On intermediate logics, II

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In our former paper [15], which we shall refer as I throughout this paper, we gave a systematic classification of the intermediate propositional logics. Here we continue a detailed study of the cone structure defined in it. Our main concern in this paper is to investigate the structure of each slice.

Each slice $\mathcal{S}_i$ can be mapped isomorphically into $\mathcal{S}_j$ $(i<j)$ by the ω-projection defined in I. This being the case, it seems necessary for us to find and locate properly the logics of $\mathcal{S}_j$ that are not available as images of logics of $\mathcal{S}_i$ $(i<j)$. Those singular logics are defined strictly in § 3 and called as proper. Some examples of proper logics are given in §§ 3-4.

In § 4, we define Θ-projection. With this projection, we deal with such logics as containing $\neg a \vee \neg\neg a$. This enabled us to find proper logics. The results in § 4 describe rather minute aspects in the ω-th slice.

In § 1, some of the results of I are extended. And in § 2, we determine those logics in which the logical connective $\vee$ is definable. Our result is that $\vee$ is only definable in the logics S_n $(1\leq n\leq\omega)$, which are represented by linear models. This would conclude the problem of definability of logical connectives in the intermediate logics.

Since this paper is a continuation of I, notations and results of I are for the most part assumed without special references. The bibliography at the conclusion is renewed from that of I.

§ 1. Remarks to the paper I.

Some of the results of the paper I will be extended in this §.

THEOREM 1.1. *Let be that* $M\in\mathcal{S}_j$ *and* $i<j\leq\omega$. *Then* $(M(i))(j)\supset\subset M$ *if and only if* $M\supset LP_i(j)$. (An extension of 5.13, I.)

PROOF. Put $N=LP_i(j)$, that is, $N\supset\subset LP_i\cap S_j\in\mathcal{S}_j$. Now,

$$\begin{aligned}(M(i))(j)&\supset\subset((M\cap S_\omega)\cup LP_i)\cap S_j\\&\supset\subset(M\cap S_\omega\cap S_j)\cup(LP_i\cap S_j)\\&\supset\subset(M\cap S_\omega)\cup N\\&\supset\subset(M\cup N)\cap(S_\omega\cup N)\\&\supset\subset(M\cup N)\cap S_j\\&\supset\subset M\cup N\,.\end{aligned}$$

On the other hand, in general, $\boldsymbol{M}\cup\boldsymbol{N}\supset\subset\boldsymbol{M}$ if and only if $\boldsymbol{M}\supset\boldsymbol{N}$. Hence the theorem is proved.

COROLLARY 1.2. *Let be that* $\boldsymbol{M}\in\mathscr{S}_{i+j}$. *Then* $\boldsymbol{M}\supset\boldsymbol{LP}_i\cap\boldsymbol{S}_{i+j}$ *if and only if there exists* $\boldsymbol{N}\in\mathscr{S}_i$ *such that* $\boldsymbol{N}(i+j)\supset\subset\boldsymbol{M}$.

Next, we will refine the results of §2 of I. The operations $\cap$ and $\cup$ will be defined new as follows. (It should be noted that these definitions are consistent with those of I.)

DEFINITION 1.3. *For* $\Lambda\neq\phi$, $\bigcap_{\lambda\in\Lambda}\boldsymbol{M}_\lambda$ *is the set of wffs valid in the model obtained as the direct product of the models* $\boldsymbol{M}_\lambda$ $(\lambda\in\Lambda)$.

REMARK. By the direct product, we mean the operation on models as defined in 1.7 of I. Though the definition 1.7 of I is for a finite k, we extend it analogously for an infinite k.

Our definition above has been introduced by models. This is sufficient, however, since every logic has a model.

DEFINITION 1.4. *For* $\Lambda\neq\phi$, $\bigcup_{\lambda\in\Lambda}\boldsymbol{M}_\lambda$ *is the set of wffs* A *for which there exists a finite sequence of wffs* $B_1, B_2, \cdots, B_n=A$ *such that, for each* i, *one of the following conditions holds*:

(i) *there exists* $\lambda\in\Lambda$ *such that* $B_i\in\boldsymbol{M}_\lambda$,

(ii) *there exist integers* j, $k<i$ *such that* $B_j=B_k\supset B_i$.

If $\Lambda=\{1, 2, \cdots, n\}$, then $\bigcap_{\lambda\in\Lambda}\boldsymbol{M}_\lambda$ and $\bigcup_{\lambda\in\Lambda}\boldsymbol{M}_\lambda$ will be written as $\boldsymbol{M}_1\cap\boldsymbol{M}_2\cap\cdots\cap\boldsymbol{M}_n$ and $\boldsymbol{M}_1\cup\boldsymbol{M}_2\cup\cdots\cup\boldsymbol{M}_n$, respectively.

Let $\mathscr{L}$ be the set of the logics. In I, it is proved that $\mathscr{L}$ forms a lattice structure with $\supset$ as the partial order relation. Now, the following theorem is immediate from the above definitions.

THEOREM 1.5. $\mathscr{L}$ *is a complete lattice.*

Further, we have the

THEOREM 1.6. *If* $\Lambda\neq\phi$, $(\bigcup_{\lambda\in\Lambda}\boldsymbol{M}_\lambda)\cap\boldsymbol{N}\supset\subset\bigcup_{\lambda\in\Lambda}(\boldsymbol{M}_\lambda\cap\boldsymbol{N})$.

