On Šilov boundaries of Siegel domains

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Introduction

The theory of the classification of homogeneous bounded domains has been developed by Piatetski-Šapiro and others (cf. [8], [5], [1]). In consequence the classification was reduced to that of affine homogeneous Siegel domains of the first kind or the second kind. In this paper we take up the problem of the classification of Siegel domains of the second kind from the point of view of their *Šilov boundaries* (For the definition see § 1). We shall consider exclusively *non-degenerate* Siegel domains (For the definition see § 2.), which form a remarkable class of Siegel domains of the second kind. The class of non-degenerate Siegel domains contain all irreducible symmetric bounded domains which are not of tube type (cf. Thoerem 2.4). Our main theorem (Theorem 2.2) is as follows:

THEOREM. *Let $\mathscr{D}$ and $\mathscr{D}'$ be non-degenerate affine homogeneous Siegel domains of the second kind in $\boldsymbol{C}^n$. Let S and S' be the Šilov boundaries of $\mathscr{D}$ and $\mathscr{D}'$, respectively. Then $\mathscr{D}$ and $\mathscr{D}'$ are holomorphically equivalent if and only if S is carried to S' by some linear transformation of $\boldsymbol{C}^n$.*

In § 1 for Siegel domains of the first kind or of the second kind we prove the existence and the uniqueness of their Šilov boundaries (cf. Theorem 1.1). These facts were implicitly proved in Piatetski-Šapiro [4]. We recall the structure of the affine automorphism group $\mathscr{G}(S)$ of Šilov boundary S (Piatetski-Šapiro [4] or Theorem 1.2). In § 2 the main theorem is proved. It follows from Theorem 2.1 and Theorem 2.2 that any non-degenerate affine homogeneous Siegel domain $\mathscr{D}$ can be reconstructed uniquely by its Šilov boundary S from the affine automorphism group $\mathscr{G}(S)$. In Theorem 2.3 we consider the decomposition of non-degenerate Siegel domains to irreducible components. In § 3 we give various examples of non-degenerate Siegel domains.

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§ 0. Notations and definitions

Let R be a real vector space of dimension n and V be an open convex cone with the vertex 0 in R which contains no entire straight lines. For brevity such a cone is called a *convex cone* in R. A linear automorphism g of R is called an *automorphism* of V if g leaves V stable. The group of all automorphisms of V is denoted by Aut V. If a convex cone V can be represented as a direct sum of two lower dimensional convex cones, then V is called *reducible*. Otherwise V is called *irreducible*.

Let W be a complex vector space of dimension m. A hermitian mapping F of $W\times W$ into R_c (= the complexification of R) is called a *V-hermitian form*, if the following two conditions are satisfied;

1) $F(u, u)\in \bar{V}$ for all $u\in W$. ($\bar{V}$ means the closure of V.)

2) $F(u, u)\neq 0$ for all $u\neq 0$.

For a convex cone V in R and a V-hermitian form F on W, we define a domain $\mathscr{D}(V, F)$ in the complex vector space $R_c\times W$ by putting

$$\mathscr{D}(V, F)=\{(x+\sqrt{-1}\ y, u)\in R_c\times W;\ y-F(u, u)\in V\}\ .$$

If $W\neq(0)$, then $\mathscr{D}(V, F)$ is called a *Siegel domain of the second kind*. If $W=(0)$, then $\mathscr{D}(V, F)$ is called a *Siegel domain of the first kind*, which is also called a tube domain over V. Hereafter, for brevity, a Siegel domain of the first kind or of the second kind will be often called a Siegel domain. The group of all complex affine transformations of the vector space $R_c\times W$ is denoted by Aff$(R_c\times W)$. An affine transformation $g\in$ Aff$(R_c\times W)$ is called an *affine automorphism* of $\mathscr{D}(V, F)$, if g leaves $\mathscr{D}(V, F)$ stable. The group of all affine automorphisms of $\mathscr{D}(V, F)$ is denoted by $\mathscr{G}_a$, which is called the *affine automorphism group* of $\mathscr{D}(V, F)$. If $\mathscr{G}_a$ is transitive on $\mathscr{D}(V, F)$, then the Siegel domain $\mathscr{D}(V, F)$ is called *affne homogeneous*. As is known in [1] or [5], if a Siegel domain $\mathscr{D}(V, F)$ is affine homogeneous, then a maximal connected $\boldsymbol{R}$-triangular subgroup $\mathscr{T}$ acts simply transitively on $\mathscr{D}(V, F)$. The group $\mathscr{T}$ is called an *Iwasawa subgroup* of $\mathscr{G}_a$ and is uniquely determined up to conjugateness in $\mathscr{G}_a$.

Let $\mathscr{D}$ be a domain in the complex number space $\boldsymbol{C}^n$. A function f is said to be *holomorphic on the closure* $\bar{\mathscr{D}}$ of $\mathscr{D}$ if f is defined on certain open set O containing $\bar{\mathscr{D}}$ and is holomorphic on O. Let E be a ring of functions which are holomorphic on $\bar{\mathscr{D}}$ and attain their maximums on $\bar{\mathscr{D}}$. A closed subset S_E of $\bar{\mathscr{D}}$ is called a *determining set* for E (cf. [9]), if for each function $f\in E$ the maximum of $|f|$ on $\bar{\mathscr{D}}$ is attained at a point of S_E. If the minimal determining set for E exists and is unique, then it is called the *Šilov boundary* of $\mathscr{D}$ with respect to E.

§1. Šilov boundaries and their affine automorphism groups

Let V be a convex cone in R and $\mathscr{D}(V, F)$ be a Siegel domain in $R_c \times W$. The closure of $\mathscr{D}(V, E) \subset R_c \times W$ and $V \subset R$ are denoted by $\overline{\mathscr{D}(V, F)}$ and $\bar{V}$, respectively. And the boundaries of $\mathscr{D}(V, F)$ and V are denoted by $\partial\mathscr{D}(V, F)$ and ∂V, respectively.

LEMMA 1.1. $\overline{\mathscr{D}(V, F)} = \{(x+\sqrt{-1}\,y, u) \in R_c \times W;\ y-F(u, u) \in \bar{V}\}$.

PROOF. Suppose that a point $(x_0+\sqrt{-1}\,y_0, u_0) \in R_c \times W$ satisfies $y_0-F(u_0, u_0) \in \bar{V}$. If we denote by $R_c(u_0)$ the complex linear subspace $\{(x+\sqrt{-1}\,y, u);\ u=u_0\} \subset R_c \times W$, then the set $\mathscr{D}_{u_0} = \mathscr{D}(V, F) \cap R_c(u_0)$ is the tube domain over the convex cone $V+F(u_0, u_0) = \{x+F(u_0, u_0) \in R;\ x \in V\}$ with the vertex $F(u_0, u_0)$. Hence the closure $\bar{\mathscr{D}}_{u_0}$ of $\mathscr{D}_{u_0}$ in $R_c(u_0)$ coincides with the "tube domain" over the closed cone $\bar{V}+F(u_0, u_0)$ with the vertex $F(u_0, u_0)$. Consequently $y_0 \in \bar{V}+F(u_0, u_0)$ implies $(x_0+\sqrt{-1}\,y_0, u_0) \in \bar{\mathscr{D}}_{u_0}$, which shows $(x_0+\sqrt{-1}\,y_0, u_0) \in \overline{\mathscr{D}(V, F)}$. q.e.d.

