On some discrete subgroups of $SL_2(R)$

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Let A be an indefinite quaternion algebra over $\boldsymbol{Q}$, i.e. a normal simple algebra over $\boldsymbol{Q}$ such that $A\otimes_{\boldsymbol{Q}}\boldsymbol{R}\cong M_2(\boldsymbol{R})$ and let O be an order of A. Denote by $\mathrm{tr}(\gamma)$ and $\mathrm{n}(\gamma)$ the reduced trace and the reduced norm in A respectively. Put

$$U=\{\varepsilon\in O\,|\,\varepsilon O=O,\ \mathrm{n}(\varepsilon)=1\}\,.$$

Then by the isomorphism $A\otimes_{\boldsymbol{Q}}\boldsymbol{R}\cong M_2(\boldsymbol{R})$, U can be identified with a discrete subgroup Γ_0 of the special linear group $SL_2(\boldsymbol{R})$. The group $SL_2(\boldsymbol{R})$ operates on the upper half plane $H=\{z\in \boldsymbol{C}\,|\,\mathrm{Im}\, z>0\}$ by the operation

$$H\ni z\mapsto g(z)=\frac{az+b}{cz+d}\in H \quad \text{for} \quad g=\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in SL_2(\boldsymbol{R})\,.$$

It is well known that Γ_0 is a Fuchsian group of the 1st kind, i.e. Γ_0 is a properly discontinuous group and its quotient space H/Γ_0 has finite volume. Let Γ be a subgroup of Γ_0 of finite index. We call such a group Γ the Fuchsian group derived from the quaternion algebra over $\boldsymbol{Q}$. In this paper we shall prove the following theorem.

THEOREM. *Let Γ be a Fuchsian group of the 1st kind. Γ is derived from a quaternion algebra over $\boldsymbol{Q}$ if and only if Γ satisfies the following condition.*
(1) $\mathrm{tr}(\gamma)$ *is a rational integer for every γ in Γ.*

In order to prove our theorem we must prepare several propositions.

PROPOSITION 1. *Let Γ be a Fuchsian group of the 1st kind in $SL_2(\boldsymbol{R})$ such that the set $\mathrm{tr}(\Gamma)$ is contained in a finite algebraic number field k. Then there exists an element g in $SL_2(\boldsymbol{R})$ and a finite algebraic number field K such that $g^{-1}\Gamma g\subseteqq SL_2(K)$.*

PROOF. Take a hyperbolic transformation γ in Γ and denote by $\mathfrak{A}_1$, $\mathfrak{A}_2$ eigenvectors of γ and by λ, λ^{-1} eigen-values of γ respectively. Since $|\mathrm{tr}(\gamma)|>2$, λ is a real number. We can choose $\mathfrak{A}_1$, $\mathfrak{A}_2$ such that their coefficients are in the field $k(\lambda)$ and $\det(\mathfrak{A}_1, \mathfrak{A}_2)>0$. Put $g_1=\frac{1}{\sqrt{\det(\mathfrak{A}_1, \mathfrak{A}_2)}}(\mathfrak{A}_1, \mathfrak{A}_2)$ and $K=k(\lambda)$. Then $g_1^{-1}\gamma g_1=\begin{pmatrix} \lambda & 0\\ 0 & \lambda^{-1}\end{pmatrix}$. Take an element $\gamma=\begin{pmatrix} a & b\\ c & d\end{pmatrix}$ of Γ such that $c>0$ and put $g_2=\begin{pmatrix} \sqrt{c}^{-1} & 0\\ 0 & \sqrt{c}\end{pmatrix}$. Then we know that $(g_1g_2)^{-1}\Gamma g_1g_2$ contains two elements

$$\gamma_0=\begin{pmatrix}\lambda & 0\\ 0 & \lambda^{-1}\end{pmatrix}\ (\lambda^2\neq 1) \text{ and } \gamma_1=\begin{pmatrix}a_1 & b_1\\ 1 & d_1\end{pmatrix}\ (b_1\neq 0)\,.$$

Take an arbitrary element $\gamma=\begin{pmatrix}a & b\\ c & d\end{pmatrix}$ of $(g_1g_2)^{-1}\Gamma g_1g_2$.

