Automorphisms of irreducible Weyl groups

By Eiichi BANNAI
(Comm. by N. Iwahori)

§ 0. Introduction.

As is well known, the symmetric group of degree 6 has a mysterious outer automorphism which transforms every transposition (an element consisting of a 2-cycle) into an element of three 2-cycles, while all the automorphisms of the symmetric group of degree different from 6 are inner. Since the symmetric group can be regarded as an example of the Weyl group of some root system, it is natural to consider the following question: to determine all the automorphisms of the irreducible Weyl groups in such a way that the above mentioned fact about the automorphisms of symmetric groups can be derived as a corollary.

In this note we give a complete determination of the automorphisms of the irreducible Weyl groups. We list our main result in Table I on page 274. In Appendix 2 we treat the automorphisms of the Coxeter groups of type $I_2(p)$, H_3 and H_4 in the notation of [1].

We had recourse to case-by-case considerations for each type in several proofs, although unified treatments are very desirable. We hope that a unified treatment should be available some time.

The author wishes to express his hearty thanks to Professors N. Iwahori and T. Yokonuma. They considerably improved the method, showing to the author several unified treatments instead of some case-by-case discussions in the author's original manuscript.

The author has also determined the automorphisms of the irreducible affine Weyl groups (i.e. the irreducible Coxeter groups of euclidean type), which will appear elsewhere.

Notations:

$\{\cdots\cdots\}$	:	the set $\cdots\cdots$.
$((\cdots\cdots))$	:	the ordered set of $\cdots\cdots$.
$\langle\cdots\cdot\rangle$	:	the subgroup generated by $\cdots\cdot$.
$C_G(x)$	:	the centralizer of an element x in a group G.
$Z(G)$	:	the center of G.

$\lvert G\rvert$	:	the order of a group G.
involution	:	an element of order two.
$\boldsymbol{Z}_n$	:	the cyclic group of order n.
$\boldsymbol{D}_n$	:	the dihedral group of order n.
$\boldsymbol{S}_n$	:	the symmetric group of degree n.
$\mathrm{Aut}(G)$	:	the automorphism group of G.
$\mathrm{Inn}(G)$	:	the inner automorphism group of G.
$i(g)$	:	the inner automorphism induced by g, i.e. $x^{i(g)}=gxg^{-1}$.
$\mathrm{Out}(G)=\mathrm{Aut}(G)/\mathrm{Inn}(G)$	:	the outer automorphism group of G.
$\cdots\simeq\cdots$	:	$\cdots$ is isomorphic to $\cdots$.
$x\sim y$	:	the elements x and y are conjugate.
$x\not\sim y$	:	the elements x and y are not conjugate.

Table I. The Automorphism Group of Weyl Groups (for the notations, see the explanations on pages 273 and 274).

type of W	W	$Z(W)$	$\mathrm{Out}(W)$	case
A_n $(n\geqslant 1)$	$(n+1)!$	2	1	$n=1$
		1	1	$n\geqslant 2$ and $n\neq 5$
		1	$\boldsymbol{Z}_2$	$n=5$
$C_n=B_n$ $(n\geqslant 2)$	$2^n\cdot n!$	2	$\boldsymbol{Z}_2$	$n=2$
		2	$\boldsymbol{Z}_2$	n : odd
		2	$\boldsymbol{Z}_2\times\boldsymbol{Z}_2$	n : even $\geqslant 4$
D_n $(n\geqslant 4)$	$2^{n-1}\cdot n!$	2	$\boldsymbol{Z}_2\times\boldsymbol{S}_3$	$n=4$
		1	1	n : odd
		2	$\boldsymbol{Z}_2\times\boldsymbol{Z}_2$	n : even $\geqslant 6$
G_2	12	2	$\boldsymbol{Z}_2$	
F_4	1152	2	$\boldsymbol{D}_8$	
E_6	51840	1	1	
E_7	2903040	2	1	
E_8	696729600	2	$\boldsymbol{Z}_2$	
H_3	120	2	$\boldsymbol{Z}_2$	
H_4	14400	2	$\boldsymbol{Z}_2\times\boldsymbol{Z}_2$	
$I_2(p)$	$2p$	2	see Appendix 2	p : even
		1		p : odd

§ 1. Weyl groups.

Let Φ be a reduced irreducible root system in the n-dimensional euclidean vector space [1]. The reduced irreducible root systems in $\boldsymbol{R}^n$ are completely classified, i.e. these are of type A_n $(n\geq 1)$, B_n $(n\geq 2)$, C_n $(n\geq 2)$, D_n $(n\geq 4)$, E_n $(n=6, 7, 8)$, G_2 and F_4 (see Bourbaki [1] Chap. VI). The explicit descriptions of these root systems are given also in [1] pages 250-275.

Let us recall some definitions:

r_α=the reflection mapping of $\boldsymbol{R}^n$ onto itself defined by $r_\alpha(x)=x-\frac{2(x, \alpha)}{(\alpha, \alpha)}\alpha$, $\alpha\in\boldsymbol{R}^n$, $x\in\boldsymbol{R}^n$.

$W(\Phi)$=the Weyl group associated with Φ, i.e. $=\langle r_\alpha;\ \alpha\in\Phi\rangle$. (Sometimes we denote it by W for brevity.)

$\alpha_1, \alpha_2, \cdots, \alpha_n$=the base of Φ given in [1] pages 250-275. (cf. Figure 1 in Appendix 1 of this note)

We call n the rank of W.

PROPOSITION 1 ([1] Chap. VI, Théorème 2). *Set* $S=\{r_{\alpha_1}, r_{\alpha_2}, \cdots, r_{\alpha_n}\}$, *then* $(W(\Phi), S)$ *is a Coxeter group. That is to say,* $W(\Phi)$ *is the group generated by the elements* $\{r_{\alpha_1}, r_{\alpha_2}, \cdots, r_{\alpha_n}\}$ *with the following defining relations* (#);

$$(\#)\quad \begin{cases} r_{\alpha_i}^2=1 \\ (r_{\alpha_i}, r_{\alpha_j})^{m_{ij}}=1,\ i\neq j,\ \textit{where}\ m_{ij}=2\ \textit{if}\ (\widehat{\alpha_i,\alpha_j})=\frac{\pi}{2} \\ m_{ij}=3\ \textit{if}\ (\widehat{\alpha_i,\alpha_j})=\frac{2}{3}\pi,\ m_{ij}=4\ \textit{if}\ (\widehat{\alpha_i,\alpha_j})=\frac{3}{4}\pi \\ \text{and}\ m_{ij}=6\ \textit{if}\ (\widehat{\alpha_i,\alpha_j})=\frac{5}{6}\pi\ . \end{cases}$$

(Sometimes we denote r_{α_i} *by* r_i *for brevity.)*

REMARK. The Weyl group of type B_n is isomorphic to the Weyl group of type C_n.

