Remarks on the theory of interpolation spaces

By Atsushi YOSHIKAWA*)
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0. Introduction.

The purpose of this paper is to give some theorems of inclusion of certain interpolation spaces connected with infinitesimal generators of semigroups. We state in § 2.5 our abstract theorems, which are based on the theory of mean spaces of Lions-Peetre [7], developed by Grisvard [3] and Komatsu [6]. As we shall see in § 4, these theorems give an operator theoretical treatment of some classical imbedding theorems due to Russian school (Besov [2], Nikol'skii [8]) originally proved by the theory of approximation of entire functions.

In order to explain our results, we here consider a simple case (Example 2.1). Let $E_0=L^p(R)$, $E_1=L^q(R)$, $1\leqslant p<q\leqslant\infty$ (R is the real line). Consider the translation in E_i (we assume here $t\geqslant 0$ for simplicity):

$$G(t)f(x)=f(x+t), \quad f\in E_i \quad (i=0, 1) .$$

$G(t)$ forms in each E_i a semigroup of bounded operators of class (C_0) (if $q=\infty$, we interprete the semigroup as the dual of that in $L^1(R)$), and the resolvent $R(\lambda)f(x)$, $\lambda>0$, of the infinitesimal generator of $G(t)$ (in E_i) is given by the formula:

$$R(\lambda)f(x)=\int_0^\infty e^{-\lambda t}f(x+t)dt, \ \lambda>0 .$$

From this we can deduce the estimate

$$\|R(\lambda)f(x)\|_{L^q(R)}\leqslant L\lambda^{\sigma-1}\|f(x)\|_{L^p(R)}$$

with $\sigma=1/p-1/q$, $L=(1-\sigma)^{1-\sigma}$, that is, $R(\lambda)$ is of the norm $\leqslant L\lambda^{\sigma-1}$ as the operator from E_0 into E_1 (cf. Definition 2.1). Applying our Theorem 2.1 to this *estimate for "resolvent"*, we can obtain an imbedding theorem of Sobolev's type:

$$B_p^{s,r}(R)\subset B_q^{t,r}(R), 1\leqslant r\leqslant\infty, t=s-\frac{1}{p}+\frac{1}{q}>0, s>0, p<q ,$$

with the continuous injection (Theorem 4.1). Here the spaces $B_p^{s,r}(R)$ are defined

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in the following way (cf. Definition 4.1):

(i) in case $s>0$ is not an integer, $B_p^{s,r}(R)\ni f$ if and only if $f\in W_p^{\langle s\rangle}(R)$ and

$$\left[\int_0^\infty t^{-(s-\langle s\rangle)r-1}\left\{\int_{-\infty}^\infty\left|\left(\frac{d}{dx}\right)^{\langle s\rangle}f(x)-\left(\frac{d}{dx}\right)^{\langle s\rangle}f(x+t)\right|^p dx\right\}^{r/p}dt\right]^{1/r}<\infty \tag{0.1}$$

where $\langle s\rangle$ is the greatest integer $<s$;

(ii) in case $s>0$ is an integer, $B_p^{s,r}(R)\ni f$ if and only if $f\in W_p^{\langle s\rangle}(R)$ and

$$\left[\int_0^\infty t^{-(r/2)-1}\left\{\int_{-\infty}^\infty\left|\left(\frac{d}{dx}\right)^{\langle s\rangle}f(x)-2\left(\frac{d}{dx}\right)^{\langle s\rangle}f(x+t)+\right.\right.\right.$$
$$\left.\left.\left.+\left(\frac{d}{dx}\right)^{\langle s\rangle}f(x+2t)\right|^p dx\right\}^{r/p}dt\right]^{1/r}<\infty \tag{0.2}$$

where $\langle s\rangle=s-1$.

If p or $r=\infty$, then the integral seminorms in (0.1) and (0.2) should be interpreted as *ess. sup.* in the corresponding variables.

In the case that many mutually commutative operators are given, for example, differential operators $\partial/\partial x^j$, $j=1,\cdots,n$, in the space $L^p(R^n)$, $1\leq p\leq\infty$, (where we denote by $(x^1,\cdots,x^n)$ the generic point of the n-dimensional Euclidean space R^n), we can obtain similar *estimates for "resolvents"* as above (cf. Examples 2.3 and 2.4) and some imbedding theorems of Nikol'skii [8] follow as a direct application of our Theorem 2.4 (cf. Theorem 4.3).

In the above considerations, we did only employ the *estimate for "resolvents"*. However, resolvents are closely related to semigroups, and sometimes we can easily obtain the *estimate for "semigroups"*. In fact, suppose that there are given semigroups of class (C_0) $G_0(t)$ and $G_1(t)$ respectively in Banach spaces E_0 and E_1, and that $G_0(t)$ and $G_1(t)$ are restrictions of another common semigroup $G(t)$ in a larger space. Then we can often obtain a certain estimate for $G(t)$, considered as an operator from E_0 into E_1:

$$\|G(t)\|_{E_0\to E_1}\leq Ct^{-\sigma},\ t>0,\ \sigma>0$$

(cf. Definition 2.2 and Example 2.2). Taibleson [10] treated some imbedding theorems in this way in case $G_0(t)$ and $G_1(t)$ are holomorphic semigroups, that is, each $G_i(t)$ has a holomorphic extension in a sector $|\arg t|<\omega_i$, $0<\omega_i<\pi/2$. But he used the holomorphicity of these semigroups as an essential tool for his proof. However, we can reduce the *estimate for "semigroup"* to *that of "resolvent"* without using holomorphicity (Theorem 2.1 and Corollary 2.1), and if $G_0(t)$ and $G_1(t)$ are holomorphic, these two estimates are, in a certain sense, equivalent (Proposition 2.3 in § 2.5).

Moreover, there exist "semigroups" that admit the *estimate for "semigroups"* but not holomorphic. For example, let $E_0 = L^p(R)$, $E_1 = L^q(R)$, $1 \leqq p < q < \infty$. If

$$G_i(t)f(x) = e^{itx^4 - tx^2} f(x), \quad t > 0, f \in E_i \quad (i=0,1)\,, \tag{0.3}$$

then it is not difficult to see that each $G_i(t)$ is a semigroup of class (C_0) in E_i, and that $G(t)$, which is defined by the right-hand side of (0.3), satisfies the following estimate:

$$\|G(t)f(x)\|_{L^p(R)} \leqq (\pi\sigma)^{\sigma/2} t^{-\sigma/2} \|f(x)\|_{L^q(R)}, \quad \sigma = \frac{1}{p} - \frac{1}{q}, \; p < q\,. \tag{0.4}$$

However, $G_i(t)$, $i=0,1$, do not have any holomorphic extension to the lower half plane in complex plane (cf. Yosida [12], Chapter 9).

We can also treat the case that the "resolvent" or the "semigroup" are "compact", and in this case we have some compact inclusions (Definitions 2.1* and 2.2*, and Theorems 2.1* and 2.2*).

The contents of this paper are as follows: § 1 is devoted to the preliminary results, which are to be required in the sequel, that is, the definitions and some properties of the mean spaces, and others. In general, propositions are without proof except some fundamental and easily provable ones. In § 2, we state our definitions (Definitions 2.1–2.4) and give illustrations of these definitions by examples (Examples 2.1–2.4). To understand our formulations, these examples are basic. In § 2.5, we state the abstract theorems (Theorems 2.1–2.4), whose proofs are given in § 3. § 4 contains some applications. Here we state some classical imbedding theorems and give their proofs. However, since the essential parts of their proofs are to be found in those of our Theorems 2.1–2.4 and in the explanations of the Examples 2.1–2.4, these proofs are fairly short.

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1. The theory of the mean space.

1.0. In this section we recall some results of the theory of the mean space of Lions-Peetre [7]. The definitions and general properties of mean spaces are given in § 1.1. The mean space of a Banach space and a domain of definition of a certain linear closed operator in this Banach space, for example, an infinitesimal generator of a semigroup of operators of class (C_0), was studied in great detail by Lions-Peetre [7], Grisvard [3], and Komatsu [6]. We state some of their results in subsequent paragraphs. § 1.0 concerns preliminary results for these paragraphs, and some fundamentals that are to be employed throughout this paper.

Let $\mathscr{E}$ and $\mathscr{F}$ be two Hausdorff linear topological vector spaces. For a linear operator L from $\mathscr{E}$ into $\mathscr{F}$, we denote by $D(L)$ the definition domain of L, which is a linear subspace of $\mathscr{E}$. In cace $\mathscr{E}=\mathscr{F}$, we may consider the powers $(L)^n$, $n=0, 1, 2, \cdots$, of L, which are defined as follows:

(i) $(L)^0=$the identity operator $I_{\mathscr{E}}$ in $\mathscr{E}$,

(ii) $(L)^1=L$,

(iii) $(L)^n$ for each $n>1$ is the operator $x\longmapsto L((L)^{n-1}x)$ with the domain $D((L)^n)=\{x\in D((L)^{n-1});\ (L)^{n-1}x\in D(L)\}$.

If $\mathscr{E}=\mathscr{F}=$a Banach space E, and L is a closed operator with the non-empty resolevent set in E, then $(L)^n$, $n=0, 1, 2, \cdots$, are closed (see Taylor [11]), and each $D((L)^n)$ becomes a Banach space with the graph norm:

$$\|x\|_{D((L)^n)}=\sum_{j=0}^{n}\|(L)^j x\|_E,\ x\in D((L)^n)\ .$$

Let L be a linear operator from a Hausdorff topological vector space $\mathscr{E}$ into another one $\mathscr{F}$, and E and F be subspaces of $\mathscr{E}$ and $\mathscr{F}$ respectively. The restriction of L in E into F means the operator $x\longmapsto Lx$ with the definition domain $\{x\in E\cap D(L);\ Lx\in F\}$. We denote by $L_{E,F}$ this restriction, and more simply L_E for $L_{E,E}$ if $E=F$, or by L in case well-understood. If $D(L_{E,F})=E$, we say that $L_{E,F}$ is the restriction of L *on* E into F. If $E=F$, we simply say the restriction of L *in* E or *on* E instead of the restriction of L in E *into* E or on E *into* E. In case we treat spaces with suffices such as E_0 or E_1 we often abbreviate L_{E_0,E_1} by $L_{0,1}$ and L_{E_i} by L_i $(i=0, 1)$ etc.

For two Hausdorff topological vector spaces $\mathscr{E}$ and $\mathscr{F}$, $\mathscr{E}\subset\mathscr{F}$ means that $\mathscr{E}$ is contained algebraically in $\mathscr{F}$ with the continuous injection, or $\mathscr{E}$ is continuously imbedded in $\mathscr{F}$.

PROPOSITION 1.1. *Let $\mathscr{E}$ and $\mathscr{F}$ be two Hausdorff topological vector spaces and E and F be two Banach spaces continuously imbedded in $\mathscr{E}$ and $\mathscr{F}$ respectively. Let L be a linear operator from $\mathscr{E}$ into $\mathscr{F}$.*

(i) *If L is a continuous linear operator on $\mathscr{E}$ into $\mathscr{F}$, then the restriction $L_{E,F}$ of L in E into F is closed;*

(ii) *If $\mathscr{E}$ and $\mathscr{F}$ are Banach spaces and L is a closed linear operator from $\mathscr{E}$ into $\mathscr{F}$, then $L_{E,F}$ is also closed;*

(iii) *Let $\mathscr{E}=\mathscr{F}$, and $E=F$. If $\mathscr{E}$, E and L are as in (i) or (ii), and if L_E is a bounded operator on E, then*

$$(L)^m_E=(L_E)^m\,,\qquad m=1,2,\cdots.$$

PROPOSITION 1.2. *Let $\mathscr{E}$ be a Hausdorff topological vector space and E be a Banach space continuously imbedded in $\mathscr{E}$. Let L be a linear operator in $\mathscr{E}$.*

(i) *If L is a continuous operator on $\mathscr{E}$ and if there exists λ such that $\lambda+L$ is an automorphism of $\mathscr{E}$ and $(\lambda+L_E)^{-1}$ is bounded on E, then*

$$D((L)^m_E)=D((L_E)^m)\,,\qquad m=1,2,\cdots;$$

(ii) *If $\mathscr{E}$ is a Banach space and L is a closed operator in $\mathscr{E}$, and if there exists λ such that $\lambda+L$ is invertible in $\mathscr{E}$ and $(\lambda+L_E)^{-1}$ is bounded on E, then*

$$D((L)^m_E)=D(L_E)^m)\,,\qquad m=1,2,\cdots.$$

PROOF. (i) We prove this proposition by the induction on m. For $m=1$, the assertion is trivial.

Now let us assume that the assertion is proved for $m\leqq n-1$. Let $x\in D((L)^n_E)$. From the resolvent equation, we have, in $\mathscr{E}$,

$$(\lambda+L)^{-n+1}(L)^n x=\{I-\lambda(\lambda+L)^{-1}\}^{n-1}Lx=\sum_{k=0}^{n-1}\binom{n-1}{k}(-1)^k\lambda^k(\lambda+L)^{-k}Lx$$
$$=Lx+\sum_{k=0}^{n-1}\binom{n-1}{k}(-1)^k\lambda^k L(\lambda+L)^{-k}x\,.$$

However, since $(\lambda+L_E)^{-k}=(\lambda+L)^{-k}_E$, we have that

$$(\lambda+L)^{-n+1}(L)^n x=(\lambda+L)^{-n+1}_E(L)^n_E x\in E \text{ and}$$
$$L(\lambda+L)^{-k}x=\{I-\lambda(\lambda+L)^{-1}\}(\lambda+L)^{-k+1}x=\{I-\lambda(\lambda+L)^{-1}_E\}(\lambda+L)^{-k+1}_E x\in E\,.$$

Hence $Lx\in E$ and $x\in D(L_E)$. Therefore we see that

$$L_E x\in D((L)^{n-1}_E)\qquad\text{and}\qquad (L)^n_E x=(L)^{n-1}_E L_E x\,.$$

Thus from the hypothesis of induction, we obtain $D((L)^n_E)\subset D((L_E)^n)$.

(ii) Since $D((L)^m_E)\subset D((L)^m)$, we can prove the assertion in the same way as in (i).

We say $(E_0, E_1, \mathscr{E})$ is an interpolation triplet if E_0 and E_1 are two Banach spaces continuously imbedded in a same Hausdorff topological vector space $\mathscr{E}$. For example, $(D(L), E, E)$ is an interpolation triplet if E is a Banach space and L is a closed linear operator in E.

When an interpolation triplet $(E_0, E_1, \mathscr{E})$ is given, we can construct two Banach spaces $E_0 \cap E_1$ and $E_0 + E_1$ as follows: $E_0 \cap E_1$ is the space of $x \in E_0 \cap E_1$ with the norm

$$\|x\|_{E_0 \cap E_1} = \max(\|x\|_{E_0}, \|x\|_{E_1})\,;$$

$E_0 + E_1$ is the space of $x = x_0 + x_1$ with $x_i \in E_i$ for $i = 0, 1$. Its norm is given by

$$\|x\|_{E_0+E_1} = \inf_{x=x_0+x_1} (\|x_0\|_{E_0} + \|x_1\|_{E_1})\,.$$

For an interpolation triplet $(E_0, E_1, \mathscr{E})$, we say that a Banach space E is an *intermediate* space of E_0 and E_1 if

$$E_0 \cap E_1 \subset E \subset E_0 + E_1\,.$$

1.1. Let R^+ be the multiplicative group of positive real numbers. We write by t its generic point and by dt/t its Haar measure. For a Banach space E, we denote by $L^p_*(E)$, $1 \leqslant p \leqslant \infty$, the space of strongly measurable E-valued functions f defined on R^+ such that

$$\|f\|_{L^p_*(E)} = \begin{cases} \left[\int_0^\infty \|f(t)\|_E^p \dfrac{dt}{t}\right]^{1/p} < \infty & \text{if } p < \infty \\ \operatorname*{ess.\,sup.}_{t>0} \|f(t)\|_E < \infty & \text{if } p = \infty\,. \end{cases} \tag{1.1}$$

$L^p_*(E)$ is a Banach space with the norm given by (1.1). Now we can define the mean space. Let $(E_0, E_1, \mathscr{E})$ be an interpolation triplet.