For a Siegel domain $\mathscr{D}(V, F)$ in $R_c \times W$, we define the subset S of $R_c \times W$ by putting,

$$S = \{(x+\sqrt{-1}\,y, u);\ y-F(u, u)=0\}\ . \tag{1.1}$$

As an immediate corollary to Lemma 1.1 we obtain the following

LEMMA 1.2. $\partial\mathscr{D}(V, F) = \{(x+\sqrt{-1}\,y, u);\ y-F(u, u) \in \partial V\}$. *In particular S is a closed subset of the boundary of $\mathscr{D}(V, F)$.*

Let a coordinate of $(z, u) \in R_c \times W$ be $(z_1, \cdots z_n, u_1, \cdots, u_m)$. We denote by E the set consisting of all functions which are holomorphic on $\overline{\mathscr{D}(V, F)}$ and satisfy the following condition (♯).

$$|f(z, u)| \to 0 \text{ if } (z, u) \in \overline{\mathscr{D}(V, F)} \text{ and } \sum_{i=1}^{n} |z_i|^2 + \sum_{i=1}^{m} |u_i|^2 \to \infty\ . \tag{♯}$$

E has a natural ring structure, and for every $f \in E$ the absolute value $|f(z, u)|$ attains its maximum on $\overline{\mathscr{D}(V, F)}$. The following lemma has been stated in Piatetski-Šapiro [4], but we give the proof for the completeness.

LEMMA 1.3. *Let $\mathscr{D}(V, F)$ be a Siegel domain in $R_c \times W$ and let $f \in E$. Then the maximum of $|f|$ on $\overline{\mathscr{D}(V, F)}$ is attained at a point of S.*

PROOF. Let $P_0 = (x_0+\sqrt{-1}\,y_0, u_0) \in \overline{\mathscr{D}(V, F)}$ be the point where the maximum of $|f|$ on $\overline{\mathscr{D}(V, F)}$ is attained. If $y_0=F(u_0, u_0)$, then we have $(x_0+\sqrt{-1}\, y_0, u_0)\in S$. Therefore we consider the case that $y_0\neq F(u_0, u_0)$. Using the notations in the proof of Lemma 1.1, we see that $(x_0+\sqrt{-1}\, y_0, u_0)\in \mathscr{D}_{u_0}\subset R_C(u_0)$. Let P be the complex line along $\sqrt{-1}\,(y_0-F(u_0, u_0))$ in $R_C(u_0)$ passing through $(\sqrt{-1}\, F(u_0, u_0), u_0)$ and P_0. Then the complex line P with the origin $(\sqrt{-1}\, F(u_0, u_0), u_0)$ is naturally identified with the complex plane. Let H be the upper half-plane in P with respect to this identification. Then we can see $H\subset \mathscr{D}_{u_0}\subset \overline{\mathscr{D}(V, F)}$. If we denote by f_1 the restriction of f to P, then f_1 is holomorphic on $\bar{H}$ and the maximum of $|f_1|$ on $\bar{H}$ is attained at $(x_0+\sqrt{-1}\, y_0, u_0)\in H$. Therefore by the maximum principle, f_1 is a constant on H. Hence the maximum of $|f_1|$ on $\bar{H}$ is attained at a point on real axis of P, which means that the maximum of $|f|$ on $\overline{\mathscr{D}(V, F)}$ is attained at a point of S. q.e.d.

Let $\mathscr{D}(V, F)$ be a Siegel domain in $R_C\times W$. Then using the V-hermitian form F we define the product of elements of $R\times W$ as follows,

$$(a', c')(a, c)=(a+a'-2\,\mathrm{Im}\, F(c, c'), c+c')\ ,$$

where $a, a'\in R$ and $c, c'\in W$. Under this multiplication $R\times W$ becomes a Lie group, which is denoted by RW. We define the action of RW on $R_C\times W$ in the following manner,

$$(1.2)\qquad (a, c)(x+\sqrt{-1}\, y, u)=(x+a+\sqrt{-1}\,(y+2F(u, c)+F(c, c)), u+c)\ .$$

Since this operation is effective, RW can be regarded as a Lie subgroup of Aff $(R_C\times W)$.

LEMMA 1.4. *The group RW is simply transitive on S. In particular S is an affine homogeneous submanifold of $R_C\times W$.*

PROOF. By easy calculations we see that S is stable under the operation of RW. For an arbitrary point $(x+\sqrt{-1}\, F(u, u), u)\in S$, take the element $(-x, -u)\in RW$. Then by (1.2) we have

$$(-x, -u)(x+\sqrt{-1}\, F(u, u), u)=(0, 0)\in S\ ,$$

which implies that RW is transitive on S. Furthermore we see that the isotropy subgroup at $(0, 0)$ of RW is reduced to the identity. q.e.d.

From now on we choose a coordinate system $(y_1, \cdots, y_n)$ of R such that the cone V is contained in the set $y_1>0, \cdots, y_n>0$. And let the coordinate system $(z_1, \cdots, z_n)$ of R_C be the natural complexification of $(y_1, \cdots, y_n)$. If we denote by $(F_1, \cdots, F_n)$ the components of the V-hermitian form F with respect to the coordinate system $(z_1, \cdots, z_n)$, then each $F_i(u, u)$ is a positive semi-definite hermitian form on W and $F_0(u, u)=\sum_{i=1}^{n} F_i(u, u)$ is positive definite on W.

LEMMA 1.5. *The function*

$$f(z, u)=\prod_{k=1}^{n}\frac{1}{z_k+\sqrt{-1}},$$

on $R_C\times W$ *belongs to* E: *The maximum of* $|f|$ *on* $\overline{\mathscr{D}(V, F)}$ *is attained at only one point* $(0, 0)\in S$.

PROOF. We put $z_k=x_k+\sqrt{-1}\,y_k$ $(1\leqq k\leqq n)$. Then

$$|f(z, u)|^2=\prod_{k=1}^{n}\frac{1}{x_k^2+(y_k+1)^2}.$$

$\overline{\mathscr{D}(V, F)}$ is contained in the set $y_k\geqq F_k(u, u)$ $(1\leqq k\leqq n)$. Therefore for any point $(z, u)\in\overline{\mathscr{D}(V, F)}$ we have $x_k^2+(y_k+1)^2\geqq 1$ for each $1\leqq k\leqq n$, which implies that $|f(z, u)|^2\leqq 1$ for every $(z, u)\in\overline{\mathscr{D}(V, F)}$. Obviously $|f(0, 0)|=1$. Suppose that $|f(z^0, u^0)|=1$ for a point $(z^0, u^0)\in\overline{\mathscr{D}(V, F)}$. And let $z^0=(z_1^0, \cdots, z_n^0)$ and let $z_k^0=x_k^0+\sqrt{-1}\,y_k^0$ $(1\leqq k\leqq n)$. Then we have $(x_k^0)^2+(y_k^0+1)^2=1$ and $y_k^0\geqq 0$ for each $1\leqq k\leqq n$. Hence $x_1^0=\cdots=x_n^0=y_1^0=\cdots=y_n^0=0$. On the other hand $y_k^0\geqq F_k(u^0, u^0)$ $(1\leqq k\leqq n)$ imply that $\sum_{k=1}^{n}y_k^0\geqq\sum_{k=1}^{n}F_k(u^0, u^0)=F_0(u^0, u^0)$. Since the left-hand side is equal to zero and F_0 is positive definite, we see $u^0=0$. Hence we have $(z^0, u^0)=(0, 0)$. It is obvious that f is holomorphic on $\overline{\mathscr{D}(V, F)}$. If $(z, u)\in\overline{\mathscr{D}(V, F)}$ and $\sum_{i=1}^{n}|z_i|^2+\sum_{i=1}^{m}|u_i|^2\to\infty$, then $|f(z, u)|\to 0$ since $\sum_{k=1}^{n}y_k\geqq F_0(u, u)$. Therefore $f\in E$. q.e.d.

