

On the group algebras of metabelian groups over algebraic number fields II

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1. Introduction

In the previous paper [5] we have investigated the group algebra of a metabelian group G which is a cyclic extension of an abelian normal subgroup and satisfies a certain condition. In this case simple components of the group algebra $\mathbb{Q}[G]$ over the rational number field are canonically expressed as cyclic algebras and their Schur indices are readily calculated.

In this paper we shall examine the group algebra of a metabelian group G which is an extension of a cyclic normal subgroup $\langle \omega \rangle$ by an abelian group such that the orders of $\langle \omega \rangle$ and $G/\langle \omega \rangle$ are relatively prime. From [5, Theorem 1] it follows that every irreducible representation of G is induced from a linear character of some subgroup which contains the centralizer of the normal subgroup $\langle \omega \rangle$. If an irreducible representation U is induced from a linear character of a subgroup $H \supset \langle \omega \rangle$, then the enveloping algebra $\text{env}_{\mathbb{Q}}(U|H)$ over \mathbb{Q} of the restriction $U|H$ of U on H is a (commutative) field and the factor group G/H is an automorphism group of this field. From this the enveloping algebra $\text{env}_{\mathbb{Q}}(U)$ of U is expressed as a crossed product whose factor set consists of roots of unity contained in the center of $\text{env}_{\mathbb{Q}}(U)$ (Theorem 1). Moreover this factor set is 'commutative'. These facts are directly and elementarily obtained from the form of the matrix algebra $\text{env}_{\mathbb{Q}}(U)$. The Schur index of U is easily calculated because the factor set has the above properties (Theorem 3). It would be worth while to mention that a hyper-elementary group (at the prime p) whose Sylow p -subgroups are abelian, is one of aforesaid metabelian groups.

We shall use the same notation as in [4] or [5].

2. Simple components of $\mathbb{Q}[G]$

Throughout this paper, G is a (non-abelian) metabelian group containing a normal cyclic subgroup $\langle \omega \rangle$ with abelian factor group. Here we assume that the order m of $\langle \omega \rangle$ and the order of $G/\langle \omega \rangle$ are relatively prime, so that G is a semi-direct product of $\langle \omega \rangle$ and an abelian subgroup A :

$$(1) \quad G = \langle \omega \rangle \cdot A, \quad \omega^m = 1, \quad (m, \#A) = 1.$$

If H_0 is the centralizer of $\langle \omega \rangle$ in G , then H_0 and G/H_0 are both abelian and $H_0 \supset \langle \omega \rangle$. From [5, Theorem 1] we know that any irreducible representation U of G is induced from some linear character ψ of a subgroup $H \supset H_0$. Note that H is a normal subgroup of G because the commutator subgroup $[G, G]$ is contained in $\langle \omega \rangle$. Now every subgroup H ($\supset H_0$) can be expressed as a semi-direct product of $\langle \omega \rangle$ and a subgroup F of A :

$$(2) \quad H = \langle \omega \rangle \cdot F, \quad F \subset A.$$

The factor group G/H can be written as direct product of cyclic groups such that

$$(3) \quad G/H = \langle \rho_1 H \rangle \times \cdots \times \langle \rho_s H \rangle, \quad \rho_i \in A, \quad \rho_i^{z_i} \in F, \quad i=1, \dots, s,$$

where z_i are respectively the orders of $\langle \rho_i H \rangle$. Set

$$(4) \quad z = z_1 \cdots z_s = \#G/H,$$

and so we have $(m, z) = 1$. Let

$$(5) \quad \rho_i^{-1} \omega \rho_i = \omega^{r_i}, \quad 1 \leq i \leq s.$$

As the commutator subgroup $[H, H]$ is contained in $\langle \omega \rangle$, we have, for some integer d ,

$$(6) \quad [H, H] = \langle \omega^d \rangle, \quad d|m.$$

Therefore the linear characters of H are given by $\psi_{\alpha, \varphi}$ such that

$$(7) \quad \psi_{\alpha, \varphi}(\omega^\nu \cdot f) = \zeta_d^{\alpha \nu} \cdot \varphi(f), \quad f \in F,$$

where ζ_d is a primitive d -th root of unity, and $0 \leq \alpha \leq d-1$, and φ ranges over the linear characters of F .

