# *On singular submodule of simple ring*

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## Introduction

Let $K$ be a division ring and $K_n$ the total matrix ring of degree $n$ over $K$. As is easily observed, every maximal left ideal of the ring $K_n$ is of dimension $n^2-n$ regarded as a left $K$-module of $K_n$, is *singular* in the sense that it consists only of non-invertible matrices and is *maximal* with respect to being a singular left $K$-submodule of $K_n$. On the other hand, $F$ being a field of characteristic 0, T. Sato proved (cf. [2]) that, except for the special linear Lie algebra $\mathfrak{sl}(n, F)$, there is no proper Lie subalgebra of the general linear Lie algebra $\mathfrak{gl}(n, F)$ of degree $n$ over $F$ which has the dimension larger than $n^2-n+1$.

Based upon these circumstances, Prof. N. Iwahori raised a conjecture that there is no singular $K$-submodule of $K_n$ of dimension larger than $n^2-n$, further the singular $K$-submodules attaining the maximal dimension $n^2-n$ are given as maximal (one-sided) ideals of the ring $K_n$. The purpose of this paper is to prove this conjecture. By Wedderburn's structure-theorem, every simple ring is isomorphic to a ring $K_n$ for a certain division ring $K$ and a positive integer $n$, so our result may have a ring-theoretical interest to some extent.

As a tool for the proof, we introduce a notion of an $\mathfrak{M}$-*stance* for a left $K$-module $\mathfrak{M}$ of $K_n$. This is an intrinsic version of the well-known procedure in linear algebra to reduce a matrix over $K$ to a so-called specified echellon type

$$\begin{pmatrix} 0\cdots 0\,1\,*\cdots *\,0\,*\cdots *\,0\,*\cdots \\ 0\cdots\cdots\cdots 0\,1\,*\cdots *\,0\,*\cdots \\ 0\cdots\cdots\cdots\cdots\cdots 0\,1\,*\cdots \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \end{pmatrix}$$

by elementary row-operations. A somewhat combinatorial operation on an $\mathfrak{M}$-stance plays an essential rôle for the investigation of the generic form of the matrices in $\mathfrak{M}$.

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## 1. $\mathfrak{M}$-stance

Let $K$ be a division ring, $I$ a set of indices, $K^I$ the left $K$-module of all mappings $\boldsymbol{a}$: $I \to K$ and $\mathfrak{M}$ a finite dimensional left $K$-submodule of $K^I$. Let $\boldsymbol{e}_k$ denote for each $k \in I$ the mapping $\boldsymbol{e}_k(i) = \delta_{ki}(i \in I)$. For each subset $J \subset I$ we denote by $\pi_J$ an endomorphism of $K^I$, or rather a projection, such that

$$\pi_J \boldsymbol{a} = \sum_{j \in J} \boldsymbol{a}(j) \boldsymbol{e}_j \text{ (for every } \boldsymbol{a} \in K^I) .$$

Definition of Stance. A *stance* of $\mathfrak{M}$, or an $\mathfrak{M}$-*stance*, is a pair $(J, \varphi)$ of a non-empty subset $J \subset I$ and a mapping (an injection in fact) $\varphi$: $J \to \mathfrak{M}$ which satisfies the following conditions:

i) $\mathfrak{M} = \sum_{j \in J} K\varphi(j)$ ,

ii) $\pi_J \varphi(j) = \boldsymbol{e}_j$ (for every $j \in J$) .

As one can see easily, the defining properties can be re-stated in the following way:

i)′ $|J| = \dim_K \mathfrak{M}$ ,

ii)′ $\operatorname{Supp}(\varphi(j) - \boldsymbol{e}_j) \cap J = \varnothing$ (for every $j \in J$),

where $|J|$ denotes the cardinal number of the set $J$ and $\operatorname{Supp}(\boldsymbol{a})$ the support of the mapping $\boldsymbol{a}$(i.e., $\operatorname{Supp}(\boldsymbol{a}) = \{i \in I \mid \boldsymbol{a}(i) \neq 0\}$).

If $(J, \varphi)$ is an $\mathfrak{M}$-stance and $L \subset J$, then $(L, \varphi|L)$ is a stance for the $K$-submodule $\sum_{j \in L} K\varphi(j)$.

If $(J, \varphi)$ is a stance of $\mathfrak{M}$, then the elements $\varphi(j)$ $(j \in J)$ of $K^I$ are linearly independent and form a basis for the $K$-module $\mathfrak{M}$, which we call *the basis associated with the stance* $(J, \varphi)$. In terms of the basis, every element $\boldsymbol{a} \in \mathfrak{M}$ can be exhibited uniquely as $\boldsymbol{a} = \sum_{j \in J} \boldsymbol{a}(j) \varphi(j)$. Accordingly, the equality $\pi_J \boldsymbol{a} = \pi_J \boldsymbol{b}$ implies $\boldsymbol{a} = \boldsymbol{b}$ provided that $\boldsymbol{a} \in \mathfrak{M}$ and $\boldsymbol{b} \in \mathfrak{M}$. Thus, if $(J, \varphi)$ and $(J, \psi)$ are two $\mathfrak{M}$-stances with the same $J$, then we have $\varphi(j) = \psi(j)$ for all $j \in J$ and consequently $\varphi = \psi$, that is, the mapping $\varphi$ and the associated basis $\{\varphi(j)\}_{j \in J}$ are uniquely determined by the set $J$. So we may call $J$ by itself a stance of $\mathfrak{M}$.

