On the irreducibility of homogeneous convex cones

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Introduction

The theory of homogeneous convex cones has been studied by M. Koecher, E. B. Vinberg, O. S. Rothaus and others. It is known that the theory is closely related to the theory of bounded homogeneous complex domains; namely every bounded homogeneous complex domain D is uniquely realized as a Siegel domain of the second kind $D(V, F)$ introduced by I. I. Pjateckiĭ-Šapiro, where V is a homogeneous convex cone and F is a V-hermitian form (cf. [3], [1]). Recently S. Kaneyuki has proved in his paper [1] that a bounded homogeneous complex domain $D=D(V, F)$ is an irreducible Kählerian manifold with respect to the Bergmann metric if and only if the convex cone V is irreducible, that is, V is not isomorphic to the direct product of two convex cones.

The aim of the present note is to give a criterion on the irreducibility of homogeneous convex cones. In his paper [2], Vinberg has established a fundamental bijective correspondence between the classes of isomorphic homogeneous convex cones and the classes of isomorphic T-algebras (see the definition in § 1 as for T-algebras). For T-algebras, we shall define in § 3 their diagrams. Then our main theorem is described as follows;

THEOREM. *A homogeneous convex cone is irreducible if and only if the diagram of the corresponding T-algebra is connected.*

In § 1, we shall recall the fundamental bijection between T-algebras and homogeneous convex cones from [2]. In § 2, it will be proved that every T-algebra is uniquely (up to order) decomposed into the direct sum of simple T-ideals. This gives us an explicit decomposition of a homogeneous convex cone into the direct product of irreducible homogeneous convex cones. In § 3, we shall define the diagrams associated to T-algebras and prove the main theorem.

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§ 1. Preliminaries.

A *matrix algebra* is a finite dimensional (not necessarily associative) algebra

$\mathfrak{A}$ over the field $\boldsymbol{R}$ of real numbers, bigraded with subspaces $\mathfrak{A}_{ij}$ $(i, j=1, 2, \cdots, m)$ such that

$$\mathfrak{A}=\sum_{i,j=1}^{m}\mathfrak{A}_{ij} \qquad \text{(direct sum)}\,,$$

$$\mathfrak{A}_{ij}\mathfrak{A}_{jk}\subset\mathfrak{A}_{ik}\,,$$

and

$$\mathfrak{A}_{ij}\mathfrak{A}_{kl}=(0) \qquad \text{for} \quad j\neq k\,.$$

For any $a, b, c\in\mathfrak{A}$, the *commutator* of a and b is denoted by

$$[a, b]=ab-ba\,,$$

and the *associator* of a, b, c by

$$[a, b, c]=a(bc)-(ab)c\,.$$

An *involution* * of a matrix algebra $\mathfrak{A}=\sum_{i,j=1}^{m}\mathfrak{A}_{ij}$ is an antiautomorphism of $\mathfrak{A}$ of order two such that

$$\mathfrak{A}_{ij}{}^{*}=\mathfrak{A}_{ji} \qquad \text{for any pair } (i, j)\,.$$

DEFINITION 1 ([2]). A matrix algebra $\mathfrak{A}=\sum_{i,j=1}^{m}\mathfrak{A}_{ij}$ with an involution * is called a *T-algebra of rank m* if the following conditions are satisfied;

(i) each subalgebra $\mathfrak{A}_{ii}$ is isomorphic to the algebra $\boldsymbol{R}$ under an isomorphism ρ,

(ii) for any $a_{ij}\in\mathfrak{A}_{ij}$,

$$a_{ii}a_{ij}=\rho(a_{ii})a_{ij}\,,$$

$$a_{ij}a_{jj}=\rho(a_{jj})a_{ij}\,,$$

(iii) for all $a, b, c\in\mathfrak{T}=\sum_{i\leq j}\mathfrak{A}_{ij}$,

$$[a, b, c]=[a, b, b^{*}]=0\,,$$

(iv) define the *trace* of an element $a=\sum_{i,j=1}^{m}a_{ij}$ $(a_{ij}\in\mathfrak{A}_{ij})$ by