LEMMA 1. *The induced representation $\psi_{\alpha, \varphi}^G$ is irreducible if and only if*

$$(8) \quad \zeta_d^\alpha \neq \zeta_d^{\alpha r_1^{z_1} \cdots r_s^{z_s}},$$

for $0 \leq \lambda_i \leq z_i - 1$ ($1 \leq i \leq s$), not all λ_i being equal to zero.

PROOF. We observe that the elements of G can be expressed uniquely in the form

$$(9) \quad g = \omega^\nu \cdot f \cdot \rho, \quad 0 \leq \nu \leq m-1, \quad f \in F, \quad \rho = \rho_1^{z_1} \cdots \rho_s^{z_s}, \quad 0 \leq \lambda_i \leq z_i - 1 \quad (1 \leq i \leq s).$$

From [1, (45.5)] it follows that $\psi_{\alpha, \varphi}^G$ is irreducible if and only if, for each $g \in H$, $\psi_{\alpha, \varphi}(\omega) \neq \psi_{\alpha, \varphi}(g^{-1} \omega g)$ or $\psi_{\alpha, \varphi}(f) \neq \psi_{\alpha, \varphi}(g^{-1} f g)$ for some $f \in F$. It is easily checked that for $g = \omega^\nu f \rho = \omega^\nu f \rho_1^{z_1} \cdots \rho_s^{z_s}$, we have $g^{-1} \omega g = f^{-1} \omega^{r_1^{z_1} \cdots r_s^{z_s}} f$, $g^{-1} f' g = f^{-1} (\rho^{-1} \omega^\nu \rho)^{-1} f' (\rho^{-1} \omega^\nu \rho) f$, $f' \in F$. Hence $\psi_{\alpha, \varphi}(g^{-1} \omega g) = \zeta_d^{\alpha r_1^{z_1} \cdots r_s^{z_s}}$, $\psi_{\alpha, \varphi}(g^{-1} f' g) =$

$\phi_{\alpha, \varphi}(f')$, $f' \in F$. The element $g = \omega^\nu f \rho_1^{\lambda_1} \cdots \rho_s^{\lambda_s}$ ($0 \leq \lambda_i \leq z_i - 1$, $1 \leq i \leq s$) belongs to H if and only if $\lambda_1 = \cdots = \lambda_s = 0$. From these facts, the assertion is easily verified.

LEMMA 2. *If the induced representation $\phi_{\alpha, \varphi}^G$ is irreducible, set*

$$(10) \quad \tau_{\lambda_1 \dots \lambda_s}(\zeta_d^\alpha) = \zeta_d^{\alpha \tau_1^{\lambda_1} \dots \tau_s^{\lambda_s}} \quad \text{for } 0 \leq \lambda_i \leq z_i - 1 \ (i=1, \dots, s).$$

Then $\{\tau_{\lambda_1 \dots \lambda_s}; 0 \leq \lambda_i \leq z_i - 1 \ (i=1, \dots, s)\}$ is a subgroup of order z in the Galois group $\mathcal{G}(\mathbb{Q}(\zeta_d^\alpha)/\mathbb{Q})$.

PROOF. We have

$$(11) \quad \zeta_d^{\alpha \tau_i^{\lambda_i}} = \zeta_d^\alpha \quad (i=1, \dots, s),$$

because $\zeta_d^{\alpha \tau_i^{\lambda_i}} = \phi_{\alpha, \varphi}(\omega^{\tau_i^{\lambda_i}}) = \phi_{\alpha, \varphi}(\rho_i^{-z_i} \omega \rho_i^{z_i}) = \phi_{\alpha, \varphi}(\omega) = \zeta_d^\alpha$. From this and Lemma 1, the assertion is obvious.