A set $J \subset I$ can be taken to be an $\mathfrak{M}$-stance if and only if the restriction $\boldsymbol{a} \mapsto \boldsymbol{a}|J$ is an isomorphism of $\mathfrak{M}$ onto $K^J$.

If a system $\{\boldsymbol{b}_q\}_{q \in Q}$ of elements $\boldsymbol{b}_q \in K^I$ satisfies the condition

$$\operatorname{Supp}(\boldsymbol{b}_q) - \bigcup_{\substack{p \neq q \\ p \in Q}} \operatorname{Supp}(\boldsymbol{b}_p) \neq \varnothing \text{ (for every } q \in Q) ,$$

then we call it *quasi-disjoint*. The associated basis is obviously quasi-disjoint. Conversely, any quasi-disjoint system $\{\boldsymbol{b}_q\}_{q \in Q}$, with a slight modification, can be taken to be an associated basis for the $K$-module $\mathfrak{M}=\sum_{q\in Q}K\boldsymbol{b}_q$ generated by the system. Exactly speaking, there exists an $\mathfrak{M}$-stance $(J, \varphi)$ with the associated basis $\{\beta_q\boldsymbol{b}_q\}_{q\in Q}$ for suitable scalars $\beta_q\in K$ $(q\in Q)$. To observe this, we select an index $j_q$ arbitrarily out of each non-empty set $\mathrm{Supp}(\boldsymbol{b}_q)-\bigcup_{\substack{p\neq q\\ p\in Q}}\mathrm{Supp}(\boldsymbol{b}_p)$ and bring them together to a set $J=\{j_q\}_{q\in Q}$. If we put $\beta_q=(\boldsymbol{b}_q(j_q))^{-1}$ and $\varphi(j_q)=\beta_q\boldsymbol{b}_q$ for each $q\in Q$, the pair $(J, \varphi)$ satisfies our demand.

Every subset of a quasi-disjoint system is also quasi-disjoint.

PROPOSITION 1.1 (Existence of Stance). *Every finite dimensional $K$-submodule $\mathfrak{M}\neq\{0\}$ of $K^I$ admits a stance.*

Our assertion is trivial in case of $\dim_K\mathfrak{M}=1$, so we suppose that $\mathfrak{M}=\mathfrak{N}\oplus K\boldsymbol{a}$ is a direct sum of two $K$-submodules $\mathfrak{N}$ and $K\boldsymbol{a}$ and that $\mathfrak{N}$ admits a stance $(L, \psi)$. Then the element $\boldsymbol{b}=\boldsymbol{a}-\sum_{l\in L}\boldsymbol{a}(l)\psi(l)$ belongs to $\mathfrak{M}$ and not to $\mathfrak{N}$ and satisfies the relation $\pi_L\boldsymbol{b}=0$. Since $\boldsymbol{b}\neq 0$, there exists an index $j_0\in I$ such that $\boldsymbol{b}(j_0)\neq 0$. Obviously $j_0\notin L$.

Now, let us define a stance $(J, \varphi)$ for the $K$-module $\mathfrak{M}$. We put

$$J=L\cup\{j_0\},$$
$$\varphi(l)=\psi(l)-(\psi(l))(j_0)(\boldsymbol{b}(j_0))^{-1}\boldsymbol{b}\ \text{(for every } l\in L),$$
$$\varphi(j_0)=(\boldsymbol{b}(j_0))^{-1}\boldsymbol{b}\,.$$

Clearly $\varphi$ maps the set $J$ into $\mathfrak{M}$ and we obtain

$$\begin{aligned}\pi_J\varphi(l)&=(\pi_L+\pi_{j_0})\,\varphi(l)\\ &=\pi_L\psi(l)-\pi_{j_0}(\psi(l)-(\psi(l))(j_0)(\boldsymbol{b}(j_0))^{-1}\boldsymbol{b})\\ &=\boldsymbol{e}_l\ \text{(for every } l\in L)\end{aligned}$$

and

$$\pi_J\varphi(j_0)=(\pi_L+\pi_{j_0})(\boldsymbol{b}(j_0))^{-1}\boldsymbol{b}=(\boldsymbol{b}(j_0))^{-1}\pi_{j_0}\boldsymbol{b}=\boldsymbol{e}_{j_0}\,.$$

Thus the pair $(J, \varphi)$ is indeed a stance for $\mathfrak{M}$.

From above, we have obtained also that, if $\mathfrak{N}\subset\mathfrak{M}$ are two finite dimensional $K$-submodules, then every $\mathfrak{N}$-stance $L$ can be extended to an $\mathfrak{M}$-stance $J$.

Conversely, if $(\{0\}\neq)\mathfrak{N}\subset\mathfrak{M}$ are two finite dimensional $K$-submodules of $K^I$, then, for every $\mathfrak{M}$-stance $J$, there exists an $\mathfrak{N}$-stance $L\subset J$.

In fact, then, the restriction $\rho_J$: $\boldsymbol{a}\mapsto\boldsymbol{a}|J$ is an isomorphism of $\mathfrak{M}$ onto $K^J$ and the set $\rho_J(\mathfrak{N})$ is a $K$-submodule of $K^J$. Hence, there exists a stance $L\subset J$ for $\rho_J(\mathfrak{N})$ so that the restriction $\rho'_L$: $\boldsymbol{b}\mapsto\boldsymbol{b}|L$ is an isomorphism of $\rho_J(\mathfrak{N})$ onto $K^L$. Thus the restriction $\rho_L=\rho'_L\circ\rho_J$: $\boldsymbol{a}\mapsto\boldsymbol{a}|L$ is an isomorphism of $\mathfrak{N}$ onto $K^L$ and $L$ is indeed a stance for $\mathfrak{N}$.