PROOF. Suppose that $A\in(\bigcup_{\lambda\in\Lambda}\boldsymbol{M}_\lambda)\cap\boldsymbol{N}$. Then there exist wffs $B_1, B_2, \cdots, B_m\in\bigcup_{\lambda\in\Lambda}\boldsymbol{M}_\lambda$ and $C_1, C_2, \cdots, C_n\in\boldsymbol{N}$ such that $(\bigwedge_{1\leq i\leq m}B_i)\supset A\in\boldsymbol{L}$ and $(\bigwedge_{1\leq j\leq n}C_j)\supset A\in\boldsymbol{L}$. Hence $((\bigwedge_{1\leq i\leq m}B_i)\vee(\bigwedge_{1\leq j\leq n}C_j))\supset A\in\boldsymbol{L}$, and so $(\bigwedge_{\substack{1\leq i\leq m\\1\leq j\leq n}}(B_i\vee C_j))\supset A\in\boldsymbol{L}$. On the other hand, as each B_i belongs to some $\boldsymbol{M}_\lambda$, each $B_i\vee C_j$ belongs to some $\boldsymbol{M}_\lambda\cap\boldsymbol{N}$. So A belongs to $\bigcup_{\lambda\in\Lambda}(\boldsymbol{M}_\lambda\cap\boldsymbol{N})$. Conversely, suppose that $A\in\bigcup_{\lambda\in\Lambda}(\boldsymbol{M}_\lambda\cap\boldsymbol{N})$. Then there exist wffs $B_1, B_2, \cdots, B_m$ such that (i) for any i, there exists $\lambda\in\Lambda$ such that $B_i\in\boldsymbol{M}_\lambda$, (ii) for any i, $B_i\in\boldsymbol{N}$, and (iii) $(\bigwedge_{1\leq i\leq m}B_i)\supset A\in\boldsymbol{L}$. By (i) and (iii), $A\in\bigcup_{\lambda\in\Lambda}\boldsymbol{M}_\lambda$. By (ii) and (iii), $A\in\boldsymbol{N}$. Hence $A\in(\bigcup_{\lambda\in\Lambda}\boldsymbol{M}_\lambda)\cap\boldsymbol{N}$.

COROLLARY 1.7. *$\mathscr{L}$ is relatively pseudo-complemented.*

This is seen from the fact that the theorem 1.6 is a necessary and sufficient condition for a complete lattice to be relatively pseudo-complemented.

REMARK. $\mathscr{L}$ is not complemented. For, if it be complemented, the complement S_2^c of S_2, for example, must satisfy the conditions that $S_2 \cup S_2^c \supset\subset S_1$ (since S_1 is the greatest element) and that $S_2 \cap S_2^c \supset\subset L$ (since L is the least element). On the other hand, $S_2 \cup M \supset\subset S_1$ if and only if $M \supset\subset S_1$. But $S_2 \cap S_1 \supset\subset S_2 \supsetneqq L$.

In order to prove the dual relation of 1.6 in an analogous way, it is needed to deal with infinitely long formulas. So, we do not know at the present if the dual relation generally holds or not.

In the paper I, we wrote that the complement M^c of a logic M might not be defined, which, in fact, was proved true in the above results. On the other hand, after we obtained 1.7, we were informed from Dr. R. A. Bull that he felt, reading § 2 of I, that he had heard something like above 1.7. The result 1.7 is itself interesting in the flow of the subject we are dealing. And we like to cite the result if we know the literature for sure. But, since we do not know yet any unfortunately, we state it here with our own proof.

In I, we defined the Δ-projection only for the axiomatic systems. It will be quite natural to extend it as follows:

DEFINITION 1.8. *Let M be a logic. $\Delta^1(M)$, or simply $\Delta(M)$, is a logic whose wffs A are those for which there exists a finite sequence of wffs $B_1, B_2, \cdots, B_n = A$ such that, for each i, at least one of the following three conditions holds:*

(i) $B_i \in L$,

(ii) *there exists a wff $C \in M$ such that B_i is a substituted case of $\Delta(C)$,*

(iii) *there exist $j, k < i$ such that $B_j = B_k \supset B_i$.*

Further, we define $\Delta^{n+1}(M)$ to be $\Delta(\Delta^n(M))$.

REMARK 1.9. It is obvious that the logic $\Delta^n(LA)$ defined in 7.1 of I is nothing but the above-defined $\Delta^n(LA)$. The results of § 7 of I also hold without any changes for the $\Delta^n(M)$'s above.

THEOREM 1.10. *Let M and N be models. $\Delta^n(M) \subset N \uparrow M$, for any $N \in \mathscr{S}_n$.*

PROOF. Let be that $A \in \Delta^n(M)$. Then there exist wffs $B_1, B_2, \cdots, B_k$ such that (i) each B_i is a substituted case of some $\Delta^n(C_i)$ where $C_i \in M$ and (ii) $\bigwedge_{1 \le i \le k} B_i \supset A \in L$. For an assignment f of $N \uparrow M$, $f(C_i) \le 1_M$, where 1_M is the value in $N \uparrow M$ which corresponds to the designated value of M. Hence $\Delta^n(C_i) = B_i \in N \uparrow M$. So, $A \in N \uparrow M$.

REMARK. $N \uparrow M$ depends on the choice of the model for the logic $\boldsymbol{M}$. But the theorem holds for any choice of the model.

Using this theorom, the theorem 7.8 of I will be extended in § 3.

REMARK 1.11. The proof of 5.3 of I might cause a misunderstanding since, in general, $N \not\ni A$ and $N \not\ni B$ do not imply $N \not\ni A \vee B$ even if A and B do not contain a propositional variable in common. So, here we give a

DETAILED PROOF of 5.3 of I. We start from that $N \not\ni A, Z$. Suppose that $N \ni A \vee Z$. There are some substituted cases $Z_1, Z_2, \cdots, Z_k$ of Z such that $\bigwedge_{1 \leq i \leq k} Z_i \supset A \in N$, since $N + Z \supset \subset S_n \supset M \in A$.
Now $N \supset \boldsymbol{L} \ni (A \vee (\bigwedge_{1 \leq i \leq k} Z_i)) \supset ((A \supset A) \supset ((\bigwedge_{1 \leq i \leq k} Z_i \supset A) \supset A))$.
By the hypothesis, $N \ni A \vee (\bigwedge_{1 \leq i \leq k} Z_i)$ and also $N \ni A \supset A$, $\bigwedge_{1 \leq i \leq k} Z_i \supset A$. Hence $N \in A$, contrary to $N \not\ni A$.