PROPOSITION 2 (see for example [1] or [2]). $Z(W)=\{1\}$, *if* W *is of type* A_n $(n\geq 2)$, D_n $(n$: *odd) or* E_6. $Z(W)=\{1, z\}$, *if* W *is of type* C_n $(=B_n)$, D_n $(n$: *even),* G_2 *or* F_4, *where* $z=(r_1\cdot r_2\cdots r_n)^{h/2}\neq 1$ *and* h *is the Coxeter number.*

DEFINITION 1. We say that the ordered subset $((x_1, x_2, \cdots, x_n))$ of n elements of W is a *canonical generator system* if (1) $\langle x_1, x_2, \cdots, x_n\rangle=W$ and (2) all the relations (#) are satisfied by $x_1, x_2, \cdots, x_n$ when the r_{α_i} are replaced by x_i, $i=1, 2, \cdots, n$ respectively.

PROPOSITION 3. *There is a bijective map between the set of canonical generator systems of* W *and the set of automorphisms of* W.

The proof is trivial.

Hence, in order to enumerate the automorphisms of W, we have only to determine all the canonical generator systems of W.

The Weyl groups of rank one or two are isomorphic to Z_2 or dihedral groups respectively, so in these cases it is a very easy task to determine the automorphisms of those groups; this will be done in Appendix 2.

Thus, from now on, we always assume that W is irreducible and that the rank of W is more than 2, unless the contrary is stated. However some of the propositions hold for W which are non-irreducible or of rank no more than 2.

§ 2. Construction of certain automorphisms and the centralizers of involutions in W.

Let $((x_1, x_2, \cdots, x_n))$ be a canonical generator system of W. Clearly we have (1) $C_W(x_i) \simeq C_W(r_i)$ for every $i=1, 2, \cdots, n$, and (2) $x_i \sim x_j$ if and only if $r_i \sim r_j$. Further we recall the following

PROPOSITION 4 ([1] Chap. VI, Proposition 3). *One has $r_i \sim r_j$ if and only if there exists a sequence $r^{(0)}, r^{(1)}, \cdots, r^{(q)}$ of elements of S such that $r_i = r^{(0)}$, $r_j = r^{(q)}$ and $r^{(t)} \cdot r^{(t+1)}$ are all of odd order for $t=0, 1, \cdots, q-1$ (cf. Figure I in Appendix 1).*

PROPOSITION A. (i) *The following maps σ_j defined on S really induce automorphisms of W. (See Figure I in Appendix.) Note that z is the element of $Z(W)$ such that $z \neq 1$ and $z = (r_1 \cdot r_2 \cdots r_n)^{h/2}$.*

1) *σ_1 = the identity automorphism of W, i.e. $r_i^{\sigma_1} = r_i$ for all $i=1, 2, \cdots, n$.*

2) *W of type C_n; $\sigma_2 : r_i^{\sigma_2} = zr_i$ $(i=1, 2, \cdots, n-1)$, $r_n^{\sigma_2} = r_n$.*

3) *W of type C_n $(n=even)$; $\sigma_3 : r_i^{\sigma_3} = zr_i$ for all $i=1, 2, \cdots, n$. $\sigma_4 : r_i^{\sigma_4} = r_i$ $(i=1, 2, \cdots, n-1)$, $r_n^{\sigma_4} = zr_n$.*

4) *W of type D_n $(n=even)$; $\sigma_2 : r_i^{\sigma_2} = zr_i$ for all $i=1, 2, \cdots, n$.*

5) *W of type F_4; $\sigma_2 : r_i^{\sigma_2} = zr_i$ for all $i=1, 2, 3, 4$. $\sigma_3 : r_1^{\sigma_3} = r_1$, $r_2^{\sigma_3} = r_2$, $r_3^{\sigma_3} = zr_3$, $r_4^{\sigma_3} = zr_4$. $\sigma_4 : r_1^{\sigma_4} = zr_1$, $r_2^{\sigma_4} = zr_2$, $r_3^{\sigma_4} = r_3$, $r_4^{\sigma_4} = zr_4$.*

6) *W of type E_8; $\sigma_2 : r_i^{\sigma_2} = zr_i$ for all $i=1, 2, \cdots, 8$.*

(ii) *The set of all σ_j of a W listed in* (i) *forms a group (subgroup of* Aut(W)) *which is isomorphic to the following groups:* (α) 1 *when W is of type A_n $(n \neq 5)$, D_n $(n=odd)$, E_6 or E_7;* (β) *Z_2 when W is of type C_n $(n=odd)$, D_n $(n=even)$ or E_8;* (γ) *$Z_2 \times Z_2$ when W is of type C_n $(n=even)$ or F_4.*

(iii) *When W is of type A_5, there exists a non trivial outer autmorphism σ_2 which transforms every transposition of S_6 (identified with W) into an element of three 2-cycles.* ([1] Chap. 4, § 1, Exercise 6.)

(iv) *σ_j is inner, if and only if $j=1$.*

PROOF. (i) Obvious, since every $((r_1^{\sigma_j}, r_2^{\sigma_j}, \cdots, r_n^{\sigma_j}))$ becomes a cononical generator system. (ii) Immediate by the direct calculations. (iv) Immediate by considering the eigen-values of the natural representation in $\boldsymbol{R}^n$ of $r_i \in S$ and $r_i^{\sigma_j}$ (since we are always assuming that the rank of W is more than 2).

We denote by k_W the number of all the automorphisms σ_j of W listed in Proposition A.