More generally, we can prove that, if $\mathfrak{M}=\mathfrak{N}_1\oplus\mathfrak{N}_2$ is a direct sum of two non-zero $K$-submodules $\mathfrak{N}_1$ and $\mathfrak{N}_2$, then, for every $\mathfrak{M}$-stance $J$, there exist an $\mathfrak{N}_1$-stance $L_1$ and an $\mathfrak{N}_2$-stance $L_2$ such that $J=L_1\cup L_2$ (necessarily a direct union). The proof is complicated a little and is omitted.

The next proposition plays the most essential rôle in this paper.

PROPOSITION 1.2 (Change of Stance). *Let* $(J, \varphi)$ *be an* $\mathfrak{M}$*-stance. If* $\varphi(j_0)(k_0)\neq 0$ *for* $j_0\in J$ *and* $k_0\in I-J$, *that is, if* $k_0\in \mathrm{Supp}(\varphi(j_0))-\{j_0\}$, *then we get a new stance for* $\mathfrak{M}$ *by replacing* $J$ *by the set* $(J\cup\{k_0\})-\{j_0\}$.

Put $\mathfrak{N}=\sum_{\substack{j\neq j_0\\ j\in J}} K\varphi(j)$ and $\boldsymbol{b}=\varphi(j_0)$, then the set $L=J-\{j_0\}$ plays a rôle of a stance for $\mathfrak{N}$ and $\pi_L\boldsymbol{b}=\boldsymbol{0}$, $\boldsymbol{b}(k_0)\neq 0$. So the argument carried out in the proof of the proposition 1.1. proves this proposition.

Permutation of $I$ and Automorphism of $K^I$. Let $\sigma$ be a permutation of $I$ (i.e. a bijection $I\to I$). The permutation $\sigma$ induces an automorphism $\boldsymbol{a}\mapsto\boldsymbol{a}\circ\sigma^{-1}$ of the module $K^I$, which we denote by the same symbol $\sigma$, i.e.,

$$\sigma\boldsymbol{a}=\boldsymbol{a}\circ\sigma^{-1} \text{ (for every } \boldsymbol{a}\in K^I).$$

In particular, we have

$$\sigma\boldsymbol{e}_i=\boldsymbol{e}_{\sigma(i)},\quad \pi_{\sigma J}=\sigma\circ\pi_J\circ\sigma^{-1} \text{ (for every } J\subset I).$$

If $\mathfrak{M}$ is a $K$-submodule of $K^I$, then $\sigma\mathfrak{M}=\{\sigma\boldsymbol{a}|\boldsymbol{a}\in\mathfrak{M}\}$ is also a $K$-submodule of $K^I$ and $\sigma$ is an isomorphism between the two.

If $(J, \varphi)$ be an $\mathfrak{M}$-stance, then $(\sigma J, \sigma\circ\varphi\circ\sigma^{-1})$ can be taken to be a stance for $\sigma\mathfrak{M}=\{\sigma\boldsymbol{a}|\boldsymbol{a}\in\mathfrak{M}\}$.

In fact, $\pi_{\sigma J}((\sigma\circ\varphi\circ\sigma^{-1})(\sigma(j)))=(\sigma\circ\pi_J\circ\sigma^{-1}\circ\sigma\circ\varphi\circ\sigma^{-1}\circ\sigma)(j)=\sigma(\pi_J\varphi(j))=\sigma(\boldsymbol{e}_j)=\boldsymbol{e}_{\sigma(j)}$ for every $\sigma(j)\in\sigma(J)$ and $\sum_{\sigma(j)\in\sigma J}K(\sigma\circ\varphi\circ\sigma^{-1})(\sigma(j))=\sum_{j\in J}K\sigma(\varphi(j))=\sigma\mathfrak{M}$.

Let $I=1, 2, \cdots, n$ and let us represent the element $\boldsymbol{a}\in K^I$ by the column $\begin{pmatrix}\boldsymbol{a}(1)\\ \vdots\\ \boldsymbol{a}(n)\end{pmatrix}$, then $\sigma\boldsymbol{a}\in K^I$ is to be represented by the column

$$\begin{pmatrix}(\sigma\boldsymbol{a})(1)\\ \vdots\\ (\sigma\boldsymbol{a})(n)\end{pmatrix}=\begin{pmatrix}\boldsymbol{a}(\sigma^{-1}(1))\\ \vdots\\ \boldsymbol{a}(\sigma^{-1}(n))\end{pmatrix}=\begin{pmatrix} & & \\ & \delta_{\sigma^{-1}(i),k} & \\ & & \end{pmatrix}\begin{pmatrix}\boldsymbol{a}(1)\\ \vdots\\ \boldsymbol{a}(n)\end{pmatrix},$$

where $(\delta_{\sigma^{-1}(i),k})$ is a permutation-matrix whose $(i, k)$-entries are $\delta_{\sigma^{-1}(i),k}$. Consequently, if $J$ is a stance of a $K$-module $\mathfrak{M}$ consisting of columns (resp. rows) of length $n$, then $\sigma J$ is a stance for the $K$-module $S\mathfrak{M}$ (resp. $\mathfrak{M}T$) for a suitable permutation-matrix $S$ (resp. $T$).