Concerning the axiom schemes for a logic in $\mathscr{S}_n$, we have the following

THEOREM 1.12. *Let be that* $\boldsymbol{L} + A \in \mathscr{S}_n$ $(n < \omega)$ *and that the number of the propositional variables contained in* A *is* k. *Then,* $k \geq n$.

PROOF. We already have lemmas as follows:
(i) $A \in \boldsymbol{S}_\omega$ if and only if $A \in \boldsymbol{S}_{k+1}$ (cf. 3 of [3] or 4.15 of [14]).
(ii) $A \in \boldsymbol{S}_n$ and $A \notin \boldsymbol{S}_{n+1}$ (cf. 4.6 of I).
Now, suppose that $k < n$. By (ii), $A \in \boldsymbol{S}_n \subset \boldsymbol{S}_{k+1}$. So, by (i), $A \in \boldsymbol{S}_\omega \subset \boldsymbol{S}_{n+1}$. This is contradictory to (ii).

REMARK. We do not know yet if there exists a wff B with just n propositional variables such that A and B are interdeducible in $\boldsymbol{L}$. As is well known, $\boldsymbol{S}_n$ and $\boldsymbol{LP}_n$, at least, are axiomatized by adding an n-variable axiom scheme to $\boldsymbol{L}$.

§ 2. Definability of logical connectives.

The logical connective $\supset$ is called definable in a logic $\boldsymbol{M}$ if and only if there exists a formula which does not contain $\supset$ and which is equivalent with $a \supset b$ in $\boldsymbol{M}$. The definability is also defined similarly for $\vee$, & and $\neg$. The problem of determining definability in the intermediate logics has been solved for $\neg$, $\supset$ and & (cf. [39]). Here we give its solution for $\vee$. The parts (i) and (ii) of the following theorem, which represent the solution for $\neg$, $\supset$ and & are included for comparison, and whose proof is borrowed from [39].

THEOREM 2.1. (i) $\neg$ *is definable in no logics.*

(ii) $\supset$ *and* & *are definable only in* $\boldsymbol{S}_1$.

(iii) $\vee$ *is definable only in* $\boldsymbol{S}_n$ $(1 \leq n \leq \omega)$.

PROOF. (i) In the model S_1, the set of value $\{1\}$ is complete with respect to $\supset$, & and $\vee$, while it is not so with respect to $\neg$.

(ii) It is well known that $\supset$ and & are definable in S_1. Let M be the model $S_1^2 \uparrow S_1$. As to the truth tables of M, we refer to the model $L_{5,2}$ shown in [14]. Then, the set of values $\{1, 2, 4, \omega\}$ (or $\{1, 2, 3, \omega\}$) is complete with respect to &, $\vee$, $\neg$ (or, $\supset$, $\vee$, $\neg$), while $2 \supset 4 = 3$ (or, $2 \& 3 = 4$). Since $M \supset\subset S_2$ and since there is no logic between S_1 and S_2, (ii) is proved.

(iii) As is known, $\vee$ is definable in S_n, since $a \vee b$ is equivalent with $((a \supset b) \supset b) \& ((b \supset a) \supset a)$ in S_ω (cf. [3]). Suppose that there exists a formula $F(a, b, c_1, c_2, \cdots, c_k)$ such that F contains at the most the propositional variables $a, b, c_1, c_2, \cdots, c_k$ and the logical connectives $\supset$, & and $\neg$, and that F is equivalent with $a \vee b$ in a logic M. What we intend is to prove that M contains the formula $(a \supset b) \vee (b \supset a)$, that is, $M \supset S_\omega$.

Let W be the set of wffs recursively defined by the two conditions, that $W \ni a \supset b$, $b \supset a$, where a and b are propositional variables, and that, if $A, B \in W$, then $A \supset B$, $A \& B$, $\neg A \in W$. By identifying intuitionistically equivalent formulas of W, we define W^* as the set of the equivalent classes. It can be ascertained by calculation that W^* contains only a finite number of elements and that the classes W_i $(1 \leq i \leq 4)$ represented by $(a \supset b) \supset (a \supset b)$, $\neg\neg(a \supset b) \supset (a \supset b)$, $\neg\neg(b \supset a) \supset (b \supset a)$ and $(\neg\neg(a \supset b) \supset (a \supset b)) \& (\neg\neg(b \supset a) \supset (b \supset a))$, respectively, are all of the classes that consist of classically valid formulas.

By the hypothesis, $((a \supset b) \vee (b \supset a)) \equiv F'$, where F' is the formula $F(a \supset b, b \supset a, a \supset b, a \supset b, \cdots, a \supset b)$, belongs to M, hence also to S_1. Since $(a \supset b) \vee (b \supset a) \in S_1$, $F' \in S_1$. Hence, $F' \in W_i$, for some i $(1 \leq i \leq 4)$.

Suppose that $F' \in W_1$. Then, $M \ni (a \supset b) \vee (b \supset a)$, since $F' \in L \subset M$.

Suppose that $F' \in W_2$. Since $F' \supset (\neg\neg(a \supset b) \supset (a \supset b)) \in L \subset M$, $((a \supset b) \vee (b \supset a)) \supset (\neg\neg(a \supset b) \supset (a \supset b)) \in M$. Substituting a by an intuitionistically provable formula, we obtain that $\neg\neg b \supset b \in M$. This means that $M \supset\subset S_1$.