$$\operatorname{Tr} a=\sum_{i=1}^{m}\rho(a_{ii})\,,$$

then

$$\operatorname{Tr}[a, b]=\operatorname{Tr}[a, b, c]=0 \qquad \text{for all } a, b, c\in\mathfrak{A}$$

and

$$\mathrm{Tr}\, aa^* > 0 \qquad \text{for all } a \in \mathfrak{A} - \{0\}\,.$$

Let e_i be the unit element of the subalgebra $\mathfrak{A}_{ii} : \rho(e_i) = 1$, then the element $e = \sum_{i=1}^{m} e_i$ is the unit element of $\mathfrak{A}$. We denote the dimension of the subspace $\mathfrak{A}_{ij}$ by n_{ij}. Note that $n_{ij} = n_{ji}$ for all $i, j = 1, 2, \cdots, m$.

DEFINITION 2. Two T-algebras $\mathfrak{A} = \sum_{i,j=1}^{m} \mathfrak{A}_{ij}$ and $\mathfrak{B} = \sum_{i,j=1}^{m} \mathfrak{B}_{ij}$ of same rank m are said to be *isomorphic* if there exists an isomorphism φ (as algebra) of $\mathfrak{A}$ onto $\mathfrak{B}$ and a permutation σ of $\{1, 2, \cdots, m\}$ such that

(i) $\qquad \varphi(a^*) = \varphi(a)^* \qquad$ for any $a \in \mathfrak{A}$,

(ii) $\qquad \varphi(\mathfrak{A}_{ij}) = \mathfrak{B}_{\sigma(i)\sigma(j)}$,

(iii) $\dim \mathfrak{B}_{ij} = 0$ for every pair (i, j) with $i < j$, $\sigma(i) > \sigma(j)$.

DEFINITION 3. A *convex cone* V in a finite dimensional real vector space R is a non-empty open subset of R satisfying the following conditions;

(i) if $x \in V$ and $\lambda > 0$, then $\lambda x \in V$,

(ii) if $x, y \in V$, then $x + y \in V$,

(iii) V contains no entire straight line.

An *automorphism* of a convex cone $V \subset R$ is a non-singular linear transformation of R which leaves V invariant. A convex cone V is *homogeneous* if the group of all automorphisms of V acts transitively on V. Two convex cones $V_i \subset R_i$ $(i = 1, 2)$ are *isomorphic* if there exists a linear isomorphism of R_1 onto R_2, by which V_1 is carried onto V_2. The *direct product* $V_1 \times V_2 \times \cdots \times V_n$ of convex cones $V_i \subset R_i$ $(i = 1, 2, \cdots, n)$ is the convex cone in the direct product $R_1 \times R_2 \times \cdots \times R_n$ consisting of vectors $(x_1, x_2, \cdots, x_n)$ with $x_i \in V_i$ $(i = 1, 2, \cdots, n)$. A convex cone is *reducible* if it is isomorphic to the direct product of two positive dimensional convex cones. A convex cone is *irreducible* if it is not reducible.

We shall close this section with a fundamental theorem of Vinberg which is essential for our later considerations.

For a T-algebra $\mathfrak{A} = \sum_{i,j=1}^{m} \mathfrak{A}_{ij}$, we put

$$\mathfrak{X} = \{x \in \mathfrak{A} : \quad x^* = x\}\,,$$

$$\mathfrak{T}(\mathfrak{A}) = \{x = \sum_{i,j=1}^{m} x_{ij}\ (x_{ij} \in \mathfrak{A}_{ij}) : \quad x_{ij} = 0 \text{ for } i > j,\ x_{ii} > 0 \text{ for all } i\}$$

and

$$V(\mathfrak{A}) = \{tt^* ; \quad t \in \mathfrak{T}(\mathfrak{A})\}\,.$$

It is known that $\mathfrak{T}(\mathfrak{A})$ is a connected Lie group acting simply transitively on $V(\mathfrak{A})$.

THEOREM A (Vinberg [2]).