DEFINITION 1.1 [7]. The mean space $(E_0, E_1)_{\theta,p}$, $0<\theta<1$, $1 \leqslant p \leqslant \infty$, is the space of means

$$x = \int_0^\infty u(t)\,\frac{dt}{t}$$

where $t^\theta u(t) \in L^p_*(E_0)$ and $t^{\theta-1}u(t) \in L^p_*(E_1)$.

PROPOSITION 1.3 [7]. $(E_0, E_1)_{\theta,p}$ is a Banach space with the norm

$$\|x\|_{(E_0,E_1)_{\theta,p}} = \inf \max(\|t^\theta u(t)\|_{L^p_*(E_0)}, \|t^{\theta-1}u(t)\|_{L^p_*(E_1)})$$

where the infinimum is taken over all $u(t)$ with $x = \int_0^\infty u(t)\dfrac{dt}{t}$ such that $t^\theta u(t) \in L^p_*(E_0)$ and $t^{\theta-1}u(t) \in L^p_*(E_1)$.

PROPOSITION 1.4 [7]. $(E_0, E_1)_{\theta,p}$ is an intermediate space of E_0 and E_1, more precisely,

$$E_0 \cap E_1 \subset (E_0, E_1)_{\theta,p} \subset (E_0, E_1)_{\theta,q} \subset E_0 + E_1 \qquad \text{if } p<q\,.$$

We have, for all $x \in E_0 \cap E_1$,

$$\|x\|_{(E_0,E_1)_{\theta,q}} \leqslant c\|x\|_{E_0}^{1-\theta}\|x\|_{E_1}^{\theta}$$

where c is a constant. Especially, if $p<\infty$, then $E_0\cap E_1$ is dense in $(E_0, E_1)_{\theta,p}$.

Example 1.1 [7]. Let Ω be a domain in the n-dimensional Euclidean space R^n. We denote by $L^p(\Omega)$ the space of functions f defined and measurable in Ω and whose p-th powers are integrable if $p<\infty$, or f are essentially bounded if $p=\infty$. $L^p(\Omega)$ is a Banach space with the norm

$$\|f\|_{L^p(\Omega)}=\begin{cases}\left[\int_\Omega |f(x)|^p dx\right]^{1/p} & \text{for } p<\infty\,;\\ \operatorname*{ess.\,sup.}_{x\in\Omega} f(x) & \text{for } p=\infty\,.\end{cases}$$

Let $1\leqslant p, q\leqslant\infty$. Then $(L^p(\Omega), L^q(\Omega), \mathscr{D}'(\Omega))$ is an interpolation triplet.

$$(L^p(\Omega), L^q(\Omega))_{\theta,r}=L^r(\Omega) \qquad \text{if } \frac{1-\theta}{p}+\frac{\theta}{q}=\frac{1}{r}\,.$$

THEOREM 1.1 [7]. *Let $(E_0, E_1, \mathscr{E})$ and $(F_0, E_1, \mathscr{F})$ be two interpolation triplets. Suppose that a linear operator L on $\mathscr{E}$ into $\mathscr{F}$ is given. If the restrictions of L on E_i into F_i are continuous with norm ω_i for $i=0,1$ respectively, then we have the restriction of L on $(E_0, E_1)_{\theta,p}$, $0<\theta<1$, $1\leqslant p\leqslant\infty$, which is continuous with the norm $\omega\leqslant\omega_0{}^{1-\theta}\omega_1$.*

THEOREM 1.1* [7] *Let $(E_0, E_1, \mathscr{E})$ and $(F_0, F_1, \mathscr{F})$ be two interpolation triplets. Suppose that a linear operator L on $\mathscr{E}$ into $\mathscr{F}$ is given.*

(i) *$E=E_0=E_1$. If both restrictions of L on E into F_0 and into F_1 are continuous and one of them is compact, then we have the restriction of L on E into $(F_0, F_1)_{\theta,p}$, $0<\theta<1$, $1\leqslant p\leqslant\infty$, which is compact.*

(ii) *$F=F_0=F_1$. If both restrictions of L on E_0 into F and on E_1 into F are continuous and one of them is compact, then we have the restriction of L on $(E_0, E_1)_{\theta,p}$, $0<\theta<1$, $1\leqslant p\leqslant\infty$, into F, which is compact.*

DEFINITION 1.2 [7]. We say that a Banach space E is of class $K_\theta(E_0, E_1)$, $0<\theta<1$, if

$$(E_0, E_1)_{\theta,1}\subset E\subset (E_0, E_1)_{\theta,\infty}$$

(cf. Proposition 1.1). We also say E_i is of class $K_i(E_0, E_1)$, $i=0,1$.

THEOREM 1.2 [7]. *Let F_0 and F_1 be two Banach spaces respectively of class $K_{\theta_0}(E_0, E_1)$ and $K_{\theta_1}(E_0, E_1)$, $0\leqslant\theta_0<\theta_1\leqslant 1$. Then*

$$(E_0, E_1)_{\lambda,p}=(F_0, F_1)_{\theta,p}\,, \qquad \lambda=(1-\theta_0)\theta+\theta_1\theta\,.$$

THEOREM 1.3 [7]. *Let E_0 and E_1 be two Banach spaces such that $E_1\subset E_0$. If* *two Banach spaces* F_0 *and* F_1 *are respectively of class* $K_{\theta_i}(E_0, E_1)$, $i=0, 1$, *and if* $\theta_0<\theta_1$, *then* $F_1\subset F_0$.

1.2. Let E be a Banach space. Let A be a closed linear operator in E such that every $\lambda>0$ belongs to the resolvent set $\rho(-A)$ of $-A$ and the estimate

$$(h) \qquad \|(\lambda+A)^{-1}\|_{E\to E}\leqq\frac{M}{\lambda}$$

holds for every $\lambda>0$, where M is a constant independent of $\lambda>0$. We call such an operator A "of type (h)" and we often say that A satisfies (h).

THEOREM 1.4 [3], [6]. *Let* A *be of type* (h) *and* m *be any positive integer.* $(E, D((A)^m))_{\theta,p}$, $0<\theta<1$, $1\leqq p\leqq\infty$, *is the space of all* $x\in E$ *such that*

$$t^{\theta m}(A(t+A)^{-1})^m x\in L^p_*(E)$$

and its norm is given by

$$\|x\|_{(E,D((A)^m))_{\theta,p}}=\|x\|_E+\|t^{\theta m}(A(t+A)^{-1})^m x\|_{L^p_*(E)} .$$

REMARK 1.1 [3], [6]. Let $x\in(E, D((A)^m))_{\theta,p}$ and

$$u(t)=c_m t^m(A)^m(t+A)^{-2m}x , \qquad t>0 , \qquad c_m=\Gamma(2m)/\Gamma(m)^2 .$$

Then we have

$$t^{m\theta}u(t)\in L^p_*(E) \qquad \text{and} \qquad t^{m\theta-m}u(t)\in L^p_*(D((A)^m))$$

and

$$x=\int_0^\infty u(t)\frac{dt}{t} .$$

Moreover we have

$$\max(\|t^{m\theta}u(t)\|_{L^p_*(E)}, \|t^{m\theta-m}u(t)\|_{L^p_*(D((A)^m))})\leqq C\|x\|_{(E,D((A)^m))_{\theta,p}}$$

where C is a positive constant independent of x (Grisvard [3], Prop. 3.1.).

THEOREM 1.5 [3], [6]. *Let* m, n *be positive integers, and* $0<\theta$, $\varphi<1$. *If* $m\theta=n\varphi$, *then*

$$(E, D((A)^m))_{\theta,p}=(E, D((A)^n))_{\varphi,p} .$$

THEOREM 1.6 [3], [6]. *Let* m, n *be positive integers. Suppose* $0<\theta-n/m<\theta<1$. *Then* $x\in(E, D((A)^m))_{\theta,p}$ *if and only if*

$$x\in D((A)^n) \qquad \text{and} \qquad (A)^n x\in(E, D((A)^m))_{\theta-n/m,p} .$$

There is an important subclass of operators of type (h), that is, the infinitesimal generators of semigroups of bounded operators (of class (C_0)). We say that a family $G(t), t\geqslant 0$, of bounded linear operators in a Banach space E is a semigroup of class (C_0) if the following three conditions are satisfied:

(i) $\|G(t)\|_{E\to E}\leqslant M, t\geqslant 0$, for some $M>0$;

(ii) $G(0)=$the identity operator in E, $G(t)G(s)=G(t+s)$ for every $t, s\geqslant 0$; and

(iii) $\|G(t)f-f\|_E\,(t>0)$ tends to 0 as t for every $f\in E$.

We define an infinitesimal generator $-A$ of a semigroup $G(t)$ of class (C_0) as an operator with the definition domain $D(-A)=\{x\in E$; strong limit of $t^{-1}(G(t)-I)x$ exists when $t\to 0\}$ and $-Ax=\lim\limits_{t\to 0} t^{-1}(G(t)-I)x$ for $x\in D(-A)$. A linear operator $-A$ in E is an infinitesimal generator of a semigroup $G(t)$ of class (C_0) if and only if $-A$ is densely defined and closed, and the resolvent set $\rho(-A)$ of $-A$ contains all $\lambda>0$ with the estimate:

$$\|(\lambda+A)^{-n}\|_{E\to E}\leqslant M\lambda^{-n}\,, \qquad n=1, 2, \cdots . \tag{1.2}$$

The resolvent of $-A$ for each $\lambda>0$ is connected with the semigroup $G(t)$ by the following formula:

$$(\lambda+A)^{-1}x=\int_0^\infty e^{-\lambda t}G(t)xdt,\ x\in E,\ \lambda>0 . \tag{1.3}$$

For more detailed discussion, see Yosida [12].

THEOREM 1.7 [6], [7]. *Let $-A$ be an infinitesimal generator of a semigroup $G(t)$ of class (C_0) in E. Then $(E, D((A)^m))_{\theta,p}$ coincides with the space of all $x\in E$ such that*

$$t^{-m\theta}(I-G(t))^m x\in L^p_*(E)$$

with the norm

$$\|x\|_E+\|t^{-m\theta}(I-G(t))^m x\|_{L^p_*(E)} .$$

REMARK 1.2. When $G(t)$ is a bounded holomorphic semigroup, then we can obtain another characterization of the mean space (Komatsu [6], and implicity in Taibleson [10]).

1.3. Let $A^1, \cdots, A^N$ be N closed linear operators in a Banach space E. Suppose that

(i) each A^j is of type (h) $(j=1, \cdots, N)$; and

(ii) $(\lambda+A^j)^{-1}(\mu+A^k)^{-1}=(\mu+A^k)^{-1}(\lambda+A^j)^{-1}$ for $\lambda, \mu>0$ and $j, k=1, \cdots, N$.

THEOREM 1.8 [3]. *For $0<\theta<1, 1\leqslant p\leqslant\infty$, we have*

$$(\bigcap_{j=1}^N D(A^j), E)_{\theta,p}=\bigcap_{j=1}^N (D(A^j), E)_{\theta,p} .$$

REMARK 1.3. Let $x \in (\bigcap_{j=1}^{N} D(A^j), E)_{\theta,p}$ and

$$u(t) = \sum_{j=1}^{N} t^N A^j (t+A^j)^{-1} \prod_{i=1}^{N} (t+A^i)^{-1} x .$$

Then we have

$$t^{1-\theta} u(t) \in L_*^p(E),\ t^{-\theta} A^k u(t) \in L_*^p(E)\ , \qquad k=1, \cdots, N\ ,$$

and

$$x = \int_0^\infty u(t) \frac{dt}{t}\ .$$

Moreover we have

$$\max\left(\|t^{1-\theta} u(t)\|_{L_*^p(E)}, \|t^{-\theta} A^k u(t)\|_{L_*^p(E)}\right) \leqslant C\|x\|_{(\bigcap_{j=1}^{N} D(A^j), E)_{\theta,p}}$$

(See Grisvard [3], Prop. 7.1).

1.4. We say for a closed linear operator A in a Banach space E that A *is of type* (h_m) or A *satisfies* (h_m) if $\rho(-A)$ contains every $\lambda \neq 0$ with $\arg \lambda = \dfrac{2k\pi}{m}$, $k=0, \cdots, m-1$, and

$$\|(\lambda + A)^{-1}\|_{E \to E} \leqslant M_m |\lambda|^{-1}$$

for $\lambda \neq 0$, $\arg \lambda = \dfrac{2k\pi}{m}$, $k=0, \cdots, m-1$ (Grisvard [3]). If A is of type (h_m), then $(A)^m$ is of type (h), since

$$(\lambda + (A)^m) x = \prod_{k=0}^{m-1} (A + \sqrt[m]{\lambda}\, e^{2k\pi i/m}) x,\ \lambda > 0,\ x \in D((A)^m)\ .$$

Thus from Theorem 1.8, we obtain

THEOREM 1.9 [3]. *Let $A^1, \cdots, A^N$ be N closed linear operators in a Banach space E. Suppose that each A^j is of type (h_{m_j}) for some positive integer m_j and that*

$$(\lambda + A^k)^{-1} (\mu + A^j)^{-1} = (\mu + A^j)^{-1} (\lambda + A^k)^{-1}$$

for $\lambda \neq 0$, $\mu \neq 0$, $\arg \lambda = \dfrac{2l\pi}{m}$, $\arg \mu = \dfrac{2l'\pi}{m}$, l, l'; integers. Then for $0 < \theta < 1$, $1 \leqslant p \leqslant \infty$, we have

$$(\bigcap_{j=1}^{N} D((A^j)^{m_j}), E)_{\theta,p} = \bigcap_{j=1}^{N} (D((A^j)^{m_j}), E)_{\theta,p}\ .$$

In §2.4, we shall give some examples of the operators satisfying the hypothesis of Theorem 1.9.

1.5. Let $(E_0, E_1, \mathscr{E})$ be an interpolation triplet. Let A be a continuous linear operator on $\mathscr{E}$ such that $\lambda+A$ for every $\lambda>0$ is an automorphism of $\mathscr{E}$. Let $A_i=A_{E_i}$ $(i=0,1)$. Each A_i is a closed linear operator in E_i. Assume both A_0 and A_1 satisfy (h).