The cases $F' \in W_3$ and $F' \in W_4$ are dealt with similarly as the case $F' \in W_2$.

q. e. d.

REMARK 2.2. Let F' and F'' be formulas such that $((a \supset b) \equiv F') \in S_1$ and $((a \& b) \equiv F'') \in S_1$. Then we have that $L + ((a \supset b) \equiv F') \supset\subset L + ((a \& b) \equiv F'') \supset\subset S_1$. On the other hand, let F be a formula such that $((a \vee b) \equiv F) \in S_1$. We only have that $L + ((a \vee b) \equiv F) \supset S_\omega$. There exists, however, such F_n that $((a \vee b) \equiv F_n) \in S_1$ and that $L + ((a \vee b) \equiv F_n) \supset\subset S_n$. An example of F_n is $(((b \supset c) \supset c) \& ((c \supset b) \supset b)) \& R_n$, which is equivalent with $b \vee c$.

§ 3. Proper elements of $\mathscr{S}_n$.

DEFINITION 3.1. *The element of* $\mathscr{S}_1$ *is proper. An element* $\boldsymbol{M}$ *of* $\mathscr{S}_n$ ($2 \leq n \leq \omega$) *is proper if, for any* $i<n$ *and for any* $N \in \mathscr{S}_i$, *the relation* $N(n) \supset\subset \boldsymbol{M}$ *does not hold.*

Using this notion, we can give a classification of each slice as follows.

DEFINITION 3.2.

$$\begin{cases} \mathscr{P}_n(n) = \{N;\ N \text{ is proper in } \mathscr{S}_n\} \ (1 \leq n \leq \omega), \\ \mathscr{P}_{n+1}(k) = \{N(n+1);\ N \in \mathscr{P}_n(k)\} \ (1 \leq k \leq n < \omega), \\ \mathscr{P}_\omega(k) = \{N(\omega);\ N \in \mathscr{P}_k(k)\} \ (1 \leq k < \omega). \end{cases}$$

$\mathscr{P}_\omega(k)$ is abbreviated as $\mathscr{P}(k)$ ($1 \leq k \leq \omega$).

LEMMA 3.3. *Let be that* $\boldsymbol{M} \in \mathscr{S}_n$ ($2 \leq n < \omega$). *Then,* $\boldsymbol{M}$ *is proper if and only if* $\boldsymbol{M} \not\supset \boldsymbol{LP}_{n-1} \cap \boldsymbol{S}_n$.

This is seen from 1.1. It is obvious that $\boldsymbol{LP}_n$ is an example of proper elements of $\mathscr{S}_n$ ($1 \leq n < \omega$). By this, it is seen that each $\mathscr{P}_n(k)$ is non-void. Other examples of proper elements will be shown below.

THEOREM 3.4. *Let* $\boldsymbol{M}$ *and* $\boldsymbol{N}$ *be models in* $\mathscr{S}_m$ *and* $\mathscr{S}_n$ ($n<\omega$), *respectively. If* $\boldsymbol{S}_{m+n} \supsetneqq \boldsymbol{M} \uparrow \boldsymbol{N}$, *then* $\mathscr{S}_{m+n} \ni \boldsymbol{M} \uparrow \boldsymbol{N} \not\supset \boldsymbol{LP}_n(m+n)$, *where we mean* ω *by* $m+n$ *when* $m=\omega$.

PROOF. Let R_ω be the wff Z. Then $\boldsymbol{LP}_n(m+n)$ can be axiomatized as $\boldsymbol{L}+P_n \vee R_{m+n}$, where we suppose that P_n and R_{m+n} do not contain a propositional variable in common. By hypothesis, $\boldsymbol{S}_{m+n} \supsetneqq \boldsymbol{M} \uparrow \boldsymbol{N} \ni R_{m+n}$. And, since even $\boldsymbol{S}_1 \uparrow \boldsymbol{N} \in \mathscr{S}_{n+1}$ does not contain P_n, there is an assignment function f of $\boldsymbol{M} \uparrow \boldsymbol{N}$ such that $f(P_n) \geq 1_N$, where 1_N is the undesignated value in $\boldsymbol{M} \uparrow \boldsymbol{N}$ which corresponds to 1_N in $\boldsymbol{N}$. Since P_n and R_{m+n} do not contain a propositional variable in common, we can also suppose that $f(R_{m+n}) \neq 1$. Since $f(P_n) \geq 1_N$, either $f(P_n) \vee f(R_{m+n}) \geq 1_N > 1$ or $f(P_n) \vee f(R_{m+n}) = f(R_{m+n}) > 1$, according as $f(R_{m+n}) \geq 1_N$ or $< 1_N$.

COROLLARY 3.5. *Under the hypothesis of* 3.4, $\boldsymbol{M} \uparrow \boldsymbol{N}$ *is proper in* $\mathscr{S}_{m+n}$ *if* $m \neq \omega$.

THEOREM 3.6. *If* $\boldsymbol{M} \in \mathscr{S}_m$ ($m<\omega$), *then* $\Delta^n(\boldsymbol{M})$ *is proper in* $\mathscr{S}_{m+n}$.

PROOF. By choosing a suitable model $\boldsymbol{N}$ for $\boldsymbol{M}$, we can suppose that $\boldsymbol{S}_n \uparrow \boldsymbol{N} \subsetneqq \boldsymbol{S}_{m+n}$ (in general, $\boldsymbol{M}^2$ suffices as $\boldsymbol{N}$ for any model $\boldsymbol{M}$). By 3.4, $\boldsymbol{S}_n \uparrow \boldsymbol{N} \not\supset \boldsymbol{LP}_m(m+n)$. Hence by 1.10, $\Delta^n(\boldsymbol{M}) \not\supset \boldsymbol{LP}_m(m+n)$.