LEMMA 3. *If the induced representation $\phi^G = \phi_{\alpha, \varphi}^G$ is irreducible, then the enveloping algebra $\text{env}_Q(\phi^G|H)$ of $\phi^G|H$ over \mathbb{Q} is a (commutative) field, and in fact*

$$(12) \quad \mathbb{Q}(\zeta_d^\alpha, \varphi(f), f \in F) \simeq \text{env}_Q(\phi^G|H).$$

Hereafter we set

$$(13) \quad K = \mathbb{Q}(\zeta_d^\alpha, \varphi(f), f \in F), \quad \tilde{K} = \text{env}_Q(\phi^G|H).$$

PROOF. If we put

$$\dot{\phi}(g) = \begin{cases} \phi(g), & g \in H, \\ 0, & g \notin H, \end{cases}$$

then the induced matrix representation ϕ'^G are given by

$$(14) \quad \phi'^G(g) = (\dot{\phi}(\rho_s^{-\lambda_s'} \cdots \rho_1^{-\lambda_1'} g \rho_1^{\lambda_1} \cdots \rho_s^{\lambda_s}))_{0 \leq \lambda_i, \lambda_i' \leq z_i - 1 \ (i=1, \dots, s)}.$$

Here the representatives $\rho_1^{\lambda_1} \cdots \rho_s^{\lambda_s}$ of G/H , which consist of $z = z_1 \cdots z_s$ elements, are arranged in suitably fixed order. It is easily verified that $\phi'^G(\omega)$ is the diagonal matrix whose diagonal elements are $\zeta_d^{\alpha \tau_1^{\lambda_1} \dots \tau_s^{\lambda_s}} = \tau_{\lambda_1 \dots \lambda_s}(\zeta_d^\alpha)$, $0 \leq \lambda_i \leq z_i - 1 \ (i=1, \dots, s)$ arranged in the above fixed order. Similarly for each $f \in F$, $\phi'^G(f)$ is the scalar matrix $\varphi(f) \cdot 1_z$. Since $(d, \#F) = 1$, it follows that

$$(15) \quad \mathbb{Q}(\zeta_d^\alpha) \cap \mathbb{Q}(\varphi(f), f \in F) = \mathbb{Q}.$$

Consequently the automorphisms $\tau_{\lambda_1 \dots \lambda_s}$ of $\mathbb{Q}(\zeta_d^\alpha)$ over \mathbb{Q} can be extended to those of K over \mathbb{Q} by setting

$$(16) \quad \tau_{\lambda_1 \dots \lambda_s}(\varphi(f)) = \varphi(f), \quad f \in F.$$

For each $\theta \in K$, denote by $\tilde{\theta}$ the diagonal matrix whose diagonal elements are $\tau_{\lambda_1 \dots \lambda_s}(\theta)$, $0 \leq \lambda_i \leq z_i - 1$, $i = 1, \dots, s$. The arrangement of them is the same as before. Then we see easily that $\tilde{K} = \{\tilde{\theta} ; \theta \in K\}$, and that the mapping $\phi: \theta \mapsto \tilde{\theta}$ gives the isomorphism of our Lemma 3.

For simplicity, write $U = \phi^G := \phi_{a, \rho}^G$ and denote by χ the character of U . Set

$$(17) \quad \epsilon_\nu = \sum_{\substack{0 \leq \lambda_i \leq z_i - 1 \\ i = 1, \dots, s}} \tau_{\lambda_1 \dots \lambda_s}(\zeta_d^\alpha)^\nu, \quad \nu = 1, 2, \dots, m.$$