(i) *For any T-algebra $\mathfrak{A}$, the set $V(\mathfrak{A})$ is a homogeneous convex cone in the real vector space $\mathfrak{X}$. Isomorphic T-algebras correspond to isomorphic convex cones.*

(ii) *For every homogeneous convex cone V, there exists the unique (up to isomorphism) T-algebra $\mathfrak{A}$ such that V is isomorphic to the cone $V(\mathfrak{A})$.*

§ 2. Irreducible convex cones.

DEFINITION 4. A *T-ideal* $\tilde{\mathfrak{A}}$ of a T-algebra $\mathfrak{A}$ is a $*$-invariant two-sided ideal; i.e., $\tilde{\mathfrak{A}}$ is a subspace of $\mathfrak{A}$ such that $\tilde{\mathfrak{A}}\mathfrak{A}\subset\tilde{\mathfrak{A}}$, $\mathfrak{A}\tilde{\mathfrak{A}}\subset\tilde{\mathfrak{A}}$ and $\tilde{\mathfrak{A}}^*=\tilde{\mathfrak{A}}$.

Obviously $\mathfrak{A}$ and (0) are always T-ideals of $\mathfrak{A}$ and are called the *trivial* T-ideals. A T-algebra is called to be *simple* if it does not contain non-trivial T-ideals. For a T-ideal $\tilde{\mathfrak{A}}$ of a T-algebra $\mathfrak{A}=\sum_{i,j=1}^{m}\mathfrak{A}_{ij}$, we put

$$\tilde{\mathfrak{A}}_{ij}=\tilde{\mathfrak{A}}\cap\mathfrak{A}_{ij}\,.$$

Then we have the relation

$$\tilde{\mathfrak{A}}=\sum_{i,j=1}^{m}\tilde{\mathfrak{A}}_{ij}\qquad\text{(direct sum)}.$$

In fact, for any element $a=\sum_{i,j=1}^{m}a_{ij}$ $(a_{ij}\in\mathfrak{A}_{ij})$, $a_{ij}=(e_i a)e_j$ where e_i is the unit element of the subalgebra $\mathfrak{A}_{ii}$. The subspace $\tilde{\mathfrak{A}}_{ii}$ is either (0) or equal to $\mathfrak{A}_{ii}$, for $\mathfrak{A}_{ii}$ is of dimension 1 from the definition.

LEMMA 1. *Let $\tilde{\mathfrak{A}}$ be a T-ideal of a T-algebra $\mathfrak{A}=\sum_{i,j=1}^{m}\mathfrak{A}_{ij}$ of rank m.*

(i) *If* $\tilde{\mathfrak{A}}_{ii}=(0)$, then $\tilde{\mathfrak{A}}_{ij}=\tilde{\mathfrak{A}}_{ji}=(0)$ for all j $(1\leqq j\leqq m)$.

(ii) *If $\tilde{\mathfrak{A}}_{ii}=\mathfrak{A}_{ii}$, then $\tilde{\mathfrak{A}}_{ij}=\mathfrak{A}_{ij}$ and $\tilde{\mathfrak{A}}_{ji}=\mathfrak{A}_{ji}$ for all j $(1\leqq j\leqq m)$.*

PROOF. To prove (i), we assume that $\tilde{\mathfrak{A}}_{ij}\neq(0)$ for some j, i.e., there exists a non-zero element $a\in\tilde{\mathfrak{A}}_{ij}$. From definition, we have $\mathrm{Tr}\,aa^*>0$ which contradicts $aa^*\in\tilde{\mathfrak{A}}_{ij}\tilde{\mathfrak{A}}_{ji}\subset\tilde{\mathfrak{A}}_{ii}=(0)$. Hence $\tilde{\mathfrak{A}}_{ij}=(0)$ for all j. Similarly we see that $\tilde{\mathfrak{A}}_{ji}=(0)$ for all j. As for (ii), for every j, we have $\mathfrak{A}_{ij}=e_i\mathfrak{A}_{ij}\subset\tilde{\mathfrak{A}}_{ij}$ and $\mathfrak{A}_{ji}=\mathfrak{A}_{ji}e_i\subset\tilde{\mathfrak{A}}_{ji}$, because $e_i\in\tilde{\mathfrak{A}}_{ii}$. (*q. e. d.*)