THEOREM 1.10 (cf. [3]). *Let $A_{\theta,p}$ be the restriction of A in $(E_0, E_1)_{\theta,p}$. Then we have*

$$D((A_{\theta,p})^m)=(D((A_0)^m),\ D((A_1)^m))_{\theta,p} \qquad \text{for} \qquad m=1,2,\cdots .$$

PROOF. Let $x\in D((A_{\theta,p})^m)$. Then there exists $y\in(E_0,E_1)_{\theta,p}$ such that $x=(\lambda+A)^{-m}y$ for $\lambda>0$. Since $y\in(E_0,E_1)_{\theta,p}$, there exists $w(t)$ such that

$$t^\theta w(t)\in L^p_*(E_0),\ t^{\theta-1}w(t)\in L^p_*(E_1)$$

and

$$y=\int_0^\infty w(t)\frac{dt}{t}\,.$$

Let $z(t)=(\lambda+A)^{-m}w(t)$. Then $z(t)=(\lambda+A_0)^{-m}w(t)=(\lambda+A_1)^{-m}w(t)$ and

$$t^\theta z(t)\in L^p_*(E_0),\ t^{\theta-1}z(t)\in L^p_*(E_1)$$

and

$$x=(\lambda+A)^{-m}y=(\lambda+A)^{-m}\int_0^\infty w(t)\frac{dt}{t}=\int_0^\infty z(t)\frac{dt}{t}\in(D((A_0)^m),\ D((A_1)^m))_{\theta,p}\,.$$

The continuity of the inclusion mapping follows immediately from the above relation. Now we are going to show the converse inclusion relation. Since $D((A_i)^m)=D((\lambda+A_i)^m)$, $\lambda>0$, we may assume that each A_i has a bounded inverse $(i=0,1)$. Let x be any element in $(D((A_0)^m),\ D((A_1)^m))_{\theta,p}$. Then there exists $E_0\cap E_1$-valued $u(t)$ such that

$$t^\theta(A_0)^m u(t)\in L^p_*(E_0),\ t^{\theta-1}(A_1)^m u(t)\in L^p_*(E_1)\ ,$$

and

$$x=\int_0^\infty u(t)\frac{dt}{t}\,.$$

Let $(A)^m u(t)=v(t)$. Then $v(t)=(A_0)^m u(t)=(A_1)^m u(t)$, and

$$y=\int_0^\infty v(t)\frac{dt}{t}\in(E_0,E_1)_{\theta,p}$$

Since $y=(A)^m x$ in $\mathscr{E}$, $x\in D((A_{\theta,p})^m)$. q.e.d.

2. Definitions, Examples, and Theorems.

2.0. First we give some notations. We denote by R^n and T^n the n-dimensional Euclidean space and the n-dimensional torus respectively. R^1 and T^1 are simply written as R and T respectively. We denote by $x=(x^1, \cdots, x^n)$ or $y=(y^1, \cdots, y^n)$ their generic points. For 1-dimensional case, x^1 is written simply as x. For x and $y \in R^n$, $x+y=(x^1+y^1, \cdots, x^n+y^n)$, $cx=(cx^1, \cdots, cx^n)$ for $c \in R$, and $|x|=(\sum_{j=1}^{n}(x^j)^2)^{1/2}$. For other notations, consult § 1. In § 2.1, we give a definition about the *estimate for "resolvent"* and its compact version (Definitions 2.1 and 2.1*), and examples (Examples 2.1 and 2.1*). Definitions concerning the *estimate for "semigroup"* (Definitions 2.2 and 2.2*) are given in § 2.2 with Examples 2.2 and 2.4*. §§ 2.3 and 2.3 are for the *estimate for many commutative "resolvents"* (Definitions 2.3 and 2.4 and Examples 2.3 and 2.4). We state our theorems in § 2.5 (Theorems 2.1, 2.1,* 2.2, 2.2,* 2.3, 2.4, and some remarks).

2.1. Consider an interpolation triplet $(E_0, E_1, \mathscr{E})$. Let A be a continuous linear operator on $\mathscr{E}$.

DEFINITION 2.1. We write $A \in (\sigma, E_0, E_1)$ if the following two conditions are satisfied:

(i) for $j=0, 1$, each A_j, the restriction of A in E_j, is of type (h) (see § 1.0 and § 1.2);

(ii) for every $\lambda>0$, $\lambda+A_{E_0+E_1}$ is invertible, and $(\lambda+A)^{-1}$, considered as an operator from E_0 into E_1, that is, $(\lambda+A)^{-1}_{0,1}$ is defined on the whole E_0, and it is continuous with the estimate

$$\|(\lambda+A)^{-1}_{0,1}\|_{E_0\to E_1} \leq L\lambda^{\sigma-1}$$

where σ and L are positive constants independent of λ.

Example 2.1. In order to illustrate the above definition, we state that

$$\frac{d}{dx} \in \left(\frac{1}{p}-\frac{1}{q}, L^p(R), L^q(R)\right), \quad 1 \leq p<q<\infty \; ; \tag{2.1}$$

and that

$$\frac{d}{dx} \in \left(\frac{1}{p}, L^p(R), C(R)\right), \quad 1 \leq p<\infty \, . \tag{2.1'}$$

Here $C(R)$ is the Banach space of uniformly continuous bounded functions f on R with the norm $\|f\|_{C(R)}=\max_{x\in R}|f(x)|$.

(i) PROOF OF (2.1). Let $E_0=L^p(R)$, $E_1=L^q(R)$, and $\mathscr{E}=\mathscr{B}'(R)$, the space of bounded distributions. Let $A=\dfrac{d}{dx}$. Consider the integral

$$R(\lambda)f(x)=\int_0^\infty e^{-\lambda t}f(x+t)dt \tag{2.2}$$

where $f\in\mathscr{B}'(R)$, $\lambda>0$ and the integral is in the sense of distribution. Consider this integral (2.1) for $f\in E_i$. Then it converges in E_i, and it gives the restriction $R(\lambda)_i$ of $R(\lambda)$ in E_i, which is a continuous linear operator on E_i with the following estimate $(i=0, 1)$,

$$\|R(\lambda)_i\|_{E_i\to E_i}\leqslant\lambda^{-1} \tag{2.3}$$

for every $\lambda>0$. To prove this inequality, first we rewrite the integral (2.2) as follows:

$$R(\lambda)f(x)=\int_{-\infty}^\infty K_\lambda(-t)f(x+t)dt \tag{2.2}$$

where

$$K_\lambda(t)=\begin{cases}0 & \text{for} \quad t>0\ ,\\ e^{\lambda t} & \text{for} \quad t<0\ .\end{cases}$$

Note that

$$K_\lambda\in L^r(R),\ 1\leqslant r\leqslant\infty, \quad \text{and} \quad \|K_\lambda\|_{L^r(R)}=(\lambda r)^{-1/r},\ \lambda>0\ . \tag{2.4}$$

Hence by Young's inequality for convolution (for example, [4]), we obtain the inequality (2.3). From the above inequality, we see, for $f\in E_0+E_1$, the integral (2.2) converges, and it gives the restriction $R(\lambda)_{E_0+E_1}$ of $R(\lambda)$ on E_0+E_1, continuous linear on E_0+E_1, with the estimate

$$\|R(\lambda)_{E_0+E_1}\|_{E_0+E_1\to E_0+E_1}\leqslant\lambda^{-1}$$

for every $\lambda>0$. On the other hand, we see, by definition of A_i etc., that

$$R(\lambda)_i f\in D(A_i) \quad \text{for any} \quad f\in E_i\ (i=0, 1)\ ,$$

$$R(\lambda)_{E_0+E_1}g\in D(A_{E_0+E_1}) \quad \text{for any} \quad g\in E_0+E_1\ ,$$

and

$$A_iR(\lambda)_i f=f-\lambda R(\lambda)_i f\ ,$$

$$A_{E_0+E_1}R(\lambda)_{E_0+E_1}g=g-\lambda R(\lambda)_{E_0+E_1}g\ .$$

Since the equation

$$(\lambda+A)f=0,\ f\in\mathscr{B}'(R)\ ,$$

implies $f=0$, $\lambda+A_{E_0+E_1}$ is invertible for every $\lambda>0$ (even for $\mathrm{Re}\,\lambda\neq0$). A fortiori $\lambda+A_i$ is invertible for $\lambda>0$ $(i=0, 1)$. Thus we see $\lambda+A_i$ and $\lambda+A_{E_0+E_1}$ map $D(A_i)$

and $D(A_{E_0+E_1})$ onto E_i and E_0+E_1 respectively in a one-to-one way ($i=0, 1$). Hence we have

$$R(\lambda)_i=(\lambda+A_i)^{-1}, \qquad i=0, 1, \tag{2.5}$$

and

$$R(\lambda)_{E_0+E_1}=(\lambda+A_{E_0+E_1})^{-1} \tag{2.6}$$

for every $\lambda>0$. We note that

$$(R(\lambda)_{E_0+E_1})_i=R(\lambda)_i$$

and

$$(A_{E_0+E_1})_i=A_i, \qquad i=0, 1.$$

Now let $f\in E_0=L^p(R)$ in the formula (2.1). Then we have, from (2.3) and Young's inequality,

$$\left\|\int_0^\infty e^{-\lambda t}f(x+t)dt\right\|_{E_1}\leqq L\lambda^{\sigma-1}\|f\|_{E_0} \tag{2.7}$$

with $\sigma=\dfrac{1}{p}-\dfrac{1}{q}$ and $L=(1-\sigma)^{1-\sigma}$.

Thus we see $(\lambda+A)_{0,1}^{-1}$ is defined on the whole E_0 for every $\lambda>0$ and it is continuous with the estimate (2.7). Hence we have (2.1).

(ii) PROOF OF (2.1′). Let $E_0=L^p(R)$, $E_1=C(R)$, and $\mathscr{E}=\mathscr{B}'(R)$. Let $A=\dfrac{d}{dx}$. Then arguing as in (i), the integral (2.2) gives the resolvent of $-A_i$ for each $i=0, 1$, $\lambda>0$. The estimate (2.3) and the relation (2.5) hold for the present choice of E_0, E_1 and A. Since, for $f\in L^p(R)$, and for $\lambda>0$,

$$|R(\lambda)f(x)-R(\lambda)f(x+h)|\leqq\left(1-\frac{1}{p}\right)^{1-1/p}\lambda^{1-1/p}\left(\int_R|f(y)-f(y+h)|^p dy\right)^{1/p} \tag{2.7′}$$

for any $h\in R$, $x\in R$; and since,

$$|R(\lambda)f(x)|\leqq\left(1-\frac{1}{p}\right)^{1-1/p}\lambda^{1-1/p}\|f\|_{L^p(R)}, \qquad (0^0=1), \tag{2.7″}$$

for any $x\in R$, we readily see (2.1′). q.e.d.

REMARK 2.1. We can show from the above proof that

$$\|R(\lambda)_i\|_{E_i\to E_i}\leqq\frac{1}{|\mathrm{Re}\,\lambda|}, \qquad i=0, 1, \tag{2.8}$$

$$\|R(\lambda)_{0,1}\|_{E_0\to E_1}\leqq L|\mathrm{Re}\,\lambda|^{\sigma-1},\ \sigma=\frac{1}{p}-\frac{1}{q},\ L=(1-\sigma)^{1-\sigma} \tag{2.9}$$

for every λ with $\operatorname{Re}\lambda>0$. For λ with $\operatorname{Re}\lambda<0$, defining

$$R(\lambda)f(x)=\int_{-\infty}^{0} e^{-\lambda t} f(x+t)dt\ ,$$

and arguing as above, we can obtain the relations (2.5) and (2.6) for such λ, and same estimates as (2.8) and (2.9).

REMARK 2.2. In Example 2.1, it is clear that $D(-A_i)$ for each i is dense in E_i, and the relation (2.1) and the estimate (2.2) show that $-A_i$ is the infinitesimal generator of the translation semigroup $G(t):f(x)\longrightarrow f(x+t)$, $t\geq 0$, in each E_i (cf. § 1.2). We also note that (2.1′) reflects a general property, namely:

PROPOSITION 2.1. *If* $A\in(\sigma, E_0, E_1)$ *and if* $D(A_0)$ *is dense in* E_0, *then the image of* E_0 *by the mapping* $(\lambda+A)^{-1}_{0,1}$, $\lambda>0$, *is contained in the closure of* $D(A_1)$ *in* E_1.

PROOF. Let $x\in E_0$. Then $(\lambda+A)^{-1}x\in E_1$, $\lambda>0$, and $\mu(\mu+A)^{-1}(\lambda+A)^{-1}x\in D(A_1)$, $\mu>0$. Since $D(A_0)$ is dense in E_0, and since A_0 is of type (h),

$$\begin{aligned}\|\mu(\mu+A)^{-1}(\lambda+A)^{-1}x-(\lambda+A)^{-1}x\|_{E_1}&=\|(\lambda+A)^{-1}(\mu(\mu+A)^{-1}x-x)\|_{E_1}\\ &\leqslant\|(\lambda+A)^{-1}\|_{E_0\to E_1}\|\mu(\mu+A)^{-1}x-x\|_{E_0}\longrightarrow 0 \text{ as } \mu\longrightarrow\infty\ . \quad \text{q.e.d.}\end{aligned}$$

DEFINITION 2.1*. We say $A\in(\sigma, E_0, E_1)^*$ if $A\in(\sigma, E_0, E_1)$ and if moreover A satisfies the following condition:

$(\lambda+A)^{-1}_{0,1}$ is a compact operator from E_0 into E_1 for every $\lambda>0$.

Example 2.1*. In order to illustrate Definition 2.1*, we 2.1*, we state that

$$\text{(2.1)*}\qquad \frac{d}{dx}\in\left(\frac{1}{p}-\frac{1}{q},\ L^p(T),\ L^q(T)\right)^*,\ 1\leqslant p<q<\infty\ ;$$

and that

$$\text{(2.1)*}'\qquad \frac{d}{dx}\in\left(\frac{1}{p},\ L^p(T),\ C(T)\right)^*,\ 1<p<\infty\ .$$

Here $C(T)$ is the Banach space of continuous functions f on T with the norm

$$\|f\|_{C(T)}=\max_{x\in T}|f(x)|\ .$$

PROOF OF (2.1)*. (i) Let $E_0=L^p(T)$, $E_1=L^q(T)$, and $\mathscr{E}=\mathscr{D}'(T)$, the space of distributions of period 1. Let $A=\frac{d}{dx}$. Consider the integral (2.2) for $f\in\mathscr{D}'(T)$. The integral is in the sense of distribution. Since f is periodic, the integral can be written as followings:

$$\int_0^\infty e^{-\lambda t} f(x+t)dt=\sum_{N=0}^\infty\int_N^{N+1} e^{-\lambda t} f(x+t)dt=\sum_{N=0}^\infty e^{-\lambda N}\int_0^1 e^{-\lambda t} f(x+t)dt$$

$$= \frac{1}{1-e^{-\lambda}} \int_0^1 e^{-\lambda t} f(x+t)dt .$$

Arguing as in Example 2.1, we obtain the continuous operator $R(\lambda)_i$, $i=0, 1$, $R(\lambda)_{E_0+E_1}$ with the same estimates as (2.3). Since the equation

$$(\lambda+A)f=0 , \quad f \in \mathscr{D}'(T) , \quad \lambda>0 ,$$

implies $f=0$, we see, as in Example 2.1, that

$$(\lambda+A_i)^{-1}=R(\lambda)_i , \qquad i=0, 1 ,$$

and

$$(\lambda+A_{E_0+E_1})^{-1}=R(\lambda)_{E_0+E_1} .$$

By a similar consideration as in Example 2.1, we have that

$$\|(\lambda+A)^{-1}_{0,1}\|_{E_0\to E_1} \leqslant L\lambda^{\sigma-1}$$

for every $\lambda>0$ with $\sigma=\frac{1}{p}-\frac{1}{q}$ and $L=(1-\sigma)^{1-\sigma}$. Thus we have shown that $A \in \left(\frac{1}{p}-\frac{1}{q}, L^p(T), L^q(T)\right)$.