Since $G \triangleright H$, it is clear that

$$(18) \quad Q(\chi) = Q(\varphi(f), f \in F, \epsilon_1, \dots, \epsilon_m).$$

Every element of $Q(\chi)$ is fixed by the automorphisms $\tau_{\lambda_1 \dots \lambda_s}$, $0 \leq \lambda_i \leq z_i - 1$, $i = 1, \dots, s$. On the other hand, it follows from facts about symmetric polynomials that $[Q(\zeta_d^\alpha) : Q(\epsilon_1, \dots, \epsilon_m)] \leq z$, so that $[K : Q(\chi)] \leq z$. Hence $Q(\chi)$ is the subfield of K which corresponds to the automorphism group $\{\tau_{\lambda_1 \dots \lambda_s} ; 0 \leq \lambda_i \leq z_i - 1, i = 1, \dots, s\} \subset \mathfrak{G}(K/Q)$ in the sense of Galois theory. In particular, $Q(\chi) = Q(\varphi(f), f \in F, \epsilon_1, \dots, \epsilon_m)$ and $[K : Q(\chi)] = z$. By the isomorphism $\phi: K \simeq \tilde{K}$, $Q(\chi)$ is mapped onto $Q(\chi) \cdot 1_z$, and so $[\tilde{K} : Q(\chi) \cdot 1_z] = z$. It is well known that $Q(\chi) \cdot 1_z$ is the center of $\text{env}_Q(U)$ and that $[\text{env}_Q(U) : Q(\chi) \cdot 1_z] = z^2$, z being equal to the degree of U . Therefore \tilde{K} is a maximal subfield of $\text{env}_Q(U)$. We can easily verify that the mapping

$$(19) \quad \tau_{\lambda_1 \dots \lambda_s} : \tilde{\theta} \mapsto U(\rho_1^{\lambda_1} \dots \rho_s^{\lambda_s})^{-1} \cdot \tilde{\theta} \cdot U(\rho_1^{\lambda_1} \dots \rho_s^{\lambda_s}), \quad \tilde{\theta} \in \tilde{K},$$

is the automorphism of \tilde{K} over $Q(\chi) \cdot 1_z$ which corresponds to $\tau_{\lambda_1 \dots \lambda_s} \in \mathfrak{G}(K/Q(\chi))$. Hence $U(\rho_1^{\lambda_1} \dots \rho_s^{\lambda_s})$, $0 \leq \lambda_i \leq z_i - 1$ ($i = 1, \dots, s$), are linearly independent over K . Thus we have

$$(20) \quad \text{env}_Q(U) = \sum_{\substack{0 \leq \lambda_i \leq z_i - 1 \\ i = 1, \dots, s}} U(\rho_1^{\lambda_1} \dots \rho_s^{\lambda_s}) \cdot \tilde{K} \quad (\text{direct sum}).$$

This is an expression of $\text{env}_Q(U)$ as crossed product. The factor set is given by

$$(21) \quad a_{\tau_{\lambda_1 \dots \lambda_s}, \tau_{\kappa_1 \dots \kappa_s}} = \varphi(\rho_1^{\lambda_1})^{\delta_1} \dots \varphi(\rho_s^{\lambda_s})^{\delta_s} \cdot 1_z,$$

$$\delta_i = \begin{cases} 1, & \lambda_i + \kappa_i \geq z_i, \\ 0, & \lambda_i + \kappa_i < z_i, \end{cases} \quad i = 1, \dots, s,$$

because

$$(22) \quad U(\rho_1^{\lambda_1} \dots \rho_s^{\lambda_s}) \cdot U(\rho_1^{\kappa_1} \dots \rho_s^{\kappa_s}) \\ = U(\rho_1^{\lambda_1 + \kappa_1 - \delta_1 z_1} \dots \rho_s^{\lambda_s + \kappa_s - \delta_s z_s}) \varphi(\rho_1^{\lambda_1})^{\delta_1} \dots \varphi(\rho_s^{\lambda_s})^{\delta_s}.$$

We sum up our results in

THEOREM 1. *Let G be a semi-direct product of a normal cyclic subgroup $\langle \omega \rangle$ of order m and an abelian subgroup A such that $(m, \#A) = 1$. Then any irreducible representation U of G is induced from a linear character ψ of a certain subgroup $H = \langle \omega \rangle \cdot F \supset H_0$, where F is a subgroup of A and H_0 is the centralizer of ω in G . Let $[H, H] = \langle \omega^d \rangle$, $d|m$. Then $\psi(\omega^\nu \cdot f) = \zeta_d^{\alpha \nu} \cdot \varphi(f)$, $f \in F$ for an integer α and some linear character φ of F . Write $G/H = \langle \rho_1 H \rangle \times \cdots \times \langle \rho_s H \rangle$, $\rho_i^{-1} \omega \rho_i = \omega^{r_i}$, $\rho_i \in A$, $\rho_i^{z_i} \in F$, $i = 1, \dots, s$, and $z = z_1 \cdots z_s = [G : H]$. Then $\text{env}_Q(U|H)$ is a maximal subfield of $\text{env}_Q(U)$. If we set $\tau_{\lambda_1 \dots \lambda_s} : \tilde{\theta} \mapsto U(\rho_1^{\lambda_1} \cdots \rho_s^{\lambda_s})^{-1} \cdot \tilde{\theta} \cdot U(\rho_1^{\lambda_1} \cdots \rho_s^{\lambda_s})$, $\tilde{\theta} \in \text{env}_Q(U|H)$, then we have $\mathfrak{G}(\text{env}_Q(U|H)/\mathbb{Q}(\chi) \cdot 1_z) = \{\tau_{\lambda_1 \dots \lambda_s}; 0 \leq \lambda_i \leq z_i - 1, i = 1, \dots, s\}$, χ being the character of U . Thus $\text{env}_Q(U)$ is expressed as a crossed product:*