COROLLARY 3.7. *If* $\boldsymbol{M} \in \mathscr{S}_2$, $\Delta^n(\boldsymbol{M}) \supsetneqq \boldsymbol{LP}_{n+1} \cap \boldsymbol{S}_{n+2}$. (An extension of 7.8, I.)

DEFINITION 3.8.

$$\begin{cases} \Omega(i, j) = \bigcup_{1 \leq k \leq \min(i, j)} \mathscr{P}_j(k), \\ \Omega(i) = \Omega(i, \omega), \end{cases}$$

LEMMA 3.9. *If* $i<j$, $\Omega(i,j)\subseteq\mathscr{S}_j$, *and if* $i\geq j$, $\Omega(i,j)=\mathscr{S}_j$.

LEMMA 3.10. $\boldsymbol{M}\in\Omega(i)$ *if and only if* $\boldsymbol{M}\in\mathscr{S}_\omega$ *and* $\boldsymbol{M}\supset\boldsymbol{LP}_i\cap\boldsymbol{S}_\omega$.

These are almost immediate.

THEOREM 3.11. *If* $i\leq j$, *then* $\Omega(i,j)$ *and* $\mathscr{S}_i$ *are isomorphic as lattices.*

This fact comes from 5.3 and 5.6 of I, and 1.1 and 1.2 of this paper.

LEMMA 3.12. $\Omega(1)\subsetneqq\Omega(2)\subsetneqq\cdots\subsetneqq\Omega(n)\subsetneqq\cdots\subsetneqq\mathscr{S}_\omega$. *And,* $\lim_{i\to\omega}\Omega(i)\subsetneqq\mathscr{S}_\omega$.

PROOF. The first part is obvious from the definition. As to the second part, $\boldsymbol{L}$ is an example of the logics which do not belong to any $\Omega(i)$.

DEFINITION 3.13. $\boldsymbol{M}\succ\boldsymbol{N}$ *if* $\boldsymbol{M}\supsetneqq\boldsymbol{N}$ *and if, for any logic* $\boldsymbol{P}$, $\boldsymbol{M}\supset\boldsymbol{P}\supset\boldsymbol{N}$ *implies either* $\boldsymbol{M}\supset\subset\boldsymbol{P}$ *or* $\boldsymbol{P}\supset\subset\boldsymbol{N}$.

REMARK. This relation corresponds to the "immediate successor or predecessor" of Troelstra [36].

COROLLARY 3.14. *If* $\boldsymbol{M},\boldsymbol{N}\in\mathscr{S}_m$ $(m<\omega)$ *and* $\boldsymbol{M}\succ\boldsymbol{N}$, *then* $\boldsymbol{M}(m+n)\succ\boldsymbol{N}(m+n)$ $(1\leq n\leq\omega)$.

§4. Θ-projection.

The Θ-projection, which is going to be defined in this §, projects each slice into $\mathscr{S}_\omega$. As a by-product, we find again proper elements.

DEFINITION 4.1. *For a wff* A, $\Theta^k(A)$ *is the formula obtained by substituting each propositional variable of* A *by its* k*-fold negation.*

$\Theta^1(A)$ will be abbreviated as $\Theta(A)$. $\Theta(a\vee\neg a)$, for example, is $\neg a\vee\neg\neg a$. It is almost obvious that $\Theta(A)$ and $\Theta^3(A)$ are intuitionistically equivalent.

THEOREM 4.2. *If* $A\in\boldsymbol{S}_1$ *and* $\Theta(A)\in\boldsymbol{L}$, *then the wff* A *necessarily contains the logical connective* $\vee$.

Proof. Suppose that A does not contain $\vee$. Then A is equivalent in $\boldsymbol{L}$ with a formula of the form $A_1\&A_2\cdots\&A_n$ where each A_i is of the form (i) $\neg B_i$, or (ii) $B_i\supset a_i$ where a_i is a propositional variable. Since $A\in\boldsymbol{S}_1$, $A_i\in\boldsymbol{S}_1$. It is easily seen that $\neg B_i\in\boldsymbol{S}_1$ implies $\neg B_i\in\boldsymbol{L}$, hence $\Theta(\neg B_i)\in\boldsymbol{L}$, and that $B_i\supset a_i\in\boldsymbol{S}_1$ implies $\Theta(B_i)\supset\neg a_i\in\boldsymbol{L}$. Hence $\Theta(A)\in\boldsymbol{L}$. This is contradictory.

DEFINITION 4.3. *For a logic* $\boldsymbol{M}$, $\Theta^k(\boldsymbol{M})$ *is the logic* $\boldsymbol{L}+\sum\Theta^k(A)$, *which is obtained by adding to* $\boldsymbol{L}$ *the formulas of the form* $\Theta^k(A)$ $(A\in\boldsymbol{M})$ *as axioms.*

Again, $\Theta^1(\boldsymbol{M})$ will be abbreviated as $\Theta(\boldsymbol{M})$. And it is almost immediate that $\Theta^{m+n}(\boldsymbol{M})\supset\subset\Theta^m(\Theta^n(\boldsymbol{M}))$. Also obvious is the following

LEMMA 4.4. $\boldsymbol{M}\supset\Theta(\boldsymbol{M})$.

THEOREM 4.5. $\boldsymbol{M}\ni\neg a\vee\neg\neg a$ *if and only if* $\Theta(\boldsymbol{M})\supset\subset\boldsymbol{L}+\neg a\vee\neg\neg a$.