From now on, let $\Omega$ be a set of $n$ elements $\{1, 2, \cdots, n\}$ and $I$ the Cartesian product $\Omega\times\Omega=\{(i, k)|i, k\in\Omega\}$. Let us denote by $K_n=K^I=K^{\Omega\times\Omega}$ the total matrix ring of degree $n$ over a division ring $K$ and by $\mathfrak{M}$ a left $K$-submodule of $K_n$.

Here we consider permutations of $I=\Omega\times\Omega$ of type $\mu=\sigma\times\tau$ only, where $\sigma$ and $\tau$ are permutations of $\Omega$ and $(\sigma\times\tau)(i, k)=(\sigma(i), \tau(k))$ for $(i, k)\in\Omega\times\Omega$.

The permutation $\mu=\sigma\times\tau$ induces a permutation of rows and columns in themselves of matrices in $K_n$:

$$A=(\alpha_{i,k})\mapsto\mu A=(\alpha_{\sigma^{-1}(i),\tau^{-1}(k)}) ,$$

which is realized by means of a both-sided multiplication by the two permutation-matrices $S$ and $T$ above mentioned:

$$\mu A=SAT \text{ (for all } A\in K_n) .$$

Thus, if $J$ is a stance of a $K$-submodule $\mathfrak{M}$ of $K_n$, then $\mu J=(\sigma\times\tau)J$ is a stance for the $K$-submodule $S\mathfrak{M}T$ which is isomorphic to $\mathfrak{M}$.

For the convenience in the sequel, we put for $i\in\Omega$ and $k\in\Omega$,

$$_iI=\{i\}\times\Omega \text{ (the } i\text{-th row)} ,$$

$$I_k=\Omega\times\{k\} \text{ (the } k\text{-th collumn)} ,$$

$$\Omega^{(i)}=\{i, i+1, \cdots, n\} ,$$

$$I^{(i)}=\Omega^{(i)}\times\Omega^{(i)} ,$$

$$\triangle=\bigcup_{i\in\Omega}\{i\}\times\Omega^{(i)}=\{(i, k)\in\Omega\times\Omega|i\leqslant k\} .$$

If $\bar{\mu}=\bar{\sigma}\times\bar{\tau}$ is a permutation of the set $I^{(2)}$, then there exists one and only one permutation $\mu=\sigma\times\tau$ of $\Omega\times\Omega$ which extends $\bar{\mu}=\bar{\sigma}\times\bar{\tau}$ and necessarily leaves $_1I$ and $I_1$ invariant (i.e., $\mu\cdot{}_1I={}_1I$, $\mu\cdot I_1=I_1$).

We note also that $(\sigma\times\tau)\,{}_iI={}_{\sigma(i)}I$, $(\sigma\times\tau)I_k=I_{\tau(k)}$.

For the $K$-submodule $\mathfrak{M}$ of $K_n$, the defining conditions of the $\mathfrak{M}$-stance $(J, \varphi)$ are as follows:

i) $\mathfrak{M}=\sum_{(i,k)\in J}K\varphi(i, k)$ ,

ii) $\pi_J\varphi(i, k)=E_{i,k}$ (for every $(i, k)\in J$) ,

where $E_{i,k}$ denotes the matrix whose $(i, k)$-entry only is non-zero and equal to 1 and $\pi_J$ the projection of $K_n$ such that

$$\pi_J A=\sum_{(i,k)\in J}\alpha_{i,k}E_{i,k} \quad\text{for}\quad A=(\alpha_{i,k})\in K_n .$$

As to the set $\triangle$, let us observe the following: if $\pi_\triangle A=E=\sum_{i=1}^{n}E_{i,i}$ (the unit matrix) (if in general $\pi_\triangle A=\text{diag}(\alpha_1, \alpha_2, \cdots, \alpha_n)$ for which $\alpha_1\alpha_2\cdots\alpha_n\neq0$), then the matrix $A$ is surely invertible. In particular, if an $\mathfrak{M}$-stance $J$ includes the set $\triangle$, then in $\mathfrak{M}$ there exists an invertible matrix $\sum_{i=1}^{n}\varphi(i, i)$.

## 2. Singular submodule

We shall call a $K$-submodule $\mathfrak{M}$ of the total matrix ring $K_n$ over $K$ to be *singular* if it consists only of non-invertible matrices. If a $K$-submodule $\mathfrak{M}$ is singular, then $\mu\mathfrak{M}$ also is singular for any permutation $\mu=\sigma\times\tau$ of $\Omega\times\Omega$.

LEMMA 2.1. *Let $J$ be a subset of $\Omega\times\Omega$ with $|J|\geqslant n^2-n+1$, then there exists a permutation $\mu=\sigma\times\tau$ of $\Omega\times\Omega$ such that $\mu J\supset\triangle$.*

We utilize the induction on $n$. The case of $n=1$ is trivial. We can assume $J\neq\Omega\times\Omega$.