To see the compactness of the operator $(\lambda+A)^{-1}_{0,1}$ for fixed $\lambda>0$, let $f_k \in E_0$ with $\|f_k\|_{E_0} \leqslant 1$, $k=1, 2, \cdots$. We are going to show that $\{(\lambda+A)^{-1}f_k\}$ forms an equibounded and equicontinuous set in E_1. Since T is compact, this implies the compactness of $(\lambda+A)^{-1}_{0,1}$ (See Yosida [12], the proof of the Fréchet-Kolmogorov theorem, p. 275-277). The equiboundedness follows immediately from the fact that $A \in (\sigma, E_0, E_1)$. In order to see the equicontinuity, let h be any small number ($|h|<1$). We have, for $h>0$,

$$\begin{aligned}
&\left\| \int_0^\infty e^{-\lambda s} (f_k(x+s) - f_k(x+s+h))ds \right\|_{E_1} \\
&\quad = \left\| \int_0^\infty e^{-\lambda s} f_k(x+s)ds - e^{\lambda h} \int_h^\infty e^{-\lambda s} f_k(x+s)ds \right\|_{E_1} \\
&\quad = \left\| (1-e^{\lambda h}) \int_h^\infty e^{-\lambda s} f_k(x+s)ds + \int_0^h e^{-\lambda s} f_k(x+s)ds \right\|_{E_1} \\
&\quad \leqslant (1-e^{\lambda h}) \left\| \int_0^\infty e^{-\lambda s} |f_k(x+s)|ds \right\|_{E_1} + \left\| \int_0^h e^{-\lambda s} |f_k(x+s)|ds \right\|_{E_1} \\
&\quad = (1-e^{\lambda h}) I_1(k) + I_2(k, h).
\end{aligned}$$

As before, $I_1(k) \leqslant L\lambda^{\sigma-1}\|f_k\|_{E_0} \leqslant L\lambda^{\sigma-1}$. Since

$$I_2(k, h) \leqslant \left\{ \int_0^1 \left[\int_0^h e^{-\lambda s} |f_k(x+s)|ds \right]^q dx \right\}^{1/q} ,$$

we obtain, arguing as in the proof of Young's inequality (see [4]),

$$I_2(k, h) \leqslant \left(\int_0^h e^{-\lambda s r}\, ds\right)^{1+1/q-1/p} \|f_k\|_{E_0} \leqslant \left(\int_0^h e^{-\lambda s r} ds\right)^{1+1/q-1/p}$$

For $h<0$, we should change the partition of integrals, and we consider $(1-e^{\lambda h})I_1(k)+e^{\lambda h}I_2(k, h)$. In both cases, since $1+\dfrac{1}{p}-\dfrac{1}{q}>0$ and $\lambda>0$, we see that

$$\left\|\int_0^\infty e^{-\lambda s}\Big(f_k(x+s)-f_k(x+s+h)\Big)ds\right\|_{E_1} \longrightarrow 0$$

as h, uniformly for f_k. Hence we have (2.1)*.

(ii) PROOF OF (2.1)*′. Let $E_0=L^p(T)$, $E_1=C(T)$, $\mathscr{E}=\mathscr{D}'(T)$, and $A=\dfrac{d}{dx}$. Then arguing as in the proofs of (2.1′) and (2.1)*, we easily see (2.1)*′. The assumption $p>1$ is needed to show the convergence of $I_2(k, h)$ when $h\longrightarrow 0$. q.e.d.

2.2. Let $(E_0, E_1, \mathscr{E})$ be interpolation triplet, and A be a continuous linear operator on $\mathscr{E}$. In this section we consider the case that each $-A_i$ $(i=0, 1)$ is an infinitesimal generator of an equibounded semigroup $G_i(t)$, $t\geqslant 0$, of linear operators on E_i. We first note the following

PROPOSITION 2.2. *Let $(E_0, E_1, \mathscr{E})$ be an interpolation triplet, and $G(t)$, $t\geqslant 0$, be a family of linear operators on $\mathscr{E}$ such that we have, in each E_i, $i=0, 1$, an equibounded semigroup $G_i(t)$, $t\geqslant 0$, as the restriction of $G(t)$ on E_i, that is, for each $i=0, 1$, $G_i(t)$, $t\geqslant 0$, satisfies*

(i) $\|G_i(t)\|_{E_i\to E_i}\leqslant M_i$;

(ii) $G_i(0)=I$, *and* $G_i(t)G_i(s)=G_i(t+s)$ *for* $t, s\geqslant 0$.

If $G_0(t)$ is a semigroup of class (C_0) and if we have, for $t>0$, the restriction $G(t)_{0,1}$ of $G(t)$ on E_0 into E_1, then the image of E_0 by the mapping $G(t)_{0,1}$ is contained in a closed subspace of E_1, and the restriction of $G_1(t)$ in this subspace forms a semigroup of class (C_0).

PROOF. Let $F=\{y\in E_1;\, G_1(t)y\longrightarrow y \text{ as } t\longrightarrow 0\}$. Then it is easy to see that F is a closed subspace of E_1, that $G_1(t)$, $t\geqslant 0$, maps F into F, and that the restriction of $G_1(t)$, $t\geqslant 0$, on F is a semigroup of class (C_0). We are going to show that $G(t)_{0,1}(E_0)\subset F$. Since $G(t)_{0,1}$, $t>0$, is defined on the whole E_0, it is continuous. Thus, for any $x\in E_0$, $s>0$, $t>0$,

$$\|G_1(s)G(t)_{0,1}x-G(t)_{0,1}x\|_{E_1}=\|G(t)_{0,1}(G_0(s)x-x)\|_{E_1}\leqq\|G(t)_{0,1}\|_{E_0\to E_1}\|G_0(s)x-x\|_{E_0}.$$

Since $G_0(s)$, $s\geqslant 0$, is a semigroup of class (C_0), the right-hand side of the above inequality tends to 0 as s. q.e.d.

Thus, we confine ourselves to the case that each $G_i(t)$, $t\geqslant 0$, $(i=0, 1)$, is a semigroup of class (C_0), and $-A_i$ is the infinitesimal generator of $G_i(t)$. Then it is easily seen that the restriction of $-A$ in E_0+E_1 generates a semigroup of class (C_0), which we denote by $G(t)$, in E_0+E_1, and that $G_i(t)$ is the restriction $G(t)_i$ of $G(t)$ in E_i $(i=0,1)$. In fact, the estimate for the resolvent of $-A_i$ for each i implies that of $-A_{E_0+E_1}$ and that $D(A_{E_0+E_1})$ is dense in E_0+E_1 follows from the fact that $D(A_i)$ is dense in E_i for each i. Thus $-A_{E_0+E_1}$ generates a semigroup $G(t)$ of class (C_0) in E_0+E_1 which takes the same values as $G_i(t)$ on E_i $(i=0,1)$. We also note that the restriction $G(t)_{E_0\cap E_1}$ of $G(t)$ forms a semigroup of class (C_0) in $E_0\cap E_1$.

DEFINITION 2.2. $G(t)\in S(\sigma, E_0, E_1)$ means that $G(t)$, considered as operators from E_0 into E_1, that is, $G(t)_{0,1}$, are defined on the whole E_0 for all $t>0$, and they are continuous with the estimate:

$$\|G(t)_{0,1}\|_{E_0\to E_1}\leqslant Kt^{-\sigma}+K', \quad K>0,\ K'\geqslant 0,\ \sigma>0\ ,$$

where K, K', and σ are constants independent of $t>0$.

Example 2.2. In order to illustrate the above definition, we state that

$$G(t)\in S\left(\frac{n}{2}\left(\frac{1}{p}-\frac{1}{q}\right),\ L^p(R^n),\ L^q(R^n)\right),\ 1\leqslant p<q<\infty\ , \tag{2.10}$$

and that

$$G(t)\in S\left(\frac{n}{2p}\ ,\ L^p(R^n),\ C(R^n)\right),\ 1\leqslant p<\infty\ . \tag{2.10$'$}$$

Here $G(t)$ is defined by

$$G(t)f(x)=\begin{cases} f(x) & \text{if} \quad t=0 \\ \dfrac{1}{(\sqrt{2\pi t})^n}\displaystyle\int_{R^n} e^{-\frac{|x-y|^2}{2t}}f(y)dy & \text{if} \quad t>0 \end{cases} \tag{2.11}$$

for $f\in\mathscr{B}'(R^n)$; and $C(R^n)$ is the Banach space of uniformly continuous, bounded functions f on R^n with the norm $\|f\|_{C(R^n)}=\max\limits_{x\in R^n}|f(x)|$.

(i) PROOF OF (2.10). Let $E_0=L^p(R^n)$, $E_1=L^q(R^n)$, $\mathscr{E}=\mathscr{B}'(R^n)$, and $A=\Delta=\dfrac{\partial^2}{(\partial x^1)^2}+\cdots+\dfrac{\partial^2}{(\partial x^n)^2}$. It is well-known that the restriction $G_i(t)=G(t)_i$, $t\geqslant 0$, $i=0, 1$, of $G(t)$ in each E_i is a semigroup of class (C_0), and that $-A_i$ is the infinitesimal generator of $G_i(t)$ (Yosida [12]). By Young's inequality for convolution, we see that $G(t)_{0,1}$ is defined on the whole E_0 for $t>0$ with the estimate:

$$\|G(t)_{0,1}\|_{E_0\to E_1}\leqslant Kt^{-\sigma}$$

where $\sigma=\dfrac{n}{2}\left(\dfrac{1}{p}-\dfrac{1}{q}\right)>0$, and $K=(2\pi)^{-\sigma}(1-\sigma)^{n/2-\sigma}$, $(K'=0)$. Hence we have (2.10).

(ii) PROOF OF (2.10)'. Let E_0, $\mathscr{E}$, A be as in the proof of (2.10). Let $E_1=C(R^n)$. Then $G_1(t)$ for the present choice of E_1 is, as is well-known (Yosida [12]), a semigroup of class (C_0) with the infinitesimal generator $-A_1$. Since, for $t>0$, and $f\in L^p(R^n)$,

$$|G(t)f(x)|\leq Kt^{-n/2p}\|f\|_{L^p(R^n)},$$

and

$$|G(t)f(x)-G(t)f(x+h)|\leq Kt^{-n/2p}\Big[\int_{R^n}|f(y)-f(y+h)|^p dy\Big]^{1/p}$$

for any $x, h\in R^n$ with $K=\Big(1-\dfrac{1}{p}\Big)^{\frac{n}{2}(1-\frac{1}{p})}(2\pi)^{-\frac{n}{2p}}$, we see (2.10)'. q.e.d.

DEFINITION 2.2*. We say $G(t)\in S(\sigma, E_0, E_1)^*$ if $G(t)\in S(\sigma, E_0, E_1)$ and if the operator $G(t)_{0,1}$ is, for each $t>0$, a compact linear operator on E_0 into E_1.

Example 2.2*. In order to illustrate Definition 2.2*, we state that

$$G(t)\in S\Big(\frac{n}{2}\Big(\frac{1}{p}-\frac{1}{q}\Big), L^p(T^n), L^q(T^n)\Big)^*, \quad 1\leq p<q<\infty, \tag{2.10)*}$$

and that

$$G(t)\in S\Big(\frac{n}{2p}, L^p(T^n), C(T^n)\Big)^*, \quad 1<p<\infty. \tag{2.10)*'}$$

Here $G(t)$ is given by

$$G(t)f(x)=\begin{cases} f(x), & t=0,\\ \displaystyle\int_{T^n} g_t(x-y)f(y)dy, & t>0,\end{cases} \tag{2.12}$$

with

$$g_t(x)=-\frac{1}{(\sqrt{2\pi t})^n}\sum_{k\in \mathbb{Z}^n} e^{-\frac{|x-k|^2}{2t}}, \quad t>0; \tag{2.13}$$

and $C(T^n)$ is the Banach space of continuous functions f on T^n with the norm $\|f\|_{C(T^n)}=\max_{x\in T^n}|f(x)|$.

(i) PROOF OF (2.10)*. Let $E_0=L^p(T^n)$, $E_1=L^q(T^n)$, and $\mathscr{E}=\mathscr{D}'(T^n)$. Let $A=\Delta=\dfrac{\partial^2}{(\partial x^1)^2}+\cdots+\dfrac{\partial^2}{(\partial x^n)^2}$. (2.12) defines a semigroup $G_i(t)$ of class (C_0) in each E_i, $i=0, 1$, as the restriction of $G(t)$. The infinitesimal generator of $G_i(t)$ is the restriction $-A_i$ of the operator $-A$. Since

$$\|g_t(x)\|_{L^1(T^n)}\leq 1, \quad \text{and} \quad \|g_t(x)\|_{L^\infty(T^n)}\leq c_\infty(1+t^{-n/2}),$$

we have, for $1<r<\infty$,

$$\|g_t(x)\|_{L^r(T^n)} \leq c_r(1+t^{-n(1-1/r)/2})$$

by the logarithmic convexity of L^p-scales (cf. Example 1.1 in § 1.1). Hence, we have, by Young's inequality for convolution,

$$\|G(t)_{0,1}\|_{E_0\to E_1} \leq c(1+t^{-n(1/p-1/q)/2}) .$$

Thus

$$G(t)\in S\Big(\frac{n}{2}\Big(\frac{1}{p}-\frac{1}{q}\Big),\ L^p(T^n),\ L^q(T^n)\Big) .$$

Now we are going to see the compactness of the operator $G(t)_{0,1}$ for each fixed $t>0$. It is enough to show that for any set $\{f_k(x)\}_{k=1,2\dots}\subset E_0$ with $\|f_k\|_{E_0}\leq 1$, $\{G(t)_{0,1}f_k\}$ is equibounded and equicontinuous in E_1 (see Yosida [12], p. 275-277). The equiboundedness is an immediate consequence of the the fact that $G(t)\in S\Big(\frac{n}{2}\Big(\frac{1}{p}-\frac{1}{q}\Big),\ L^p(T^n),\ L^q(T^n)\Big)$. Let $h\in T^n$. Since

$$\begin{aligned}&\Big(\int_{T^n}|G(t)f_k(x+h)-G(t)f_k(x)|^q dx\Big)^{1/q}\\ &=\Big(\int_{T^n}\Big|\int_{T^n}(g_t(x+h-y)-g_t(x-y))f(y)dy\Big|^q dx\Big)^{1/q}\\ &\leq\Big(\int_{T^n}|g_t(x+h)-g_t(x)|^r dx\Big)^{1/r}\Big(\int_{T^n}|f(y)|^p dy\Big)^{1/p},\quad \frac{1}{r}=\frac{1}{q}-\frac{1}{p}-1,\end{aligned}$$

for every fixed $t>0$, the equicontinuity follows immediately. Therefore, (2.10)* has been shown.

(ii) PROOF OF (2.10)*′. Let $E_0=L^p(T^n)$, $E_1=C(T^n)$. Let $\mathscr{E}$ and A be same as in the proof of (2.10)*. Then (2.12) defines a semigroup $G_i(t)$ of class (C_0) in each E_i with the infinitesimal generator $-A_i$. As before, it is easily seen that

$$G(t)\in S\Big(\frac{n}{2p},\ L^p(T^n),\ C(T^n)\Big) .$$

In order to show the compactness of $G(t)_{0,1}$, consider $f_k\in E_0$, $\|f_k\|_{E_0}\leq 1$. It is clear that $\{G(t)_{0,1}f_k\}$ is equibounded in $C(T^n)$. The equicontinuity of $\{G(t)_{0,1}f_k\}$ follows from the following inequality:

$$|G(t)f_k(x+h)-G(t)f_k|\leq\Big[\int|g_t(y+h)-g_t(y)|^{p'}dy\Big]^{1-1/p}\Big[\int|f_k(y)|^p dy\Big]^{1/p}$$

for $x, h\in T^n$, and $p'=p/(p-1)$. Thus we have (2.10)*′. q.e.d.

2.3. Let $(E_0, E_1, \mathscr{E})$ be an interpolation triplet. Let A^j, $j=1,\cdots,N$, be mutually commutative continuous linear operators on $\mathscr{E}$. Let E_{kj}, $k=1,\cdots,N$;

$j=0, \cdots, N$ be Banach spaces continuously imbedded in $\mathscr{E}$, and $E_{k0}=E_0$, $E_{kN}=E_1$ for $k=1, \cdots, N$. Let $\sigma_1, \cdots, \sigma_N$ be positive numbers.