PROOF. The if-part comes from 4.4. Conversely, suppose that $\boldsymbol{M}\ni\neg a\vee\neg\neg a$.

Since $\Theta(M)\ni\neg\neg\neg a\vee\neg\neg\neg\neg a$ and since $\neg a\vee\neg\neg a$ and $\neg\neg\neg a\vee\neg\neg\neg\neg a$ are equivalent in L, $\Theta(M)\ni\neg a\vee\neg\neg a$, that is, $\Theta(M)\supset L+\neg a\vee\neg\neg a$. Next, suppose that $\Theta(M)\ni A$. Then there exists a wff $B\in M$ such that $\Theta(B)\supset A\in L$. Since B also belongs to S_1, there is some substituted cases $C_1, C_2, \cdots, C_n$ of $a\vee\neg a$ such that $(C_1\,\&\,C_2\,\&\cdots\&\,C_n)\supset B\in L$. Hence,
$(\Theta(C_1)\,\&\,\Theta(C_2)\,\&\cdots\&\,\Theta(C_n))\supset\Theta(B)\in L$. So,
$(\Theta(C_1)\,\&\,\Theta(C_2)\,\&\cdots\&\,\Theta(C_n))\supset A\in L$. This means that $A\in L+\neg a\vee\neg\neg a$.

LEMMA 4.6. *If $M\supset N$, then $\Theta(M)\supset\Theta(N)$.*

This is immediate from the definition. By 4.5, it is denied that, if $M\supsetneqq N$, then $\Theta(M)\supsetneqq\Theta(N)$.

LEMMA 4.7. *For any logic M, $\Theta(M)\subset L+\neg a\vee\neg\neg a$.*

PROOF. Since $S_1\supset M$, $\Theta(S_1)\supset\Theta(M)$. On the other hand, $\Theta(S_1)\supset\subset L+\neg a\vee\neg\neg a$.

LEMMA 4.8. *Let M be a model whatever. Then, $\neg a\vee\neg\neg a\in M\uparrow S_1$.*

PROOF. In $M\uparrow S_1$, let ω be the greatest value, which corresponds to $\neg 1$. Then, by the definition of $M\uparrow S_1$, $\neg v=1$ if and only if $v=\omega$, and $\neg v=\omega$ if $v\neq\omega$. Hence, $\neg v\vee\neg\neg v=1$ for any value v.

COROLLARY 4.9. *Let M be a model whatever. Then, $M\uparrow S_1\supset L+\neg a\vee\neg\neg a$.*

COROLLARY 4.10. *For any logic M, $\Theta(M)\in\mathscr{S}_\omega$.*

DEFINITION 4.11. $Q=(\neg a\supset b)\supset((\neg\neg a\supset b)\supset b)$.

$$LQ_n=L+Q+P_n \qquad (1\leq n<\omega)\,.$$

$$(LQ_n\supset\subset LQ\cup LP_n)\,.$$

The three lemmas which follow are almost immediate.

LEMMA 4.12. *Q and $\neg a\vee\neg\neg a$ are interdeducible in L.*

LEMMA 4.13. *$LQ\in\mathscr{S}_\omega$ and $LQ_n\in\mathscr{S}_n$.*

LEMMA 4.14. $LQ_1\supset\subset S_1$.

THEOREM 4.15. $LQ_2\supset\subset S_2$.

PROOF. Let P be the formula obtained from P_2 by replacing a_0, a_1, a_2 by $b\,\&\,\neg b, b, Z$, respectively. Then, it is easily seen that the formula $(\neg b\vee\neg\neg b)\supset(P\supset Z)$ is provable in L. Hence $LQ_2\ni Z$. This means that $LQ_2\supset S_2$. The converse part comes from 4.13.

THEOREM 4.16. *If $i\geq 3$, LQ_i does not have a finite model.*

PROOF. Let M_i be $S_{i-2}\uparrow S_1^\omega\uparrow S_1\in\mathscr{S}_i$. It is obvious that $M_i\ni Q, P_i$. But M_i does not contain any X_n's (cf. 6.10 of I). Hence LQ_i does not contain any X_n's since $M_i\supset LQ_i$. By 6.8 of I, LQ_i does not have a finite model.

COROLLARY 4.17. *If $i\geq 3$, $S_i\supsetneqq LQ_i$.*

THEOREM 4.18. *If* $i \geq 3$, $\boldsymbol{LQ}_i$ *is proper in* $\mathscr{S}_i$.

PROOF. Let $\boldsymbol{M}_i$ be $\boldsymbol{S}_{i-2} \uparrow \boldsymbol{S}_1^2 \uparrow \boldsymbol{S}_1 \in \mathscr{S}_i$. Since $\boldsymbol{M}_i \ni Q, P_i$, $\boldsymbol{M}_i \supset \boldsymbol{LQ}_i$. But, $\boldsymbol{M}_i \ni P_{i-1} \vee R_i$, where we suppose that P_{i-1} and R_i do not contain a propositional variable in common. Hence $\boldsymbol{LQ}_i \not\supset \boldsymbol{LP}_{i-1} \cap \boldsymbol{S}_i$. By 1.1, $\boldsymbol{LQ}_i$ is proper.

DEFINITION 4.19 *For a logic* $\boldsymbol{M}$, $\boldsymbol{M}^+$ *is the set of the wffs of* $\boldsymbol{M}$ *such that they do not contain the logical connective* $\neg$.

LEMMA 4.20. *In* $\boldsymbol{M} \uparrow \boldsymbol{N}$, *the set of the values which formerly belonged to* $\boldsymbol{M}$ *is closed with respect to* $\supset$, & *and* $\vee$.

COROLLARY 4.21. $\boldsymbol{M}^+ \supset (\boldsymbol{M} \uparrow \boldsymbol{N})^+$.