LEMMA 2. *A T-ideal of a T-algebra is a T-algebra.*

PROOF. Let $\tilde{\mathfrak{A}}$ be a T-ideal of a T-algebra $\mathfrak{A}=\sum_{i,j=1}^{m}\mathfrak{A}_{ij}$ and $\tilde{\Delta}$ be the set of indices i satisfying $\tilde{\mathfrak{A}}_{ii}=\mathfrak{A}_{ii}$. From Lemma 1, we can identify $\tilde{\mathfrak{A}}$ with $\sum_{i,j\in\tilde{\Delta}}\mathfrak{A}_{ij}$

which can be considered to be a T-algebra. (*q.e.d.*)

A T-ideal of a T-algebra is *simple* if it is a simple T-algebra.

Let $\mathfrak{A}=\sum_{i,j=1}^{m}\mathfrak{A}_{ij}$ be a T-algebra of rank m. A non-empty subset $\tilde{\Delta}$ of the set $\Delta=\{1, 2, \cdots, m\}$ is *admissible* if $\mathfrak{A}_{ij}=(0)$ for $(i, j)\in\tilde{\Delta}\times(\Delta-\tilde{\Delta})$. Furthermore, an admissible subset is called to be *irreducible* if it does not properly contain an admissible subset. Note that the complementary subset of an admissible subset is also admissible. Hence Δ is always represented as disjoint union of irreducible admissible subsets.

PROPOSITION 1. *There exists a bijection of the set of all T-ideals of a given T-algebra $\mathfrak{A}=\sum_{i,j=1}^{m}\mathfrak{A}_{ij}$ of rank m to the set of all admissible subsets of the set $\Delta=\{1, 2, \cdots, m\}$ of indices. Especially, the simple T-ideals are mapped onto the irreducible admissible subsets.*

PROOF. We define the mapping μ by associating to each T-ideal $\tilde{\mathfrak{A}}$ the following subset of Δ:

$$\Delta(\tilde{\mathfrak{A}})=\{i\in\Delta:\ \tilde{\mathfrak{A}}_{ii}=\mathfrak{A}_{ii}\}\ .$$

From Lemma 1, we see that $\Delta(\tilde{\mathfrak{A}})$ is admissible. Let $\tilde{\mathfrak{A}}$ and $\tilde{\mathfrak{A}}'$ be two T-ideals of $\mathfrak{A}$ such that $\Delta(\tilde{\mathfrak{A}})=\Delta(\tilde{\mathfrak{A}}')$. Again, from Lemma 1, we obtain the relations $\tilde{\mathfrak{A}}_{ij}=\tilde{\mathfrak{A}}'_{ij}$ for all i, j; hence $\tilde{\mathfrak{A}}=\tilde{\mathfrak{A}}'$. This implies that μ is injective. Next, let $\tilde{\Delta}$ be any admissible subset of Δ. We put $\tilde{\mathfrak{A}}=\sum_{i,j=1}^{m}\tilde{\mathfrak{A}}_{ij}$ where $\tilde{\mathfrak{A}}_{ij}=\mathfrak{A}_{ij}$ for $i, j\in\tilde{\Delta}$ and $\tilde{\mathfrak{A}}_{ij}=(0)$ otherwise. It is easily checked that $\tilde{\mathfrak{A}}$ is a T-ideal of $\mathfrak{A}$ satisfying $\Delta(\tilde{\mathfrak{A}})=\tilde{\Delta}$. It follows that μ is surjective. For the second part, it is sufficient to note the following result; let $\tilde{\mathfrak{A}}$ be a T-ideal of $\mathfrak{A}$ and $\tilde{\mathfrak{B}}$ be a T-ideal of the T-algebra $\tilde{\mathfrak{A}}$, then $\tilde{\mathfrak{B}}$ is a T-ideal of $\mathfrak{A}$. (*q.e.d.*)

THEOREM 1. *Any T-algebra is uniquely (up to order) decomposed into the direct sum of simple T-ideals.*

