L^p-theory for characterizing the domain of the fractional powers of $-\Delta$ in the half space

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§ 1. Introduction.

It is an open problem to characterize the domain of the fractional powers of, say, an infinitesimal generator A of an analytic semigroup in a Banach space E. The problem is solved when E is an L^2-space over a smooth domain in $\boldsymbol{R}^n$ and A is an elliptic partial differential operator considered under some boundary conditions (J. L. Lions [6], P. Grisvard [3] and D. Fujiwara [2]). However, if we treat in L^p ($p\neq 2$, $1<p<\infty$) space, the problem is still open. In this case the real interpolation method of J. L. Lions and J. Peetre [7] does not fit the problem. In this note we shall use the complex interpolation method of A. P. Calderón [1] and following the author's previous paper [2] we shall characterize the domain of the fractional powers of minus Laplacian with the third boundary condition in the half space.

In this note, we use the following notations: For any Banach space E, we denote by $\|u\|_E$ the norm of an element u in E. If there is little fear of confusion, we simply write this as $\|u\|$. We denote by $x=(x_1, \cdots, x_n)$ the generic point of $\boldsymbol{R}^n$. $\boldsymbol{R}^n_+$ means the upper half space $\boldsymbol{R}^n_+=\{x\in \boldsymbol{R}^n; x_n>0\}$. For any open set Ω in $\boldsymbol{R}^n$ with smooth boundary $\partial\Omega$, $H^{2,p}(\Omega)$, $1<p<\infty$, is the space of functions whose distribution derivatives of order ≤ 2 all belong to $L^p(\Omega)$ and $H^{s,p}(\Omega)$ is the complex interpolation space $[H^{2,p}(\Omega), L^p(\Omega)]_\theta$ with $s=2(1-\theta)$. It is well known that $H^{1,p}(\Omega)$ coincides with the space of functions whose derivatives of order ≤ 1 all belong to $L^p(\Omega)$. Let $C^2(\bar{\Omega})$ be the space of functions whose derivatives of order ≤ 2 are all continuous in $\bar{\Omega}$ and uniformly bounded in $\bar{\Omega}$. We denote by ν the outer unit normal to $\partial\Omega$. Then, the linear mapping γ_1 defined by $\gamma_1 f=\left.\frac{\partial f}{\partial\nu}\right|_{\partial\Omega}$ for any f in $C^2(\bar{\Omega})$ can be extended from $H^{s,p}(\Omega)$ to $H^{s-1/p-1,p}(\partial\Omega)$ if $s>1+1/p$, and the restriction mapping f to $f|_{\partial\Omega}$ can be extended from $H^{s,p}(\Omega)$ to $H^{s-\frac{1}{p},p}(\partial\Omega)$. We denote again by γ_0 and γ_1 these extensions. Consider the following function spaces: $H_0^{s,p}(\Omega)=\{u\in H^{s,p}(\Omega): \gamma_0 u=0\}$ for $s>1/p$, $H_1^{s,p}(\Omega)=\{u\in H^{s,p}(\Omega): \gamma_1 u=0\}$ for $s>1+1/p$, and $H_\alpha^{s,p}(\Omega)=\{u\in H^{s,p}(\Omega): \gamma_1+\alpha\gamma_0 u=0\}$ for $s>1+1/p$, where α is a smooth function on $\alpha\gamma$.

The aim of the present paper is to characterize with some exceptions the complex interpolation spaces $[H_0^{s,p}(\Omega), L^p(\Omega)]_\theta$, $[H_1^{s,p}(\Omega), L^p(\Omega)]_\theta$ and $[H_\alpha^{s,p}(\Omega), L^p(\Omega)]_\theta$, for $1<p<\infty$. We use it to characterize the domain of fractional powers of $-\Delta$ considered in the space $L^p(\boldsymbol{R}_+^n)$ under one of the Dirichlet, Neumann and the third boundary condition.

§ 2. A characterization of some complex interpolation spaces.

