1	-1	-1	1
χ_8	1	1	-1	-1	- i	-1	i	i	1	- i
χ_9	2	-2	-2	2	0	0	0	0	0	0
χ_{10}	2	-2	2	-2	0	0	0	0	0	0

If we redefine $K_6^{[2]}=K_2$ and $K_6^{[3]}=K_6$ and leave the exponentiation of the other classes unaltered, then we obtain a pseudo-group such that $\{K_1, K_3\}$ is S - and T -normal but not U -normal.

For, the verification that the pseudo-group axioms are satisfied is straightforward and since $\{K_1, K_3\}^\perp = \{\chi_1, \chi_3, \chi_5, \chi_7, \chi_{10}\}$ and $\{K_1, K_3\}^{\perp\perp} = \{K_1, K_3\}$ we conclude that $\{K_1, K_3\}$ is R and S -normal. Moreover, the \sim equivalence relation on \mathfrak{R}

defined as in Lemma 4.6 by means of $\{K_1, K_3\}$ gives rise to the following equivalence classes: $\{K_1, K_3\}$, $\{K_2, K_4\}$, $\{K_6, K_9\}$, $\{K_5, K_8\}$ and $\{K_7, K_{10}\}$. Thus we obtain the following table:

(2)

	\tilde{K}_1	\tilde{K}_4	\tilde{K}_5	\tilde{K}_6	\tilde{K}_7
χ_1	1	1	1	1	1
χ_3	1	1	-1	1	-1
χ_5	1	1	1	-1	-1
χ_7	1	1	-1	-1	1
χ_{10}	2	-2	0	0	0

where \tilde{K} denotes the \sim equivalence class containing K ; this is the character table of the two non-Abelian non-isomorphic groups of order 8. Hence we can define an exponentiation on the \sim equivalence classes which turns (2) into a pseudo-group. Thus $\{K_1, K_3\}$ is S and T -normal. However, $K_6 \sim K_9$ but $K_6^{[2]} = K_2$ and $K_9^{[2]} = K_1$ are not equivalent under \sim and thus $\{K_1, K_3\}$ is not U -normal.

There is also a method for "fitting together" two pseudo-groups which satisfy certain simple relations to obtain a pseudo-group having order equal to the product of the orders of the two original pseudo-groups. With a few added hypotheses on the two pseudo-groups, the newly constructed pseudo-group cannot come from a group.

To this effect, let $G = \{\mathfrak{K} = \{K_i\}, \mathfrak{E} = \{\chi_i\}, w, \{e_n\} = E\}$ and $H = \{\mathfrak{L} = \{L_j\}, X = \{\phi_j\}, v, \{f_n\} = F\}$ be two pseudo-groups of orders g and h respectively. Assume also that h is the square of a positive rational integer b , $h = b^2$, and that every class of H has order dividing b .

Without loss of generality, we assume that $\mathfrak{K} \cap \mathfrak{L} = \emptyset$ and that 1_M is an object not in $\mathfrak{K} \cup \mathfrak{L}$. We shall now construct a pseudo-group M of order gh .

Set $\mathfrak{N} = \{K \in \mathfrak{K} | K \neq 1_G\} \cup \{1_M\} \cup \{L \in \mathfrak{L} | L \neq 1_H\}$, $\mathfrak{E}^* = \{\chi \in \mathfrak{E} | \chi \neq 1\}$ and $X^* = \{\phi \in X | \phi \neq 1\}$. Define w' on \mathfrak{N} by: $w'(K) = \frac{w(K)}{h}$, $w'(L) = v(L)$ and $w'(1_M) = \frac{1}{gh}$. Define exponentiation functions $e'_n: \mathfrak{N} \rightarrow \mathfrak{N}$ for all $n \in Z$ by:

$$e'_n(K) = \begin{cases} e_n(K) & \text{if } e_n(K) \neq 1_G \\ 1_M & \text{if } e_n(K) = 1_G \end{cases}, \quad e'_n(L) = \begin{cases} f_n(L) & \text{if } f_n(L) \neq 1_H \\ 1_M & \text{if } f_n(L) = 1_H \end{cases}$$

and $e'_n(1_M) = 1_M$ for all $n \in Z$. For $\chi \in \mathfrak{E}^*$ define $\chi': \mathfrak{N} \rightarrow C$ by: $\chi'(K) = b\chi(K)$, $\chi'(L) = 0$ and $\chi'(1_M) = b\chi(1_G)$ and for $\phi \in X^*$ define $\phi': \mathfrak{N} \rightarrow C$ by: $\phi'(K) = \phi'(K) = \phi(1_H)$, $\phi'(L) = \phi(L)$ and $\phi'(1_M) = \phi(1_H)$. Setting $\mathcal{A} = \{1_M\} \cup \{\chi' | \chi \in \mathfrak{E}^*\} \cup \{\phi' | \phi \in X^*\}$ where 1_M

denotes the constant function $1_{\mathfrak{N}}(X)=1$ for all $X \in \mathfrak{N}$, we claim that the system $\{\mathfrak{N}, \Delta, w', \{e'_n\}\} = M$ forms a pseudo-group. For, the verification of pseudo-group Axioms I-IV and VI is straightforward. To verify Axiom V, the only non-obvious checking occurs in the following: if $L \in \mathfrak{U} \cap \mathfrak{N}$ is of order s in M (and hence in H), if $Z_s = \langle \sigma \rangle$ is a cyclic group of order s with irreducible characters $\{\lambda_1, \dots, \lambda_s\}$ and if the embedding ζ is such that $\zeta(\sigma^n) = f_n(L)$ for all $n \in \mathbb{Z}$ then $\phi' \circ \zeta$ is a character of Z_s for $\phi \in X^*$ because H is a pseudo-group and $\chi' \circ \zeta = \frac{b}{s} \chi(1_G) (\sum_{i=1}^s \lambda_i)$ is a character of Z_s since $s|b$.

When G is the pseudo-group of an Abelian group, when H satisfies the two conditions mentioned above and when $g = \text{order } G \geq h = \text{order } h$ then M is not a group pseudo-group. For, under these conditions, $Z(M) = \{1_M\} \cup (\mathfrak{N} \cap \mathfrak{N})$ contains g classes and if $L \in \mathfrak{U} \cap \mathfrak{N}$ then $c'(L) = \sum_{\gamma \in \mathcal{A}} \eta(L) \overline{\eta(L)} \leq h \leq g = \text{card } Z(M)$ which cannot occur in a finite group.

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(Received September 18, 1969)