PROOF. Let $\mathfrak{A}=\sum_{i,j=1}^{m}\mathfrak{A}_{ij}$ be a T-algebra of rank m and let the set $\Delta=\{1, 2, \cdots, m\}$ be represented as disjoint union of irreducible admissible subsets $\Delta_1, \cdots, \Delta_r$. By $\mathfrak{A}_k$ we denote the simple T-ideal corresponding to Δ_k $(1\leq k\leq r)$ in Proposition 1. Then $\mathfrak{A}$ is the direct sum of simple T-ideals $\mathfrak{A}_1, \cdots, \mathfrak{A}_r$. Next, let $\mathfrak{A}=\sum_{k=1}^{s}\tilde{\mathfrak{A}}_k$ be a decomposition into the direct sum of simple T-ideals. Then we have the relation $\Delta=\bigcup_{k=1}^{s}\Delta(\tilde{\mathfrak{A}}_k)$ (disjoint), where $\Delta(\tilde{\mathfrak{A}}_k)$ is the irreducible admissible subset corresponding to T-ideal $\tilde{\mathfrak{A}}_k$ under the bijection in Proposition 1. Since

Δ is uniquely decomposed into disjoint union of irreducible admissible subsets, the uniqueness in this theorem is proved by using Proposition 1. (*q.e.d.*)

Let us define the *direct product* of two T-algebras $\mathfrak{A}=\sum_{i,j=1}^{m}\mathfrak{A}_{ij}$ and $\mathfrak{B}=\sum_{i,j=1}^{n}\mathfrak{B}_{ij}$. We denote by $\mathfrak{C}=\mathfrak{A}\times\mathfrak{B}$ the direct product of algebras. Now define the subspaces $\mathfrak{C}_{ij}$ of $\mathfrak{C}$ as follows:

$$\mathfrak{C}_{ij}=\begin{cases}\mathfrak{A}_{ij} & (1\leqq i,j\leqq m)\,,\\ \mathfrak{B}_{i-m,\,j-m} & (m<i,j\leqq m+n)\,,\\ (0) & (\text{otherwise})\,.\end{cases}$$

Then $\mathfrak{C}$ is bigraded by $\mathfrak{C}_{ij}$ $(1\leqq i,j\leqq m+n)$: $\mathfrak{C}=\sum_{i,j=1}^{m+n}\mathfrak{C}_{ij}$ (direct sum). We define an involution of $\mathfrak{C}$ by $(a,b)^*=(a^*,b^*)$ $(a\in\mathfrak{A},b\in\mathfrak{B})$. One can easily verify that the algebra $\mathfrak{C}$ with the involution $*$ is a T-algebra.

LEMMA 3. *Let $\mathfrak{C}$ be the direct product of two T-algebras $\mathfrak{A}$ and $\mathfrak{B}$. Then the convex cone $V(\mathfrak{C})$ is the direct product of the convex cones $V(\mathfrak{A})$ and $V(\mathfrak{B})$.*

PROOF. Since the group $\mathfrak{T}(\mathfrak{C})$ is the direct product of the groups $\mathfrak{T}(\mathfrak{A})$ and $\mathfrak{T}(\mathfrak{B})$, the homogeneous convex cone $V(\mathfrak{C})$ is the direct product of $V(\mathfrak{A})$ and $V(\mathfrak{B})$.

THEOREM 2. *Let $\mathfrak{A}$ be a T-algebra and $V(\mathfrak{A})$ the homogeneous convex cone corresponding to $\mathfrak{A}$. Then $V(\mathfrak{A})$ is irreducible if and only if $\mathfrak{A}$ is simple.*

PROOF. If $\mathfrak{A}$ is non-simple, then $V(\mathfrak{A})$ is reducible from Lemma 3. Conversely, if $V(\mathfrak{A})$ is reducible, then there exist convex cones $V_i(i=1,2)$ such that $V(\mathfrak{A})$ is isomorphic to the direct product of V_1 and V_2. Then V_i is homogeneous (cf. [2]). Let $\mathfrak{A}_i$ be the T-algebra associated to V_i, then $V(\mathfrak{A}_1\times\mathfrak{A}_2)$ is isomorphic to $V(\mathfrak{A})$ from Lemma 3. Hence $\mathfrak{A}$ is isomorphic to the direct product $\mathfrak{A}_1\times\mathfrak{A}_2$ which is a non-simple T-algebra. (*q.e.d.*)

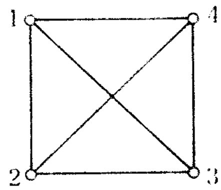
§ 3. Diagrams of T-algebras.