For any given point $x=(x_1, x_2, \cdots, x_n)=(x', x_n)$, we denote by $\hat{x}$ the point $(x_1, x_2, \cdots, -x_n)$. It is clear that the associated map σ in $L^p(\boldsymbol{R}^n)$ defined by $(\sigma f)(x)=f(\hat{x})$ is an involutive isometric mapping. Define X (resp. Y) as the eigen-space of σ associated to the eigen-value 1 (-1). Since σ is also an isometry in $H^{2,p}(\boldsymbol{R}^n)$, we have the topological direct decomposition:

$$(1)\qquad \begin{aligned} &L^p(\boldsymbol{R}^n)=X\oplus Y\,, \\ &H^{2,p}(\boldsymbol{R}^n)=(H^{2,p}(\boldsymbol{R}^n)\cap X)\oplus(H^{2,p}(\boldsymbol{R}^n)\cap Y)\,. \end{aligned}$$

The projections P and Q from $L^p(\boldsymbol{R}^n)$ to X and $L^p(\boldsymbol{R}^n)$ to Y are given by $P=\frac{1}{2}(I+\sigma)$, $Q=\frac{1}{2}(I-\sigma)$. Using interpolation method, we can easily prove the following topological direct decomposition: For $0<s<2$,

$$(2)\qquad H^{s,p}(\boldsymbol{R}^n)=H^{s,p}(\boldsymbol{R}^n)\cap X\oplus H^{s,p}(\boldsymbol{R}^n)\cap Y\,,$$

$$(3)\qquad \begin{aligned} &H^{s,p}(\boldsymbol{R}^n)\cap X=[H^{2,p}(\boldsymbol{R}^n)\cap X, X]_\theta\,, \\ &H^{s,p}(\boldsymbol{R}^n)\cap Y=[H^{2,p}(\boldsymbol{R}^n)\cap Y, Y]_\theta\,, \end{aligned}$$

with $2(1-\theta)=s$.

Now let us consider some extension mappings from $L^p(\boldsymbol{R}_+^n)$ to $L^p(\boldsymbol{R}^n)$. For an arbitrary f in $L^p(\boldsymbol{R}_+^n)$, we define the following two mappings τ and χ:

$$(4)\qquad (\tau f)(x', x_n)=\begin{cases} f(x', x_n) & x_n\geq 0\,, \\ -f(x', -x_n) & x_n<0\,, \end{cases}$$

$$(\chi f)(x', x_n)=\begin{cases} f(x', x_n) & x_n\geq 0\,, \\ f(x', -x_n) & x_n<0\,, \end{cases}$$

clearly $\tau L^p(\boldsymbol{R}_+^n)=Y$, $\chi L^p(\boldsymbol{R}_+^n)=X$. Further we see

PROPOSITION 1.

(i) *χ gives an isomorphism from $[H_1^{2,p}(\boldsymbol{R}_+^n), L^p(\boldsymbol{R}_+^n)]_\theta$ to $H^{2(1-\theta),p}(\boldsymbol{R}^n)\cap X$. And for any u in $[H_1^{2,p}(\boldsymbol{R}_+^n), L^p(\boldsymbol{R}_+^n)]_\theta$, $\|\chi u\|^p_{H^{2(1-\theta),p}(\boldsymbol{R}^n)}\leq 2\|u\|^p_{[H_1^{2,p}(\boldsymbol{R}_+^n), L^p(\boldsymbol{R}_+^n)]_\theta}$.*

(ii) *τ gives an isomorphism from $[H_0^{2,p}(\boldsymbol{R}_+^n), L^p(\boldsymbol{R}_+^n)]$ to $H^{2(1-\theta),p}(\boldsymbol{R}^n)\cap Y$. And for any u in $[H_0^{2,p}(\boldsymbol{R}_+^n), L^p(\boldsymbol{R}_+^n)]_\theta$, $\|\tau u\|^p_{H^{2(1-\theta),p}(\boldsymbol{R}^n)}\leq 2\|u\|^p_{[H_0^{2,p}(\boldsymbol{R}_+^n), L^p(\boldsymbol{R}_+^n)]_\theta}$.*

In both cases, the restriction mapping ρ from $L^p(\boldsymbol{R}^n)$ to $L^p(\boldsymbol{R}_+^n)$ is the inverse *mapping.*