DEFINITION 2.3. We write $\{A^1, \cdots, A^N\} \in (\{\sigma_1, \cdots, \sigma_N\}, E_0, E_1)$ if the following two conditions are satisfied:

(i) $A^j \in (\sigma_j, E_{k\mu(j-k)-1}, E_{k\mu(j-k)})$ for $j, k=1, \cdots, N$ where

$$\mu(n)=\begin{cases} n+1 & \text{if} \quad n\geqslant 0\,, \\ n+N+1 & \text{if} \quad n<0\,; \end{cases} \tag{2.14}$$

(ii) the restriction of $(\lambda+A^j)^{-1}$, $\lambda>0$, in E_{kl} are mutually commutative, that is

$$(\nu+A^i_{kl})^{-1}(\lambda+A^j_{kl})^{-1}=(\lambda+A^j_{kl})^{-1}(\nu+A^i_{kl})^{-1}$$

for $\nu>0$ and $\lambda>0$, $i, j=1, \cdots, N$; $k=1, \cdots, N$; $l=0, \cdots, N$.

Example 2.3. In order to illustrate the above definition, we state that

$$\left\{\frac{\partial}{\partial x^1}, \cdots, \frac{\partial}{\partial x^n}\right\} \in \Big(\underbrace{\Big\{\frac{1}{p}-\frac{1}{q}, \cdots, \frac{1}{p}-\frac{1}{q}\Big\}}_{n}, L^p(R^n), L^q(R^n)\Big) \tag{2.15}$$

where $1\leqslant p<q<\infty$.

PROOF OF (2.14). Let $E_0=L^p(R^n)$, $E_1=L^q(R^n)$, $\mathscr{E}=\mathscr{D}'(R^n)$, and $A^j=\dfrac{\partial}{\partial x^j}$, $j=1, \cdots, n$. The operators A^j are mutually commutative, and continuous on $\mathscr{E}$. We are going to introduce the spaces E_{kj}. Let π be a permutation of $(1, \cdots, n)$. Let $\rho=(r_1, \cdots, r_n)$ be a set of real numbers such that $1\leqslant r_n\leqslant \cdots \leqslant r_1<\infty$. We denote by $L^{\pi\rho}$ the space of functions f, measurable in R^n, which satisfy the following condition:

(2.16)

$$\|f\|_{\pi\rho}=\left[\int_{-\infty}^{\infty} \cdots \left\{\left(\int_{-\infty}^{\infty} |f(\pi x)|^{r_{\pi(1)}} dx^{\pi(1)}\right)^{r_{\pi(2)}/r_{\pi(1)}} dx^{\pi(2)}\right\}^{r_{\pi(3)}/r_{\pi(2)}} \cdots dx^{\pi(n)}\right]^{1/r_{\pi(n)}}<\infty$$

where $\pi x=(x^{\pi(1)}, \cdots, x^{\pi(n)})$. $L^{\pi\rho}$ is a Banach space with the norm given by (2.16), and it is continuously imbedded in $\mathscr{D}'(R^n)$. By Fubini's theorem, $L^{\pi\rho}=L^r(R^n)$ if $r_1=\cdots=r_n=r$. Let $\pi_k=\begin{pmatrix}1 & 2 & \cdots & n-k+1 & n-k+2 & \cdots & n \\ k & k+1 & \cdots & n & 1 & \cdots & k-1\end{pmatrix}$ and $\rho_j=(\overbrace{p \cdots p}^{j}\ \overbrace{q \cdots q}^{n-j})$. We set $E_{kj}=L^{\pi_k\rho_j}$, $k=1, \cdots, n$; $j=0, \cdots, n$. For such spaces, see, for example, Benedek-Panzoni [1]. Consider the integral

$$R^j(\lambda)f(x)=\int_0^\infty e^{-\lambda t} f(x+te_j)dt \qquad \text{for} \qquad j=1, \cdots, n \tag{2.17^j}$$

where $\lambda>0$, $e_j=(0, \cdots, 0, \overset{j}{1}, 0, \cdots, 0)$, $f\in\mathscr{E}$, and the integral is in the sense of distribution. Considering these integrals in E_{kl}, we obtain the restrictions $R^j(\lambda)_{kl}$

of $R^j(\lambda)$ in E_{kl}. Now we show the continuity of $R^j(\lambda)_{1l}$, that is, let us show that

$$(2.18)\quad \left[\int_{R^{n-l}}\left\{\int_{R^l}\left|\int_0^\infty e^{-\lambda t}f(x+te_j)dt\right|^q dx^1\cdots dx^l\right\}^{q/p}dx^{l+1}\cdots dx^n\right]^{1/p} \leqslant \frac{1}{\lambda}\left[\int_{R^{n-l}}\left\{\int_{R^l}|f(x)|^q dx^1\cdots dx^l\right\}^{p/q}dx^{l+1}\cdots dx^n\right]^{1/p}.$$

In fact, let $\varphi(x^1,\cdots,x^l)\in L^{q'}(R)$ where $q'=q/(q-1)$. For almost every $(x^{l+1},\cdots,x^n)$,

$$\int_{R^l}\overline{\varphi(x^1,\cdots,x^l)}\left\{\int_0^\infty e^{-\lambda t}f(x+te_j)dt\right\}dx^1\cdots dx^l$$
$$\leqslant\int_0^\infty e^{-\lambda t}\,dt\left\{\int_{R^l}|f(x+te_j)|\,|\varphi(x^1\cdots x^l)|\,dx^1\cdots dx^l\right\}$$
$$\leqslant\int_0^\infty e^{-\lambda t}\,dt\left\{\left[\int_{R^l}|f(x+te_j)|^q dx^1\cdots dx^l\right]^{1/q}\times\left[\int_{R^l}|\varphi(x^1,\cdots,x^l)|^{q'}dx^1\cdots dx^l\right]^{1/q'}\right\}.$$

Thus for almost every $(x^{l+1},\cdots,x^n)$, we have

$$\left[\int_{R^l}\left|\int_0^\infty e^{-\lambda t}f(x+te_j)dt\right|^q dx^1\cdots dx^l\right]^{1/q}\leqslant\int_0^\infty e^{-\lambda t}\left[\int_{R^l}|f(x+te_j)|^q dx^1\cdots dx^l\right]^{1/q}dt.$$

If $p=1$, (2.18) follows immediately by Fubini's theorem. For $p>1$, let $\psi(x^{l+1},\cdots,x^n)\in L^{p'}(R^{n-l})$ where $p'=p/(p-1)$. Then

$$\int_{R^{n-l}}\overline{\psi(x^{l+1},\cdots,x^n)}\left[\int_{R^l}\left|\int_0^\infty e^{-\lambda t}f(x+te_j)dt\right|^q dx^1\cdots dx^l\right]^{1/q}dx^{l+1}\cdots dx^n$$
$$\leqslant\int_0^\infty e^{-\lambda t}\,dt\left\{\int_{R^{n-l}}|\psi(x^{l+1},\cdots,x^n)|\left[\int_{R^l}|f(x+te_j)|^q dx^1\cdots dx^l\right]^{1/q}dx^{l+1}\cdots dx^n\right\}$$
$$\leqslant\int_0^\infty e^{-\lambda t}\,dt\left\{\left[\int_{R^{n-l}}|\psi(x^{l+1},\cdots,x^n)|^{p'}dx^{l+1}\cdots dx^n\right]^{1/p'}\times\left[\int_{R^{n-l}}\left[\int_{R^l}|f(x+te_j)|^q dx^1\cdots dx^l\right]^{p/q}dx^{l+1}\cdots dx^n\right]^{1/p}\right\}.$$

Hence we obtain (2.18).

In the same way, we have that $R^j(\lambda)_{kl}$, $j=1,\cdots,n$ are continuous on every E_{kl}, $k=1,\cdots,n$; $l=0,\cdots,n$, with the estimate

$$(2.19)\quad \|R^j(\lambda)_{kl}\|_{E_{kl}\to E_{kl}}\leqslant\frac{1}{\lambda}$$

for each fixed $\lambda>0$.

The commutativity of $R^j(\lambda)_{kl}$ and $R^i(\lambda)_{kl}$, $i, j=1, \cdots, n$; $k=1, \cdots, n$; $l=0, \cdots, n$; $\lambda>0, \nu>0$, follows immediately from that of $R^j(\lambda)$ and $R^i(\lambda)$ in $\mathscr{E}$. The latter is obvious from the formula $(2.17)^j$ and $(2.17)^i$. Since for each $j=1, \cdots, n$ the equation

$$(\lambda+A^j)f=0, \quad f\in \mathscr{B}'(R^n), \lambda>0$$

implies $f=0$, $\lambda+A^j_E$ is invertible for every $\lambda>0$. Here $E=\sum E_{kl}$. Hence, a fortiori $\lambda+A^j_{kl}$ is invertible for every $\lambda>0$, and as in Example 2.1, we have

$$(\lambda+A^j_{kl})^{-1}=R^j(\lambda)_{kl} \tag{2.20}$$

and

$$(\lambda+A^j_E)^{-1}=R^j(\lambda)_E$$

for $j=1, \cdots, n$.

Now we are going to show that $(\lambda+A^j)^{-1}_{1j-1,1j}$ is continuous for each $\lambda>0$, and for each $j=1, \cdots, n$. We have shown in the above that

$$\left[\int_{R^{j-1}}\left|\int_0^\infty e^{-\lambda t}f(x+te_j)dt\right|^q dx^1 \cdots dx^{j-1}\right]^{1/q} \leqslant \int_0^\infty e^{-\lambda t}\,dt\left\{\left[\int_{R^{j-1}}|f(x+te_j)|^q dx^1 \cdots dx^{j-1}\right]^{1/q}\right\}.$$

From this, we can show as in Example 2.1 that

$$\left[\int_{R^j}\left|\int_0^\infty e^{-\lambda t}f(x+te_j)dt\right|^q dx^1 \cdots dx^{j-1}dx^j\right]^{1/q} \leqslant L\lambda^{\sigma-1}\left\{\int_{-\infty}^\infty\left[\int_{R^{j-1}}|f(x)|^q dx^1 \cdots dx^{j-1}\right]^{p/q}dx^j\right\}^{1/p}$$

where $\sigma=\dfrac{1}{p}-\dfrac{1}{q}$ and $L=(1-\sigma)^{1-\sigma}$. Integrating in $x^{j+1}, \cdots, x^n$, the p-th powers of the both sides of this inequality, we obtain that

$$\|R^j(\lambda)_{1j-1,1j}\|_{E_{1j-1}\to E_{1j}}\leqslant L\lambda^{\sigma-1}\,. \tag{2.21}$$

In this way, we see that each $R^j(\lambda)_{k\mu(j-k)-1,k\mu(j-k)}$ is defined on the whole $E_{k\mu(j-k)-1}$ and it is continuous with the estimate

$$\|R^j(\lambda)_{k\mu(j-k)-1,k\mu(j-1)}\|_{E_{k\mu(j-k)-1}\to E_{k\mu(j-k)}}\leqslant L\lambda^{\sigma-1} \tag{2.22}$$

where $\sigma=\dfrac{1}{p}-\dfrac{1}{q}$ and $L=(1-\sigma)^{1-\sigma}$. Hence we have $\left\{\dfrac{\partial}{\partial x_1}, \cdots, \dfrac{\partial}{\partial x^n}\right\}\in$

$(\{\sigma, \cdots, \sigma\}, L^p(R^n), L^q(R^n))$ with $\sigma=\dfrac{1}{p}-\dfrac{1}{q}$ and $1\leqq p<q<\infty$. For the sake of facile understanding of the above considerations, we give a diagram for the case $n=3$. In this case, $\pi_1=\begin{pmatrix}1&2&3\\1&2&3\end{pmatrix}$, $\pi_2=\begin{pmatrix}1&2&3\\2&3&1\end{pmatrix}$, $\pi_3=\begin{pmatrix}1&2&3\\3&1&2\end{pmatrix}$, $\rho_0=(p\ p\ p)$, $\rho_1=(q\ p\ p)$, $\rho_2=(q\ q\ p)$, $\rho_3=(q\ q\ q)$. Let $L^{p\,q\,r}_{i\,j\,k}$ correspond to $\rho=(p\ q\ r)$ and $\pi=\begin{pmatrix}1&2&3\\i&j&k\end{pmatrix}$, that is, $L^{p\,q\,r}_{i\,j\,k}=L^{\pi\rho}$. Then $E_0=E_{10}=E_{20}=E_{30}$, $E_1=E_{13}=E_{23}=E_{33}$, $E_{11}=L^{q\,p\,p}_{123}$, $E_{12}=L^{q\,q\,p}_{123}$, $E_{21}=L^{q\,p\,p}_{231}$, $E_{22}=L^{q\,q\,p}_{231}$, $E_{31}=L^{q\,p\,p}_{312}$, $E_{32}=L^{q\,q\,p}_{312}$. The diagram becomes as follows:

$$\begin{array}{ccccccc}
E_{10} & \xrightarrow{(\lambda+A^1)^{-1}} & E_{11} & \xrightarrow{(\mu+A^2)^{-1}} & E_{12} & \xrightarrow{(\nu+A^3)^{-1}} & E_{13} \\
\| & & & & & & \| \\
E_{20} & \xrightarrow{(\mu+A^2)^{-1}} & E_{21} & \xrightarrow{(\nu+A^3)^{-1}} & E_{22} & \xrightarrow{(\lambda+A^1)^{-1}} & E_{23} \\
\| & & & & & & \| \\
E_{30} & \xrightarrow{(\nu+A^3)^{-1}} & E_{31} & \xrightarrow{(\lambda+A^1)^{-1}} & E_{32} & \xrightarrow{(\mu+A^2)^{-1}} & E_{33}
\end{array}$$

REMARK 2.3. It is not difficult to see that each $D(A^j_{kl})$ is dense in E_k, $j, k=1, \cdots, n$; $l=0, \cdots, n$, in the above example. From the relations $(2.17)^j$, (2.20), and from the estimate (2.19), we see that each $-A^j_{kl}$ generates a semigroup of class (C_0) in E_{kl}, the translation semigroup in the j-th coordinate $G^j(t)$: $f(x)\longrightarrow f(x+te_j)$.

2.4. Let the spaces $E_0, E_1, E_{jk}, \mathscr{E}$, and the operators A^j be as in § 2.3 $(j=1, \cdots, N;\ k=0, \cdots, N)$. Let m_j be positive integers, and σ_j be positive numbers $(j=1, \cdots, N)$.

DEFINITION 2.4. We write $\{A^1, \cdots, A^N\}\in(\{\sigma_1, \cdots, \sigma_N\}, \{m_1, \cdots, m_N\}, E_0, E_1)$ if the following two conditions are satisfied:

(i) the restrictions A^j_{kl} of A^j in E_{kl} are of type (h_{m_j}) (see § 1.4), and

$$(\lambda+A^i_{kl})^{-1}(\nu+A^j_{kl})^{-1}=(\nu+A^j_{kl})^{-1}(\lambda+A^i_{kl})^{-1}$$

for $\lambda\neq0$, $\nu\neq0$, $\arg\lambda=2r\pi/m_i$, $\arg\nu=2r'\pi/m_j$, (r, r': integers), $j, k=1, \cdots, N$; $l=0, \cdots, N$.