$$(23) \quad \text{env}_Q(U) = \sum_{\substack{0 \leq \lambda_i \leq z_i - 1 \\ i = 1, \dots, s}} U(\rho_1^{\lambda_1} \cdots \rho_s^{\lambda_s}) \cdot \text{env}_Q(U|H) \quad (\text{direct sum}) \\ = (a_{\tau_{\lambda_1 \dots \lambda_s}, \tau_{\kappa_1 \dots \kappa_s}}, \text{env}_Q(U|H)/\mathbb{Q}(\chi) \cdot 1_z),$$

whose factor set is given by

$$(24) \quad a_{\tau_{\lambda_1 \dots \lambda_s}, \tau_{\kappa_1 \dots \kappa_s}} = \varphi(\rho_1^{z_1})^{\delta_1} \cdots \varphi(\rho_s^{z_s})^{\delta_s} \cdot 1_z,$$

$$(25) \quad \delta_i = \begin{cases} 1, & \lambda_i + \kappa_i \geq z_i, \\ 0, & \lambda_i + \kappa_i < z_i, \end{cases} \quad i = 1, \dots, s.$$

This crossed product is isomorphic to the crossed product:

$$(26) \quad (a_{\tau_{\lambda_1 \dots \lambda_s}, \tau_{\kappa_1 \dots \kappa_s}}, \mathbb{Q}(\zeta_d^\alpha, \varphi(f), f \in F)/\mathbb{Q}(\chi)),$$

where

$$\tau_{\lambda_1 \dots \lambda_s}(\zeta_d^\alpha) = \zeta_d^{\alpha \tau_1^{z_1} \cdots \tau_s^{z_s}}, \quad \tau_{\lambda_1 \dots \lambda_s}(\varphi(f)) = \varphi(f), \quad f \in F,$$

$$\mathfrak{G}(\mathbb{Q}(\zeta_d^\alpha, \varphi(f), f \in F)/\mathbb{Q}(\chi)) = \{\tau_{\lambda_1 \dots \lambda_s}; 0 \leq \lambda_i \leq z_i - 1, i = 1, \dots, s\},$$

$$a_{\tau_{\lambda_1 \dots \lambda_s}, \tau_{\kappa_1 \dots \kappa_s}} = \varphi(\rho_1^{z_1})^{\delta_1} \cdots \varphi(\rho_s^{z_s})^{\delta_s}, \quad \delta_i (i = 1, \dots, s) \text{ being given by (25).}$$

REMARK 1. The above factor set $(a_{\tau_{\lambda_1 \dots \lambda_s}, \tau_{\kappa_1 \dots \kappa_s}})$ consists of roots of unity, and for each pair $\tau_{\lambda_1 \dots \lambda_s}, \tau_{\kappa_1 \dots \kappa_s}$, the order of $a_{\tau_{\lambda_1 \dots \lambda_s}, \tau_{\kappa_1 \dots \kappa_s}}$ in the multiplicative group C^\times is relatively prime to d , $(d|m)$. Moreover, $a_{\tau_{\lambda_1 \dots \lambda_s}, \tau_{\kappa_1 \dots \kappa_s}} \in \mathbb{Q}(\chi)$, i.e. every $a_{\tau_{\lambda_1 \dots \lambda_s}, \tau_{\kappa_1 \dots \kappa_s}}$ is fixed by all the automorphisms of $\mathfrak{G}(\mathbb{Q}(\zeta_d^\alpha, \varphi(f), f \in F)/\mathbb{Q}(\chi))$. Further this factor set is 'commutative', i.e., for each pair $\tau_{\lambda_1 \dots \lambda_s}, \tau_{\kappa_1 \dots \kappa_s}$, $a_{\tau_{\lambda_1 \dots \lambda_s}, \tau_{\kappa_1 \dots \kappa_s}} = a_{\tau_{\kappa_1 \dots \kappa_s}, \tau_{\lambda_1 \dots \lambda_s}}$.