PROPOSITION 5. *Let $((x_1, x_2, \cdots, x_n))$ be a canonical generator system of W. Then there exists an automorphism σ_j $(1 \leqslant j \leqslant k_W)$ listed in Proposition A such that $x_i \sim r_i^{\sigma_j}$ hold for all $i=1, \cdots, n$.*

PROOF. 1) W is of type A_n: We first prove that, if $C_W(x) \cong C_W(r_1)$ holds for an involution x in W, then $x \sim r_1$, or $n=5$ and $x \sim r_1^{\sigma_2}$. Let the cycle type of x in S_{n+1} be $(1)^{n+1-2k}(2)^k$, and let $x \not\sim r_1$ and $x \not\sim r_1^{\sigma_2}$ (if $n=5$). Thus $k \geqslant 2$ since r_1 is of type $k=1$. Let $k=2$ (thus $n \geqslant 3$), then $|C_W(r_1)| = |C_W(x)|$ implies $(n-1)(n-2) = 4$, since $|C_W(r_1)| = 2 \cdot (n+1-2k)!$ and $|C_W(x)| = 2^k \cdot k!(n+1-2k)!$, a contradiction. Let $k=3$ (thus $n \geqslant 6$, since if $k=3$, $n=5$ then $x \sim r_1^{\sigma_2}$), then $|C_W(r_1)| = |C_W(x)|$ implies $(n-1)(n-2)(n-3)(n-4) = 2^2 \cdot 3!$, a contradiction. Let $k \geqslant 4$ (thus $n \geqslant 7$), then, using $l! \cdot m! \leqslant (l+m)!$ $(l \geqslant 0,\ m \geqslant 0)$ and $2^{l-1} \leqslant l!$ $(l>0)$, $|C_W(r_1)| = 2^k \cdot k! \cdot (n+1-2k)!$ $\leqslant 2 \cdot 2^2 \cdot 2^2 \cdot 2^{k-5} \cdot (n+1-k)! < 2(n-1)(n-2) \cdot 2^{k-5} \cdot (n+1-k) \leqslant 2(n-1)(n-2) \cdot (k-4)!\,(n+1-k)! \leqslant 2(n-1)(n-2) \cdot (n-3)! = 2 \cdot (n-1)! = |C_W(r_1)|$, a contradiction. Proposition 4, together with the remark preceding Proposition 4, shows that the assertion of the proposition is true for type A_n.

2) W is of type C_n: We can naturally regard the group W as the group of all the monomial matrices of degree n whose non-zero coefficients are all 1 or -1. Let x be an involution in W, then by choosing a suitable element $y \in W$, we have

$$y^{-1}xy = \begin{pmatrix} \boxed{\begin{matrix}0&1\\1&0\end{matrix}} & & & & & & \\ & \ddots & & & & & \\ & & \boxed{\begin{matrix}0&1\\1&0\end{matrix}} & & & & \\ & & & 1 & & & \\ & & & & \ddots & & \\ & & & & & 1 & \\ & & & & & & -1 \\ & & & & & & & \ddots \\ & & & & & & & & -1 \end{pmatrix} \quad \begin{matrix} k\ \ 2\times 2 \text{ matrices} \\ l \\ m \end{matrix}, \text{ for some } k,\ l,\ m$$

such that $2k+l+m=n$. Especially r_1 is of type $k=1$, $l=n-2$, $m=0$, zr_1 is of type $k=1$, $l=0$, $m=n-2$, r_n is of type $k=0$, $l=n-1$, $m=1$ and zr_n is of type $k=0$, $l=1$, $m=n-1$. $|C_W(r_1)|=2^n\cdot(n-2)!$, $|C_W(r_n)|=2^n\cdot(n-1)!$ and $|C_W(x)|=2^n\cdot k!\cdot l!\cdot m!$. Let $C_W(x)\cong C_W(r_n)$ hold for an involution x in W, and let $x\not\sim r_n$ and $x\not\sim zr_n$ (thus $k\geqq 1$, or $k=0$, $l<n-1$, $m>1$, or $k=0$, $l>1$, $m<n-1$). Then we have immediately $|C_W(x)|<|C_W(r_n)|$, a contradiction. Let $C_W(x)\cong C_W(r_1)$ hold for an involution x, and let $x\not\sim r_1$ and $x\not\sim zr_1$ (thus $k\neq 1$, or $k=1$, $l>0$, or $k=1$, $m>0$). Let $k=0$, then $|C_W(x)|=|C_W(r_1)|$ holds if and only if $m=3$ and $l=5$, or $m=5$ and $l=3$ (thus $n=8$). But in this exceptional case, $C_W(x)$ has an element of order 15, while $C_W(r_1)$ has no element of order 15, a contradiction. Let $k=1$, then $|C_W(x)|=2^n\cdot l!\cdot m!<2^n\cdot(l+m)!=2^n\cdot(n-2)!=|C_W(r_1)|$, a contradiction. Let $k=2$ (thus $n\geqq 4$). Let $n\geqq 5$. Then $|C_W(x)|=2^n\cdot 2\cdot l!\cdot m!\leqq 2^n\cdot 2(l+m)!=2^n\cdot 2\cdot(n-4)!<2^n\cdot 2\cdot(n-2)!$, a contradiction. Let $k=2$ and $n=4$. Then $|Z(C_W(x))|=4$, while $|Z(C_W(r_1))|=8$, a contradiction. Let $k\geqq 3$. Then $|C_W(x)|=2^n\cdot k!\cdot l!\cdot m!\leqq 2^n\cdot(k+l+m)!=2^n\cdot(n-3)!<2^n\cdot(n-2)!=|C_W(r_1)|$, a contradiction. Proposition 4, together with the remark preceding Proposition 4, shows that there occur only the following four cases, (i) $x_1\sim r_1$ and $x_n\sim r_n$, (ii) $x_1\sim zr_1$ and $x_n\sim r_n$, (iii) $x_1\sim r_1$ and $x_n\sim zr_n$, (iv) $x_1\sim zr_1$ and $x_n\sim zr_n$.

We remark that there exist three normal subgroups of index 2 in W. The first (we denote it by W_1) is the kernel of the linear representations of W such that $r_i\to 1$ $(i=1, 2, \cdots, n-1)$, $r_n\to -1$. The second is the kernel of the linear representation such that $r_i\to -1$ $(i=1, 2, \cdots, n-1)$, $r_n\to 1$. The third (we denote it by W_3) is the kernel of the linear representation such that $r_i\to -1$ $(i=1, 2, \cdots, n-1)$, $r_n\to -1$.