(ii) for every $j, k=1, \cdots, N$, there exists a ray $\lambda\neq0$, $\arg\lambda=2r\pi/m_i$ (with a certain positive integer r), such that $(\lambda+A^j)^{-1}$, considered as operators from $E_{k\mu(j-k)-1}$ into $E_{k\mu(j-k)}$, that is, $(\lambda+A^j)^{-1}_{k\mu(j-k)-1,k\mu(k-j)}$ are defined on the whole $E_{k\mu(j-k)-1}$ for all λ on this ray, and they are continuous with the estimate:

$$\|(\lambda+A^j)^{-1}_{k\mu(j-k)-1,k\mu(j-k)}\|_{E_{k\mu(j-k)-1}\to E_{k\mu(j-k)}}\leqq L_{jk}|\lambda|^{\sigma_j^{-1}}$$

where L_{jk} is a positive constant independent of λ. $\mu(n)$ is the function defined by (2.15).

Example 2.4. In order to illustrate the above definition, we state that

$$\left\{c_1 \frac{\partial}{\partial x^1}, \cdots, c_n \frac{\partial}{\partial x^n}\right\} \in (\{\underbrace{\sigma, \cdots, \sigma}_{n}\}, \{m_1, \cdots, m_n\}, L^p(R^n), L^q(R^n)) , \tag{2.23}$$

where $\sigma = \frac{1}{p} - \frac{1}{q}$, $1 \leqslant p < q < \infty$, and c_j, $j=1, \cdots, n$, are suitable complex numbers such that $-\arg c_j + 2k\pi/m_j \not\equiv \pi/2$ (mod. π).

Proof of (2.23). Let the spaces $E_0, E_1, E_{jk}, \mathscr{E}$ and the operators be same as in Example 2.3, $j=1, \cdots, n$; $k=0, \cdots, n$. By the argument in Remark 2.1 and the proof of (2.14), we can see that

$$\|(\lambda + A^j_{kl})^{-1}\|_{E_{kl} \to E_{kl}} \leqslant \frac{1}{|\mathrm{Re}\,\lambda|} \tag{2.24}$$

for every λ with $\mathrm{Re}\,\lambda \neq 0$ ($j=1, \cdots, n; k=1, \cdots, n; l=0, \cdots, n$; and $(\lambda + A^j)^{-1}_{k\mu(j-k)-1, k\mu(j-k)}$ is defined on the whole $E_{k\mu(j-k)-1}$, and it is continuous with the estimate;

$$\|(\lambda + A^j)^{-1}_{k\mu(j-k)-1, k\mu(j-k)}\|_{E_{k\mu(j-k)-1} \to E_{k\mu(j-k)}} \leqslant L|\mathrm{Re}\,\lambda|^{\sigma-1} \tag{2.25}$$

for λ with $\mathrm{Re}\,\lambda \neq 0$, where $\sigma = \frac{1}{p} - \frac{1}{q}$ and $L = (1-\sigma)^{1-\sigma}$. The commutativity of $(\lambda + A^i_{kl})^{-1}$ and $(\nu + A^j_{kl})^{-1}$, $\mathrm{Re}\,\lambda \neq 0$, $\mathrm{Re}\,\nu \neq 0$, $i, j, k=1, \cdots, n$; $l=0, \cdots, n$, can be shown as in Example 2.3. Let m_j, $j=1, \cdots, n$, be certain positive integers. Let $c_j = c(m_j) \neq 0$ be suitable complex numbers such that $-\arg c_j + 2k\pi/m_j \not\equiv \pi/2$ (mod. π). Then from the above considerations, (2.23) follows immediately. q.e.d.

2.5. Here we collect our main theorems and related results. All these propositions are to be proved in § 3. In the statements of the theorems, m or m_j, $j=1, \cdots, N$, are certain positive integers. For Theorem 2.x or Theorem 2.x*, $x=1, 2, 3, 4$, the notations, definitions, and examples are given in § 2.x.

THEOREM 2.1. *Let* $(E_0, E_1, \mathscr{E})$ *be an interpolation triplet, and* A *be a continuous linear operator on* $\mathscr{E}$. *If* $A \in (\sigma, E_0, E_1)$ *then*

$$(E_0, D((A_0)^m))_{\theta+\sigma/m, p} \subset (E_1, D((A_1)^m))_{\theta, p}$$

where $0 < \theta < \theta + \frac{\sigma}{m} < 1$ and $1 \leqslant p \leqslant \infty$.

THEOREM 2.1*. *Let* $(E_0, E_1, \mathscr{E})$ *be an interpolation triplet, and* A *be a continuous linear operator on* $\mathscr{E}$. *If* $A \in (\sigma, E_0, E_1)^*$, *then we have, for* $0 < \theta < \theta + \frac{\sigma}{m} + \frac{\delta}{m} < 1$, $\delta \geqslant 0$, *and for* $1 \leqslant p \leqslant \infty$,

$$(E_0, D((A_0)^m))_{\theta+\sigma/m+\delta/m, p} \subset (E_1, D((A_1)^m))_{\theta, p} .$$

If $\delta>0$ and $p<\infty$, the injection is compact.

THEOREM 2.2. *Let $(E_0, E_1, \mathscr{E})$ be an interpolation triplet, and $-A$ be a continuous linear operator on $\mathscr{E}$ whose restriction $-A_i$ in each E_i $(i=0, 1)$ generates a semigroup $G_i(t)$, $t\geq 0$, of class (C_0) which turns out to be the restriction on E_i of a semigroup $G(t)$, $t\geq 0$, in E_0+E_1. If $G(t)\in S(\sigma, E_0, E_1)$, then*

$$(E_0, D((A_0)^m))_{\theta+\sigma/m,p}\subset(E_1, D((A_1)^m))_{\theta/p}$$

where $0<\theta<\theta+\dfrac{\sigma}{m}<1$ *and* $1\leq p\leq\infty$.

COROLLARY 2.1. *If $0<\sigma<1$ in the above theorem, then we have that $k_0+A\in(\sigma, E_0, E_1)$ for some k_0 and that $(\lambda+k_0+A)^{-1}_{0,1}$, $\mathrm{Re}\,\lambda>0$, is a continuous linear operator on E_0 into E_1 with the estimate:*

$$\|(\lambda+k_0+A)^{-1}_{0,1}\|_{E_0\to E_1}\leq M'|\mathrm{Re}\,\lambda|^{\sigma-1},\quad \mathrm{Re}\,\lambda>0\ ,$$

for some $M'>0$.

REMARK 2.4. Let $(E_0, E_1, \mathscr{E})$ be an interpolation triplet, and A be a continuous linear operator on $\mathscr{E}$. Consider the case that each $-A_i$ $(i=0, 1)$ is an infinitesimal generator of a holomorphic semigroup $G_i(t)$ in E_i, that is, each $G_i(t)$ is a semigroup of class (C_0), and has a holomorphic extension to the sector $|\arg t|<\dfrac{\pi}{2}-\omega_i$ for some ω_i, $0<\omega_i<\dfrac{\pi}{2}$. Then, for each $i=0, 1$, the resolvent set of $-A_i$, $\rho(-A_i)$, contains each λ, $\lambda\neq 0$, $|\arg\lambda|<\pi-\omega_i$ and

$$\|(\lambda+A_i)^{-1}\|_{E_i\to E_i}\leq M_i(\arg\lambda)|\lambda|^{-1},\ \lambda\neq 0,\ |\arg\lambda|<\pi-\omega_i$$

(see Yosida [12], Chapter 9). It is easy to see that $D(-A_{E_0+E_1})$ is dense in E_0+E_1, that $\rho(-A_{E_0+E_1})$ contains each λ, $\lambda\neq 0$, $|\arg\lambda|<\pi-\omega$, $\omega=\max(\omega_0, \omega_1)$, and that

$$\|(\lambda+A_{E_0+E_1})^{-1}\|_{E_0+E_1\to E_0+E_1}\leq M(\arg\lambda)|\lambda|^{-1},\ \lambda\neq 0,\ |\arg\lambda|<\pi-\omega\ .$$

Hence $-A_{E_0+E_1}$ generates a holomorphic semigroup $G(t)$ in E_0+E_1, and $G(t)_i=G(t)$, $i=0, 1$. Then we have the following

PROPOSITION 2.3. *Let E_0, E_1, A be as in Remark 2.4. If $(\lambda+A)^{-1}$, $\lambda\neq 0$, $|\arg\lambda|<\pi-\omega$, are continuous as operators from E_0 into E_1, and if*

$$\|(\lambda+A)^{-1}_{0,1}\|_{E_0\to E_1}\leq M'(\arg\lambda)|\lambda|^{\sigma-1},\ \sigma>0\ ,$$

for such λ, then we have

$$G(t)\in S(\sigma, E_0, E_1)\ .$$

THEOREM 2.2*. *Let $(E_0, E_1, \mathscr{E})$ be an interpolation triplet, and $-A$ be a continuous linear operator on $\mathscr{E}$ whose restriction $-A_i$ on each E_i $(i=0, 1)$*

generates a semigroup $G_i(t)$, $t\geq 0$, *of class* (C_0) *which turns out to be the restriction on* E_i *of a semigroup* $G(t)$, $t\geq 0$, *of a class* (C_0) *in* E_0+E_1. *If* $G(t)\in S(\sigma, E_0, E_1)^*$, *then, for* $0<\theta<\theta+\frac{\sigma}{m}+\frac{\delta}{m}<1$, $\delta\geq 0$, *and for* $1\leq p\leq\infty$,

$$(E_0, D((A_0)^m))_{\theta+\sigma/m+\delta/m,p}\subset(E_1, D((A_1)^m))_{\theta,p}\ .$$

The injection is compact for $\delta>0$ and $p<\infty$.

THEOREM 2.3. *Let* $(E_0, E_1, \mathscr{E})$ *be an interpolation triplet, and* A^j, $j=1,\cdots,N$, *be continuous linear operators on* $\mathscr{E}$. *If* $\{A^1,\cdots,A^N\}\in(\{\sigma_1,\cdots,\sigma_N\}, E_0, E_1)$, *and if* $\sum_{j=1}^N\sigma_j<1$, *then*

$$\bigcap_{j=1}^N(E_0, D(A_0{}^j))_{\theta+\sum_{j=1}^N\sigma_j,p}\subset\bigcap_{j=1}^N(E_1, D(A^j{}_1))_{\theta,p}$$

where $0<\theta<\theta+\sum_{j=1}^N\sigma_j<1$ *and* $1\leq p\leq\infty$.

THEOREM 2.4. *Let* $(E_0, E_1, \mathscr{E})$ *be an interpolation triplet, and* A^j, $j=1,\cdots,N$, *be continuous linear operator on* $\mathscr{E}$. *If*

$$\{A^1,\cdots,A^N\}\in(\{\sigma_1,\cdots,\sigma_N\}, \{m_1,\cdots,m_N\}, E_0, E_1),$$

and if $\sum_{j=1}^N\sigma_jm_j{}^{-1}<1$, *then*

$$\bigcap_{j=1}^N(E_0, D((A^j{}_0)^{m_j}))_{\theta+\sum_{j=1}^N\sigma_jm_j^{-1},p}\subset\bigcap_{j=1}^N(E_1, D((A_1{}^j)^{m_j}))_{\theta,p}$$

where $0<\theta<\theta+\sum_{j=1}^N\sigma_jm_j{}^{-1}<1$ *and* $1\leq p\leq\infty$.

3. Proofs of Theorems.

3.0. In this section we give the proofs of our Theorems 2.1–2.4 in §2.5. For the notations consult the indication given at the beginning of §2.5. Proposition 1.2 which makes legitimate our formulations is implicitly employed, and we often omit the subscripts such as i, j, in A_i, $A_{i,j}$, $G(t)_i$ etc. in the following proofs.

3.1. PROOF OF THEOREM 2.1. Let $a\in(E_0, D((A_0)^m))_{\theta+\sigma/m,p}$ and set

$$u_k(t)=\frac{\Gamma(2k)}{\Gamma(k)^2}t^k(A)^k(t+A)^{-2k}a,\quad t>0,\ k\geq m\ .$$

We note, by our Theorem 1.5 and Remark 1.1, that

$$t^{m\theta+\sigma}u_k(t)\in L^p_*(E_0),\ t^{m\theta+\sigma-k}(A)^ku_k(t)\in L^p_*(E_0)$$

and

$$\int_0^\infty u_k(t)\frac{dt}{t}=a \quad \text{in} \quad E_0 .$$

From our hypothesis,

$$\|u_{m+1}(t)\|_{E_1}\leqslant C_1t^\sigma\|u_m(t)\|_{E_0} \quad (C_1=2(2m+1)L/m) ,$$

and

$$\|(A)^m u_{m+1}(t)\|_{E_1}\leqslant C_1t^\sigma\|(A)^m u_m(t)\|_{E_0} .$$

Therefore we have

$$t^{m\theta}u_{m+1}(t)\in L_*^p(E_1),\ t^{m\theta-m}(A)^m u_{m+1}(t)\in L_*^p(E_1)$$

with

$$\|t^{m\theta}u_{m+1}(t)\|_{L_*^p(E_1)}\leqslant C_1\|t^{m\theta+\sigma}u_m(t)\|_{L_*^p(E_0)}$$

and

$$\|t^{m\theta-m}(A)^m u_{m+1}(t)\|_{L_*^p(E_1)}\leqslant C_1\|t^{m\theta+\sigma-m}(A)^m u_m(t)\|_{L_*^p(E_0)} .$$

Hence $b=\int_0^\infty u_{m+1}(t)\frac{dt}{t}\in(E_1, D((A_1)^m))_{\theta,p}$, and since $a=b$ in E_0+E_1, $a=b\in (E_1,D((A_1)^m))_{\theta,p}$. Finally from our proof and Remark 1.1,

$$\|x\|_{(E_1,D((A_1)^m))_{\theta,p}}\leqslant C_1\|x\|_{(E_0,D((A_0)^m))_{\theta+\sigma/m,p}} . \qquad \text{q.e.d.}$$