THEOREM 1.1. *Let* $\mathscr{D}(V, F)$ *be a Siegel domain in* $R_C\times W$. *And let* $S=\{(x+\sqrt{-1}\,y, u)\in R_C\times W;\ y=F(u, u)\}$ *and* E *be the ring of functions which are holomorphic on* $\overline{\mathscr{D}(V, F)}$ *and satisfy the condition* (#). *Then the Šilov boundary* S_E *of* $\mathscr{D}(V, F)$ *with respect to* E *exists and is unique. Furthermore* $S_E=S$.

PROOF. It is sufficient to prove that S is the unique minimal determining set for E. By Lemma 1.2 and Lemma 1.3 S is a determining set for E. Since the group RW leaves $\mathscr{D}(V, F)$ stable, $\overline{\mathscr{D}(V, F)}$ is also stable under RW. Therefore it follows that the group RW acts on the ring E. Let p_0 be an arbitrary point of S. By Lemma 1.4 there exists $g\in RW$ such that $g\cdot p_0=(0, 0)$. Hence we see from Lemma 1.5 that $f\cdot g\in E$ and that the maximum of $|f\cdot g|$ on $\overline{\mathscr{D}(V, F)}$ is attained at only one point p_0. This shows that S is the unique minimal determining set for E. q.e.d.

From now on we shall call S the *Šilov boundary* of $\mathscr{D}(V, F)$. And we define the group $\mathscr{G}(S)$ by putting

$$\mathscr{G}(S)=\{g\in \mathrm{Aff}\,(R_C\times W);\ gS=S\}.$$

The group $\mathscr{G}(S)$ is a closed subgroup of $\mathrm{Aff}\,(R_C\times W)$ and is called the *affine* *automorphism group of S.* As is known in [4], [1], the affine automorphism group $\mathscr{G}_a$ of $\mathscr{D}(V, F)$ is written in the following way

$$\mathscr{G}_a = \mathscr{H} \cdot RW \qquad \text{(semi-direct)}, \tag{1.3}$$

where $\mathscr{H}$ is the isotropy subgroup of $\mathscr{G}_a$ at $(0, 0)$. Then we have the following theorem of Piatetski-Šapiro [4].

THEOREM 1.2. *Let $\mathscr{D}(V, F)$ be a Siegel domain in $R_C \times W$, S be the Šilov boundary of it and $\tilde{\mathscr{H}}$ be the isotropy subgroup of $\mathscr{G}(S)$ at $(0, 0) \in S$. Then*

1) *there exist the representation ρ of $\tilde{\mathscr{H}}$ into $GL(R)$ and the representation σ of $\tilde{\mathscr{H}}$ into $GL(W)$ such that*

$$\rho(h)F(u, v) = F(\sigma(h)u, \sigma(h)v) \qquad h \in \tilde{\mathscr{H}}.$$

2) $\mathscr{G}(S) = \tilde{\mathscr{H}} \cdot RW$ (semi-direct)

3) $\mathscr{G}(S)$ acts on $R_C \times W$ as follows

$$(h, a, c)(x + \sqrt{-1}\, y, u)$$
$$= (\rho(h)x + a + \sqrt{-1}\,(\rho(h)y + 2F(\sigma(h)u, c) + F(c, c)), \sigma(h)u + c),$$

where $h \in \tilde{\mathscr{H}}$, $a \in R$, $c \in W$.

4) $\tilde{\mathscr{H}} \supset \mathscr{H}$, *in particular*, $\mathscr{G}(S) \supset \mathscr{G}_a$

5) $h \in \tilde{\mathscr{H}}$ *belongs to* $\mathscr{H}$ *if and only if* $\rho(h) \in \mathrm{Aut}\, V$.

In the next section we shall study under what condition $\mathscr{G}(S)$ coincides with $\mathscr{G}_a$.

§ 2. Non-degenerate Siegel domains

Let $\mathscr{D}(V, F)$ be a Siegel domain in $R_C \times W$. As was seen in § 1, the group RW is uniquely determined by $\mathscr{D}(V, F)$. The underlying vector space of the Lie algebra of RW is naturally identified with $R + W$, W being considered as the real vector space, and bracket relations in $R + W$ are given (cf. [1], [8]) by

$$[R, R] = [R, W] = 0$$

$$[c, c'] = -4\,\mathrm{Im}\, F(c', c) \qquad c, c' \in W.$$

In particular

$$F(u, u) = \frac{1}{4}[ju, u], \tag{2.1}$$

where j is the real linear endomorphism of W corresponding to $\sqrt{-1}$-multiplication of the complex vector space W.

Therefore $[ju, u] \neq 0$ for all $u \neq 0$, $u \in W$, since F is V-hermitian form. In particular we see

$$(0) \neq [W, W] \subset R, \text{ if } W \neq (0) . \tag{2.2}$$

DEFINITION 2.1. A Siegel domain $\mathscr{D}(V, F)$ is called *non-degenerate* if $[W, W]=R$ is satisfied. Otherwise $\mathscr{D}(V, F)$ is called *degenerate.*

DEFINITION 2.2. A homogeneous bounded domain $\mathscr{D}$ in C^n is called *non-degenerate* if $\mathscr{D}$ is holomorphically equivalent to a non-degenerate affine homogeneous Siegel domain.

We denote by L the linear closure in R of the set $\{F(u, u); u \in W\}$. Then L is regarded as a subspace of the Lie algebra $R+W$ of the group RW corresponding to $\mathscr{D}(V, F)$.

LEMMA 2.1. $[W, W]=L$.

PROOF. (2.1) shows that $L \subset [W, W]$. Conversely for $u, v \in W$

$$\begin{aligned}[u, v] &= -4 \operatorname{Im} F(v, u) \\ &= -\operatorname{Im} \{F(v+u, v+u)-F(v-u, v-u)+\sqrt{-1}\, F(v+\sqrt{-1}\, u, v+\sqrt{-1}\, u) \\ &\quad -\sqrt{-1}\, F(v-\sqrt{-1}\, u, v-\sqrt{-1}\, u)\} \\ &= F(v-\sqrt{-1}\, u, v-\sqrt{-1}\, u)-F(v+\sqrt{-1}\, u, v+\sqrt{-1}\, u) ,\end{aligned}$$

which shows $[u, v] \in L$. q.e.d.

LEMMA 2.2. *If $L=R$, then there exists $x_0 \in V$ such that*

$$x_0 = \sum_{i=1}^{n} a_i F(u_i, u_i) ,$$

where $a_i > 0, u_1, \cdots, u_n \in W$ and $n = \dim R$.