Since $J=\bigcup_{i=1}^{n}(J\cap{}_iI)=\bigcup_{K=1}^{n}(J\cap I_k)$ are disjoint unions, we have

$$n^2>|J|=\sum_{i=1}^{n}|J\cap{}_iI|=\sum_{k=1}^{n}|J\cap I_k|\geqslant n^2-n+1=n(n-1)+1\,.$$

Therefore, there exist indices $i_1$ and $k_1$ such that

$$|J\cap{}_{i_1}I|=n\,,\quad |J\cap I_{k_1}|\leqslant n-1\,.$$

Let us take the transpositions $\sigma_1=(1, i_1)$ and $\tau_1=(1, k_1)$ of $\Omega$ and define a permutation $\mu_1=\sigma_1\times\tau_1$ of $\Omega\times\Omega$. Then we have

$$|\mu_1\cdot J\cap{}_1I|=|\mu_1\cdot J\cap{}_{\sigma_1(i_1)}I|=|\mu_1(J\cap{}_{i_1}I)|=|J\cap{}_{i_1}I|=n\,,\quad \text{i.e.,}\ \mu_1\cdot J\supset{}_1I$$

and

$$|\mu_1\cdot J\cap I_1|=|J\cap I_{k_1}|\leqslant n-1\,.$$

Now, we have

$$\begin{aligned}|\mu_1\cdot J\cap I^{(2)}|&=|\mu_1\cdot J-({}_1I\cup I_1)|\\&=|\mu_1\cdot J|-|\mu_1\cdot J\cap{}_1I|-|\mu_1\cdot J\cap I_1|+|\mu_1\cdot J\cap{}_1I\cap I_1|\\&\geqslant(n^2-n+1)-n-(n-1)+1=(n-1)^2-(n-1)+1\,,\end{aligned}$$

which satisfies our hypothesis for $n-1$. By the assumption of induction, there exists a permutation $\bar{\mu}_2=\bar{\sigma}_2\times\bar{\tau}_2$ of $I^{(2)}=\Omega^{(2)}\times\Omega^{(2)}$ such that

$$\bar{\mu}_2(\mu_1\cdot J\cap I^{(2)})\supset\bigcup_{i=2}^{n}\{i\}\times\{i, i+1, \cdots, n\}\,.$$

Let $\mu_2$ be the unique extention of $\bar{\mu}_2$ and $\mu=\mu_2\circ\mu_1$, then we obtain the inclusion relation

$$\mu J=\mu_2(\mu_1\cdot J)\supset\mu_2({}_1I\cup(\mu_1\cdot J\cap I^{(2)}))=\mu_2\cdot{}_1I\cup\bar{\mu}_2(\mu_1\cdot J\cap I^{(2)})$$
$$\supset{}_1I\cup(\bigcup_{i=2}^{n}\{i\}\times\{i,\ i+1,\ \cdots,\ n\}),$$

which completes the proof.

We can now prove the

THEOREM 2.2. *The dimension of any singular left (or right) K-submodule* $\mathfrak{M}$ *of the total matrix ring* $K_n$ *of degree n over a division ring K is at most* $n^2-n$.

PROOF. Let $\mathfrak{M}$ be a singular $K$-submodule of $K_n$ and $(J, \varphi)$ an $\mathfrak{M}$-stance. If we assume $\dim_K\mathfrak{M}\geqslant n^2-n+1$, then we must have $|J|\geqslant n^2-n+1$ and, by virtue of the preceding lemma, there exists a permutation $\mu=\sigma\times\tau$ of $\Omega\times\Omega$ such that $\mu J\supset\triangle$. Since the set $\mu J$ is a stance of the $K$-submodule $\mu\mathfrak{M}$, there exists an invertible matrix in $\mu\mathfrak{M}$ which is singular. This contradiction proves the theorem.

LEMMA 2.3. *Let* $(J, \varphi)$ *be a stance of a singular K-submodule* $\mathfrak{M}$ *of* $K_n$. *If there exists an index* $(i_0, k_0)\in J$ *satisfying the conditions*

$$\pi_{{}_{i_0}I}\,\varphi(i_0,\ k_0)=E_{i_0,k_0},$$

$$\pi_{{}_{i_0}I}\,\varphi(p,\ q)=0 \text{ (the zero matrix) } (\textit{for every } (p,\ q)\in J-({}_{i_0}I\cup I_{k_0}))$$

(resp. *the same conditions for* $\pi_{I_{k_0}}$), *then we have*

$$|({}_{i_0}I\cup I_{k_0})-J|\leqslant((n^2-n)-\dim_K\mathfrak{M})+1.$$

In fact, if we assume

$$|({}_{i_0}I\cup I_{k_0})-J|\geqslant((n^2-n)-\dim_K\mathfrak{M})+2,$$

then we must have

$$\begin{aligned}|J-({}_{i_0}I\cup I_{k_0})|&=|J|+|({}_{i_0}I\cup I_{k_0})-J|-|{}_{i_0}I\cup I_{k_0}|\\&\geqslant\dim_K\mathfrak{M}+(((n^2-n)-\dim_K\mathfrak{M})+2)-(2n-1)\\&=(n-1)^2-(n-1)+1.\end{aligned}$$

By virtue of the lemma 2.1., there exists a permutation $\mu=\sigma\times\tau$ of $\Omega\times\Omega$ such that

$$\sigma(i_0)=1,\ \tau(k_0)=1,$$
$$\mu(J-({}_{i_0}I\cup I_{k_0}))=\mu J-({}_1I\cup I_1)\supset\bigcup_{i=2}^{n}\{i\}\times\{i,\ i+1,\ \cdots,\ n\}.$$

