On B-group properties of pseudo-semi-dihedral groups

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This note is a continuation of Kanazawa-Enomoto [1], where it is proved that a semi-dihedral group of order 2^{n+2} is a B-group if $n>2$, while in case $n=2$ it is not a B-group. In this paper we shall prove the following:

THEOREM. *Let H be a pseudo-semi-dihedral group of order 2^{n+2} with $n\geqq 2$, i.e. H is a group generated by two elements x, y with the defining relations $x^{2m}=y=e$, $y^{-1}xy=x^{m+1}$, where $m=2^n$. Then if $n>2$, H is a B-group. For $n=2$, H is not a B-group.*

We note here that '*pseudo-semi-dihedral groups*' form one of the six types of 2-groups which have a cyclic subgroup of index two. For the other five types, all B-groups among them have already been determined (see [1]).

1. Let H be a pseudo-semi-dihedral group of order $4m$ as in the above theorem. Then the center C of H is a cyclic subgroup of order m generated by x^2. Let h, k be in H. Then h and k commute if and only if one of h, k or hk^{-1} is an element of C. Otherwise one has $hk=hkz$, where $z=x^m$ is the unique central involution in H. Now we decompose an element $\sigma=\sum_{h\in H}a_h\cdot h$ in the group ring ZH into four parts as follows ($\sigma_{(i)}$ is called the *i-th part* of σ):

$$\sigma_{(1)}=\sum_{h\in C}a_h\cdot h\,,\qquad \sigma_{(2)}=\sum_{h\in zC}a_h\cdot h\,,$$

$$\sigma_{(3)}=\sum_{h\in yzC}a_h\cdot h\,,\qquad \sigma_{(4)}=\sum_{h\in yC}a_h\cdot h\,.$$

Also we put $\sigma'=\sigma_{(1)}+\sigma_{(2)}$ and $\sigma''=\sigma_{(3)}+\sigma_{(4)}$. Then for σ, τ in ZH, we have $\sigma_{(i)}\tau_{(j)}=\tau_{(j)}\sigma_{(i)}z$ if $i\neq 1$, $j\neq 1$ and $i\neq j$. Otherwise $\sigma_{(i)}\tau_{(j)}=\tau_{(j)}\sigma_{(i)}$ holds. For h in H and σ in ZH, we denote by $c(h;\sigma)$ the coefficient of h appearing in σ, i.e. $\sigma=\sum_{h\in H}c(h;\sigma)\cdot h$. For any mapping f of Z into Z, we define a mapping f^* of ZH into ZH by $c(h;f^*(\sigma))=f(c(h;\sigma))$ for σ in ZH, i.e. $f^*(\sigma)=\sum_{h\in H}f(a_h)\cdot h$ for $\sigma=\sum_{h\in H}a_h\cdot h$. For σ in ZH we define the support $S(\sigma)$ of σ by

$$S(\sigma)=\{h\in H;\ c(h;\sigma)\neq 0\}$$

and the dual σ^* of σ as $c(h;\sigma^*)=c(h^{-1};\sigma)$. (As to the definitions and basic properties of B-groups or S-rings, see [1] or [2]).

2. In order to show that H is a B-group for n>2, it is enough to show that any primitive S-ring over H is trivial (see (1.4) in [1]). Now let $\mathfrak{B}$ be a non-

trivial primitive S-ring over H and we shall derive a contradiction in case $n>2$.

LEMMA 1. *$\mathfrak{B}$ is a commutative ring.*

PROOF. For σ, τ in $\mathfrak{B}$, put $\rho=\sigma\tau-\tau\sigma$. Then we have

$$\rho=(\sigma_{(3)}\tau_{(4)}+\sigma_{(4)}\tau_{(3)}+\sigma_{(2)}\tau_{(4)}+\sigma_{(4)}\tau_{(2)}+\sigma_{(2)}\tau_{(3)}+\sigma_{(3)}\tau_{(2)})(e-z),$$

and so we have $\rho z=-\rho$. Now we shall define a mapping f of $\boldsymbol{Z}$ into $\boldsymbol{Z}$ by $f(a)=|a|$. Then $f^*(\rho)z=f^*(\rho)$ holds, and therefore $f^*(\rho)\in \boldsymbol{Z}\cdot \underline{H}$, where $\underline{H}=\sum_{h\in H} h\in \boldsymbol{Z}H$ (see (1.3) in [1]). But $\rho_{(1)}=0$ implies $(f^*(\rho))_{(1)}=0$, and therefore we have $f^*(\rho)=0$, hence $\rho=0$. This means $\sigma\tau=\tau\sigma$ for any σ, τ in $\mathfrak{B}$, that is, $\mathfrak{B}$ is a commutative ring, and the lemma is proved.