PROOF. Being $L=R$, there exist $u_1, \cdots, u_n \in W$ such that the system $\{F(u_1, u_1), \cdots, F(u_n, u_n)\}$ is a base of R. We denote by K the convex closure of the points $0, F(u_1, u_1), \cdots, F(u_n, u_n)$. Then we have $K \subset \bar{V}$ since $\bar{V}$ is convex. Furthermore $F(u_1, u_1), \cdots, F(u_n, u_n)$ are linearly independent and so the interior K^i of K is not empty. Using the fact that V is convex, we can see that the interior of $\bar{V}$ coincides with V. Hence $K^i \subset V$ holds. If we choose a point $x_0 \in K^i$, then x_0 is a point satisfying the conditions of the lemma. q.e.d.

THEOREM 2.1. *Let $\mathscr{D}(V, F)$ be an affine homogeneous Siegel domain and S be the Šilov boundary of it. Let $\mathscr{G}_a$ and $\mathscr{G}(S)$ be the affine automorphism groups of $\mathscr{D}(V, F)$ and S, respectively. Then $\mathscr{G}(S)=\mathscr{G}_a$ if and only if $\mathscr{D}(V, F)$ is non-degenerate.*

PROOF. Suppose that $\mathscr{D}(V, F)$ is non-degenerate. In order to prove $\mathscr{G}(S)=\mathscr{G}_a$ it is sufficient to prove $\mathscr{H}=\tilde{\mathscr{H}}$. By Lemma 2.1 and Lemma 2.2 there exists $x_0 \in V$ such that

$$x_0 = \sum a_i F(u_i, u_i) , \qquad a_i > 0 .$$

Using Theorem 1.2 we have

$$\rho(h)x_0 = \sum a_i \rho(h) F(u_i, u_i) = \sum a_i F(\sigma(h)u_i, \sigma(h)u_i) ,$$

for every $h \in \tilde{\mathscr{H}}$. Since $a_i > 0$ and $F(\sigma(h)u_i, \sigma(h)u_i) \in \bar{V}$, we see $\rho(h)x_0 \in \bar{V}$. By the invariance of the interior points we obtain $\rho(h)x_0 \in V$ for every $h \in \tilde{\mathscr{H}}$. On the other hand $\rho(\mathscr{H})$ is transitive on V since $\mathscr{D}(V, F)$ is affine homogeneous (cf. [1], [4], [8]). Therefore for an arbitrary $x \in V$, there exists $h' \in \mathscr{H}$ such that $\rho(h')x_0 = x$. Hence we have

$$\rho(h)x = \rho(h)\rho(h')x_0 = \rho(hh')x_0 \in V ,$$

for every $h \in \tilde{\mathscr{H}}$, which implies $\rho(h) \in \operatorname{Aut} V$. Taking account of Theorem 1.2 we see $\tilde{\mathscr{H}} = \mathscr{H}$.

To prove the converse it is not necessary to assume that $\mathscr{D}(V, F)$ is affine homogeneous. Suppose that $\mathscr{D}(V, F)$ is degenerate. Then $[W, W] \subsetneqq R$. The subspace $[W, W]$ is invariant under $\rho(\tilde{\mathscr{H}})$. In fact, $[W, W] = L$ by Lemma 2.3 and

$$\rho(h)F(u, u) = F(\sigma(h)u, \sigma(h)u) \in L ,$$

for each $h \in \tilde{\mathscr{H}}$. We choose a base of R with respect to which $\rho(h)$ can be represented by the following matrix

$$\rho(h) = \begin{pmatrix} \rho_L(h) & * \\ 0 & \rho_{R/L}(h) \end{pmatrix} ,$$

where ρ_L and $\rho_{R/L}$ are representations on L and R/L induced by ρ, respectively. For every real number t we define a linear transformation $A(t)$ of R by putting

$$A(t) = \begin{pmatrix} \rho_L(h) & * \\ 0 & \rho_{R/L}(h) + t \cdot 1_{n-s} \end{pmatrix} ,$$

where $n = \dim R$ and $s = \dim L$. Then $A(t) \in GL(R)$ except finitely many t say, except $t = t_1, \cdots, t_k$. Furthermore if $t \neq t_1, \cdots, t_k$, then the linear transformation $h(t)$ on $R_C \times W$

$$h(t);\ \begin{cases} z \longrightarrow A(t)z & z \in R_C \\ u \longrightarrow \sigma(h)u & u \in W , \end{cases}$$

belongs to $\tilde{\mathscr{H}}$. In fact we have

$$A(t)F(u, v) = \rho_L(h)F(u, v) = \rho(h)F(u, v) = F(\sigma(h)u, \sigma(h)v) ,$$

since $F(u, v)$ can be written as a complex linear combination of the elements of L. Hence S is stable under $h(t)$ and $(0, 0)\in S$ is left fixed by $h(t)$, from which it follows that $h(t)\in\tilde{\mathscr{H}}$ for each $t\neq t_1, \cdots, t_k$. We denote by $(x_1, \cdots, x_n)$ the component of $x\in R$ with respect to the base of R. Then there exists $x_0=(x_1^0, \cdots, x_n^0)\in V$ such that at least one of $x_{s+1}^0, \cdots, x_n^0$ are not equal to zero. The orbit $\{A(t)x_0;\ t\in \boldsymbol{R}, t\neq t_1, \cdots, t_k\}$ is a dense subset of a straight line in R. But this set is contained in V since $\tilde{\mathscr{H}}=\mathscr{H}$. Therefore V contains an entire straight line, which is a contradiction. Consequently we have $[W, W]=R$. q.e.d.

Let X and X' be complex vector spaces. Subsets $M\subset X$ and $M'\subset X'$ are said to be *linearly equivalent* if there exists a complex linear isomorphism of X onto X' by which M is carried to M'.

THEOREM 2.2. *Let $\mathscr{D}(V, F)$ and $\mathscr{D}(V', F')$ be non-degenerate affine homogeneous Siegel domains, and let S and S' be Šilov boundaries of $\mathscr{D}(V, F)$ and $\mathscr{D}(V', F')$, respectively. Then $\mathscr{D}(V, F)$ and $\mathscr{D}(V', F')$ are holomorphically equivalent if and only if S and S' are linearly equivalent.*

PROOF. If $\mathscr{D}(V, F)\subset R_C\times W$ and $\mathscr{D}(V', F')\subset R'_C\times W'$ are holomorphically equivalent, then they are linearly equivalent (cf. [1], [8]). We denote by φ the linear isomorphism under which $\mathscr{D}(V, F)$ is carried to $\mathscr{D}(V', F')$. Then, as is seen in the proof of Theorem 6.1 of [1], there exist the linear isomorphism φ_1 of R onto R' and φ_2 of W onto W' such that

$$\varphi(x+\sqrt{-1}\,y, u)=(\varphi_1(x)+\sqrt{-1}\,\varphi_1(y), \varphi_2(u)) \qquad x, y\in R, u\in W \tag{2.3}$$

$$\varphi_1 F(u, v)=F'(\varphi_2(u), \varphi_2(v))\ . \tag{2.4}$$

Let $(x+\sqrt{-1}\,y, u)\in S$. Then

$$\varphi_1(y)-F'(\varphi_2(u), \varphi_2(u))=\varphi_1(y-F(u, u))=0\ ,$$

which implies that $\varphi(S)=S'$.