PROOF. First note that for any f in $L^p(\mathbf{R}_+^n)$,

$$\|\tau f\|_{L^p(\mathbf{R}^n)}^p = \|\chi f\|_{L^p(\mathbf{R}^n)}^p = 2\|f\|_{L^p(\mathbf{R}_+^n)}^p \ . \tag{8}$$

When f belongs to $H^{1,p}(\mathbf{R}_+^n)$, we have

$$D_j\tau f = \tau D_j f\ , \qquad j=1, 2, \cdots, n-1\ , \tag{9}$$

$$D_n\tau f = \chi D_n f + \gamma_0 f \otimes \delta(x_n)\ , \tag{10}$$

$$D_j\chi f = \chi D_j f\ , \qquad j=1, 2, \cdots, n-1\ , \tag{11}$$

$$D_n\chi f = \tau D_n f\ , \tag{12}$$

where and hereafter we denote by $D_j u$ the distribution derivative $\dfrac{\partial}{\partial x_j} u$ of u, $1 \leq j \leq n$, and $\delta(x_n)$ is the Dirac's distribution in x_n-space. For any f in $H^{2,p}(\mathbf{R}_+^n)$, we have

$$D_k D_j \tau f = \tau D_k D_j f, \qquad j, k \neq n\ , \tag{13}$$

$$D_k D_n \tau f = D_n D_k \tau f = \chi D_n D_j f + D_k(\gamma_0 f) \otimes \delta(x_n)\ , \qquad k \neq n\ , \tag{14}$$

$$D_n^2 \tau f = \tau D_n^2 f + \gamma_0 f \otimes D_n \delta(x_n)\ , \tag{15}$$

$$D_k D_j \chi f = \chi D_k D_j f\ , \qquad j, k \neq n\ , \tag{16}$$

$$D_k D_n \chi f = D_n D_k \chi f = \tau D_n D_k f\ , \qquad k \neq n\ , \tag{17}$$

$$D_n^2 \chi f = \chi D_n^2 f + \gamma_1 f \otimes \delta(x_n)\ . \tag{18}$$

Therefore, when f is in $H_0^{2,p}(\mathbf{R}_+^n)$, we have

$$D_k D_j \tau f = \tau D_k D_j f\ , \qquad j, k \neq n\ ,$$

$$D_k D_n \tau f = \chi D_n D_k f\ , \qquad k \neq n\ ,$$

$$D_n^2 \tau f = \tau D_n^2 f\ .$$

So that $\tau f \in H^{2,p}(\mathbf{R}^n) \cap Y$ and

$$\|\tau f\|_{H^{2,p}(\mathbf{R}^n)}^p = 2\|f\|_{H_0^{2,p}(\mathbf{R}_+^n)}^p\ . \tag{20}$$

The equality (20) implies that τ is a continuous 1:1 mapping. τ is a surjection, because the mapping γ_0 is continuous on $H^{2,p}(\mathbf{R}^n) \cap Y$. By the theory of interpolation, $[H_0^{2,p}(\mathbf{R}_+^n), L^p(\mathbf{R}^n)]_\theta$ is isomorphic to $[H^{2,p}(\mathbf{R}^n) \cap Y, Y]_\theta = H^{2(1-\theta),p}(\mathbf{R}^n) \cap Y$. Thus we have proved (i). The proof of (ii) is similar.

Let π be the mapping $L^p(\mathbf{R}_+^n) \to L^p(\mathbf{R}^n)$ defined by

$$\pi f(x', x_n) = \begin{cases} f(x' x_n)\ , & x_n > 0\ , \\ 0 \quad , & x_n < 0\ . \end{cases} \tag{21}$$

Using just the same argument as in the proof of Proposition 1, we can prove

PROPOSITION 2.