REMARK 2. $\mathfrak{G}(\mathbb{Q}(\zeta_d^\alpha, \varphi(f), f \in F)/\mathbb{Q}(\chi)) \simeq G/H$.

THEOREM 2. *The notation being as in Theorem 1, we assume that the factor group G/H_0 is cyclic. Then G/H is also cyclic, so that we have*

$$\begin{aligned} \text{env}_Q(U) &= \sum_{i=0}^{z-1} U(\rho^i) \text{env}_Q(U|H) \\ &\simeq (\varphi(\rho^z), \mathbf{Q}(\zeta_{d^a}, \varphi(f) \in F)/\mathbf{Q}(Z), \tau) \quad (\text{cyclic algebra}) \end{aligned}$$

where $\rho = \rho_1$, $G/H = \langle \rho H \rangle$, $z = z_1 = [G:H]$, $\tau(\zeta_{d^a}) = \zeta_{d^{a'}} 1$, $\tau(\varphi(f)) = \varphi(f)$, $f \in F$. Thus every simple component of $\mathbf{Q}[G]$ is canonically expressed as a cyclic algebra. The local index of this cyclic algebra at any finite prime \mathfrak{p} of $\mathbf{Q}(Z)$ is equal to

$$(27) \quad \left(e_{\mathfrak{p}}, \frac{q-1}{v} \right),$$

where $e_{\mathfrak{p}}$ is the ramification exponent of $\mathbf{Q}(\zeta_{d^a}, \varphi(f) \in F)/\mathbf{Q}(Z)$ at \mathfrak{p} , and $q = N_{\mathbf{Q}(Z)/\mathbf{Q}}(\mathfrak{p})$, and v is the order of $\varphi(\rho^z)$ in C^* (namely, $\varphi(\rho^z)$ is a primitive v -th root of unity).

The former part of this Theorem is obvious from Theorem 1. The latter part can be proved by the same argument as in [4].

REMARK 1. Let $G = \langle \omega \rangle \cdot P$ be a hyper-elementary group (at the prime p), where the order of the normal subgroup $\langle \omega \rangle$ is equal to m and P is an abelian Sylow p -subgroup and $(m, p) = 1$. If the Sylow p -subgroup of the multiplicative group of integers modulo m is cyclic, then the assumption of Theorem 2 is satisfied (that is, G/H_0 is cyclic). For instance, if $m = l^a$ is a power of a prime $l (\neq 2)$, then the multiplicative group of integers modulo m itself is cyclic.

REMARK 2. The above numbers $e_{\mathfrak{p}}$ and q are easily calculated (cf. [4]).

3. The Schur index

Throughout this section, the notation is the same as in Theorem 1. For simplicity we set

$$(1) \quad K = \mathbf{Q}(\zeta_{d^a}, \varphi(f), f \in F), \quad k = \mathbf{Q}(Z), \quad \mathfrak{A} = (a_{\tau_{\lambda_1} \dots \lambda_s}, \tau_{\lambda_1} \dots \lambda_s}, K/k).$$