Conversely, suppose that S is linearly equivalent to S'. Then there exists a linear isomorphism φ of $R_C\times W$ onto $R'_C\times W'$ which carries S to S'. Hence it follows that $\varphi\cdot\mathscr{G}(S)\cdot\varphi^{-1}=\mathscr{G}(S')$. Let $\mathscr{G}_a$ and $\mathscr{G}_a'$ be the affine automorphism group of $\mathscr{D}(V, F)$ and $\mathscr{D}(V', F')$, respectively. Since $\mathscr{D}(V, F)$ and $\mathscr{D}(V', F')$ are non-degenerate, we have $\mathscr{G}(S)=\mathscr{G}_a$ and $\mathscr{G}(S')=\mathscr{G}_a'$ (cf. Theorem 2.1). Therefore we obtain $\varphi\cdot\mathscr{G}_a\cdot\varphi^{-1}=\mathscr{G}_a'$. Since $\mathscr{D}(V, F)$ is non-degenerate, there exists $x_0\in V$ such that $x_0=\sum_i a_iF(u_i, u_i)$, where $a_i>0$ (cf. Lemma 2.2). Then $z_0=(\sqrt{-1}\,x_0, 0)\in\mathscr{D}(V, F)$. From the arguments in the proof of Theorem 1 in Chapter 1 of [4], it follows that there exist the linear isomorphisms φ_1 of R onto R' and φ_2 of W onto W' satisfying (2.3) and (2.4). Using (2.4) we have $\varphi_1(x_0)\in V'$; which shows $\varphi(z_0)=(\sqrt{-1}\,\varphi_1(x_0), 0)\in\mathscr{D}(V', F')$. Therefore

$$\varphi(\mathscr{D}(V,F))=\varphi(\mathscr{G}_a\cdot z_0)=(\varphi\mathscr{G}_a\varphi^{-1})(\varphi(z_0))=\mathscr{D}(V',F') .$$

Hence $\mathscr{D}(V,F)$ is linearly equivalent and of course holomorphically equivalent to $\mathscr{D}(V',F')$ under φ.

Let $\mathscr{D}$ be a homogeneous bounded domain in C^n. If $\mathscr{D}$ is not holomorphically equivalent to the direct product of two homogeneous bounded domains, then $\mathscr{D}$ is called *irreducible*. It is known in [1] that every homogeneous bounded domain is uniquely decomposed into the direct product of irreducible domains.

THEOREM 2.3. *Let $\mathscr{D}$ be a homogeneous bounded domain in C^n. Then $\mathscr{D}$ is non-degenerate if and only if each irreducible component of $\mathscr{D}$ is non-degenerate.*

PROOF. Let $\mathscr{D}=\mathscr{D}_1\times\cdots\times\mathscr{D}_s$ be the irreducible decomposition of $\mathscr{D}$. And let $\mathscr{D}(V,F)\subset R_c\times W$ and $\mathscr{D}(V_i,F_i)\subset R_{ic}\times W_i$ $(1\leqslant i\leqslant s)$ be affine homogeneous Siegel domains which are holomorphically equivalent to $\mathscr{D}$ and $\mathscr{D}_i$ $(1\leqslant i\leqslant s)$, respectively. Then there correspond to $\mathscr{D}(V,F)$ and $\mathscr{D}(V_i,F_i)$ the groups RW and R_iW_i, respectively. From Proposition 6.1 and the proof of Proposition 6.2 and Proposition 5.1 in [1], we see that

$$RW\cong R_1W_1\times\cdots\times R_sW_s$$

$$R=R_1+\cdots+R_s \qquad \text{(direct sum)}$$

$$W=W_1+\cdots+W_s \qquad \text{(direct sum)} .$$

Hence we obtain $[W,W]=\sum_{i=1}^{s}[W_i,W_i]$. On the other hand we know

$$[W,W]\subset R\,,\quad [W_i,W_i]\subset R_i \quad (1\leqslant i\leqslant s) .$$

Suppose that $\mathscr{D}$ is non-degenerate. Then there exists a non-degenerate Siegel domain which is holomorphically equivalent to $\mathscr{D}$. Therefore by uniqueness of realizations (cf. Theorem 6.1 [1]) $\mathscr{D}(V,F)$ is also non-degenerate, which shows $[W,W]=R$. Consequently we have $[W_i,W_i]=R_i$ $(1\leqslant i\leqslant s)$. This means that each $\mathscr{D}(V_i,F_i)$ is non-degenerate. The "if" part of the theorem is easily proved. q.e.d.

Let V be a convex cone in R and $\mathscr{G}$ be a subgroup of Aut V. The group $\mathscr{G}$ is called *separable*, if R is the direct sum of two $\mathscr{G}$-invariant proper subspaces.

LEMMA 2.3. (Vinberg [7]) *Let V be a convex cone in R, $\mathscr{G}$ be a connected closed subgroup of Aut V and $\mathscr{N}$ be the normalizer of $\mathscr{G}$ in $GL(R)$. If $\mathscr{G}$ acts transitively on V, then the connected component $\mathscr{N}^0$ of $\mathscr{N}$ is a subgroup of Aut V.*

Using Lemma 2.3 we obtain the following two lemmas, which are generalizations of results of Rothaus [6].

LEMMA 2.4. *Let V and $\mathscr{G}$ be the same as in Lemma 2.3. If $\mathscr{G}$ is separable, then V is reducible.*

LEMMA 2.5. *Let V be irreducible and $\mathscr{G}$ be the same as in Lemma 2.3. Then the centralizer of $\mathscr{G}$ in $GL(R)$ consists of all scalar matrices with non-zero coefficients.*

COROLLARY 2.1. *Let V be an irreducible convex cone in R and a subgroup $\mathscr{G}$ of $Aut\ V$ be the connected component of an algebraic subgroup of $GL(R)$. If $\mathscr{G}$ acts transitively on V, then the center of $\mathscr{G}$ consists of all scalar matrices with positive coefficients.*

PROOF. Under the assumptions of the corollary it is known in Vinberg [7] that $\mathscr{G}$ contains the group of all scalar matrices with positive coefficients. Since V is a convex cone, the center of $\mathscr{G}$ contains no scalar matrix with negative coefficients. Hence the corollary is an immediate consequence of Lemma 2.5. q.e.d.

A homogeneous bounded domain $\mathscr{G}$ in C^n is said to be *of tube type* if $\mathscr{D}$ is holomorphically equivalent to an affine homogeneous Siegel domain of the first kind.

THEOREM 2.4. *Let $\mathscr{D}$ be a symmetric bounded domain. If each irreducible component of $\mathscr{D}$ is not of tube type, then $\mathscr{D}$ is non-degenerate.*