PROOF OF THEOREM 2.1*. In the proof we denote by $\|\ \|_0$ and $\|\ \|_1$ the norms in $(E_0, D((A_0)^m))_{\theta+\sigma m^{-1}+\delta m^{-1},p}$ and in $(E_1, D((A_1)^m))_{\theta,p}$ respectively. Let $a_n\in (E_0, D((A_0)^m))_{\theta+\sigma m^{-1}+\delta m^{-1},p}$ with $\|a_n\|_0\leqslant M$ for $n=1, 2, \cdots$. We show that there is a subsequence $\{a_n''\}$ of $\{a_n\}$ such that $\{a_n''\}$ converges in $(E_1, D((A_1)^m))_{\theta,p}$. Let

$$u_I(t)=c_{m+1}t^{m+1}(A)^{m+1}(t+A)^{-2m-2}a_I \quad \text{for} \quad t>0 \quad \left(c_{m+1}=\frac{\Gamma(2m+2)}{\Gamma(m+1)^2}\right)$$

where I corresponds to the indices occurring in the proof. Since $(1+A)^{-1}$ is compact on E_0 into E_1, and since the operators $(A)^{m+1}(1+A)^{-2m-1}$ and $(A)^{2m+1}(1+A)^{-2m-1}$ are bounded on E_1, we can choose a subsequence $\{a_{n1}\}$ of $\{a_n\}$ such that

$$(1+A)^{-1}a_{n1}\longrightarrow b_1 \text{ in } E_1 \text{ for some element } b_1\in E_1 ;$$

$$\|u_{n1}(1)-c_{m+1}(A)^{m+1}(1+A)^{-2m-1}b_1\|_{E_1}<\frac{M}{n} ,$$

and

$$\|(A)^m u_{n1}(1)-c_{m+1}(A)^{2m+1}(1+A)^{-2m-1}b_1\|_{E_1}<\frac{M}{n} .$$

Since $(t+A)^{-1}a_{n1} \longrightarrow b_t=(1+A)(t+A)^{-1}b_1$ in E_1 by the resolvent equations, and since the operators $t^m(A)^{m+1}(t+A)^{-2m-1}$ and $(A)^{2m+1}(t+A)^{-2m-1}$ are bounded on E_1 for $t>0$, we can choose a subsequence $\{a_{nk}\}$ of $\{a_{nk-1}\}$ $(k=1, 2, \cdots)$ satisfying

$$\|u_{nk}(k)-c_{m+1}k^{m+1}(A)^{m+1}(k+A)^{-2m-1}b_k\|_{E_1}<\frac{Mk^{-\alpha}}{n}$$

and

$$\|(A)^m u_{nk}(k)-c_{m+1}k^{m+1}(A)^{2m+1}(k+A)^{-2m-1}b_k\|_{E_1}<\frac{Mk^{m-\alpha}}{n}$$

where $\alpha>m+1$. Hence we obtain a subsequence $\{a'_n;\ a'_n=a_{nn}\}$ of $\{a_n\}$ with the property

$$\|u'_n(k)-c_{m+1}k^{m+1}(A)^{m+1}(k+A)^{-2m-1}b_k\|_{E_1}<\frac{Mk^{-\alpha}}{n} \tag{3.1$'$}$$

and

$$\|(A)^m u'_n(k)-c_{m+1}k^{m+1}(A)^{2m+1}(k-A)^{-2m-1}b_k\|_{E_1}<\frac{Mk^{m-\alpha}}{n}\ , \tag{3.2$'$}$$

for $k\leqslant n$; $k, n=1, 2, \cdots$. Here $u'_n=u_{nn}$. Similarly we can choose a subsequence $\{a''_n\}$ of $\{a'_n\}$ satisfying

$$\|u''_n(r)-c_{m+1}r^{m+1}(r+A)^{-2m-1}b_k\|_{E_1}<\frac{Mk^{-\alpha}}{n}\ ,\quad r=\frac{1}{k}\ , \tag{3.1$''$}$$

and

$$\|(A)^m u''_n(r)-c_{m+1}r^{m+1}(A)^{2m+1}(r+A)^{-2m-1}b_r\|_{E_1}<\frac{Mk^{m-\alpha}}{n}\ ,\ r=\frac{1}{k}\ , \tag{3.2$''$}$$

for $k\leqslant n$; $k, n=1, 2, \cdots$. Here u'' corresponds to a''. Evidently (3.1′) and (3.2′) hold for $u''_n(k)$, $k\leqslant n$. Now we are going to show that $\{a''_n\}$ coverges in $(E_1, D((A_1)_m))_{\theta,p}$. Since for $k\leqslant l$,

$$\begin{aligned}(A)^k(t+A)^{-l}x&=\{(s+A)(t+A)^{-1}\}^l(A)^k(s+A)^{-l}x\\&=\{I+(s-t)(t+A)^{-1}\}^l(A)^k(s+A)^{-l}x,\ x\in E_1\ ,\end{aligned}$$

we have

$$\begin{aligned}\|(A)^k(t+A)^{-l}x\|_{E_1}&\leqslant\left(1+\left|\frac{s-t}{t}\right|M_1\right)^l\|(A)^k(s+A)^{-l}x\|_{E_1}\\&\leqslant(\max(1, M_1))^l\left(\frac{s}{t}\right)^l\|(A)^k(s+A)^{-l}x\|_{E_1},\quad t<s\ ,\end{aligned}$$

where M_1 is the constant for type (h) corresponding to A_1. Hence we have

$$\|t^{m+1}(A)^{m+1}(t+A)^{-2m-2}x\|_{E_1}\leqslant C_2\left(\frac{s}{t}\right)^{m+1}\|s^{m+1}(A)^{m+1}(s+A)^{-2m-2}x\|_{E_1} \tag{3.3}$$

and

$$\|t(A)^{2m+1}(t+A)^{-2m-2}x\|_{E_1} \leqslant C_2\left(\frac{s}{t}\right)^{2m} \|s(A)^{2m+1}(s+A)^{-2m-2}x\|_{E_1} \tag{3.4}$$

if $t<s$ ($C_2=(\max(1, M_1))^{2m+2}$). Therefore for $x\in(E_0, D((A_0)^m))_{\theta+\sigma/m+\delta/m,p}$ we have, from our hypothesis,

$$\left(\frac{t}{s}\right)^{m+1}\|t^{m+1}(A)^{m+1}(t+A)^{-2m-2}x\|_{E_1} \leqslant C_2' s^\sigma \|s^m(A)^m(s+A)^{-2m}x\|_{E_0} .$$

Multiplying both sides by $s^{m\theta+\delta}$ and integrating their p-th powers from t to the infinity in ds/s, we obtains, by Remark 1.1,

$$t^{m\theta}\|t^{m+1}(A)^{m+1}(t+A)^{-2m-2}x\|_{E_1} \leqslant C_3 t^{-\delta}\|x\|_0 . \tag{3.5}$$

Similarly,

$$t^{m\theta}\|t(A)^{2m+1}(t+A)^{-2m-2}x\|_{E_1} \leqslant C_3' t^{-\delta}\|x\|_0 . \tag{3.6}$$

Let $x=a_k''-a_l''$. Then we have, from Theorems 1.3 and 2.1,

$$\|x\|_1 \leqslant \max\{\|t^{m\theta}(u_k''(t)-u_l''(t))\|_{L_*^p(E_1)}, \|t^{m\theta-m}(A)^m(u_k''(t)-u_l''(t))\|_{L_*^p(E_1)}\} , \ p<\infty .$$

Note that the quantities in the above braces are integrals of those in the left-hand sides of (3.6) and (3.7). For any $\varepsilon>0$, let n be large enough that

$$2c_{m+1}\max(C_3, C_3'')M(p\delta)^{-1/p}n^{-\delta} < \frac{\varepsilon}{4}$$

and

$$2c_{m+1}\max(C_5, C_5'')M(p(m\theta+\sigma))^{-1/p}n^{-m\theta-\sigma} < \frac{\varepsilon}{4}$$

where

$$\|t^{m+1}(A)^{m+1}(t+A)^{-2m-2}x\|_{E_1} \leqslant C_4 t^\sigma \|x\|_{E_0} \leqslant C_5 t^\sigma \|x\|_0$$

and

$$\|t(A)^{2m+1}(t+A)^{-2m-2}x\|_{E_1} \leqslant C_4' t^\sigma \|x\|_{E_0} \leqslant C_5' t^\sigma \|x\|_0 .$$

Choose k and l large enough that $k, l>n$ and

$$4C_2M\left(\frac{1}{k}+\frac{1}{l}\right)((m\theta p)^{-1}\sum_{j=1}^\infty j^{p(m\theta-\sigma)})^{-1/p} < \frac{\varepsilon}{4} .$$

Then

$$\int_0^\infty t^{m\theta p}\|u_k''(t)-u_l''(t)\|_{E_1}^p\frac{dt}{t} = \int_0^{1/n} + \sum_{j=1}^{n-1}\int_{1/(j+1)}^{1/j} + \sum_{j=1}^{n-1}\int_j^{j+1} + \int_n^\infty = I_1+I_2+I_3+I_4 .$$

By the choice of n and (3.5),

$$I_1 \leqslant (c_{m+1})^p \int_0^{1/n} t^{m\theta p} \|t^{m+1}(A)^{m+1}(t+A)^{-2m-2}x\|_{E_1}^p \frac{dt}{t}$$
$$\leqslant (c_{m+1}C_5)^p \|x\|_\delta^p \int_0^{1/n} t^{m\theta p+\sigma p-1}dt \leqslant (2MC_5c_{m+1})^p \frac{n^{-(m\theta p+\sigma p)}}{m\theta p+\sigma p} < \left(\frac{\varepsilon}{4}\right)^p ,$$

$$I_4 \leqslant (c_{m+1})^p \int_n^\infty t^{m\theta p} \|t^{m+1}(A)^{m+1}(t+A)^{-2m-2}x\|_{E_1}^p \frac{dt}{t}$$
$$\leqslant (C_3c_{m+1})^p \|x\|_\delta^p \int_n^\infty t^{-\delta p-1}dt \leqslant (2MC_3c_{m+1})^p \frac{n^{-p\delta}}{p\delta} < \left(\frac{\varepsilon}{4}\right)^p .$$

By (3.3) and (3.1′),

$$I_3 \leqslant (c_{m+1})^p \sum_{j=1}^{n-1} \int_j^{j+1} t^{m\theta p} \|t^{m+1}(A)^{m+1}(t+A)^{-2m-2}x\|_{E_1}^p \frac{dt}{t}$$
$$\leqslant (C_2c_{m+1})^p \sum_{j=1}^{n-1} \int_j^{j+1} t^{m\theta p} \left(\frac{j+1}{j}\right)^p \|(j+1)^{m+1}(A)^{m+1}(j+1+A)^{-2m-2}x\|_{E_1}^p \frac{dt}{t}$$
$$\leqslant (2C_2)^p \sum_{j=1}^{n-1} \int_j^{j+1} t^{m\theta p-1}dt \|u_k''(j+1)-u_l''(j+1)\|_{E_1}^p$$
$$\leqslant \left(2C_2M\left(\frac{1}{k}+\frac{1}{l}\right)\right)^p \frac{1}{m\theta p} \sum_{j=1}^{n-1} ((j+1)^{m\theta p}-j^{m\theta p})(j+1)^{-p\alpha}$$
$$\leqslant \left(4C_2M\left(\frac{1}{k}+\frac{1}{l}\right)\right)^p \frac{1}{m\theta p} \sum_{j=2}^{\infty} j^{m\theta p-p\alpha} < \left(\frac{\varepsilon}{4}\right)^p .$$

Similarly by (3.3) and (3.1′), $I_2 \leqslant \left(\frac{\varepsilon}{4}\right)^p$. Hence

$$\|t^{m\theta}(u_k''(t)-u_l''(t))\|_{L_*^p(E_1)} < \varepsilon .$$

In the same way, from (3.4), (3.2′), (3.2″), (3.6) and the choice of n, k,

$$\|t^{m\theta-m}((A)^m u_k''(t)-(A)^m u_l''(t))\|_{L_*^p(E_1)} < \varepsilon .$$

Therefore, for k and l large enough, we have

$$\|a_k''-a_l''\|_1 < \varepsilon . \qquad \text{q.e.d.}$$

3.2. PROOF OF THEOREM 2.2. We shall reduce the theorem to Theorem 2.1. Since the restriction of $G(t)$ in $E_0 \cap E_1$ forms a semigroup of class (C_0), the domain of the restriction of A in $E_0 \cap E_1$, and a fortiori $D(A_0) \cap D(A_1)$ are dense in $E_0 \cap E_1$. Since the restriction of A in $(E_0, E_1)_{\theta,q}$ has the domain $(D(A_0), D(A_1))_{\theta,q}$ (Theorem 1.10), we immediately see that this restriction generates a semigroup of class (C_0) in $(E_0, E_1)_{\theta,q}$ for $q<\infty$. This semigroup is the restriction of $G(t)$ in $(E_0, E_1)_{\theta,q}$. Since

$$(Kt^{-\sigma}+K')^{\theta}<K^{\theta}t^{-\sigma\theta}+K'^{\theta},\ 0<\theta<1;\ K,t>0,\ K'\geqslant 0\ ,$$

we have, by Theorems 1.1 and 1.2,

$$G(t)\in S(\theta\sigma,\ E_0,\ (E_0,\ E_1)_{\theta,q})\ ,$$

$$G(t)\in S((1-\theta)\sigma,\ (E_0,\ E_1)_{\theta,q},\ E_1)$$

and

$$G(t)\in S((\theta'-\theta)\sigma,\ (E_0,\ E_1)_{\theta',q},\ (E_0,\ E_1)_{\theta,q}),\ 0<\theta<\theta'<1\ .$$

Choose $0=\theta_0<\theta_1\ \cdots\ <\theta_N=1$ such that $\sigma_j=(\theta_j-\theta_{j-1})<1$, $j=1,\cdots,N$ and $\sum_{j=1}^{N}\sigma_j=1$. Setting $E^j=(E_0,E_1)_{\theta_j,q}$, $q<\infty$, $(E_0=E^0,\ E_1=E^N)$, we see

$$G(t)\in S(\sigma_j,\ E^{j-1},\ E^j)\ ,\qquad j=1,\cdots,N\ .$$

Hence we may assume that

$$G(t)\in S(\sigma,\ E_0,\ E_1)\quad\text{with}\quad 0<\sigma<1\ .$$

Now in E_0+E_1,

$$(\lambda+A)^{-1}x=\int_0^{\infty}e^{-\lambda t}\,G(t)x\,dt,\ \lambda>0\ .$$

However, if $x\in E_0$, then the right-hand side converges in E_1 since $0<\sigma<1$, and

$$\|(\lambda+A)^{-1}x\|_{E_1}\leqslant(K\Gamma(\sigma-1)\lambda^{\sigma-1}+K'\lambda^{-1})\|x\|_{E_0}\leqslant C_6\left(\lambda^{\sigma-1}+\frac{K'}{C_6}\,\lambda^{-1}\right)\|x\|_{E_0}$$

$$\leqslant 2C_6\lambda^{\sigma-1}\|x\|_{E_0}\quad\text{for}\quad\lambda\geqslant k_0=\left(\frac{K'}{C_6}\right)^{1/\sigma}\ .$$

Hence,

$$\|(\lambda+k_0+A)^{-1}\|_{E_0\to E_1}\leqslant 2C_6\lambda^{\sigma-1},\ \lambda>0\ .$$

Since the operators k_0+A_i, $i=0,1$, satisfy (h), as is easily seen, we have

$$k_0+A\in(\sigma,\ E_0,\ E_1)\ .$$

Since $D((A_i)^m)=D((k_0+A_i)^m)$, $i=0,1$; $m=1,2,\cdots$, we obtain the theorem by Theorem 2.1. q.e.d.

PROOF OF COROLLARY 2.1. This is an immediate consequence of the above proof of Theorem 2.2.