Elements of $\mathfrak{S}(K/k)$ are often denoted by τ, τ' , etc. In order to determine the Schur index of U , we must calculate the local indices of the crossed product \mathfrak{A} at all the places \mathfrak{p} of k . Let \mathfrak{p} and \mathfrak{P} be prime ideals of k and K respectively, such that $\mathfrak{P}|\mathfrak{p}$. Since K/\mathbf{Q} is abelian, we denote by $\mathfrak{S}_{\mathfrak{p}}$ the decomposition group of \mathfrak{P} in K/k and write $K_{\mathfrak{P}} = K^{\mathfrak{p}}$, so that $\mathfrak{S}_{\mathfrak{p}} = \mathfrak{S}(K^{\mathfrak{p}}/k_{\mathfrak{p}})$. Then

$$(2) \quad \mathfrak{A}_{\mathfrak{p}} = (a_{\tau, \tau'}, K/k) \otimes_k k_{\mathfrak{p}} \sim ((a_{\tau, \tau'})_{\mathfrak{S}_{\mathfrak{p}}}, K^{\mathfrak{p}}/k_{\mathfrak{p}}),$$

where $(a_{\tau, \tau'})_{\mathfrak{S}_{\mathfrak{p}}}$ denotes the factor set of $\mathfrak{S}_{\mathfrak{p}}$ with $\tau, \tau' \in \mathfrak{S}_{\mathfrak{p}} \subset \mathfrak{S}(K/k)$. If \mathfrak{p} is not

ramified in K^p/k_p , then the index of \mathfrak{A}_p is equal to 1, because the factor set $(a_{\tau,\tau'})_{\mathfrak{G}_p}$ consists of roots of unity. Recall that if p does not divide d ($d|m$), then p is not ramified in K/k .

Now assume that p is ramified in K/k . We must compute the order of the cocycle $(a_{\tau,\tau'})_{\mathfrak{G}_p}$ in the 2-cohomology group $H^2(\mathfrak{G}_p, (K^p)^\times)$. Since $[K:k]=z$, $(z, d)=1$, and $p|d$, it follows that K^p/k_p is a tamely ramified extension. The inertia group $\mathfrak{H} \subset \mathfrak{G}_p$ is cyclic, and \mathfrak{G}_p is a cyclic extension of \mathfrak{H} . Now there exists an abelian extension L over k_p (possibly $L=K^p$) with following properties: (i) L contains subfields L_u, L_r such that L_u/k_p is unramified, and L_r/k_p is totally and tamely ramified, (ii) $L=L_u \cdot L_r$, (iii) $L \supset K^p$. For instance, if E is the unramified extension of k_p of degree $z_p=[K^p:k_p]$, then $L=E \cdot K^p$ is a field with required properties. We fix such extension L/k_p . Denote by $\text{Inf}(a_{\tau,\tau'})_{\mathfrak{G}_p}$ the image of the cocycle $(a_{\tau,\tau'})_{\mathfrak{G}_p}$ by the inflation map: $H^2(\mathfrak{G}_p, (K^p)^\times) \rightarrow H^2(\mathfrak{G}(L/k_p), L^\times)$. Since the inflation map is injective, the order of $(a_{\tau,\tau'})_{\mathfrak{G}_p}$ in $H^2(\mathfrak{G}_p, (K^p)^\times)$ is equal to that of $\text{Inf}(a_{\tau,\tau'})_{\mathfrak{G}_p}$ in $H^2(\mathfrak{G}(L/k_p), L^\times)$. Furthermore, the factor set $\text{Inf}(a_{\tau,\tau'})_{\mathfrak{G}_p}$ consists of roots of unity contained in k_p , and is 'commutative', because $(a_{\tau,\tau'})_{\mathfrak{G}_p}$ has these properties. Therefore our problem to determine the local indices of the crossed product \mathfrak{A} is reduced to the following