(i) τ: $[H_0^{1,p}(\boldsymbol{R}_+^n), L^p(\boldsymbol{R}_+^n)]_\theta \to H^{1-\theta,p}(\boldsymbol{R}^n)\cap Y$ *is an isomorphism.*

(ii) χ: $[H_0^{1,p}(\boldsymbol{R}_+^n), L^p(\boldsymbol{R}_+^n)]_\theta \to H^{1-\theta,p}(\boldsymbol{R}^n)\cap X$ *is an isomorphism.*

(iii) π: $[H_0^{1,p}(\boldsymbol{R}_+^n), L^p(\boldsymbol{R}_+^n)]_\theta \to H^{1-\theta,p}(\boldsymbol{R}^n)$ *is continuous.*

The following lemma is essential for our discussion.

LEMMA 3. (E. Shamir [9]).

$$[H_0^{1,p}(\boldsymbol{R}_+^n), L^p(\boldsymbol{R}_+^n)]_\theta = \begin{cases} H_0^{1-\theta,p}(\boldsymbol{R}_+^n), & \text{when } \frac{1}{p}<1-\theta\leq 1 \\ H^{1-\theta,p}(\boldsymbol{R}_+^n), & \text{when } 0<1-\theta<\frac{1}{p}. \end{cases}$$

Now we can prove

THEOREM 4.

$$[H_0^{2,p}(\boldsymbol{R}_+^n), L^p(\boldsymbol{R}_+^n)]_\theta = \begin{cases} H_0^{2(1-\theta),p}(\boldsymbol{R}_+^n), & \text{when } \frac{1}{p}<2(1-\theta)\leqq 2 \\ H^{2(1-\theta),p}(\boldsymbol{R}_+^n), & \text{when } 0\leqq 2(1-\theta)<\frac{1}{p}, \end{cases}$$

$$[H_n^{2,p}(\boldsymbol{R}_+^n), L^p(\boldsymbol{R}_+^n)]_\theta = \begin{cases} H_n^{2(1-\theta),p}(\boldsymbol{R}_+^n), & \text{when } 1+\frac{1}{p}<2(1-\theta)<2 \\ H^{2(1-\theta),p}(\boldsymbol{R}_+^n), & \text{when } 0\leqq 2(1-\theta)<1+\frac{1}{p}. \end{cases}$$

PROOF.

(i) Since we have the following isomorphisms

$$[H_0^{2,p}(\boldsymbol{R}_+^n), L^p(\boldsymbol{R}_+^n)]_{1/2} \xrightarrow{\tau} H^{1,p}(\boldsymbol{R}^n)\cap Y \xrightarrow{\rho} H_0^{1,p}(\boldsymbol{R}_+^n),$$

it follows from Lemma 3 and Théorème de réitération that

$$[H_0^{2,p}(\boldsymbol{R}_+^n), L^p(\boldsymbol{R}_+^n)]_\theta = \begin{cases} H_0^{2(1-\theta),p}(\boldsymbol{R}_+^n), & \text{for } \frac{1}{p}<2(1-\theta)\leqq 1 \\ H^{2(1-\theta),p}(\boldsymbol{R}_+^n), & \text{for } 2(1-\theta)<\frac{1}{p}. \end{cases}$$

For $1\leqq 2(1-\theta)\leqq 2$ the inclusion $[H_0^{2,p}(\boldsymbol{R}_+^n), L^p(\boldsymbol{R}_+^n)]_\theta \subset H_0^{2(1-\theta),p}(\boldsymbol{R}_+^n)$ is clear. Inverse inclusion is proved in the following manner. Let f be an arbitrary function in $H_0^{1+s,p}(\boldsymbol{R}_+^n)$, $2\geqq 1+s=2(1-\theta)\geqq 1$, then $D_j\tau f=\tau D_j f$ for $j\neq n$, and $D_n\tau f=\chi D_n f$. Since $D_j f\in H^s(\boldsymbol{R}_+^n)$, $D_n f\in H^s(\boldsymbol{R}_+^n)$, it follows from Propotition 2 that $\tau D_j f$ and $\chi D_n f\in H^s(\boldsymbol{R}^n)$. And we obtain that $\tau f\in H^{1+s}(\boldsymbol{R}^n)\cap Y$. This is equivalent to $f\in\rho(H^{1+s}(\boldsymbol{R}^n)\cap Y)=[H_0^{2,p}(\boldsymbol{R}_+^n), L^p(\boldsymbol{R}_+^n)]_\theta$.