Then, $(1, 1)=\mu(i_0, k_0)\in\mu J$ since $(i_0, k_0)\in J$ by hypothesis, and for $\psi=\mu\circ\varphi\circ\mu^{-1}$ we have

$$\begin{aligned}\pi_{{}_1I}\,\psi(1,\ 1)&=\pi_{\mu\cdot{}_{i_0}I}(\mu\circ\varphi\circ\mu^{-1})(1,\ 1)=\mu(\pi_{{}_{i_0}I}\,\varphi(i_0,\ k_0))\\&=\mu(E_{i_0,k_0})=E_{\sigma(i_0),\tau(k_0)}=E_{1,1},\end{aligned}$$

and similarly for $(p, q)\in\mu J-({}_1I\cup I_1)$ we have $\mu_{{}_1I}\psi(p, q)=0$, in particular

$$\pi_{{}_1I}\psi(p, p)=0 \text{ for } p=2, 3, \cdots, n$$

(the notation $\psi(p, p)$ has surely a meaning since $(p, p)\in\mu J$). Of course, $\pi_{\mu J}\psi(p, p)=E_{p,p}$ for every $p$. In the sequel, if we define a matrix $A\in\mu\mathfrak{M}$ by setting

$$A=\sum_{i=1}^{n}\psi(i, i),$$

then, since $\triangle\subset{}_1I\cup\mu J$ and $\pi_\triangle=\pi_\triangle\circ\pi_{{}_1I\cup\mu J}=\pi_\triangle\circ(\pi_{{}_1I}+\pi_{\mu J}-\pi_{{}_1I}\circ\pi_{\mu J})$, we have

$$\pi_\triangle A=\sum_{i=1}^{n}E_{i,i}=E,$$

which implies the existence of an invertible matrix in the singular $K$-submodule $\mu\mathfrak{M}$.

COROLLARY 2.4. *If there exists an index* $i_0\in\Omega$ (resp. $k_0\in\Omega$) *such that* ${}_{i_0}I\subset J$ (resp. $I_{k_0}\subset J$), *then for any* $k\in\Omega$ (resp. *for any* $i\in\Omega$) *we have*

$$|I_k-J|\leqslant((n^2-n)-\dim_K\mathfrak{M})+1$$
$$(\text{resp. } |{}_iI-J|\leqslant((n^2-n)-\dim_K\mathfrak{M})+1),$$

*provided that* $\mathfrak{M}$ *and* $J$ *are a singular* $K$-*submodule of* $K_n$ *and its stance.*

Now, we can characterize the singular $K$-submodule attaining the maximal dimension.

THEOREM 2.5. *Every singular left* (*or right*) $K$-*submodule of the simple ring* $K_n=K^{\Omega\times\Omega}$ *of dimension* $n^2-n$ *is a maximal* (*one sided*) *ideal of the ring* $K_n$.

PROOF. Since the permutation $\mu=\sigma\times\tau$ of $\Omega\times\Omega$ induces the both-sided multiplication $\mathfrak{M}\to S\mathfrak{M}T$ by fixed permutation-matrices $S$ and $T$ and the statement in the theorem is not affected by such an operation, we can assume the order of rows or columns in themselves freely for our convenience.

Let $\mathfrak{M}$ be a singular left $K$-submodule of $K_n$ of dimension $n^2-n$. The degree $n$ is assumed to be $\geqslant 3$; for, otherwise, the proof is trivial.

Let $(J, \varphi)$ be an $\mathfrak{M}$-stance. If $J$ includes a row, for instance the first row ${}_1I$, then, owing to the corollary 2.4., we have $|I_k-J|\leqslant 1$ for every column-index $k\in\Omega$. While $\sum_{k=1}^{n}|I_k-J|=|I-J|=n^2-(n^2-n)=n$, we obtain

$$(1)\qquad |I_k-J|=1$$

for all $k\in\Omega$. Moreover, we can prove for every $(i, k)\in J-{}_1I$ the inclusion relation

(2) $$\mathrm{Supp}(\varphi(i, k)) \subset I_k$$

provided ${}_1I \subset J$. In fact, if it should be the case that $\mathrm{Supp}(\varphi(i, k)) \ni (p, q)$ with $q \neq k$, then we could take the index $(p, q)$ into $J$ in place of $(i, k)$, so that the new stance $(J \cup \{(p, q)\}) - \{(i, k)\}$, since $I_q - J = \{(p, q)\}$ by (1) and $(i,k) \in {}_1I$, should include the column $I_q$ together with the row ${}_1I$. This contradicts the equality (1).

*Case* 1. We assume first that there exists an $\mathfrak{M}$-stance $J$ which includes at least two rows. For expediency we assume ${}_1I \subset J$ and ${}_nI \subset J$. Then, by the remark just stated, the relation (2) holds for every $(i, k) \in (J - {}_1I) \cup (J - {}_nI)$, that is, for all $(i, k) \in J$. Since $|I_k - J| = 1$ on the other hand, we can put

$$I - J = \{(i_1, 1), (i_2, 2), \cdots, (i_n, n)\}$$
$$= \{(i_k, k) | 1 \leqslant k \leqslant n\}$$

and

(3) $$\varphi(i, k) = E_{i,k} + \beta_{i,k} E_{i_k,k}$$

for every $(i, k) \in J = \{(i, k) | i \neq i_k\}$, where $\{\varphi(i, k)\}$ is the basis associated with the stance $J$. We shall show that we can assume all the $i_k$ identified.