Now suppose that the simple basis of $\mathfrak{B}$ is $\{\tau_1, \cdots, \tau_s, \tau_{s+1}, \cdots, \tau_r, \tau_{s+1}^*, \cdots, \tau_r^*\}$, where $\tau_i^*=\tau_i$ $(i=1, 2, \cdots, s)$ and $\tau_j^*\neq\tau_j$ $(j=s+1, \cdots, r)$. If we put $\rho_i=\tau_i$ $(i=1, 2, \cdots, s)$, $\rho_j=\tau_j+\tau_j^*$ $(j=s+1, \cdots, r)$ and $\mathfrak{W}=\sum_{i=1}^{r} \boldsymbol{Z}\rho_i$, it is easily checked that $\mathfrak{W}$ is a non-trivial primitive S-ring over H and $\sigma=\sigma^*$ holds for any σ in $\mathfrak{W}$. Therefore in the following we may and shall assume that the simple basis of $\mathfrak{B}$ is $\{\tau_1, \cdots, \tau_r\}$ and $\tau_i^*=\tau_i$ holds for $i=1, \cdots, r$.

LEMMA 2. *Let σ be in $\mathfrak{B}$. If we have $\sigma''z=\sigma''$, then σ is contained in the trivial S-ring $\mathfrak{B}_0$.*

PROOF. $\sigma_{(4)}z=\sigma_{(4)}$ implies $(\sigma^2)_{(2)}\equiv(\sigma^2)_{(3)}\equiv 0$ (mod. $2\boldsymbol{Z}H$). Therefore if we define a mapping f of $\boldsymbol{Z}$ into $\boldsymbol{Z}$ by $f(x)\equiv x$ (mod. 2), where $f(x)$ is equal to 0 or 1, then the support $S(f^*(\sigma^2))$ of $f^*(\sigma^2)$ is contained in a proper subgroup $\langle x^2, y\rangle$ of H, since $(f^*(\sigma^2))_{(2)}=(f^*(\sigma^2))_{(3)}=0$. Because $\mathfrak{B}$ is primitive, we have $S(f^*(\sigma^2))\subset\{e\}$ and therefore we may write

$$\sigma^2=a\cdot e+2\cdot\rho \tag{2.1}$$

with $\rho\in \boldsymbol{Z}H$ and $a\in \boldsymbol{Z}$. On the other hand $(\sigma^2)_{(1)}$ is equal to $\sum_{i=1}^{4}(\sigma_{(i)})^2$ and $\sigma''z=\sigma''$ implies $(\sigma'')^2\equiv 0$ (mod. $2\boldsymbol{Z}H$). Therefore we have

$$(\sigma^2)_{(1)}\equiv(\sigma_{(1)})^2+(\sigma_{(2)})^2\equiv \sum_{h\in S(\sigma')} h^2 \pmod{2\boldsymbol{Z}H}.$$

Hence (2.1) implies

$$\sum_{h\in S(\sigma')} h^2\equiv a\cdot e \pmod{2\boldsymbol{Z}H}. \tag{2.2}$$

For h, k in $\langle x\rangle$, the equation $h^2=k^2$ implies $h=k$ or $h=kz$. Hence (2.2) implies that if $h\ (\neq e, z)\in S(\sigma')$ then $hz\in S(\sigma')$ with $c(h;\sigma)=c(hz;\sigma)$. Therefore we may write

$$\sigma'=b\cdot e+c\cdot z+\alpha\cdot(e+z),\quad b,c\in \boldsymbol{Z},\quad \alpha\in \boldsymbol{Z}H.$$

Then $(\sigma'-(b-c)\cdot e)z=\sigma'-(b-c)\cdot e$ holds, and therefore $(\sigma-(b-c)\cdot e)z=\sigma-(b-c)\cdot e$, since $\sigma''z=\sigma''$. This implies $\sigma-(b-c)\cdot e\in \boldsymbol{Z}\cdot H$, hence $\sigma\in\mathfrak{B}_0$ (see (1.3) in [1]).