These are almost immediate from the definition.

The following theorem shows an interesting feature of $\boldsymbol{LQ}$. This is seen also in McKay's [25] with a different proof.

THEOREM 4.22. $(\boldsymbol{LQ})^+ \supset\subset \boldsymbol{L}^+$.

PROOF. Let $\boldsymbol{J}_n$ be the model defined by Jaśkowski. By 4.21 and 4.10, for any n,

$$(\boldsymbol{J}_n)^+ \supset (\boldsymbol{J}_n \uparrow \boldsymbol{S}_1)^+ \supset (\boldsymbol{LQ})^+ .$$

Since $\bigcap_{1 \leq n < \omega} (\boldsymbol{J}_n) \supset\subset \boldsymbol{L}$, $\boldsymbol{L}^+ \supset (\boldsymbol{LQ})^+$. The converse part is obvious.

REMARK. By 4.22, $\boldsymbol{LQ}$ cannot be axiomatized by adding to $\boldsymbol{L}$ an axiom scheme without the logical connective $\neg$. The axiom scheme Q is one of the simplest in the sense that it contains only $\supset$ and $\neg$.

LEMMA 4.23. If $\boldsymbol{M}^+ \supset\subset \boldsymbol{L}^+$, *then* $\boldsymbol{M} \in \mathscr{S}_\omega$.

PROOF. If $\boldsymbol{M} \in \mathscr{S}_n$ $(n < \omega)$, the formula P_n, which does not contain $\neg$, belongs to $\boldsymbol{M}$, contrary to $\boldsymbol{M}^+ \supset\subset \boldsymbol{L}^+$.

THEOREM 4.24. $\boldsymbol{LQ}$ *is proper in* $\mathscr{S}_\omega$.

PROOF. Suppose that $\boldsymbol{LQ}$ is not proper, that is, $\boldsymbol{LQ} \in \Omega(i)$ for some i. Then, by 3.10, $\boldsymbol{LQ} \supset \boldsymbol{LP}_i \cap \boldsymbol{S}_\omega \ni P_i \vee Z$, which is a wff not containing $\neg$ and not belonging to $\boldsymbol{L}$. This is contradictory.

LEMMA 4.25. *If* $\boldsymbol{M} \in \Omega(i)$, *then there exists a wff* A *such that* (i) A *contains only the logical connective* $\supset$ *and that* (ii) $A \in \boldsymbol{M}$ *but* $A \notin \boldsymbol{L}$.

PROOF. Let A be the wff $(P_i \supset d) \supset ((Z \supset d) \supset d)$. Then, A is interdeducible with $P_i \vee Z$ in $\boldsymbol{L}$. By 3.10, $A \in \boldsymbol{M}$.

THEOREM 4.26. *If* $i \geq 3$, $\boldsymbol{LQ}_i(\omega) \supset\subset \boldsymbol{LQ} \cup \boldsymbol{LP}_i(\omega)$ *and* $\boldsymbol{LQ}_i(\omega) \supsetneqq \boldsymbol{LP}_i(\omega)$.

PROOF. $\boldsymbol{LQ}_i(\omega) = \boldsymbol{LQ}_i \cap \boldsymbol{S}_\omega \supset\subset (\boldsymbol{LQ} \cup \boldsymbol{LP}_i) \cap \boldsymbol{S}_\omega \supset\subset (\boldsymbol{LQ} \cap \boldsymbol{S}_\omega) \cup (\boldsymbol{LP}_i \cap \boldsymbol{S}_\omega) \supset\subset \boldsymbol{LQ} \cup \boldsymbol{LP}_i(\omega)$. Next, suppose that $\boldsymbol{LP}_i(\omega) \supset\subset \boldsymbol{LQ}_i(\omega)$, which is the relation that holds if and only if $\boldsymbol{LP}_i(\omega) \supset \boldsymbol{LQ}$. Hence $\boldsymbol{LP}_2 \ni Q$. This means that $\boldsymbol{LP}_2 \cup \boldsymbol{LQ} \supset\subset \boldsymbol{LP}_2$, contrary to 4.15.

THEOREM 4.27. $\boldsymbol{LQ}_n \not\subset \boldsymbol{LP}_2 \cap \boldsymbol{S}_n$ $(n \geq 3)$.

PROOF. If $LP_2 \cap S_n \supset LQ_n$, then $LP_2 \cap S_2 \ni Q$. Hence, $LP_2 \supset LQ$, that is, $LP_2 \cup LQ \supset \subset LP_2$. This is contrary to 4.15.

By this theorem and 4.18, we know that LQ_n and $LP_2 \cap S_n$ are incomparable by the relation $\supset$.

LEMMA 4.28. *If* $M \in \mathscr{P}_m(m)$ *and* $N \in \mathscr{P}_n(n)$ *and* $k = \max(m, n)$, *then* $M \cap N \in \mathscr{P}_k(k)$.

This is almost immediate.

THEOREM 4.29. *Let* $PQ(m, n)$ *be* $LP_m(\omega) \cap LQ_n(\omega)$. *Then, for* $m=1$ *and* $k \geq 1$, *and for* $m \geq 2$ *and* $k \geq 0$, $PQ(m, m+k) \supsetneq PQ(m, m+k+1)$.

PROOF. It is obvious that $PQ(m, m+k) \supset PQ(m, m+k+1)$. By 4.28, the strict inclusion is ascertained.

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Bibliography

[1] R. A. Bull: The implicational fragment of Dummett's LC, J. Symbolic Logic, **27** (1962), 189-194.

[2] R. A. Bull: Some results for implicational calculi, J. Symbolic Logic, **29** (1964), 33-39.