(ii) If $\alpha\equiv 1$, using the isomorphism χ instead of τ, a parallel discussion proves our assertion. To prove (ii) for general α, we use the following technique: Let φ be a function in $C^\infty(\boldsymbol{R}^1)$ satisfying $\frac{1}{2}\leqq\varphi(t)\leqq 2$ for any $t\in\boldsymbol{R}^1$ and $\varphi(t)=t$ in some neighborhood of $t=1$. And define $\Phi(x)=\varphi(e^{-\frac{1-\alpha(x')}{\alpha(x')}}x_n)$. Then, since $\Phi(x)\in C^2(\overline{\boldsymbol{R}^n_+})$ and $\frac{1}{2}\leqq\Phi(x)\leqq 2$, the mapping $T: f\rightarrow\Phi f$ is an isomorphism both in $L^p(\boldsymbol{R}^n_+)$ and in $H^{2,p}(\boldsymbol{R}^n_+)$. Therefore, by interpolation, T is an isomorphism from $[H_\alpha{}^{2,p}(\boldsymbol{R}^n_+), L^p(\boldsymbol{R}^n_+)]_\theta$ to $[H_1{}^{2,p}(\boldsymbol{R}^n_+), L^p(\boldsymbol{R}^n_+)]_\theta$ $(0\leqq\theta\leqq 1)$. Thus we have proved that

$$[H_\alpha{}^{2,p}(\boldsymbol{R}^n_+), L^p(\boldsymbol{R}^n_+)]_\theta=\begin{cases} H_\alpha{}^{2(1-\theta),p}(\boldsymbol{R}^n_+)\,, & \text{for } 1+\frac{1}{p}<2(1-\theta)<2 \\ H^{2(1-\theta),p}(\boldsymbol{R}^n_+)\,, & \text{for } 0\leqq 2(1-\theta)<1+\frac{1}{p}\,. \end{cases}$$

Now let us consider the case that Ω is a bounded domain with smooth boundary $\partial\Omega$. Let $\hat{\Omega}$ be a copy of Ω and let us denote by κ the natural diffeomorphism from $\bar{\Omega}$ to $\hat{\Omega}$. Then we obtain a compact manifold Ω_1 without boundary by gluing Ω with $\hat{\Omega}$ along the boundary that is, Ω_1 is the quotient manifold obtained from the union $\Omega\cup\hat{\Omega}$ by identifying any point in $\partial\Omega$ with its image by κ. We have the following imbeddings $\mu: \bar{\Omega}\rightarrow\Omega_1$, $\nu: \hat{\Omega}\rightarrow\Omega_1$ which satisfy the relation $\mu|_{\partial\Omega}=\nu\circ\kappa|_{\partial\Omega}$. We can define an involutive automorphism σ' on Ω_1 by $\sigma'(\mu(x))=\nu\circ\kappa(x)$ for $x\in\bar{\Omega}$ and by $\sigma'(\nu(x))=\mu\circ\kappa^{-1}(x)$ for $x\in\hat{\Omega}$. Using σ' instead of σ above we can adapt the discussion above to this case and prove

THEOREM 5.

$$[H_0{}^{2,p}(\Omega), L^p(\Omega)]_\theta=\begin{cases} H_0{}^{2(1-\theta),p}(\Omega)\,, & \textit{for } \frac{1}{p}<2(1-\theta)\leqq 2\,, \\ H^{2(1-\theta),p}(\Omega)\,, & \textit{for } 0\leqq 2(1-\theta)<\frac{1}{p}\,. \end{cases}$$

$$[H_\alpha{}^{2,p}(\Omega), L^p(\Omega)]_\theta=\begin{cases} H_\alpha{}^{2(1-\theta),p}(\Omega)\,, & \textit{for } 1+\frac{1}{p}<2(1-\theta)<2\,, \\ H^{2(1-\theta),p}(\Omega)\,, & \textit{for } 0\leqq 2(1-\theta)<1+\frac{1}{p}\,, \end{cases}\qquad 1<p<\infty\,.$$

§3. The domains of fractional powers of $-\Delta$.