Assume $i_1 = 2$. If $i_2 \neq i_1 = 2$, then, let us consider the matrix $\varphi(2, 2) = E_{2,2} + \beta_{2,2} E_{i_2,2}$. When $\beta_{2,2} \neq 0$ i.e. $(i_2, 2) \in \mathrm{Supp}(\varphi(2, 2))$, we can take $(i_2, 2)$ into the stance in place of $(2, 2)$, so that $I_2 - J$ becomes to be $\{(2, 2)\}$ and $i_1 = i_2 = 2$ for the new stance $J$. The contrary case, however, does never stand. Because, if it should be the case that $\beta_{2,2} = 0$ i.e. $\pi_{I_2} \varphi(2, 2) = E_{2,2}$, we should obtain the relations

$$(2, 2) \in J, \ \pi_{I_2} \varphi(2, 2) = E_{2,2},$$
$$\pi_{I_2} \varphi(p, q) = 0 \ \text{(for every } (p, q) \in J - ({}_2I \cup I_2))$$

(cf. (3)), and hence, by the lemma 2.3., $|({}_2I \cup I_2) - J| \leqslant 1$. This inequality contradicts the fact $\{(2, 1), (i_2, 2)\} \subset ({}_2I \cup I_2) - J$.

Thus, step by step we reform the stance if necessary, until we obtain a stance of the form $J = I - {}_2I$ or, what is the same thing, $I - J = {}_2I = \{2\} \times \Omega$. Then the matrices in the associated basis are expressible in the forms

$$\varphi(i, k) = E_{i,k} + \beta_{i,k} E_{2,k} \quad (i = 1, 3, 4, \cdots, n).$$

We shall show that for every fixed row-index $i \neq 2$ all the coefficients $\beta_{i,k}$ $(1 \leqslant k \leqslant n)$ are identical. This is, however, almost self-evident. Since the singular $K$-submodule $\mathfrak{M}$ contains the matrix

$$\varphi(1, 1)+\varphi(1, 2)+\sum_{i=3}^{n}\varphi(i, i)=\begin{pmatrix} 1 & 1 & 0 & \cdots & 0 \\ \beta_{1,1} & \beta_{1,2} & \beta_{3,3} & \cdots & \beta_{n,n} \\ & & 1 & & 0 \\ & 0 & & \ddots & \\ & & 0 & & 1 \end{pmatrix},$$

it must be that $\beta_{1,1}=\beta_{1,2}$. Similarly $\beta_{1,1}=\beta_{1,2}=\cdots=\beta_{1,n}$ and we can conclude in general $\beta_{i,1}=\beta_{i,2}=\cdots=\beta_{i,n}=\lambda_i$ for every $i\neq 2$.

Here, all the scalars $\lambda_i$ $(i=1, 3, 4, \cdots, n)$ are contained in the centre $C$ of the division ring $K$, because, for instance, the matrix

$$\varphi(1, 1)+\lambda\varphi(1, 2)+\sum_{i=3}^{n}\varphi(i, i)=\begin{pmatrix} 1 & \lambda & 0 & \cdots & 0 \\ \lambda_1 & \lambda\lambda_1 & \lambda_3 & \cdots & \lambda_n \\ & & 1 & & 0 \\ & 0 & & \ddots & \\ & & 0 & & 1 \end{pmatrix}$$

should be non-invertible as a member of the singular left $K$-submodule $\mathfrak{M}$ for any $\lambda$ in $K$.

Thus, we can write down the generic form of the matrices $A$ in $\mathfrak{M}$, provided that there exists an $\mathfrak{M}$-stance including at least two rows. It is as follows:

$$(4) \qquad \begin{aligned} A &= \sum_{i\neq p}\sum_{k}\alpha_{i,k}\varphi(i, k)=\sum_{i\neq p}\sum_{k}\alpha_{i,k}(E_{i,k}+\lambda_i E_{p,k}) \\ &=(E+\sum_{q\neq p}\lambda_q E_{p,q})(\sum_{i\neq p}\sum_{k}\alpha_{i,k}E_{i,k}), \end{aligned}$$

where $p$ is a fixed row-index which we have assumed to be 2 for convenience, $\lambda_i$ $(i\in\Omega,\ i\neq p)$ are fixed scalars in the centre $C$ of $K$ and $\alpha_{i,k}$ are arbitrary scalars in $K$.

To complete the proof in this case, we define a matrix $Q$ by setting

$$Q=\sum_{q\neq p}(-\lambda_q E_{1,q})+E_{1,p} \quad (\neq 0).$$

Since $Q(E+\sum_{q\neq p}\lambda_q E_{p,q})=E_{1,p}$, we have $QA=E_{1,p}(\sum_{i\neq p}\sum_{k}\alpha_{i,k}E_{i,k})=0$ for all $A$ in $\mathfrak{M}$. On the other hand, the set $\mathfrak{J}$ of all matrices $A$ satisfying $QA=0$ forms a proper right ideal of the ring $K_n$. It is at the same time a singular left $K$-submodule of $K_n$ since $Q$ commutes with every scalar in $K$. Hence the inclusion relation $\mathfrak{M}\subset\mathfrak{J}$ and the maximality of $\mathfrak{M}$ with respect to being a singular left $K$-submodule lead the equality $\mathfrak{M}=\mathfrak{J}$. The ideal $\mathfrak{M}$ is of course maximal with respect to being a proper right ideal of the ring $K_n$.

In the similar way, we can conclude that $\mathfrak{M}$ is a maximal left ideal of $K_n$ if there exists an $\mathfrak{M}$-stance including at least two columns. In this case, $\mathfrak{M}$ consists of all matrices $A \in K_n$ satisfying $AP=0$ for a certain matrix $P$, where $P$, unlike $Q$, does not necessarily commute with all scalars in $K$ (Note that $\mathfrak{M}$ is assumed to be a *left* $K$-submodule).