LEMMA 3. *Let σ be in $\mathfrak{B}$. Then $\sigma_{(1)}=0$ implies $\sigma=0$ if $n>2$.*

PROOF. $\sigma_{(1)}=0$ implies $(\sigma^2)_{(i)}z=(\sigma^2)_{(i)}$ $(i=2, 3, 4)$, since

$$\sigma^2=(\sigma_{(2)})^2+(\sigma_{(3)})^2+(\sigma_{(4)})^2+(\sigma_{(3)}\sigma_{(4)}+\sigma_{(4)}\sigma_{(2)}+\sigma_{(2)}\sigma_{(3)})(e+z)\,.$$

Then by Lemma 2 we may put

$$\sigma^2=a\cdot e+b\cdot \underline{H}\quad a,\ b\in \boldsymbol{Z}\,.$$

In our case we may assume that σ is simple, and put

$$t=|S(\sigma_{(2)})|\,,\ \ u=|S(\sigma_{(3)})|\,,\ \ v=|S(\sigma_{(4)})|\,.$$

Then considering the coefficient of e and the sum of coefficients in each part of σ^2, we have

$$t+u+v=a+b\,, \tag{3.1}$$

$$t^2+u^2+v^2=a+bm\,, \tag{3.2}$$

$$2uv=2vt=2tu=bm\,. \tag{3.3}$$

If $tuv\neq 0$, then (3.3) implies $t=u=v$, and we have $t^2=a$ from (3.2) and (3.3). Then (3.1) implies

$$3t-t^2=b>0\,.$$

This means $t=1$ or $t=2$, but then we have $m=1$ or $m=4$ respectively, contradicting to our assumption $n>2$. Therefore $tuv=0$, and so at least two of t, u and v are equal to zero. Hence $\langle S(\sigma)\rangle$ is a proper subgroup of H, and so σ must be equal to zero.

LEMMA 4. *Let g be an odd integer and σ an element of $\mathfrak{B}$. Then $\mathfrak{B}$ contains uniquely an element $\sigma^{[g]}$ such that $c(h^g;\,\sigma^{[g]})=c(h;\,\sigma)$ for any h in C.*

PROOF. The uniqueness part follows from Lemma 3. To show the existence of $\sigma^{[g]}$, we choose a prime number p such that $p>2\cdot\operatorname{Max}_{h\in C}|c(h;\,\sigma)|$ and $p\equiv g$ (mod. m), and put

$$\sigma_{(0)}=\sigma_{(2)}+\sigma_{(3)}+\sigma_{(4)}\,.$$

Then $\sigma_{(0)}$ and $\sigma_{(1)}$ commute, and so we have

$$\sigma^p\equiv(\sigma_{(1)})^p+(\sigma_{(0)})^p\equiv\sum_{h\in C}c(h;\,\sigma)\cdot h^g+(\sigma_{(0)})^p\ \ (\text{mod. } pZH)\,. \tag{4.1}$$

Now we shall consider the product of p elements of $S(\sigma_{(0)})$. Suppose a_i of them are elements of $S(\sigma_{(i)})$ $(i=2, 3, 4)$. Then the product is in C if and only if each a_i is odd. Now suppose a_i of $\{j_1, j_2, \cdots, j_p\}$ is equal to i $(i=2, 3, 4)$, and each a_i is odd. The we have

$$\sigma_{(j_1)}\sigma_{(j_2)}\cdots\sigma_{(j_p)}=\sigma_{(j_2)}\cdots\sigma_{(j_p)}\sigma_{(j_1)}z^{a_1+a_2+a_3-a_{j_1}}$$
$$=\sigma_{(j_2)}\cdots\sigma_{(j_p)}\sigma_{(j_1)}\,.$$

Continuing this process, we have

$$\sigma_{(j_1)}\cdots\sigma_{(j_p)}=\sigma_{(j_2)}\cdots\sigma_{(j_p)}\sigma_{(j_1)}=\cdots=\sigma_{(j_p)}\sigma_{(j_1)}\cdots\sigma_{(j_{p-1})}\,.$$