Let α be a C^∞ function defined on $\partial\boldsymbol{R}^n_+=\boldsymbol{R}^{n-1}$, satisfying the condition that either $\alpha\neq 0$ or $\alpha\equiv 0$. Then the operator $A_\alpha=-\Delta+1$ restricted to $H_\alpha{}^{2,p}(\boldsymbol{R}^n_+)$ is an infinitesimal generator of an analytic semi-group in $L^p(\boldsymbol{R}^n_+)$. We shall denote by $D(A_\alpha{}^\theta)$ the domain of the fractional power $A_\alpha{}^\theta$ of A_α. It is possible to determine $D(A_\alpha{}^\theta)$.

THEOREM 6. $$D(A_\alpha{}^\theta)=[D(A_\alpha),\ L^p(\boldsymbol{R}^n_+)]_{1-\theta}\ .$$

PROOF. We shall denote by B (B_0 and B_1 resp.) the operator $-\Delta+I$ restricted to $H^{2,p}(\boldsymbol{R}^n)$ ($H^{2,p}(\boldsymbol{R}^n)\cap Y$ and $H^{2,p}(\boldsymbol{R}^n)\cap X$ resp.). It is clear that

$$A_0 f=\rho B_0\tau f \text{ for any } f \text{ in } D(A_0) \text{ and} \tag{23}$$

$$A_1 f=\rho B_1\tau f \text{ for any } f \text{ in } D(A_1)\ . \tag{24}$$

Since τ is the isomorphism from $L^p(\boldsymbol{R}^n_+)$ to Y and ρ is its inverse,

$$D(A_0{}^\theta)=\rho D(B_0{}^\theta)\ ,\quad 0<\theta<1\ . \tag{25}$$

Similarly,

$$D(A_1{}^\theta)=\rho D(B_1{}^\theta)\ . \tag{26}$$

On the other hand projections P and Q commute with B, so that we have

$$D(B_0{}^\theta)=QD(B^\theta)=D(B^\theta)\cap Y \text{ and}$$

$$D(B_1{}^\theta)=PD(B^\theta)=D(B^\theta)\cap X\ .$$

And by definition,

$$D(B^\theta)=[D(B),\ L^p(\boldsymbol{R}^n)]_{1-\theta}=H^{2\theta,p}(\boldsymbol{R}^n)\ .$$

Therefore,

$$D(A_0{}^\theta)=\rho D(B_0{}^\theta)=(H^{2\theta,p}(\boldsymbol{R}^n)\cap Y)=[D(A_0),\ L^p(\boldsymbol{R}^n_+)]_{1-\theta}\ .$$

$$D(A_1{}^\theta)=\rho D(B_1{}^\theta)=\rho([D(B),\ L^p(\boldsymbol{R}^n)]_{1-\theta}\cap X)=[D(A_1),\ L^p(\boldsymbol{R}^n_+)]_{1-\theta}\ .$$

Finally, we shall determine $D(A_\alpha{}^\theta)$ for general α. Let us denote by $\hat{A}$ the following operator:

$$\hat{A}f=T^{-1}A_1Tf \text{ for any } f \text{ in } D(A_\alpha)\ . \tag{27}$$

Then, we have $\hat{A}=A_\alpha+K$, where K is a differential operator of order ≤ 1 with coefficienet in $C^\infty(\boldsymbol{R}^n_+)$. It is known that for any $\varepsilon>0$, $D(A_\alpha{}^{3/4})\subset H^{3/2-\varepsilon,p}(\boldsymbol{R}^n_+)\subset D(K)$[1]. Therefore, it follows from Theorem A in the appendix that

$$D(A_\alpha{}^\theta)=D(\hat{A}^\theta)=T^{-1}D(A_1{}^\theta)=T^1[D(A_1),\ L^p(\boldsymbol{R}^n_+)]_{1-\theta}=[D(A_\alpha),\ L^p(\boldsymbol{R}^n_+)]_{1-\theta}\ .$$