By the following relation

$$(2)\qquad \begin{pmatrix}\lambda & 0\\ 0 & \lambda^{-1}\end{pmatrix}\begin{pmatrix}a & b\\ c & d\end{pmatrix}=\begin{pmatrix}\lambda a & \lambda b\\ \lambda^{-1}c & \lambda^{-1}d\end{pmatrix}$$

$a+b$ and $\lambda a+\lambda^{-1}d$ are contained in k. Hence a and d are contained in K. Especially a_1, d_1 are contained in K. Since $\det(\gamma_1)=a_1d_1-b_1=1$, b_1 is also contained in K. On the other hand, by the following relation

$$(3)\qquad \begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}a_1 & b_1\\ 1 & d_1\end{pmatrix}=\begin{pmatrix}aa_1+b & ab_1+bd_1\\ ca_1+d & cb_1+dd_1\end{pmatrix}$$

aa_1+b and cb_1+dd_1 are contained in K. Hence b and c are also contained in K.

PROPOSITION 2. *Let the assumption be the same as in Proposition* 1. *Put* $k_0=\boldsymbol{Q}(\mathrm{tr}(\gamma)|\gamma\in\Gamma)$ *and* $A=k_0[\Gamma]=\{\sum_{i=1}^{d} a_i\gamma_i|a_i\in k_0, \gamma_i\in\Gamma\}$. *Then A is a quaternion algebra over* k_0.

PROOF. By the Proposition 1 we can assume that Γ contains two elements $\gamma_0=\begin{pmatrix}\lambda & 0\\ 0 & \lambda^{-1}\end{pmatrix}(\lambda^2\neq 1)$, $\gamma_1=\begin{pmatrix}a_1 & b_1\\ 1 & d_1\end{pmatrix}(b_1\neq 0)$ and $\Gamma\subseteqq SL_2(K_0)$ where $K_0=k_0(\lambda)$ is equal to either k_0 or a quadratic extension over k_0. Hence we have $A\subseteqq M_2(K_0)$ and $1\lneqq\dim_{k_0}(A)\leqq 8$. We shall show first that the radical R of the algebra A is trivial. Since $R^e=\{0\}$ for some integer e, we have $\mathrm{tr}(\gamma)=\det(\gamma)=0$ for every element $\gamma=\begin{pmatrix}a & b\\ c & d\end{pmatrix}$ in R. By (2) we have $a+d=0$ and $a\lambda+d\lambda^{-1}=0$. Hence $a=d=0$. Moreover, by (3) we have $b=c=0$. This shows $R=\{0\}$.

Let Z be the center of the algebra A. We shall show that $Z=k_0\cdot 1_2$. Take an arbitrary element $\gamma=\begin{pmatrix}a & b\\ c & d\end{pmatrix}$ of Z. By the fact that γ commutes with γ_0, we have $b=c=0$. Since γ commutes with γ_1, we have $a=d$. Hence $\gamma=a\cdot 1_2$. By (1) a is in k_0. Hence A is a normal simple algebra over k_0. Put $r=\dim_{k_0}A$. Then $1\lneqq r\leqq 8$ and r is a square number. Hence $r=4$. This completes the proof.

PROPOSITION 3. *Let Γ be a Fuchsian group of the 1st kind in $SL_2(\boldsymbol{R})$ such that $\mathrm{tr}(\Gamma)$ is contained in the ring of integers O_k of a finite algebraic number field k. Put* $k_0=\boldsymbol{Q}(\mathrm{tr}(\gamma)|\gamma\in\Gamma)$,

$$A=k_0[\Gamma]=\{\sum_{i=1}^{d} a_i\gamma_i|a_i\in k_0,\ \gamma_i\in\Gamma\}$$

and

$$O=O_{k_0}[\Gamma]=\{\sum_{i=1}^{d} a_i\gamma_i|a_i\in O_{k_0},\ \gamma_i\in\Gamma\}\,.$$

Then O is an order of the quaternion algebra A.