PROOF. Since $\mathscr{D}$ is symmetric, each irreducible component is also symmetric. Therefore by Theorem 2.3 we can assume without loss of generality that $\mathscr{D}$ is irreducible. Let $\mathscr{D}(V, F)\subset R_C\times W$ be an affine homogeneous Siegel domain which is holomorphically equivalent to $\mathscr{D}$. We may assume that $\mathscr{D}(V, F)$ is the Cayley transform of $\mathscr{D}$ due to Korányi-Wolf [2]. Since $\mathscr{D}$ is not of tube type, we have $W\neq(0)$. The connected component $\mathscr{G}_a{}^0$ of the affine automorphism group of $\mathscr{D}(V, F)$ can be decomposed as follows (cf. (1.3)),

$$\mathscr{G}_a{}^0=\mathscr{H}^0\cdot RW \qquad \text{(semi-direct)}\,,$$

where $\mathscr{H}^0$ is the connected component of $\mathscr{H}$. According to Korányi-Wolf [2], $\mathscr{H}^0$ is reductive. R and W can be regarded as subspaces of the Lie algebra G_a of $\mathscr{G}_a{}^0$. Then R is an abelian ideal of G_a and W is invariant under $\mathrm{Ad}_{G_a}\mathscr{H}$ (cf. [1], [8]). Furthermore $\mathrm{Ad}_R\mathscr{H}^0=\mathrm{Ad}_R\mathscr{G}_a{}^0$ holds and $\mathrm{Ad}_R\mathscr{H}^0$ acts transitively on V (cf. [1], [8]). Since $\mathscr{G}_a{}^0$ is the connected component of an algebraic group (cf. [1],]2]), $\mathrm{Ad}_R\mathscr{H}^0$ is the connected component of an reductive real algebraic group. Furthermore V is irreducible, since $\mathscr{D}(V, F)$ is irreducible (cf. [1]). Hence from Corollary 2.1 (or [2], [6]) we see that the center of $\mathrm{Ad}_R\mathscr{H}^0$

consists of semi-simple endomorphism. Therefore by a theorem of [3], $\mathrm{Ad}_R \mathscr{H}^0$ is fully reducible. Suppose that $\mathrm{Ad}_R \mathscr{H}^0$ is not irreducible. Then by Lemma 2.2 V is reducible. This is a contradiction. Hence $\mathrm{Ad}_R \mathscr{H}^0$ is irreducible. On the other hand $W \neq (0)$ implies $[W, W] \neq (0)$ (cf. (2.2)). Furthermore $[W, W]$ is invariant under $\mathrm{Ad}_R \mathscr{H}^0$ ([1], [8]), which shows $[W, W]=R$. q.e.d.

§ 3. Examples of non-degenerate Siegel domains

Piatetski-Šapiro [4] has constructed some examples of affine homogeneous Siegel domains corresponding to irreducible classical self-dual cones. We shall give some examples of non-degenerate Siegel domains corresponding to such cones. We may study that by means of the theory of the root system of j-algebra (cf. [5], [8]), but we shall check here whether the linear closure of $\{F(u, u);\ u \in W\}$ is R or not. We shall use the following notations for irreducible self-dual cones.

a) The cone $H^+(p, \boldsymbol{R})$, $p \geq 1$.

Let R be the real vector space $H(p, \boldsymbol{R})$ of all real symmetric matrices of degree p. We denote by $H^+(p, R)$ the set of all positive definite matrices in $H(p, \boldsymbol{R})$. It is known that $H^+(p, \boldsymbol{R})$ is an irreducible self-dual cone. We define a coordinate system of R in the following way. For each matrix $Y=(y_{kt}) \in R$, the coordinate of Y is defined to be $(y_{1,1}, \cdots, y_{p,p}, y_{2,1}, y_{3,1}, y_{3,2}, \cdots, y_{p,1}, \cdots, y_{p,p-1})$. $\dim H(p, \boldsymbol{R})=\frac{1}{2}p(p+1)$.

b) The cone $H^+(p, \boldsymbol{C})$, $p \geq 1$.

Let R be the real vector space $H(p, \boldsymbol{C})$ of all hermitian matrices of degree p. We denote by $H^+(p, \boldsymbol{C})$ the set of all positive definite matrices in $H(p, \boldsymbol{C})$. It is known that $H^+(p, \boldsymbol{C})$ is an irreducible self-dual cone. We define a coordinate system of R in the following way. For each matrix $Y=(y_{k,t}) \in R$, the coordinate of Y is defined to be $(y_{1,1}, \cdots, y_{p,p}, \mathrm{Re}\, y_{2,1}, \mathrm{Im}\, y_{2,1}, \mathrm{Re}\, y_{3,1}, \mathrm{Im}\, y_{3,1}, \mathrm{Re}\, y_{3,2}, \mathrm{Im}\, y_{3,2}, \cdots, \mathrm{Re}\, y_{p,p-1}, \mathrm{Im}\, y_{p,p-1})$. $\dim H(p, \boldsymbol{C})=p^2$.

c) The cone $H^+(p, \boldsymbol{K})$, $p \geq 1$.

Let R be the real vector space $H(p, \boldsymbol{K})$ of all hermitian matrices of degree $2p$ satisfying a relation

$$YJ=J\bar{Y},$$

where

$$J=\begin{pmatrix} j & & 0 \\ & \ddots & \\ 0 & & j \end{pmatrix} \text{ and } j=\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{3.1}$$

We denote by $H^+(p, \boldsymbol{K})$ the set of all positive definite matrices in $H(p, \boldsymbol{K})$.

It is known that $H^+(p, \boldsymbol{K})$ is an irreducible self-dual cone. We shall write Y in the form $Y=(y_{k,t})$, $k, t=1, \cdots, p$, where $y_{k,t}$'s are 2×2 minor matrices. Then from (3.1) we obtain ${}^t y_{k,t}=y_{t,k}$, $y_{k,t}j=j\bar{y}_{k,t}$. Hence it follows

$$y_{k,k}=\begin{pmatrix} u_{k,k} & 0 \\ 0 & u_{k,k} \end{pmatrix}, \qquad y_{k,t}=\begin{pmatrix} u_{k,t} & v_{k,t} \\ -\bar{v}_{k,t} & \bar{u}_{k,t} \end{pmatrix}, \qquad k<t .$$

We put $u_{k,t}=a_{k,t}+\sqrt{-1}\,b_{k,t}$, $v_{k,t}=c_{k,t}+\sqrt{-1}\,d_{k,t}$, where $a_{k,t}, b_{k,t}, c_{k,t}, d_{k,t}$ are real. We define a coordinate system of R in the following way. For each matrix $Y=(y_{k,s})\in R$, the coordinate of Y is defined to be $(u_{1,1}, \cdots, u_{p,p}, a_{1,2}, b_{1,2}, c_{1,2}, d_{1,2}, \cdots, a_{1,p}, b_{1,p}, c_{1,p}, d_{1,p}, a_{2,3}, b_{2,3}, c_{2,3}, d_{2,3}, \cdots, a_{p-1,p}, b_{p-1,p}, c_{p-1,p}, d_{p-1,p})$. $\dim H(p, \boldsymbol{K})=p(2p-1)$.

Let $E_{i,j}$ be a square matrix such that its (i, j) element is equal to one and others are equal to zero.

1. Let $V=H^+(p, \boldsymbol{R})$ and s be a positive integer and $r(t)$ be a non-decreasing integer-valued function defined on an interval $[1, s]$ such that $0<r(1)$, $r(s)\leqq p$. Consider the complex vector space W of all $p\times s$ complex matrices $\mathfrak{U}=(u_{k,t})$ such that $u_{k,t}=0$ for $k>r(t)$. Obviously $\dim W=m=\sum_{t=1}^{s} r(t)$. We put $F(\mathfrak{U}, \mathfrak{V})=\frac{1}{2}(\mathfrak{U}{}^t\bar{\mathfrak{V}}+\mathfrak{V}{}^t\bar{\mathfrak{U}})$ for $\mathfrak{U}, \mathfrak{V}\in W$. It is known in [4] that $F(\mathfrak{U}, \mathfrak{V})$ is a V-hermitian form and that the corresponding Siegel domain $\mathscr{D}(V, F)$ is affine homogeneous.