Combining Theorems 4 and 6, we have proved

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1) $D(A_\alpha{}^\theta)$ is a space of K_θ class in the sense of J. Lions and J. Peetre [7] that is for any $u\in D(A_\alpha)$. $\|B_\alpha{}^\theta\leqslant C\|A_\alpha u\|^\theta\|u\|^{1-\theta}$. On the other hand $[D(A_\alpha),\ L^p(\boldsymbol{R}^n_+)]_{1-\theta}$ is a K_θ class. Therefore $D(A_\alpha{}^\theta)\subset[D(A_\alpha),\ L^p(\boldsymbol{R}^n_+)]_{1-\theta+\varepsilon}$ (Cf. J. Lions and J. Peetre [7] and H. Komatsu [5] or P. Grisvard [4]).

THEOREM 7.

$$D(A_0{}^\theta)=\begin{cases} H_0{}^{2\theta,p}(R_+^n)\,, & \text{for } \frac{1}{p}<2\theta\leqslant 2\,,\\ H^{2\theta,p}(R_+^n)\,, & \text{for } 0\leqslant 2\theta<\frac{1}{p}\,.\end{cases}$$

$$D(A_\alpha{}^\theta)=\begin{cases} H_\alpha{}^{2\theta,p}(R_+^n)\,, & \text{for } 1+\frac{1}{p}<2\theta\leqslant 2\,,\\ H^{2\theta,p}(R_+^n)\,, & \text{for } 0\leqslant 2\theta<1+\frac{1}{p}\,.\end{cases}$$

§ 4. Appendix

In § 3 we used the following fact: Let E be an abstract Banach space, and $-H$ be an infinitesimal generator of an analytic semi-group in E. Further we assume K is a closed operator in E. Then

THEOREM A (see [8°]). *If for some* δ: $0<\delta<1$, $D(H^\delta)\subset D(K^\delta)$, *then*

(i) $-H_1=-(H+K)$ *generates an analytic semi-group and*

(ii) $D(H_1{}^\theta)=D((H+K)^\theta)=D(H^\theta)$ *for* $0\leqslant\theta\leqslant 1$.

PROOF. We may assume that for any λ, $\mathrm{Re}\,\lambda>-1$, $(\lambda+H)^{-1}$ exists and is a bounded operator. Using S. Banach's closed graph theorem, we have

(28) $$\|Ku\|\leqslant C\|H^\delta u\| \quad \text{for any } u \text{ in } D(H^\delta)\,.$$

Set $K_\lambda=K(H+\lambda)^{-1}$ for $\mathrm{Re}\,\lambda>-1$. Then for any $0<\theta\leqslant\delta$, we have

(29) $$\begin{aligned}\|KH^{-\theta}u\|&\leqslant C\|H^\delta(H+\lambda)^{-1}H^{-\theta}u\|\\ &\leqslant C\|H(H+\lambda)^{-1}u\|^{\delta-\theta}\|(H+\lambda)^{-1}u\|^{1-\delta+\theta}\,{}^{2)}\leqslant C(1+|\lambda|)^{\delta-\theta-1}\|u\|\,.\end{aligned}$$

And for $\delta<\theta<1$, there holds an estimate

(30) $$\|K_\lambda H^{-\theta}u\|\leqslant C\|H^\delta H^{-\theta}(H+\lambda)^{-1}u\|\leqslant C(1+|\lambda|)^{-1}\|u\|\,.$$

Thus there is a positive number λ_0 such that for any λ, with $\mathrm{Re}\,\lambda\geqslant\lambda_0$, $\|K_\lambda\|<\frac{1}{2}$ and for these λ,

(31) $$(\lambda+H+K)^{-1}=(\lambda+H)^{-1}+(\lambda+H)^{-1}M_\lambda\,,$$

where

(32) $$M_\lambda=\sum_{j=1}^{\infty}(-K_\lambda)^j\,.$$

Therefore, for these λ, there holds

2) See footnote 1).

(33) $$\|(\lambda+H+K)^{-1}\|\leq C(1+|\lambda|)^{-1}.$$

(33) implies that $-H_1=-(H+K)$ generates an analytic semi-group.