First of all, we shall introduce the diagrams of T-algebras. By use of them, we can easily find when homogeneous convex cones are irreducible.

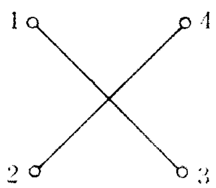
DEFINITION 5. For every T-algebra $\mathfrak{A}=\sum_{i,j=1}^{m}\mathfrak{A}_{ij}$ of rank m, we associate to $\mathfrak{A}$ the *diagram* $D(\mathfrak{A})$ consisting of m vertices, denoted by $1,2,\cdots,m$ and of line segments joining vertices i and j whenever $\mathfrak{A}_{ij}\neq(0)$.

Note that $\mathfrak{A}_{ij}\neq(0)$ is equivalent to $\mathfrak{A}_{ji}\neq(0)$.

EXAMPLES. By $H^+(n,\boldsymbol{R})$ we denote the homogeneous convex cone of all positive definite real symmetric matrices of degree n. Let $\mathfrak{A}^{(n)}$ be a T-algebra associated to $H^+(n, \boldsymbol{R})$. Then $\mathfrak{A}^{(n)}$ is isomorphic with the T-algebra of all real matrices of degree n. Thus $n_{ij}=1$ for all i, j. Hence $\mathfrak{A}^{(4)}$ has the following diagram



Let $\mathfrak{A}_0$ be a T-algebra of rank 4 such that $n_{13}=n_{24}=1$ and $n_{12}=n_{14}=n_{23}=n_{34}=0$. It is easy to see $\mathfrak{A}_0 \cong \mathfrak{A}^{(2)} \times \mathfrak{A}^{(2)}$; thus $V(\mathfrak{A}_0) \cong H^+(2, \boldsymbol{R}) \times H^+(2, \boldsymbol{R})$. The diagram of $\mathfrak{A}_0$ is as follows;



The diagram of a T-algebra is called to be *connected* if for any vertices i and j there exists a series of vertices $i=i_0, i_1, \cdots, i_s=j$ such that the vertices i_{k-1} and i_k are joined by a line segment for all k $(1 \leqq k \leqq s)$. In the above examples, $D(\mathfrak{A}^{(4)})$ is connected but $D(\mathfrak{A}_0)$ is disconnected. We can see easily that there exists a bijection of the set of all connected components of $D(\mathfrak{A})$ onto the set of all irreducible admissible subsets of Δ. Hence, from Proposition 1, we have

THEOREM 3. *A T-algebra $\mathfrak{A}$ is simple if and only if the diagram $D(\mathfrak{A})$ is connected.*

Thus we obtain our main theorem by use of Theorem A, Theorem 2 and Theorem 3.

THEOREM 4. *A homogeneous convex cone V is irreducible if and only if the diagram of a T-algebra associated to V is connected.*

Let V be a convex cone in an n-dimensional real vector space. Then we call n the *dimension of* V.

COROLLARY. *Let V be a homogeneous convex cone and m be the rank of the associated T-algebra. If the dimension of V is at most* $2(m-1)$, *then V is reducible.*

PROOF. Let $\mathfrak{A}=\sum_{i,j=1}^{m}\mathfrak{A}_{ij}$ be the associated T-algebra to V; $V=V(\mathfrak{A})$. Since the dimension of $V(\mathfrak{A})$ is equal to the dimension of the subspace $\sum_{i\leqq j}\mathfrak{A}_{ij}$ of $\mathfrak{A}$, the number of the pairs (i, j) satisfying dim $\mathfrak{A}_{ij}\neq 0$ and $i<j$ is at most $m-2$ from the assumptions. This implies that the number of line segments in the diagram $D(\mathfrak{A})$ are at most $m-2$. Hence the diagram $D(\mathfrak{A})$ can not be connected. From Theorem 4, the cone V is reducible. (*q.e.d.*)

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