Hence we have $((\sigma_{(0)})^p)_{(1)}\equiv 0$ (mod. pZH), and then (4.1) implies

$$(\sigma^p)_{(1)}\equiv\sum_{h\in G}c(h;\sigma)\cdot h^g \ (\text{mod. } pZH)\,.$$

Now we define a mapping f of $\boldsymbol{Z}$ into $\boldsymbol{Z}$ by $f(a)\equiv a$ (mod. p), $|f(a)|<\frac{p}{2}$, and put $\sigma^{[g]}=f^*(\sigma^p)$. Then $\sigma^{[g]}$ is an element of $\mathfrak{B}$ and $\sigma^{[g]}\equiv\sigma^p$ (mod. pZH). Therefore $c(h^g;\sigma^{[g]})=c(h;\sigma)$, and the lemma is proved.

The following lemmas are proved in the same way as in [1].

LEMMA 5. (1) *If g, g' are odd integers, then $(\sigma^{[g]})^{[g']}=\sigma^{[gg']}$, and if $g\equiv g'$ (mod. m) then $\sigma^{[g]}=\sigma^{[g']}$.*

(2) $\sum_{h\in G}c(h;\sigma)=\sum_{h\in G}c(h;\sigma^{[g]})$ *and* $|S(\sigma_{(1)})|=|S((\sigma^{[g]})_{(1)})|$.

(3) *For any σ in $\mathfrak{B}$, there is an element ρ in $\mathfrak{B}$ such that $\sigma=\rho^{[g]}$.*

LEMMA 6. *For g odd, the mapping φ_g of $\mathfrak{B}$ into $\mathfrak{B}$, defined by $\varphi_g(\sigma)=\sigma^{[g]}$, is a ring-automorphism of $\mathfrak{B}$.*

LEMMA 7. *Let g be odd, and $\{\sigma_1,\cdots,\sigma_r\}$ be the simple basis of $\mathfrak{B}$. Then $(\sigma_1^{[g]},\cdots,\sigma_r^{[g]})$ is a permutation of $(\sigma_1,\cdots,\sigma_r)$, and $|S(\sigma_i)|=|S(\sigma_i^{[g]})|$.*

Now we shall define a rational element of $\mathfrak{B}$ as in [1]: an element σ in $\mathfrak{B}$ is called rational if and only if it coincides with $\sigma^{[g]}$ for all odd integers g. All the rational elements in $\mathfrak{B}$ form a primitive S-subring $\mathfrak{B}'$ of $\mathfrak{B}$. We can show that $\mathfrak{B}'=\mathfrak{B}_0$ implies $\mathfrak{B}=\mathfrak{B}_0$ as in [1], and therefore we may and shall assume $\mathfrak{B}=\mathfrak{B}'$ in the following.

Now suppose $\mathfrak{B}\neq\mathfrak{B}_0$. Then there is a simple element τ in $\mathfrak{B}$ such that $S(\tau)\not\ni e, z$. By the rationality of $\mathfrak{B}$, we have $\tau_{(1)}z=\tau_{(1)}$, and then $(\tau^2)_{(i)}z=(\tau^2)_{(i)}$ ($i=2, 3, 4$) hold. Hence by Lemma 2 we may put

(8.1) $$\tau^2=a\cdot e+b\cdot \underline{H}\quad a, b\in\boldsymbol{Z}\,.$$

Now put $s=|S(\tau_{(1)})|$, $t=|S(\tau_{(2)})|$, $u=|S(\tau_{(3)})|$, $v=|S(\tau_{(4)})|$. Then considering the coefficient of e and the sum of coefficients in each part of σ^2, we have

(8.2) $$s+t+u+v=a+b\,,$$

(8.3) $$s^2+t^2+u^2+v^2=a+bm\,,$$

(8.4) $$2(st+uv)=2(su+tv)=2(sv+tu)=bm\,.$$

From (8.2) and (8.3), we know that $b(m-1)$ is even and so b is even, since $m-1$ is odd. Therefore (8.1) implies

(8.5) $$\tau^2 \equiv a \cdot e \pmod{2ZH}.$$

Now we shall consider the first part of τ^2. The solutions h of $h^2=x^{4i}$ are x^{2i}, $x^{2i}z$, $yx^{2i}w$ and $yx^{2i}wz$, where $z=x^m$ and $w=x^{m/2}$. Therefore (8.5) implies that $h\in S(\tau_{(4)})$ is equivalent to $hz\in S(\tau_{(4)})$ if $h\neq y$, yz, since $\tau_{(1)}z=\tau_{(1)}$. We shall show that this equivalence holds even if $h=y$ or yz.