To prove (ii) we may further assume that $\lambda_0=-1$. First we shall prove that $D(H_1^{\delta'})\subset D(K)$ if δ' is sufficiently near to 1. Note that for any u in $D(H_1)$,

(34) $$\begin{aligned}\|K_\lambda H_1^{-\theta}u\|&=\|KH_1^{-1}H_1^{1-\theta}(\lambda+H_1)^{-1}u\|\\&\leq C\|H_1(\lambda+H_1)^{-1}u\|^{1-\theta}\|(\lambda+H_1)^{-1}u\|^{\theta}\leq C(1+|\lambda|)^{-\theta}\|u\|.\end{aligned}$$

For any u in $D(H_1)$, $H^\delta u$ is defined by

$$\begin{aligned}H^\delta u&=\frac{\pi}{\sin\pi\delta}\int_0^\infty t^{\delta-1}H(t+H)^{-1}udt.\\&=\frac{\pi}{\sin\pi\delta}\int_0^\infty t^{\delta-1}(H_1-K)((t+H_1)^{-1}-(t+H)^{-1}M_t)udt.\end{aligned}$$

So that we have

$$\begin{aligned}\left\|\frac{\pi}{\sin\pi\delta}(H^\delta u-H_1u)\right\|\leq C\Big[&\int_0^\infty t^{\delta-1}\|K(t+H)^{-1}H_1^{-\delta'}H_1^{\delta'}u\|dt\\&+\int_0^\infty t^{\delta-1}\|H_1(t+H_1)^{-1}M_t\cdot H_1^{-\delta'}H_1^{\delta'}u\|dt\Big].\end{aligned}$$

Since $\|M_\lambda H_1^{-\delta'}v\|\leq C(1+|\lambda|)^{-\delta'}\|v\|$, we obtain

(35) $$\|H^\delta u-H_1^\delta u\|\leq C_{\delta'}\|H_1^{\delta'}n\|\quad\text{if}\quad\delta'>\delta.$$

Now assume that a sequence $\{u_n\}$ of elements in $D(H_1^{\delta'})$ converges to u and $H_1^{\delta'}u_n$ tends to $H_1^{\delta'}u$. Then it is clear that $H_1^\delta u$ tends to $H_1^\delta u$. So that it follows from (35) that $H^\delta u_n$ converges to $H^\delta u$. This proves that $D(H_1^\delta)\subset D(H^\delta)\subset D(K)$.

So far we have proved that the assumption of Theorem A is symmetric in H and H_1. To complete the proof of (ii) we have only to prove that $D(H^\theta)\subset D(H_1^\theta)$. We can prove the estimate

(36) $$\|H^\theta u-H_1^\theta u\|\leq C\|H^\theta u\|\quad\text{for any } u \text{ in } D(H).$$

In fact, using (29) or (30), we obtain

$$\begin{aligned}\frac{\sin\pi\theta}{\pi}H_1^\theta u&=\int_0^\infty t^{\theta-1}H_1(t+H_1)^{-1}udt\\&=\int_0^\infty t^{\theta-1}(H+K)\{(t+H)^{-1}+(t+H)^{-1}M_t\}H^{-\theta}H^\theta udt\end{aligned}$$

and

$$\frac{\sin\pi\theta}{\pi}\|H_1^\theta u-H^\theta u\|\leq\int_0^\infty t^{\theta-1}\|K_tH^{-\theta}H^\theta u\|dt+\int_0^\infty t^{\theta-1}\|M_tH^{-\theta}H^\theta u\|dt.$$

Since

$$\|M_t H^{-\theta} v\| \leqslant C(1+|t|)^{\delta-\theta-1} \quad \text{for} \quad 0 \leqslant \theta \leqslant \delta \,, \quad \text{and}$$

$$\|M_t H^{-\theta} v\| \leqslant C(1+|t|)^{-1} \quad \text{for} \quad \delta < \theta \,,$$

we have (36).

So that if a sequence $\{u_n\}$ in $D(H)$ converges to an element $u \in D(H^\theta)$ in the graph topology of H^θ. Then $H_1{}^\theta u$ also converges. This implies that the domain of H^θ is included by the domain of $H_1{}^\theta$.

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