Let n be odd. Then the case (iii) is impossible, because if the case (iii) holds, by Proposition 4 all the x_i $(i=1, \cdots, n)$ are contained in W_1. Also the case (iv) is impossible, because if the case (iv) holds, by Proposition 4 all x_i $(i=1, \cdots, n)$ are contained in W_3. Thus the assertion of the proposition is true for type C_n.

3) W is of type D_n: We can naturally regard W as the subgroup W_1 of the Weyl group of type C_n (we denote the Weyl group of type C_n by $\mathfrak{W}$ only in this subsection 3)). Thus W is identified with the group consisting of all the monomial matrices of degree n whose non-zero entries are all 1 or -1 and the number of whose (-1)-entries are even. Let x be an involution of W. Then choosing a suitable $y\in W$, we have either

$$y^{-1}xy=\begin{pmatrix} \boxed{\begin{matrix}0&1\\1&0\end{matrix}} & & & & & & & \\ & \ddots & & & & & & \\ & & \boxed{\begin{matrix}0&1\\1&0\end{matrix}} & & & & & \\ & & & 1 & & & & \\ & & & & \ddots & & & \\ & & & & & 1 & & \\ & & & & & & -1 & \\ & & & & & & & \ddots \\ & & & & & & & & -1 \end{pmatrix}\ (k\ 2\times 2 \text{ matrices},\ l,\ m), \text{ for some } k,\ l,\ m$$

$2k+l+m=n$, $l+m\geqq 1$, $m=$even, or (ii) $n=$even and

$$y^{-1}xy=\begin{pmatrix} \boxed{\begin{matrix}0&1\\1&0\end{matrix}} & & \\ & \ddots & \\ & & \boxed{\begin{matrix}0&1\\1&0\end{matrix}} \end{pmatrix}\ (k) \quad \text{or} \quad y^{-1}xy=\begin{pmatrix} \boxed{\begin{matrix}0&1\\1&0\end{matrix}} & & & \\ & \ddots & & \\ & & \boxed{\begin{matrix}0&1\\1&0\end{matrix}} & \\ & & & \boxed{\begin{matrix}0&-1\\-1&0\end{matrix}} \end{pmatrix}\ (k-1)$$

In case (i) we have $|C_W(x)|=\frac{1}{2}|C_{\mathfrak{W}}(x)|$, and in case (ii) we have $|C_W(x)|=|C_{\mathfrak{W}}(x)|$. All the same argument as in 2) shows that, if $C_W(x)\cong C_W(r_1)$ holds for an involution x, then $x\sim r_1$, or $n=$even and $x\sim zr_n$. Proposition 4, together with the remark preceding Proposition 4, shows that the assertion of the proposition is true for type D_n.

4) W is of type F_4: It is easily checked by making use of the determination of the conjugacy classes (or the character table) of the Weyl group of type F_4 in [6], that, if $|C_W(x)|\cong|C_W(r_1)|$ holds for an involution x in W, then either $x\sim r_1$ or $x\sim zr_1$ or $x\sim r_4$ or $x\sim zr_4$. Proposition 4, together with the remark preceding Proposition 4, and the same argument as in the case of type C_n ($n=$odd) in 2) show that the assertion of the proposition is true for type F_4.

5) W is of type E_n ($n=6, 7, 8$): It is easily checked by making use of the determination of the conjugacy classes (or the character tables) of the Weyl groups of type E_n ($n=6, 7, 8$) in [4], [5] or [3], that, if $|C_W(x)|\cong|C_W(r_1)|$ holds for an involution x in W of type E_6, then $x\sim r_1$, and that, if $C_W(x)\cong C_W(r_1)$

holds for an involution x in W of type E_7 or E_8, then $x\sim r_1$ or $x\sim zr_1$. But when W is of type E_7, $x_1\sim zr_1$ is impossible, otherwise Proposition 4 and the remark preceding Proposition 4 show that all x_i ($i=1, \cdots, n$) are contained in the proper subgroup of index 2 in W (i.e. in the even subgroup of W), a contradiction. Proposition 4, together with the remark preceding Proposition 4, shows that the assertion of the proposition is true for type E_n, q.e.d.

REMARK.[1] One can also treat the cases $W\cong W(F_4)$, $W\cong W(E_n)$ by computing the number of elements in the conjugacy class containing the given involution x, exploiting the fact that x is a product of mutually commuting reflections.

§ 3. Estimation of the upper bound of $|\mathrm{Out}(W)|$.

DEFINITION 2. We say that a canonical generator system $((x_1, x_2, \cdots, x_n))$ is *admissible* when $x_i\sim r_i$ hold for all $i=1, 2, \cdots, n$.

In the preceding section we proved that every canonical generator system is transformed onto an admissible canonical generator system by some σ_j^{-1} $(1\leqslant j \leqslant k_W)$.

We denote by $\bar{\Phi}$ the set of non-oriented roots, i.e. the set obtained from Φ by identifying α and $-\alpha$, $\alpha\in\Phi$.

DEFINITION 3. We say that the ordered subset $((\bar{\beta}_1\ \bar{\beta}_2, \cdots, \bar{\beta}_n))$ of n elements of $\bar{\Phi}$ is a (*)-*system*, if the following two conditions are satisfied; (1) $\bar{\beta}_i\neq\bar{\beta}_j$, if $i\neq j$, (2) $|\cos(\widehat{\beta_i,\beta_j})|=\cos\dfrac{\pi}{m_{ij}}$, if $i\neq j$.

W acts on the set of (*)-systems.