[3] M. Dummett: A propositional calculus with denumerable matrix, J. Symbolic Logic, **24** (1959), 97-106.

[4] M. Dummett and E. J. Lemmon: Modal logics between S4 and S5, Zeitschr. f. math. Logik und Grundlagen d. Math., **5** (1959), 250-264.

[5] K. Gödel: Zum intuitionistischen Aussagenkalkül, Akademie der Wissenschaften in Wien, Mathematisch-naturwissenschaftliche Klasse, Anzeiger, **69** (1932), 65-66.

[6] M. Hanazawa: A characterization of axiom schema playing the rôle of Tertium non Datur in intuitionistic logic, Proc. Japan Acad., **42** (1966), 1007-1010.

[7] R. Harrop: On the existence of finite models and decision procedures for propositional calculi, Proc. Camb. Phil. Soc., **54** (1958), 1-13.

[8] T. Hosoi: The separation theorem on the classical system, J. Fac. Sci., Uuiv. Tokyo, Sec. I, **12** (1966), 223-230.

[9] T. Hosoi: On the separation theorem of intermediate propositional calculi, Proc. Japan Acad., **42** (1966), 535-538.

[10] T. Hosoi: Algebraic proof of the separation theorem on Dummett's LC, Proc. Japan Acad., **42** (1966), 693-695.

[11] T. Hosoi: The axiomatization of the intermediate propositional systems S_n of Gödel, J. Fac. Sci., Univ. Tokyo, Sec. I, **13** (1966), 183-187.

[12] T. Hosoi: The separable axiomatization of the intermediate propositional systems S_n of Gödel, Proc. Japan Acad., **42** (1966), 1001-1006.

[13] T. Hosoi: A criterion for the separable axiomatization of Gödel's S_n, Proc. Japan Acad., **43** (1967), 365-368.

[14] T. Hosoi: On the axiomatic method and the algebraic method for dealing with propositional logics, J. Fac. Sci., Univ. Tokyo, Sec. I, **14** (1967), 131-169.

[15] T. Hosoi: On intermediate logics I, J. Fac. Sci., Univ. Tokyo, Sec. I, **14** (1967), 293-312.

[16] S. Jaśkowski: Recherches sur le système de la logique intuitioniste, Actualités scientifiques et industrielles, **393** (1936), Paris, 58-61.
[17] J. Kalicki: Note on truth-tables, J. Symbolic Logic, **15** (1950), 174-181.
[18] J. Kalicki: A test for the existence of tautologies according to many-valued truth-tables, J. Symbolic Logic, **15** (1950), 182-184.
[19] J. Kalicki: A test for the equality of truth-tables, J. Symbolic Logic, **17** (1952), 161-163.
[20] J. Kalicki: An undecidable problem in the algebra of truth-tables, J. Symbolic Logic, **19** (1954), 172-176.
[21] G. Kreisel and H. Putnam: Eine unableitbarkeitsbeweismethode für den intuitionischen Aussagenkalkül, Arch. f. Math. Log., **3** (1957), 74-78.
[22] C.G. McKay: A note on the Jaśkowski sequence, Zeitschr. f. math. Logic und Grundlagen d. Math., **13** (1967), 95-96.
[23] C.G. McKay: On finite logics, Indag. Math., **29** (1967), 363-365.
[24] C.G. McKay: The non-separability of a certain finite extension of Heyting's propositional logic, Indag. Math., **30** (1968), 312-315.
[25] C.G. McKay: The decidability of certain intermediate propositional logics, J. Symbolic Logic, **33** (1968), 258-264.
[26] J.C.C. McKinsey and A. Tarski: Some theorems about the sentential calculi of Lewis and Heyting, J. Symbolic Logic, **13** (1948), 1-15.
[27] C.A. Meredith: Postulates for implicational calculi, J. Symbolic Logic, **31** (1966), 7-9.
[28] S. Miura: A remark on the intersection of two logics, Nagoya Math. J., **26** (1966), 167-171.
[29] S. Miura and S. Nagata: Certain method for generating a series of logics, Nagoya Math. J., **31** (1968), 125-129.
[30] S. Nagata: A series of successive modifications of Peirce's rule, Proc. Japan Acad., **42** (1966), 859-861.
[31] I. Nishimura: On formulas of one variable in intuitionistic propositional calculus, J. Symbolic Logic, **25** (1960), 327-331.
[32] A.N. Prior: Two additions to positive implication, J. Symbolic Logic, **29** (1964), 31-32.
[33] G.F. Rose: Propositional calculus and realizability, Transactions of the Amer. Math. Soc., **75** (1953), 1-19.
[34] K. Segerberg: Propositional logics related to Heyting's and Johansson's, Theoria, **34** (1968), 26-61.
[35] I. Thomas: Finite limitations on Dummett's LC, Notre Dame J. of Formal Logic, **3** (1962), 170-174.
[36] A.S. Troelstra: On intermediate propositional logics, Indag. Math., **27** (1965), 141-152.
[37] T. Umezawa: Über Zwischensysteme der Aussagenlogik, Nagoya Math. J., **9** (1955), 181-189.
[38] T. Umezawa: On intermediate many-valued logics, J. Math. Soc. Japan, **11** (1959), 116-128.
[39] T. Umezawa: On intermediate propositional logics, J. Symbolic, Logic, **24** (1959), 20-36.
[40] T. Umezawa: On logics intermediate between intuitionistic and classical predicate logic, J. Symbolic Logic, **24** (1959), 141-153.
[41] T. Umezawa: On some properties of intermediate logics, Proc. Japan Acad., **35**

(1959), 575-577.
[42] T. Umezawa: On an application of intermediate logics, Nagoya Math. J., **16** (1960), 119-133.

(Received July 24, 1968).

(Revised November 22, 1968).