THEOREM 3. *Let K^p be a finite abelian extension of a p -adic number field k_p . Assume that K^p contains subfields L_u and L_r such that L_u/K_p is unramified, L_r/k_p is totally and tamely ramified of degree $e=e_p$, and $K^p=L_u \cdot L_r$. Let $\mathfrak{G}(K^p/L_u)=\langle \xi \rangle$ and $\mathfrak{G}(K^p/L_r)=\langle \eta \rangle$. Let $(a_{\tau,\tau'}, K^p/k_p) = \sum_{\tau \in \mathfrak{G}} u_\tau K^p$, $u_\tau u_{\tau'} = u_{\tau\tau'} a_{\tau,\tau'}$ be a crossed product whose factor set $(a_{\tau,\tau'})$ consists of roots of unity contained in the ground field k_p and satisfies the condition $a_{\tau,\tau'} = a_{\tau',\tau}$ for every pair τ, τ' of elements in $\mathfrak{G} = \mathfrak{G}(K^p/k_p)$. Set $q = N_{k_p}/\mathbb{Q}_p(v)$. If $a_\xi = a_{\xi,1} a_{\xi,\xi} \cdots a_{\xi,\xi^{q-1}}$ is a primitive v -th root of unity such that $p \nmid v$, then the index of the crossed product $(a_{\tau,\tau'}, K^p/k_p)$ is equal to*

$$(3) \quad \frac{e_p}{\left(e_p, \frac{q-1}{v} \right)}.$$

PROOF. Let $[L_u:k_p]=f$. Then the elements u_ξ, u_{η^μ} , $0 \leq \nu \leq e-1$, $0 \leq \mu \leq f-1$ induce precisely the automorphisms of K^p over k_p , and so they are linearly independent over K^p . Hence we have

$$\begin{aligned} (a_{\tau,\tau'}, K^p/k_p) &= \sum_{\tau \in \mathfrak{G}} u_\tau K^p = \sum_{\substack{0 \leq \nu \leq e-1 \\ 0 \leq \mu \leq f-1}} u_\xi u_{\eta^\mu} K^p && (\text{direct sum}) \\ &\simeq \left(\sum_{0 \leq \nu \leq e-1} u_\xi u_{\eta^\nu} L_r \right) \otimes_{k_p} \left(\sum_{0 \leq \mu \leq f-1} u_{\eta^\mu} L_u \right) \end{aligned}$$

$$=(a_{\xi^{\nu}, \xi^{\nu'}}, L_r/k_p) \otimes_{k_p} (a_{\eta^{\mu}, \eta^{\mu'}}, L_u/k_f),$$

because each element of L_r is fixed by the automorphisms γ_i^{μ} ($0 \leq \mu \leq f-1$), and each one of L_u is fixed by ξ^{ν} ($0 \leq \nu \leq e-1$), and $u_{\xi^{\nu}} u_{\eta^{\mu}} = u_{\eta^{\mu}} u_{\xi^{\nu}}$.

As the factor set $(a_{\eta^{\mu}, \eta^{\mu'}})$ of $\mathcal{B}(L_u/k_f)$ consists of units, we have $(a_{\eta^{\mu}, \eta^{\mu'}}, L_u) \sim 1$. As L_r/k_p is cyclic, we have

$$(4) \quad (a_{\xi^{\nu}, \xi^{\nu'}}, L_r/k_p) = (a_{\xi}, L_r/k_p, \xi), \quad a_{\xi} = a_{\xi, 1} a_{\xi, \xi} \cdots a_{\xi, \xi^{e-1}},$$

where the right side is a cyclic algebra. The index of the cyclic algebra is the order of the norm residue symbol $(a_{\xi}, L_r/k_p)$. Recall that a_{ξ} is a primitive v -th root of unity and $v \nmid v$. Therefore, by the same argument as in [4, § 4] we see that the order of $(a_{\xi}, L_r/k_p)$ is equal to

$$\frac{e_p}{\left(e_p, \frac{q-1}{v}\right)}.$$

REMARK 1. For our case, we may assume that K^p/Q_p is cyclotomic. Hence the above e_p and $q=N_{k_f/Q_p}(v)$ are elementarily computed (cf. [4]).

REMARK 2. Going back to the notation of (1), the local index of the crossed product \mathfrak{A} at an infinite place p_{∞} of k is easily determined. Namely, let ι be the automorphism of K which maps each element of K to its complex conjugate. Then the index of $\mathfrak{A}_{p_{\infty}}$ is equal to 2 if k is totally real and $a_{\iota, 1} \cdots a_{\iota, \iota} = a_{\iota, \iota} = -1$. Otherwise, the index of $\mathfrak{A}_{p_{\infty}}$ is equal to 1.

REMARK 3. $e_p | p-1$ ($v | p$).

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