Let $x\in W$ and $x\sim r_i$, for some $r_i\in S$. Then there exists a unique $\bar{\alpha}\in\bar{\Phi}$ such that $x=r_\alpha$. Set $\varphi(x)=\bar{\alpha}\in\bar{\Phi}$. Thus we have the map $\bar{\varphi}$ from the set of admissible canonical generator systems to the set of (*)-systems. Clearly this

[1] We illustrate the method for $W(E_8)$. Since all x_i $(i=1, 2, \cdots, 8)$ of a canonical generator system are conjugate by Proposition 4 and the remark preceding Proposition 4, x_1 must not be an element of even subgroup. Thus, if $x_1\not\sim r_1$ and $x_1\not\sim zr_1$, then one of the x_1 and zx_1 must consist of three mutually orthogonal reflections. Thus we have only to prove that, if x is an involution which consists of three mutually orthogonal reflections, then $|C_W(x)|\neq|C_W(r_1)|=120$. In the set Φ of non-oriented roots, we consider the subset $\bar{\beta}_1$, $\bar{\beta}_2$, $\bar{\beta}_3$ such that $(\widehat{\beta_i,\beta_j})=\pi/2$ $(i\neq j)$. An explicit calculation shows that W is transitive on the set of all the sets $\{\bar{\beta}_1, \bar{\beta}_2, \bar{\beta}_3\}$ of above type, and that there exist more than 120 such sets which contain the fixed non-oriented root $\bar{\beta}_0$ (e.g. $\bar{\beta}_0=\bar{\alpha}_1$). But, for the two such sets $\{\bar{\beta}_0, \bar{\beta}_1, \bar{\beta}_2\}$ and $\{\bar{\beta}_0, \bar{\beta}_3, \bar{\beta}_4\}$, $r_{\beta_0}r_{\beta_1}r_{\beta_2}=r_{\beta_0}r_{\beta_3}r_{\beta_4}$, if and only if the subspaces $\boldsymbol{R}\beta_1+\boldsymbol{R}\beta_2$ and $\boldsymbol{R}\beta_3+\boldsymbol{R}\beta_4$ coincide. Furthermore $\boldsymbol{R}\beta_1+\boldsymbol{R}\beta_2=\boldsymbol{R}\beta_3+\boldsymbol{R}\beta_4$, if and only if $\{\bar{\beta}_1, \bar{\beta}_2\}=\{\bar{\beta}_3, \bar{\beta}_4\}$, since all roots have the same length. Thus the assertion is true.

map $\tilde{\varphi}$ is injective. $\tilde{\varphi}$ is also surjective as we shall see later in the proof of Proposition 6.

DEFINITION 4. We say that an automorphism of W is a *particular automorphism* when it stabilizes the subset S. An outer particular automorphism is the automorphism which is particular and outer.

Let p_1 (=the identity automorphism), $p_2, \cdots, p_h, p_{h+1}, \cdots, p_l$ be all the particular automorphisms of W among which $p_2, \cdots, p_h$ are outer and $p_{h+1}, \cdots, p_l$ are inner. (See Figure I in Appendix 1.)

In the following we put $h=h_W$, $l=l_W$.

PROPOSITION B. *Suppose that the group W admits a non-trivial particular automorphism which is inner. Then $Z(W)=1$. Conversely, if $Z(W)=1$, then every particular automorphism of W is inner. More precisely, we have* (i) *$h_W=l_W=1$, when W is of type C_n $(n\geqq 3)$, E_7 or E_8.* (ii) *$h_W=1$ and $l_W=2$, when W is of type A_n, E_6 or D_n $(n=odd)$.* (iii) *$h_W=l_W=2$, when W is of type D_n $(n=even\geqq 6)$ or F_4.* (iv) *$h_W=l_W=6$, when W is of type D_4 (See Figure I.).*

PROOF. l_W is equal to the number of automorphisms of the Coxeter diagram, and the number coincides with the value of l_W given in the proposition. (i) Obvious. (ii) The unique non identity particular automorphism p_2 of W is inner, since $p_2=i((r_n r_{n-1}\cdots r_2 r_1 r_2\cdots r_{n-1}r_n)(r_{n-2}\cdots r_3\cdots r_{n-2})(r_{n-4}\cdots r_5\cdots r_{n-4})\cdots(r_{\frac{n+1}{2}}))$, when W is of type $A_n(n=odd)$; $p_2=i((r_n\cdots r_1\cdots r_n)(r_{n-2}\cdots r_3\cdots r_{n-2})\cdots(r_{\frac{n}{2}+1}r_{\frac{n}{2}}r_{\frac{n}{2}+1}))$, when W is of type A_n $(n=even)$; $p_2=i((r_1\cdots r_{n-2}r_{n-1}r_n r_{n-2}\cdots r_1)(r_2\cdots r_{n-1}r_n r_{n-2}\cdots r_2)(r_3\cdots r_{n-1}r_n r_{n-2}\cdots r_3)\cdots(r_{n-1}r_n))$, when W is of type D_n $(n=odd)$; $p_2=i((r_6r_5r_4r_3r_1r_3r_4r_5r_6)(r_5r_4r_3r_4r_5)(r_4)(r_{\alpha_0}))$, where α_0 is the highest root, when W is of type E_6. (iii) and (iv)[2] We have only to prove that, if $Z(W)\neq 1$, then every non-trivial particular automorphism of W is outer. Let a non-trivial particular automorphism p be inner. Then there exists a $w\in W$ such that $wr_iw^{-1}=r_{\nu(i)}$ for $i=1,2,\cdots,n$, where ν is a permutation of $1,2,\cdots,n$. Thus $w(\alpha_i)=\varepsilon\alpha_{\nu(i)}$, where $\varepsilon=1$ or -1, and we see that from the connectedness of the Dynkin diagram that ε is independent of the i's. If $\varepsilon=1$, then $w=1$ ([1] Chap. IV n°1.5)). If $\varepsilon=-1$ and $|Z(W)|\neq 1$, then $w=z$, since the element $w\in W$ such that $w\{\alpha_1,\cdots,\alpha_n\}=-\{\alpha_1,\cdots,\alpha_n\}$ is unique and $z(\alpha_i)=-\alpha_i$ $(1\leqq i\leqq n)$ ([1]). This contradicts the assumption that p is non-trivial, q. e. d.

PROPOSITION 6. *For every admissible canonical generator system $((x_1, x_2, \cdots, x_n))$ of W, there exist an inner automorphism $i(w)$ and an identity or outer particular automorphism p_u $(1\leqq u\leqq h_W)$ of W such that* $((x_1^{i(w)\cdot p_u}, x_2^{i(w)\cdot p_u}, \cdots,$

2) This unified proof is due to Iwahori.

$x_n^{i(w)\cdot p_u}))=((r_1, r_2, \cdots, r_n))$.