PROPOSITION 3.1. $\mathscr{D}(V, F)$ is non-degenerate if and only if $r(s)=p$. In particular $\mathscr{G}(S)=\mathscr{G}_a$ is valid in this case.

PROOF. Suppose that $r(s)\neq p$. Then each $\mathfrak{U}=(u_{k,t})\in W$ must satisfy the relation $u_{p,t}=0$ for $t=1, \cdots, s$. Hence the p-th coordinate of $F(\mathfrak{U}, \mathfrak{U})$ with respect to the coordinate system chosen in a) is zero, which implies that the linear closure of the set $\{F(\mathfrak{U}, \mathfrak{U});\, \mathfrak{U}\in W\}$ is not R. Consequently $\mathscr{D}(V, F)$ is degenerate.

To prove the converse suppose that $r(s)=p$. It is sufficient to see that the set $\{F(\mathfrak{U}, \mathfrak{U});\ \mathfrak{U}=(u_{k,t})\in W,\ u_{k,t}=0 \text{ for } 1\leqq k\leqq p, 1\leqq t\leqq s-1\}$ contains $\frac{1}{2}p(p+1)$ linearly independent vectors in R. For such a $\mathfrak{U}$ we have

$$F(\mathfrak{U}, \mathfrak{U})=\begin{pmatrix} |u_{1,s}|^2 & & & * \\ \operatorname{Re} u_{2,s}\bar{u}_{1,s} & |u_{2,s}|^2 & & \\ \cdots & \cdots & \cdots & \cdots \\ \operatorname{Re} u_{p,s}\bar{u}_{1,s} & \operatorname{Re} u_{p,s}\bar{u}_{2,s} & \cdots & |u_{p,s}|^2 \end{pmatrix} \tag{3.2}$$

Put

$$\mathfrak{U}_i=E_{i,s}, \quad 1\leqq i\leqq p$$
$$\mathfrak{U}_{i,j}=E_{i,s}+E_{j,s}, \quad 1\leqq i<j\leqq p .$$

Then the system $\{F(\mathfrak{U}_i, \mathfrak{U}_i)\,(1\leqq i\leqq p),\ F(\mathfrak{U}_{i,j}, \mathfrak{U}_{i,j})\,(1\leqq i<j\leqq p)\}$ is a base of R. In

fact by (3.2) their coordinates are given by

$$\begin{aligned}
&F(\mathfrak{U}_1, \mathfrak{U}_1)=(1, 0, 0, \cdots, 0, 0)\\
&F(\mathfrak{U}_2, \mathfrak{U}_2)=(*, 1, 0, \cdots, 0, 0)\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&F(\mathfrak{U}_{p-1,p}, \mathfrak{U}_{p-1,p})=(*, *, *, \cdots, *, 1)\,.
\end{aligned}$$

Hence the linear closure of the set $\{F(\mathfrak{U}, \mathfrak{U});\ \mathfrak{U}\in W\}$ is R, which shows that $\mathscr{D}(V, F)$ is non-degenerate. q.e.d.

2. Let $V=H^+(p, \boldsymbol{C})$ and s_1, s_2 be two positive integers and $r_i(t)$ $(i=1, 2)$ be non-decreasing integer-valued functions defined on intervals $[1, s_i]$ $(i=1, 2)$ such that $0<r_i(1)$, $r_i(s_i)\le p$ $(i=1, 2)$. Consider the complex vector space W_i of all $p\times s_i$ complex matrices $\mathfrak{U}^{(i)}=(u^{(i)}_{k,t})$ such that $u^{(i)}_{k,t}=0$ for $k>r_i(t)$ $(i=1, 2)$. Put $W=W_1+W_2$ (direct sum as vector space). Then W is a complex vector space of dimension $m=\sum_{t=1}^{s_1} r_1(t)+\sum_{t=1}^{s_2} r_2(t)$. For $\mathfrak{U}^{(1)}, \mathfrak{V}^{(1)}\in W_1$ and $\mathfrak{U}^{(2)}, \mathfrak{V}^{(2)}\in W_2$, we put $F_1(\mathfrak{U}^{(1)}, \mathfrak{V}^{(1)})=\frac{1}{2}\mathfrak{U}^{(1)}\,{}^t\bar{\mathfrak{V}}^{(1)}$ and $F_2(\mathfrak{U}^{(2)}, \mathfrak{V}^{(2)})=\frac{1}{2}\mathfrak{V}^{(2)}\,{}^t\mathfrak{U}^{(2)}$. Let $F(\mathfrak{U}, \mathfrak{V})=F_1(\mathfrak{U}^{(1)}, \mathfrak{V}^{(1)})+F_2(\mathfrak{U}^{(2)}, \mathfrak{V}^{(2)})$ for $\mathfrak{U}=(\mathfrak{U}^{(1)}, \mathfrak{U}^{(2)})$, $\mathfrak{V}=(\mathfrak{V}^{(1)}, \mathfrak{V}^{(2)})$. It is known in [4] that $F(\mathfrak{U}, \mathfrak{V})$ is a V-hermitian form and that the corresponding Siegel domain $\mathscr{D}(V, F)$ is affine homegeneous.

PROPOSITION 3.2. $\mathscr{D}(V, F)$ is non-degenerate if and only if $r_1(s_1)=p$ or $r_2(s_2)=p$. In particular $\mathscr{G}(S)=\mathscr{G}_a$ is valid in this case.

PROOF. Suppose that $r(s_1)\neq p$ and $r(s_1)\neq p$. Then each $\mathfrak{U}=(\mathfrak{U}^{(1)}, \mathfrak{U}^{(2)})=((u^{(1)}_{k,t}), (u^{(2)}_{l,q}))\in W$ must satisfy the relations $u^{(1)}_{k,p}=0$, $1\le k\le s_1$, and $u^{(2)}_{l,p}=0$, $1\le l\le s_2$. Hence the p-th coordinate of $F(\mathfrak{U}, \mathfrak{U})$ with respect to the coordinate system chosen in b) is zero, which implies that the linear closure of the set $\{F(\mathfrak{U}, \mathfrak{U});\ \mathfrak{U}\in W\}$ is not R. Consequently $\mathscr{D}(V, F)$ is degenerate.

To prove the converse suppose that $r_1(s_1)=p$. It is sufficient to see that the set $\{F(\mathfrak{U}, \mathfrak{U});\ \mathfrak{U}=(\mathfrak{U}^{(1)}, (0))=((u^{(1)}_{k,t}), (0))\in W,\ u^{(1)}_{k,t}=0$ for $1\le k\le p, 1\le t\le s_1-1\}$ contains p^2 linearly independent vectors in R. For such a $\mathfrak{U}$ we have

$$F(\mathfrak{U}, \mathfrak{U})=\frac{1}{2}\begin{pmatrix} |u^{(1)}_{1,s_1}|^2 & & * \\ u^{(1)}_{2,s_1}\bar{u}^{(1)}_{1,s_1} & |u^{(1)}_{2,s_1}|^2 & \\ \cdots\cdots & \cdots\cdots & \cdots\cdots \\ u^{(1)}_{p,s_1}u^{(1)}_{1,s_1} & \cdots\cdots & |u^{(1)}_{p,s_1}|^2 \end{pmatrix} \tag{3.3}$$