By way of contradiction suppose $\tau_{(4)}=\rho+y$, $\rho\in ZH$, $S(\rho)\ni y$ and $\rho z=\rho$. Then

$$\begin{aligned}(\tau^2)_{(2)} &= 2\tau_{(1)}\tau_{(2)}+\tau_{(3)}\tau_{(4)}(e+z)\\ &\equiv \tau_{(3)}y(e+z) \pmod{2ZH}.\end{aligned}$$

Therefore (8.5) implies $\tau_{(3)}y(e+z)\equiv 0 \pmod{2ZH}$, and then we have $\tau_{(3)}z=\tau_{(3)}$, since $\tau_{(3)}$ is simple. Similarly we have $\tau_{(2)}z=\tau_{(2)}$ and therefore we may write

$$\tau=\tau'+y$$

with $\tau' z=\tau'$, $S(\tau')\ni y$. Then we have $a=1$, since

$$b=c(z;\tau^2)=|S(\tau')|=|S(\tau)|-1=a+b-1.$$

But we have $(a+b)^2=a+4bm$ from (8.2) through (8.4), and so $b=4m-2$. Therefore $|S(\tau)|=a+b=4m-1$. On the other hand $|S(\tau)|\leqq 4m-2$, since $S(\tau)\ni e, z$. This is a contradiction and thus we have obtained the following

LEMMA 8. $\tau_{(4)}z=\tau_{(4)}$.

Now we can complete the proof of our theorem just in the same manner as in [1], exchanging the 3rd part for the 4th part.

3. We shall show here that a pseudo-semi-dihedral group of order 16 is not a B-group. It is shown by the same idea as in [1], but we shall give here another type of example.

Let G be the Weyl group of the simple Lie algebra of type (D_5). Then G is the semi-direct product of S and D, where S and D are isomorphic to the symmetric group $\mathfrak{S}_5$ on five letters and the elementary abelian group $\boldsymbol{Z}_2\times\boldsymbol{Z}_2\times\boldsymbol{Z}_2\times\boldsymbol{Z}_2$ of order 16 respectively. They are described in the matrix form as follows: S consists of all permutation matrices of degree five, and D consists of all diagonal matrices of degree five having ± 1 on its diagonal and whose determinant is one.

Now put

$$x=\begin{pmatrix}0&0&0&-1&0\\1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&0&-1\end{pmatrix},\qquad y=\begin{pmatrix}-1&0&0&0&0\\0&1&0&0&0\\0&0&-1&0&0\\0&0&0&1&0\\0&0&0&0&1\end{pmatrix}.$$

Then the group $H=\langle x, y\rangle$ generated by x and y is isomorphic to a pseudo-semi-dihedral group of order 16, and $S\cap H$ consists of identity matrix alone. Therefore we have $G=SH=HS$ and we may regard G as a permutation group on H with H as a regular subgroup. The isotropy subgroup of e is S, and H is decomposed into three S-orbits K_1, K_2 and K_3, according to the number of negative entries:

$$K_1=\{e\},$$

$$K_2=\{x^3, x^4, x^5, yx^5, yx^7\},$$

$$K_3=\{x, x^2, x^6, x^7, y, yx, yx^2, yx^3, yx^4, yx^6\}.$$

Since $K_iS=SK_i$ $(i=1, 2, 3)$, and $\langle K_2\rangle=\langle K_3\rangle=H$, S is a maximal subgroup of G; i.e. G is primitive as a permutation group on H. Therefore H is not a B-group.

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References

[1] M. Kanazawa and H. Enomoto; On B-group properties of semi-dihedral groups, J. Fac. Sci., Univ. Tokyo, Sec. I, **15** (1968), 71-85.
[2] H. Wielandt; "Finite Permutation Groups," Academic Press, New York and London, 1964.

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