To prove the proposition we use the following

LEMMA.[3] *Let the subset* $\{\beta_1, \beta_2, \cdots, \beta_n\}$ *of* Φ *satisfy the relations* $(\beta_i, \beta_j)/\sqrt{(\beta_i, \beta_i)(\beta_j, \beta_j)}=(\alpha_i, \alpha_j)/\sqrt{(\alpha_i, \alpha_i)(\alpha_j, \alpha_j)}$ *for all* $i, j=1, 2, \cdots, n$. *Then there exists a* $w \in W$ *such that the set* $\{\beta_1^w, \beta_2^w, \cdots, \beta_n^w\}$ *is equal to the set* $\{\alpha_1, \alpha_2, \cdots, \alpha_n\}$.

PROOF OF LEMMA. Since $\beta_1, \beta_2, \cdots, \beta_n$ are linearly independent in $\boldsymbol{R}^n$ and $2(\beta_i, \beta_j)/(\beta_j, \beta_j)$ are non positive integers for all $i, j=1, 2, \cdots, n, i\neq j$, there exists a unique root system Φ_0 in $\boldsymbol{R}^n$ with $\beta_1, \beta_2, \cdots, \beta_n$ as its base. Furthermore, $\Phi_0=\{w(\beta_i);\, w\in\langle r_{\beta_1}, r_{\beta_2}, \cdots, r_{\beta_n}\rangle, i=1, 2, \cdots, n\}$. We have $\Phi\supset\Phi_0$. From our assumption the angle $\widehat{\beta_i, \beta_j}$ is equal to the angle $\widehat{\alpha_i, \alpha_j}$. Hence the Dynkin diagram of $((\beta_1, \cdots, \beta_n))$ coincides with that of $((\alpha_1, \cdots, \alpha_n))$ or of $((\alpha_1^*, \cdots, \alpha_n^*))$ $\left(\text{where } \alpha^*=\dfrac{2\alpha}{(\alpha, \alpha)}\right)$. Therefore, Φ and Φ_0 consist of the same number of roots. Thus we have $\Phi=\Phi_0$. Hence there must exist $w\in W$ such that $\{\beta_1^w, \beta_2^w, \cdots, \beta_n^w\}=\{\alpha_1, \alpha_2, \cdots, \alpha_n\}$, q. e. d.

PROOF OF PROPOSITION 6. The ordered subset $((\varphi(x_1), \varphi(x_2), \cdots, \varphi(x_n)))$ of $\bar{\Phi}$ becomes a (*)-system. We can choose $\gamma_i \in \Phi$ such that $\varphi(x_i)=\bar{\gamma}_i$ for every $i=1, 2, \cdots, n$, so that the relations $(\gamma_i, \gamma_j)/\sqrt{(\gamma_i, \gamma_i)(\gamma_j, \gamma_j)}=(\alpha_i, \alpha_j)/\sqrt{(\alpha_i, \alpha_i)(\alpha_j, \alpha_j)}$ may hold for all $i, j=1, 2, \cdots, n$. This is indeed possible since every Coxeter diagram associated with W is a tree. Owing to the lemma, there exists a $w\in W$ such that $\{\gamma_1^w, \gamma_2^w, \cdots, \gamma_n^w\}=\{\alpha_1, \alpha_2, \cdots, \alpha_n\}$. The assertion of the proposition is obvious from the fact that $x=r_\alpha$ if and only if $x^{i(w)}=r_{w(\alpha)}$.

From Proposition 5, we can choose for every canonical generator system $((x_1, x_2, \cdots, x_n))$ of W a σ_j $(1\leqslant j\leqslant k_W)$, an $i(w)$ and a p_u $(1\leqslant u\leqslant h_W)$ such that $((x_1^{\sigma_j^{-1}\cdot i(w)\cdot p_u}, x_2^{\sigma_j^{-1}\cdot i(w)\cdot p_u}, \cdots, x_n^{\sigma_j^{-1}\cdot i(w)\cdot p_u}))=((r_1, r_2, \cdots, r_n))$. Thus we have

PROPOSITION 7. $$|\mathrm{Out}(W)|\leqslant k_W\cdot h_W\,.$$

§ 4. Main Theorem.

THEOREM. *The* $k_W\cdot h_W$ *automorphisms* $\{p_u\cdot\sigma_j\}$ $(u=1, 2, \cdots, h_W$ *and* $j=1, 2, \cdots, k_W)$ *of* W *are all distinct modulo* Inn(W). *Thus* $|\mathrm{Out}(W)|=k_W\cdot h_W$. *Furthermore* $\mathrm{Out}(W)=1$, *when* W *is of type* A_n $(n\neq 5)$, D_n $(n=odd)$, E_6 *or* E_7; $\mathrm{Out}(W)\cong\boldsymbol{Z}_2$, *when* W *is of type* A_5, C_n $(n=odd)$ *or* E_8; $\mathrm{Out}(W)\cong\boldsymbol{Z}_2\times\boldsymbol{Z}_2$, *when* W *is of type* C_n $(n=even)$ *or* D_n $(n=even\geqslant 6)$; $\mathrm{Out}(W)\cong\boldsymbol{Z}_2\times S_3\cong\boldsymbol{D}_{12}$, *when* W *is of type* D_4; $\mathrm{Out}(W)\cong\boldsymbol{D}_8$, *when* W *is of type* F_4.

[3] The author has learned the lemma from Yokonuma. The proof given here is due to Iwahori. In the original manuscript the author proved the counterpart of the lemma by explicit calculations for each type.

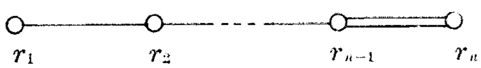
PROOF. The assertion is obvious when $h_W=1$ by Proposition A. When W is of type D_n ($n=even>6$), $\{p_u\cdot\sigma_j\}$ ($u=1,2$ and $j=1,2$) from a group (subgroup of Aut(W)) isomorphic to $\boldsymbol{Z}_2\times\boldsymbol{Z}_2$, since $p_2\cdot\sigma_2=\sigma_2\cdot p_2$. Now if $p_u\cdot\sigma_j$ is inner, one has $u=1$ and $j=1$, so the assertion is true. When W is of type F_4, $\{p_u\cdot\sigma_j\}$ ($u=1,2$ and $j=1,2,3,4$) form a group isomorphic to $\boldsymbol{D}_8$, since $p_2\cdot\sigma_2=\sigma_2\cdot p_2$, $p_2\cdot\sigma_3=\sigma_4\cdot p_2$ and $p_2\cdot\sigma_4=\sigma_3\cdot p_2$. Moreover, if $p_u\cdot\sigma_j$ is inner, we have $u=1$ and $j=1$, so the assertion is true. When W is of type D_4, $\{p_u\cdot\sigma_j\}$ ($u=1,2,3,4,5,6$ and $j=1,2$) form a group isomorphic to $\boldsymbol{Z}_2\times\boldsymbol{S}_3\cong\boldsymbol{D}_{12}$, since $p_u\cdot\sigma_j=\sigma_j\cdot p_u$ for all u and j. Now if $p_u\cdot\sigma_j$ is inner, one has $u=1$ and $j=1$, hence the assertion is true, q. e. d.