Put

$$\begin{aligned}
&\mathfrak{U}^{(1)}_i=E_{i,s_1} \text{ for } 1\le i\le p\,,\\
&\mathfrak{U}^{(1)}_{i,j}=E_{i,s_1}+E_{j,s_1} \text{ for } 1\le i<j\le s_1\,,\\
&\mathfrak{V}^{(1)}_{i,j}=E_{i,s_1}+\sqrt{-1}\,E_{j,s_1} \text{ for } 1\le i<j\le s_1\,.
\end{aligned}$$

Then we can verify that the system $\{F(\mathfrak{U}_i^{(1)}, \mathfrak{U}_i^{(1)})\ (1 \leqq i \leqq p),\ F(\mathfrak{U}_{i,j}^{(1)}, \mathfrak{U}_{i,j}^{(1)})\ (1 \leqq i < j \leqq p),$ $F(\mathfrak{V}_{i,j}^{(1)}, \mathfrak{V}_{i,j}^{(1)})\ (1 \leqq i < j \leqq p)\}$ is a base of R. In fact using (3.3) we have

$$F(\mathfrak{U}_1^{(1)}, \mathfrak{U}_1^{(1)}) = \left(\frac{1}{2}, 0, 0, \cdots, 0, 0\right)$$
$$F(\mathfrak{U}_2^{(1)}, \mathfrak{U}_2^{(1)}) = \left(*, \frac{1}{2}, 0, \cdots, 0, 0\right)$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$F(\mathfrak{V}_{p-1,p}^{(1)}, \mathfrak{V}_{p-1,p}^{(1)}) = \left(*, *, *, \cdots, *, \frac{1}{2}\right).$$

Hence the linear closure of the set $\{F(\mathfrak{U}, \mathfrak{U});\ \mathfrak{U} \in W\}$ is R, which shows that $\mathscr{D}(V, F)$ is non-degenerate.

In the case of $r_2(s_2) = p$ the proof is quite analogous and so will be omitted.

3. Let $V = H^+(p, \boldsymbol{K})$ and s be a positive integer and $r(t)$ be a non-decreasing integer-valued function defined on an interval $[1, s]$ such that $0 < r(1)$, $r(s) \leqq 2p$. Consider the complex vector space W of all $2p \times s$ complex matrices $\mathfrak{U} = (u_{k,t})$ such that $u_{k,t} = 0$ for $k > r(t)$. $\dim W = m = \sum_{t=1}^{s} r(t)$. For $\mathfrak{U}, \mathfrak{V} \in W$ we put $F(\mathfrak{U}, \mathfrak{V}) = \frac{1}{2}(\mathfrak{U}{}^t\bar{\mathfrak{V}} + J\bar{\mathfrak{V}}{}^t\mathfrak{U}\,{}^tJ)$ for $\mathfrak{U}, \mathfrak{V} \in W$. It is known in [4] that $F(\mathfrak{U}, \mathfrak{V})$ is a V-hermitian form and that the corresponding Siegel domain $\mathscr{D}(V, F)$ is affine homogeneous.

PROPOSITION 3.3. $\mathscr{D}(V, F)$ is non-degenerate if and only if $r(s) = 2p-1$ or $2p$. In particular $\mathscr{S}(S) = \mathscr{S}_a$ is valid in this case.

PROOF. Suppose that $r(s) \leqq 2p-2$. Then each $\mathfrak{U} = (u_{k,t}) \in W$ must satisfy the relation $u_{p-1,t} = u_{p,t} = 0$ for $1 \leqq t \leqq s$. Hence the p-th coordinate of $F(\mathfrak{U}, \mathfrak{U})$ with respect to the coordinate system chosen in c) is zero, which implies that the linear closure of the set $\{F(\mathfrak{U}, \mathfrak{U});\ \mathfrak{U} \in W\}$ is not R. Consequently $\mathscr{D}(V, F)$ is degenerate.

To prove the converse suppose that $r(s) = 2p-1$ or $2p$. It is sufficient to see that the set

$$\{F(\mathfrak{U}, \mathfrak{U});\ \mathfrak{U} = (u_{k,t}) \in W,\ u_{k,t} = 0 \text{ for } 1 \leqq k \leqq p, 1 \leqq t \leqq s-1\},$$

contains $p(2p-1)$ linearly independent vectors in R. For such a $\mathfrak{U}$, $F(\mathfrak{U}, \mathfrak{U}) = (y_{k,t}) \in H(p, \boldsymbol{K})$ —$y_{k,t}$'s are 2×2 minor matrices— is given by

$$y_{k,k} = \frac{1}{2}\begin{pmatrix} |u_{2k-1,s}|^2 + |u_{2k,s}|^2 & 0 \\ 0 & |u_{2k-1,s}|^2 + |u_{2k,s}|^2 \end{pmatrix} \tag{3.4}$$

$$y_{k,t} = \frac{1}{2}\begin{pmatrix} u_{2k-1,s}\bar{u}_{2t-1,s} + \bar{u}_{2k,s}u_{2t,s} & u_{2k-1,s}\bar{u}_{2t,s} - \bar{u}_{2k,s}u_{2t-1,s} \\ u_{2k,s}\bar{u}_{2t-1,s} - \bar{u}_{2k-1,s}u_{2t,s} & u_{2k,s}\bar{u}_{2t,s} + \bar{u}_{2k-1,s}u_{2t-1,s} \end{pmatrix} \tag{3.5}$$

Put

$$\mathfrak{U}_i = E_{2i-1,s} \text{ for } 1 \leq i \leq p\,,$$
$$\mathfrak{U}_{i,j} = E_{2i-1,s} + E_{2j-1,s} \text{ for } 1 \leq i < j \leq p\,,$$
$$\mathfrak{V}_{i,j} = \sqrt{-1}\,E_{2i-1,s} + E_{2j-1,s} \text{ for } 1 \leq i < j \leq p\,,$$
$$\mathfrak{U}'_{i,j} = E_{2i,s} + E_{2j-1,s} \text{ for } 1 \leq i < j \leq p\,,$$
$$\mathfrak{V}'_{i,j} = \sqrt{-1}\,E_{2i,s} + E_{2j-1,s} \text{ for } 1 \leq i < j \leq p\,.$$

Then the system $\{F(\mathfrak{U}_i, \mathfrak{U}_i)\,(1 \leq i \leq p),\ F(\mathfrak{U}_{i,j}, \mathfrak{U}_{i,j})\,(1 \leq i < j \leq p),\ F(\mathfrak{V}_{i,j}, \mathfrak{V}_{i,j})\,(1 \leq i < j \leq p),\ F(\mathfrak{U}'_{i,j}, \mathfrak{U}'_{i,j})\,(1 \leq i < j \leq p),\ F(\mathfrak{V}'_{i,j}, \mathfrak{V}'_{i,j})\,(1 \leq i < j \leq p)\}$ is a base of R. In fact using (3.4) and (3.5) we have

$$F(\mathfrak{U}_1, \mathfrak{U}_1) = \left(\frac{1}{2}, 0, 0, \cdots, 0, 0\right),$$
$$F(\mathfrak{U}_2, \mathfrak{U}_2) = \left(*, \frac{1}{2}, 0, \cdots, 0, 0\right),$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$F(\mathfrak{V}'_{p-1,p}, \mathfrak{V}'_{p-1,p}) = \left(*, *, *, \cdots, \frac{1}{2}\right).$$

Hence the linear closure of the set $\{F(\mathfrak{U}, \mathfrak{U});\ \mathfrak{U} \in W\}$ is R, which shows that $\mathscr{D}(V, F)$ is non-degenerate.

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