PROOF. It is trivial that O is a ring and generates the algebra A over k_0. We have only to show that the ring O is a finitely generated O_{k_0}-module. By the preceding propositions we may assume that the group Γ contains two elements

$$\gamma_0=\begin{pmatrix}\lambda & 0\\ 0 & \lambda^{-1}\end{pmatrix}\ (\lambda^2\neq 1) \text{ and } \gamma_1=\begin{pmatrix}a_1 & b_1\\ 1 & d_1\end{pmatrix}\ (b_1\neq 0),$$

and that Γ is contained in $SL_2(K_0)$ where $K_0=k_0(\lambda)$. Take an arbitrary element $\gamma=\begin{pmatrix}a & b\\ c & d\end{pmatrix}$ of O. Then by the condition (2) $a+d$ and $\lambda a+\lambda^{-1}d$ are in O_{k_0}. Hence a and d are in the ideal $\frac{1}{\lambda^2-1}O_{k_0}$. By (3) aa_1+b and cb_1+dd_1 are also in $\frac{1}{\lambda^2-1}O_{k_0}$. Thus we know that all coefficients of γ in Γ are contained in an ideal of K_0. Hence O is a finitely generated O_{k_0}-module.

PROOF OF THEOREM. It is trivial that a group Γ derived from a quaternion algebra over $\boldsymbol{Q}$ satisfies the condition (1). We must prove the converse. Put $A=\boldsymbol{Q}[\Gamma]$ and $O=\boldsymbol{Z}[\Gamma]$. Then by the preceding propositions we know that A is a quaternion algebra over $\boldsymbol{Q}$ and that the ring O is an order of A and that the group Γ is a subgroup of the unit group of O. We must see that the quaternion algebra A is indefinite. Since $\boldsymbol{R}[A]=\boldsymbol{R}[\Gamma]\subseteq M_2(\boldsymbol{R})$ and $\boldsymbol{R}[\Gamma]$ is a quaternion algebra over $\boldsymbol{R}$, we have $\boldsymbol{R}[A]=M_2(\boldsymbol{R})$. Hence $A\otimes_{\boldsymbol{Q}}\boldsymbol{R}\cong\boldsymbol{R}[\Gamma]=M_2(\boldsymbol{R})$. The unit group of O can be identified with a Fuchsian group Γ_0 and Γ is a subgroup of Γ_0. Hence Γ is of finite index in Γ_0. This completes the proof of our theorem.

COROLLARY TO THE THEOREM. *Let Γ be a Fuchsian group of the 1st kind contained in $SL_2(\boldsymbol{Q})$. Then Γ is commensurable with the unimodular group $SL_2(\boldsymbol{Z})$ if and only if* $\mathrm{tr}(\gamma)$ *is a rational integer for every element γ in Γ.*

PROOF. If Γ is commensurable with $SL_2(\boldsymbol{Z})$, by the lemma in [1], we know that Γ satisfies the condition (1). Conversely, we assume that Γ satisfies the condition (1). Put $A=\boldsymbol{Q}[\Gamma]$, $O=\boldsymbol{Z}[\Gamma]$. Then A is a quaternion algebra over $\boldsymbol{Q}$ and is contained in $M_2(\boldsymbol{Q})$. Hence $A=M_2(\boldsymbol{Q})$. O is an order of $M_2(\boldsymbol{Q})$. It is well known that there exists an element g in $GL_2^+(\boldsymbol{Q})$ such that $O\subseteq g^{-1}M_2(\boldsymbol{Z})g$. Hence group Γ is contained in $g^{-1}SL_2(\boldsymbol{Z})g$ and is of finite index. Since $g^{-1}SL(\boldsymbol{Z})g$ is commensurable with $SL_2(\boldsymbol{Z})$ group Γ is commensurable with the unimodular group $SL_2(\boldsymbol{Z})$. This proves the corollary.

In the paper [1], we needed three conditions to show that group Γ is commensurable with the unimodular group $SL_2(\boldsymbol{Z})$. Now this corollary shows that we need the only one condition.

Let k be a totally real algebraic number field and let A be a quaternion algebra over k such that $A\otimes_{Q}\boldsymbol{R}\cong M_2(\boldsymbol{R})\times\boldsymbol{K}\times\cdots\times\boldsymbol{K}$, where $\boldsymbol{K}$ is Hamilton's quaternion algebra. Let O be an order of A. Put $U=\{\varepsilon\in O|\varepsilon O=O,\ n(\varepsilon)=1\}$. Then U can be identified with the Fuchsian group Γ. $\mathrm{Tr}(\gamma)$ is an integer in k for every γ in Γ. If we could prove the converse, this would be a generalization of our theorem. However, this is not true because we can find some counter examples in [2] which are well known as Hecke's group.

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References

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