*Case* 2. Secondly, we consider the case that an $\mathfrak{M}$-stance $J$ includes one and only one row or column. If we assume for instance ${}_1I\subset J$, then for every $i\neq 1$ we have $|{}_iI-J|\geqq 1$. While $\sum_{i=1}^{n}|{}_iI-J|=|I-J|=n$, all the summands but one are equal to 1 and the one is equal to 2. So we can assume for instance that

$$|{}_iI-J|=1 \text{ for } i=2, 3, \cdots, n-1, \text{ and } |{}_nI-J|=2\,.$$

On the other hand, because of ${}_1I\subset J$, we have $|I_k-J|=1$ for every column-index $k$. Hence we can assume for instance

$$I-J=\left(\bigcup_{k=1}^{n-1}\{(k+1, k)\}\right)\cup\{(n, n)\}\,.$$

Now, let us define a matrix $A$ in $\mathfrak{M}$ by setting

$$A=\sum_{i=1}^{n-3}\varphi(i, i)+\varphi(n-2, n)+\varphi(n-1, n-1)=\left(\begin{array}{ccc|ccc} 1 & & & & & \\ * & \ddots & & & & \\ & \ddots & 1 & & & \\ \hline & & * & 0 & 0 & 1 \\ & & & \alpha & 1 & 0 \\ & & & 0 & \beta & \gamma \end{array}\right)$$

and we take also the matrix $\varphi(n, n-2)$ out of $\mathfrak{M}$. As was observed at the beginning of the proof (cf. (2)), we can write as $\varphi(n, n-2)=E_{n,n-2}+\delta E_{n-1,n-2}$. When $\delta\neq 0$, we can reduce this case to the Case 1 by a change of stance. If it were that $\delta=0$ and $\varphi(n, n-2)=E_{n,n-2}$, then the singular $K$-submodule $\mathfrak{M}$ should contain the invertible matrix

$$F=A+\xi\varphi(n, n-2)=A+\xi E_{n,n-2},$$

where $\xi$ is an element $\neq\beta\alpha$ of $K$. ($\alpha=0$ in reality.)

As to the column, we can discuss in the similar way.

*Case* 3. Finally, we come to investigate the case that $J$ does include neither a row nor a column. Then we can assume for instance that

$$I-J=\bigcup_{p=1}^{n}\{(p, p)\}\,.$$

If an index $(p, p)$ is contained in a certain $\mathrm{Supp}(\varphi(i, k))((i, k)\in J)$, $(p, p)$ can be taken into $J$ instead of $(i, k)$, so that the new stance $J$ becomes to be of type already considered. If it were that $(p, p) \notin \mathrm{Supp}(\varphi(i, k))$ for all $p \in \Omega$ and $(i, k) \in J = \{(i, k) | i \neq k\}$, then $\varphi(i, k) = E_{i,k}$ for every $(i, k) \in J$ and $\mathfrak{M}$ should contain the invertible matrix $\sum_{i=1}^{n-1} \varphi(i, i+1) + \varphi(n, 1) = \sum_{i=1}^{n-1} E_{i,i+1} + E_{n,1}$.

Thus the proof is complete.

REMARK. Let us denote by $\mathrm{diag}(\alpha_1, \alpha_2, \cdots, \alpha_n)$ the diagonal matrix with diagonal entries $\alpha_1, \alpha_2, \cdots, \alpha_n$. Now, let $P_1 = \mathrm{diag}(1, 0, \cdots, 0)$ and $Q_n = \mathrm{diag}(0, \cdots, 0, 1)$; and let $\mathfrak{M}_1$ be the left ideal of $K_n$ consisting of all matrices $A \in K_n$ such that $AP_1 = 0$, and $\mathfrak{M}_n$ the right ideal of $K_n$ consisting of all $A \in K_n$ such that $Q_n A = 0$. As is readily verified, for any invertible matrix $S \in K_n$ with entries in the centre $C$ of $K$, $S\mathfrak{M}_n$ is a maximal singular left $K$-submodule of $K_n$ of dimension $n^2 - n$; and for any invertible $T \in K_n$, $\mathfrak{M}_1 T$ is also such a $K$-submodule. Conversely, by Theorem 2.5., every maximal singular left $K$-submodule of $K_n$ of dimension $n^2 - n$ has one of the two forms above (cf. (4)). They are all mutually isomorphic as left $K$-modules.

Now, let us exhibit a sequence of maximal singular (two sided) $K$-submodules of $K_n$. As one can verify without difficulty, the set $\mathfrak{M}_k$ consisting of all matrices $A$ such that $Q_k A P_k = 0$ for $P_k = \mathrm{diag}(1, \cdots, 1, 0, \cdots, 0)$ (1 repeated $k$ times) and $Q_k = \mathrm{diag}(0, \cdots, 0, 1, \cdots, 1)$ (0 repeated $k-1$ times) forms in fact such a $K$-submodule of $K_n$ of dimension $n^2 - k(n-k+1)$ for every $k = 1, 2, \cdots, n$. Accordingly, we obtain a family of such $K$-submodules; that is, the family $\{S\mathfrak{M}_k T\}$, where $S$ and $T$ are matrices mentioned above and $k = 1, 2, \cdots, n$. In this paper, we characterized the two extremals $\mathfrak{M}_1$ and $\mathfrak{M}_n$ in this sequence as representatives of the maximal singular one-sided $K$-submodules attaining the maximal dimension. The determination of all maximal singular $K$-submodules of the simple ring $K_n$ remains still unsolved for us. However, it seems to us to be an interesting question to settle whether the family exhausts all the maximal singular left $K$-submodules of the ring $K_n$.

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