Appendix 1. Coxeter diagram associated with W and the description of some automorphisms of W.

Figure I.

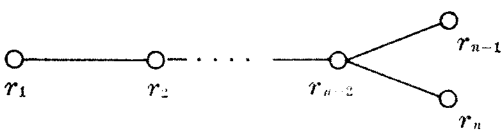
(1) 

(A_n),

(2) 

(C_n),

$r_i^{p_2}=r_2$ $(i=1,2,\cdots,n)$.

(3) 

(D_n),

$r_i^{p_2}=r_i$ $(i=1,2,\cdots,n-2)$, $r_{n-1}^{p_2}=r_n$, $r_n^{p_2}=r_{n-1}$; further

$r_1^{p_3}=r_3$, $r_2^{p_3}=r_2$, $r_3^{p_3}=r_1$, $r_4^{p_3}=r_4$,

$r_1^{p_4}=r_4$, $r_2^{p_4}=r_2$, $r_3^{p_4}=r_3$, $r_4^{p_4}=r_1$,

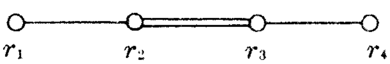
$r_1^{p_5}=r_3$, $r_2^{p_5}=r_2$, $r_3^{p_5}=r_4$, $r_4^{p_5}=r_1$,

$r_1^{p_6}=r_4$, $r_2^{p_6}=r_2$, $r_3^{p_6}=r_1$, $r_4^{p_6}=r_3$,

for $n=4$.

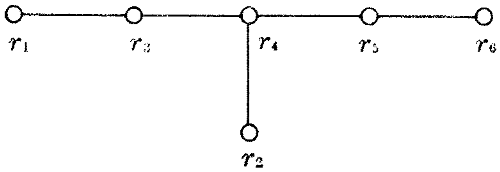
(4) 

(G_2).

(5) 

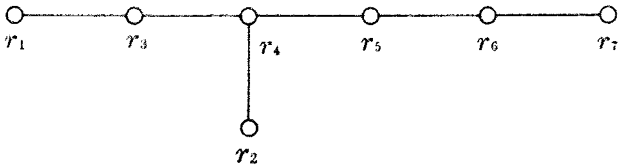
(F_4),

$r_i^{p_2}=r_{4-i}^{p_2}$ $(i=1,2,3,4)$.

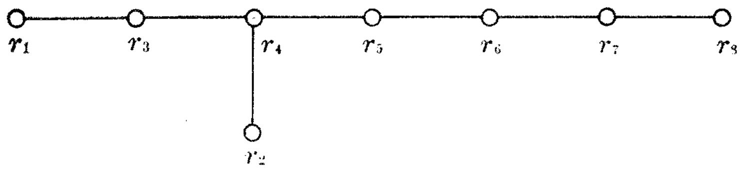
(6) 

(E_6).

$r_1^{\rho_2}=r_6,\ r_2^{\rho_2}=r_2,\ r_3^{\rho_2}=r_5,\ r_4^{\rho_2}=r_4,\ r_5^{\rho_2}=r_3,\ r_6^{\rho_2}=r_1.$

(7) 

(E_7).

(8) 

(E_8).

Appendix 2. Automorphisms of Coxeter groups of type $I_2(p)$, H_3 and H_4 (see Fig. II).

(1) W is of type $I_2(p)$. (W is the dihedral group of order $2p$.)

(i) $p=odd$. There exist $\varphi(p)/2$ (where φ is the Euler function) canonical generator systems:

$$(r_1, \underbrace{r_2r_1 \cdots r_2r_1r_2r_1r_2 \cdots r_1r_2}_{k\ r_2\text{'s}}), \qquad (k, p)=1, \quad k\leqslant p/2\ .$$

Set $$\sigma_k : r_1^{\sigma_k}=r_1,\ r_2^{\sigma_k}=\underbrace{r_2r_1 \cdots r_2r_1r_2r_1r_2 \cdots r_1r_2}_{k\ r_2\text{'s}}\ .$$

Then $\{\sigma_k; (k, p)=1, k\leqslant p/2\}$ form complete left coset representatives of $\mathrm{Aut}(W)/\mathrm{Inn}(W)$. Further $\mathrm{Out}(W)\cong \boldsymbol{Z}_{\varphi(p)}$.

(ii) $p=even\geqslant 4$. There exist $\varphi(p)$ canonical generator systems:

$$(r_1, \underbrace{r_2r_1 \cdots r_2r_1}_{\frac{k-1}{2}\ r_2\text{'s}}r_2\underbrace{r_1r_2 \cdots r_1r_2}_{\frac{k-1}{2}\ r_2\text{'s}}), \qquad (k, p)=1, \quad k\leqslant p/2\ ,$$

and $$(r_2, \underbrace{r_1r_2 \cdots r_1r_2}_{\frac{k'-1}{2}\ r_2\text{'s}}r_1\underbrace{r_2r_1 \cdots r_2r_1}_{\frac{k'-1}{2}\ r_2\text{'s}}), \qquad (k', p)=1, \quad k'\leqslant p/1\ .$$

Set
$$\sigma_k : r_1^{\sigma_k}=r_1,\ r_2^{\sigma_k}=\underbrace{r_2r_1\cdots r_2r_1}_{\frac{k-1}{2}\ r_2\text{'s}}r_2\underbrace{r_1r_2\cdots r_1r_2}_{\frac{k-1}{2}\ r_2\text{'s}},$$
$$\sigma_{k'} : r_1^{\sigma_k}=r_2,\ r_2^{\sigma_k}=\underbrace{r_1r_2\cdots r_1r_2}_{\frac{k'-1}{2}\ r_2\text{'s}}r_1\underbrace{r_2r_1\cdots r_2r_1}_{\frac{k'-1}{2}\ r_2\text{'s}}.$$

Then $\{\sigma_k, \sigma_{k'}; (k, p)=1, k\leqslant p/2, (k', p)=1, k'\leqslant p/2\}$ form complete left coset representatives of $\mathrm{Aut}(W)/\mathrm{Inn}(W)$. Further $\mathrm{Out}(W)\cong \boldsymbol{Z}_2\times \boldsymbol{Z}_{\varphi(p)/2}$.

(2) W is of type H_3 or H_4.

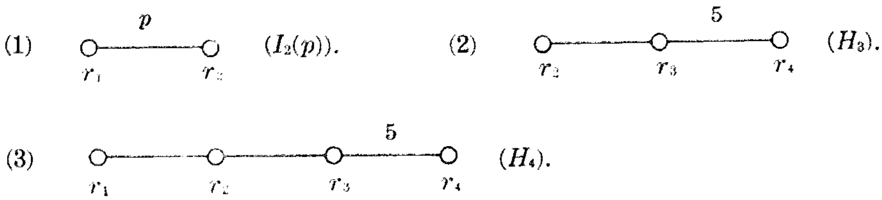
Set α_i $(i=1, 2, 3, 4)\in \boldsymbol{R}^4$ as follows: $\alpha_1=(\tau^{-1}, -1, 0, \tau)$, $\alpha_2=(0, 2, 0, 0)$, $\alpha_3=(0, -1, -\tau, -\tau^{-1})$ and $\alpha_4=(0, 0, 2, 0)$, where $\tau=(\sqrt{5}+1)/2$. Then $(\widehat{\alpha_1, \alpha_2})=\frac{2}{3}\pi$, $(\widehat{\alpha_1, \alpha_3})=\frac{\pi}{2}$, $(\widehat{\alpha_1, \alpha_4})=\frac{\pi}{2}$, $(\widehat{\alpha_2, \alpha_3})=\frac{2}{3}\pi$, $(\widehat{\alpha_2, \alpha_4})=\frac{\pi}{2}$, $(\widehat{\alpha_3, \alpha_4})=\frac{4}{5}\pi$. (From now on, we denote r_{α_i} by r_i for brevity.)

Set $W(H_3)=\langle r_2, r_3, r_4\rangle$ and $W(H_4)=\langle r_1, r_2, r_3, r_4\rangle$. Then $(W(H_3), \{r_2, r_3, r_4\})$ and $(W(H_4), \{r_1, r_2, r_3, r_4\})$ are Coxeter groups of order 120 and 14400 respectively. $W(H_3)$ contains 15 reflections which are reflections with respect to the following 30 vectors: 6 vectors which are permutations of $(\pm 2, 0, 0)$, 24 vectors which are even permutations of $(\pm 1, \pm\tau, \pm\tau^{-1})$. $W(H_4)$ contains 60 reflections which are reflections with respect to the following 120 vectors: 8 vectors which are permutations of $(\pm 2, 0, 0, 0)$, 16 vectors $(\pm 1, \pm 1, \pm 1, \pm 1)$, 96 vectors which are even permutations of $(0, \pm 1, \pm\tau, \pm\tau^{-1})$. Set $\alpha_1'=(\tau, -1, \tau^{-1}, 0)$, $\alpha_2'=\alpha_2$, $\alpha_3'=\alpha_2$ and $\alpha_4'=(0, 0, 0, 2)$. Note that $(\widehat{\alpha_3', \alpha_4'})=\frac{3}{5}\pi$. (We denote $r_i'=r_{\alpha_i'}$ for brevity.)

(i) The $((r_2', r_3', r_4'))$ is a canonical generator system of $W(H_3)$, and the map ρ such that $r_i^\rho=r_i'$ $(i=2, 3, 4)$ is indeed an outer automorphism of $W(H_3)$, because the eigenvalues of $r_3\cdot r_4$ and $r_3'\cdot r_4'$ are different. A similar argument in the previous sections shows (together with a little calculation) that every canonical generator system of $W(H_3)$ is transformed by an inner automorphism onto $((r_2, r_3, r_4))$ or $((r_2', r_3', r_4'))$. Thus we have $\mathrm{Out}(W)\cong \boldsymbol{Z}_2$.

(ii) The $((r_1', r_2', r_3', r_4'))$ is a canonical generator system of $W(H_4)$, and the map ρ such that $r_i^\rho=r_i'$ $(i=1, 2, 3, 4)$ is indeed an outer automorphism of $W(H_4)$. The map $\sigma_2 : r_i\mapsto zr_i$ $(i=1, 2, 3, 4)$ induces an outer automorphism of $W(H_4)$, where $z(x)=-x$, $x\in \boldsymbol{R}^4$ and $z=(r_1r_2r_3r_4)^{15}$. A similar argument in the previous sections shows (together with a little calculation) that every canonical generator system of $W(H_4)$ is transformed by an inner automorphism, or by an inner automorphism and σ_2, onto either $((r_1, r_2, r_3, r_4))$ or $((r_1', r_2', r_3', r_4'))$. Thus we have $\mathrm{Out}(W)\cong \boldsymbol{Z}_2\times \boldsymbol{Z}_2$.

Figure II.



Thus we can easily accomplish Table I.

University of Tokyo

References

[1] N. Bourbaki, Groupes et algèbres de Lie, chap. 4, 5 et 6, Hermann, Paris 1968.
[2] H. S. M. Coxeter and W. Moser, Generators and relations for discrete groups, 2nd ed. Springer, Berlin 1965.
[3] H. Enomoto, Character table of the Weyl group of type E_8 (unpublished).
[4] J. S. Frame, The classes and representations of the groups of 27 lines and 28 bitangents, Annali di Math. **32** (1951), 83-119.
[5] ———, The characters of the Weyl group E_8 (to appear).
[6] T. Kondo, The characters of the Weyl group of type F_4, J. Fac. Sci. Univ. Tokyo, Sec. I, **11** (1965), 145-153.

(Received September 29, 1969)