PROOF OF PROPOSITION 2.3. As is well-known, a holomorphic semigroup $G(t)$ is expressed by the following formula:

$$G(t)x=\frac{1}{2\pi i}\int_\Gamma e^{\lambda t}(\lambda+A)^{-1}x d\lambda,\ x\in E_0+E_1,\ t>0\,,$$

where we take path of integration $\Gamma=\{\lambda(s);\ s\in R\}\subset\rho(-A)$ such that

$$\frac{\pi}{2}<|\arg\lambda(s)|<\pi-\omega \quad \text{for } |s| \text{ large,}$$

and

$$\operatorname{Re}\lambda(s)>0 \quad \text{for } |s| \text{ not large}$$

(see Yosida [12], Chapter 9). From the holomorphicity of the integrand, it follows that

$$G(t)x=\frac{1}{2\pi i}\frac{1}{t}\int_{t\Gamma} e^{\mu}\left(\frac{\mu}{t}+A\right)^{-1}xd\mu=\frac{1}{2\pi i}\frac{1}{t}\int_{\Gamma} e^{\mu}\left(\frac{\mu}{t}+A\right)^{-1}d\mu\,.$$

Let $x\in E_0$. Then this integral converges in E_1, and we obtain

$$\|G(t)x\|_{E_1}\leqslant\frac{1}{2\pi}\left|\int_\Gamma e^{\operatorname{Re}\mu}|\mu|^{\sigma-1}|d\mu|\right|t^{-\sigma}\|x\|_{E_0}\,. \quad \text{q.e.d.}$$

PROOF OF THEOREM 2.2*. Employing Theorem 1.1 and 1.1*, we may assume, as in the proof of Theorem 2.1, that $0<\sigma<1$. We are going to show that the compactness of $G(t)_{0,1}$ for each $t>0$ implies that of $(\lambda+A)^{-1}_{0,1}$ for each $\lambda>0$. Let $a_n\in E_0$ with $\|a_n\|_{E_0}\leqslant 1$ for $n=1,2,\cdots$. We shall show that, for each fixed $\lambda>0$, we can choose a subsequence $\{a'_n\}$ of $\{a_n\}$ such that $\{(\lambda+A)^{-1}a'_n\}$ converges in E_1. We begin by choosing a subsequence $\{a_{n11}\}$ of $\{a_n\}$ satisfying

$$\|G(1)a_{n11}-b_1\|_{E_1}<\frac{1}{n} \quad \text{for some } b_1\in E_1\,.$$

Choose, repeatedly, subsequence $\{a_{n1k}\}$ of $\{a_{n1k-1}\}$ such that

$$\|G(k)a_{n1k}-G(k-1)b_1\|_{E_1}<\frac{1}{n} \quad (k=1,2,\cdots)\,.$$

Here we have used the compactness of $G(t)_{0,1}$, $t>0$. Thus we obtain a subsequence $\{a_n^{(1)};\ a_n^{(1)}=a_{n1n}\}$ of $\{a_n\}$ such that

$$\|G(k)a_n^{(1)}-G(k-1)b_1\|_{E_1}<\frac{1}{n} \quad \text{for} \quad k\leqslant n\,.$$

Again by the compactness of the operator $G(t)_{0,1}$, $t>0$, we can choose a subsequence $\{a_{n21}\}$ of $\{a_n^{(1)}\}$ such that $G\left(\frac{1}{2}\right)a_{n21}\longrightarrow b_2$ in E_1 for some $b_2\in E_1$. As before we obtain a subsequence $\{a_n^{(2)}\}$ of $\{a_{n21}\}$ satisfying

$$\left\| G\left(k+\frac{1}{2}\right)a_n^{(2)} - G(k)b_2 \right\|_{E_1} < \frac{1}{n} \quad \text{for} \quad k \leqslant n .$$

Repeating this procedure, we finally obtain a subsequence $\{a'_n; a'_n = a_n^{(n)}\}$ of $\{a_n\}$ with the property:

$$\left\| G\left(j+\frac{1}{k}\right)a'_n - G(j)b_k \right\|_{E_1} < \frac{1}{n} \quad \text{for} \quad j \leqslant n \quad \text{and} \quad k \leqslant n , \tag{3.7}$$

where $j=0, 1, 2, \cdots$; $k, n=1, 2, \cdots$ and b_k are certain elements in E_1. We shall show that $\{(\lambda+A)^{-1}a'_n\}$ converges in E_1. For $x \in E_0$,

$$(\lambda+A)^{-1}x = \int_0^\infty e^{-\lambda t}\, G(t)x dt \quad \text{in} \quad E_0+E_1 \tag{3.8}$$

for every fixed $\lambda>0$ and this integral converges in E_1. Let ε be any positive number. Take k large enough that

$$2\left(\frac{Kk^{\sigma-1}}{1-\sigma}+\frac{K}{k}\right) < \frac{\varepsilon}{3} \quad \text{and} \quad 2\left(K\int_{k+1/k}^\infty e^{-\lambda t}\, t^{-\sigma} dt + \frac{K'}{\lambda}\, e^{-\lambda(k+1/k)}\right) < \frac{\varepsilon}{3} .$$

Let n and m be large enough that $n, m>k$ and

$$\left(\frac{1}{n}+\frac{1}{m}\right)M_1/\lambda < \frac{\varepsilon}{3} \quad \text{where} \quad \|G(t)\|_{E_1\to E_1} \leqslant M_1 .$$

Let $x=a'_n-a'_m$ in (3.8), then

$$\|(\lambda+A)^{-1}x\|_{E_1} \leqslant \int_0^{1/k} e^{-\lambda t}\, \|G(t)x\|_{E_1} dt + \sum_{j=0}^{k-1} \int_{j+1/k}^{j+1+1/k} e^{-\lambda t} \|G(t)x\|_{E_1} dt +$$

$$+ \int_{k+1/k}^{0} e^{-\lambda t} \|G(t)x\|_{E_1} dt = I_1+I_2+I_3$$

By the choice of k,

$$I_1 \leqslant 2\left(\frac{Kk^{\sigma-1}}{1-\sigma}+\frac{K}{k}\right) < \frac{\varepsilon}{3} ,$$

and

$$I_3 \leqslant 2\left(K\int_{k+1/k}^\infty e^{-\lambda t}\, t^{-\sigma} dt + \frac{K'}{k}\, e^{-\lambda(k+1/k)}\right) < \frac{\varepsilon}{3} .$$

Since $G(t)x = G\left(t-j-\frac{1}{k}\right) G\left(j+\frac{1}{k}\right)x$ for $t>j+\frac{1}{k}$, we have by (3.7)

$$\|G(t)x\|_{E_1} \leqslant M_1 \left\| G\left(j+\frac{1}{k}\right)x \right\|_{E_1} \leqslant M_1\left(\frac{1}{n}+\frac{1}{m}\right) .$$

Thus

$$I_2=\sum_{j=0}^{k-1}\int_{j+1/k}^{j+k+1/k} e^{-\lambda t}\|G(t)x\|_{E_1}dt$$
$$\leqslant M_1\Big(\frac{1}{n}+\frac{1}{m}\Big)\sum_{j=0}^{k-1}\int_{j+1/k}^{j+1+1/k} e^{-\lambda t}\,dt\leqslant\frac{M_1}{\lambda}\Big(\frac{1}{n}+\frac{1}{m}\Big)<\frac{\varepsilon}{3}\,.$$

Hence

$$\|(\lambda+A)^{-1}(a'_n-a'_m)\|_{E_1}<\varepsilon\,,$$

that is, $(\lambda+A)_{0,1}^{-1}$ is compact for every fixed $\lambda>0$. q.e.d.

3.3. PROOF OF THEOREM 2.3. We remark that

$$\bigcap_{j=1}^{N}(E_0, D(A^j{}_0))_{\varphi,p}\subset\bigcap_{j=1}^{N}(E_{j1}, D(A^j_{E_{j1}}))_{\varphi-\sigma_j,p} \tag{3.9}$$

by Theorem 2.1 where $\varphi=\theta+\sum_{j=1}^{N}\sigma_j$. Let $a\in\bigcap_{j=1}^{N}(E_0, D(A^j{}_0))_{\theta,p}$ and set

$$u(t)=\sum_{j=1}^{N}t^N A^j(A^j+t)^{-1}\prod_{i=1}^{N}(A^i+t)^{-1}a\,.$$

Then

$$\|u(t)\|_{E_1}\leqslant\sum_{j=1}^{N}\|t^N\prod_{i=1}^{N}(A^i+t)^{-1}A^j(A^j+t)^{-1}a\|_{E_1}\leqslant C_7t^{\sum_{j=1}^{N}\sigma_j}\sum_{j=1}^{N}\|A^j(A^j+t)a\|_{E_0}\,,$$

and, for $k=1,\cdots,N$,

$$\|A^k u(t)\|_{E_1}\leqslant\sum_{i=1}^{N}\|t^N A^j(A^j+t)\prod_{i\neq k}(A^i+t)^{-1}A^k(A^k+t)^{-1}a\|_{E_1}$$
$$\leqslant C_8t^{1+\sum_{j\neq k}\sigma_j}\|A^k(A^k+t)^{-1}a\|_{E_{k1}}\,.$$

By our assumption and (3.9), we have

$$t^\theta u(t)\in L^p_*(E_1)\quad\text{and}\quad t^{\theta-1}A^k u(t)\in L^p_*(E_1)$$

for $k=1,\cdots,N$. Hence

$$b=\int_0^\infty u(t)\frac{dt}{t}\in\bigcap_{j=1}^{N}(E_1, D(A^j{}_1))_{\theta,p}$$

and from Remark 1.3,

$$\|b\|_1\leqslant C_9\|a\|_0\,.$$

Here we mean by $\|\ \|_0$ and $\|\ \|_1$ the norms of $\bigcap_{j=1}^{N}(E_0, D(A^j{}_0))_{\varphi,p}$ and $\bigcap_{j=1}^{N}(E_1, D(A^j{}_1))_{\theta,p}$ respectively. Since, by Remark 1.3,

$$a=\int_0^\infty u(t)\frac{dt}{t} \quad \text{in } E_0 ,$$

we have $a=b$ in E_0+E_1. Therefore $a=b\in \bigcap_{j=1}^{N} (E_1, D(A^j{}_1))_{\theta,p}$. q.e.d.

3.4. PROOF OF THEOREM 2.4. Since

$$(3.10) \qquad (\lambda+(A^i)^{m_i})^{-1}=\prod_{n=0}^{m_i-1}(A^i-\lambda^{1/m_i}\omega_{m_i}^n)^{-1} , \qquad i=1, \cdots, N ,$$

for $\lambda>0$ on each space E_{kj}, $k=1, \cdots, N$; $j=0, \cdots, N$, and on $E=\sum E_{kj}$ with $\omega_m=e^{2\pi i/m}$, and since A^i are of type (h_{m_i}) on each of these spaces, we see that $(A^i)^{m_i}$ are of type (h) in each space. Next, we may assume that, for $\rho>0$,

$$\|(\rho+A^i)^{-1}_{k\mu(i-k)-1,k\mu(i-k)}\|_{E_{k\mu(i-k)-1}\to E_{k\mu(i-k)}}\leqslant L_{ik}\rho^{\sigma_i-1} .$$

Then from (3.10) in E and from that A^i are of type (h_{m_j}) in $E_{k\mu(i-k)-1}$, it follows that, for $x\in E_{k\mu(i-k)-1}$,

$$\begin{aligned}\|(\lambda+(A^i)^{m_i})^{-1}x\|_{E_{k\mu(i-k)}} &\leqslant L_{ik}\lambda^{\sigma_i/m_i-1/m_i}\prod_{n=1}^{m_i-1}\|(A^i+\lambda^{1/m_i}\omega_{m_i}^n)^{-1}x\|_{E_{k\mu(i-k)-1}}\\ &\leqslant L_{ik}\lambda^{\sigma_i/m_i-1/m_i}(M^{i,k}\lambda^{-1/m_i})^{m_i-1}\|x\|_{E_{k\mu(i-k)-1}}\\ &\leqslant C_{ik}\lambda^{\sigma_i/m_i-1}\|x\|_{E_{k\mu(i-k)-1}} .\end{aligned}$$

In the above calculation, $M^{i,k}$ are the constants for type (h_{m_i}) for $A^i_{k\mu(i-k)-1}$. Hence we see that

$$\{(A^1)^{m_1}, \cdots, (A^N)^{m_N}\}\in\left(\left\{\frac{\sigma_1}{m_1}, \cdots, \frac{\sigma_N}{m_N}\right\}, E_0, E_1\right) ,$$

and the theorem follows immediately from Theorem 2.3. q.e.d.

4. Applications.

4.0. Finally we give some applications of our theorems in § 2.5. We begin by defining the function spaces of Besov [2], Nikol'skii [8] and others, which we are going to treat. We note our notations are found in §§ 1 and 2.

4.1. In the space $L^p(R)$, $1\leqslant p<\infty$, consider the operator $A=\dfrac{d}{dx}$ with the domain $D(A)=\left\{f\in L^p(R) : \dfrac{d}{dx}f \text{ in the sense of distribution belongs } L^p(R)\right\}$. Then as we have seen in Example 2.1, A is a closed linear operator in $L^p(R)$ of type (h) (see § 1.2), and $-A$ is an infinitesimal generator of class (C_0), the translation semigroup in this space (see Remark 2.2).

DEFINITION 4.1. For $1\leqq r\leqq\infty$ and for $s>0$,

$$B_p^{s,r}(R)=(L^p(R),\ D((A)^m))_{s/m,r} \tag{4.1}$$

where m is any fixed integer $>s$. As seen from Theorem 1.5, $B_p^{s,r}(R)$ does not depend on the choice of m, and from Theorems 1.6 and 1.7 and Remark 2.2, we have another characterization of $B_p^{s,r}(R)$: since $D((A)^m)=W_p^m(R)$, $m=1, 2, \cdots$, by definition,

(i) in case $s>0$ is not an integer, $B_p^{s,r}(R)\ni f$ if and only if $f\in W_p^{\langle s\rangle}(R)$ and

$$\left\{\int_0^\infty t^{-(s-\langle s\rangle)r-1}\left[\int_{-\infty}^\infty\left|\left(\frac{d}{dx}\right)^{\langle s\rangle}f(x)-\left(\frac{d}{dx}\right)^{\langle s\rangle}f(x+t)\right|^p dx\right]^{r/p}dt\right\}^{1/r}<\infty \tag{4.2}$$

where $\langle s\rangle$ is the greatest integer $<s$;

(ii) in case $s>0$ is an integer, $B_p^{s,r}(R)\ni f$ if and only if $f\in W_p^{\langle s\rangle}(R)$ and

$$\left\{\int_0^\infty t^{-r/2-1}\left[\int_{-\infty}^\infty\left|\left(\frac{d}{dx}\right)^{\langle s\rangle}f(x)-2\left(\frac{d}{dx}\right)^{\langle s\rangle}f(x+t)+\left(\frac{d}{dx}\right)^{\langle s\rangle}f(x+2t)\right|^p dx\right]^{r/p}dt\right\}^{1/r}<\infty \tag{4.3}$$

where $\langle s\rangle=s-1$. If $r=\infty$, then the integral norms in (4.2) and (4.3) should be interpreted as *ess. sup.* in the variable t.

DEFINITION 4.1′. For $1\leqq r\leqq\infty$ and for $s>0$, we define

$$B_\infty^{s,r}(R)=(C(R),\ C^m(R))_{s/m,r} \tag{4.1′}$$

where m is any fixed integer $>s$. Consider the operator $A=\dfrac{d}{dx}$ in the space $C(R)$, with the domain of definition $D(A)=\left\{f\in C(R);\ f \text{ is differentiable and } \dfrac{d}{dx}f \text{ belongs to } C(R)\right\}$. As seen in Example 2.1′ and Remark 2.2, $-A$ is the infinitesimal generator of the translation semigroup in $C(R)$, and since $C^m(R)=D((A)^m)$, $B_\infty^{s,r}(R)$ is expressed more concretely, by Theorems 1.5, 1.6 and 1.7, as follows:

(i) in case $s>0$ is not an integer, $B_\infty^{s,r}(R)\ni f$ if and only if $f\in C^{\langle s\rangle}(R)$ and

$$\left\{\int_0^\infty t^{-(s-\langle s\rangle)r-1}\left(\sup_{x\in R}\left|\left(\frac{d}{dx}\right)^{\langle s\rangle}f(x)-\left(\frac{d}{dx}\right)^{\langle s\rangle}f(x+t)\right|\right)^r dt\right\}^{1/r}<\infty \tag{4.2′}$$

where $\langle s\rangle$ is the greatest integer $<s$;

(ii) in case $s>0$ is an integer, $B_\infty^{s,r}(R)\ni f$ if and only if $f\in C^{\langle s\rangle}(R)$ and

$$(4.3') \qquad \left\{\int_0^\infty t^{-(r/2)-1}\left(\sup_{x\in R}\left|\left(\frac{d}{dx}\right)^{\langle s\rangle} f(x)-2\left(\frac{d}{dx}\right)^{\langle s\rangle} f(x+t)+\right.\right.\right.$$
$$\left.\left.\left.+\left(\frac{d}{dx}\right)^{\langle s\rangle} f(x+2t)\right|\right)^r dt\right\}^{1/r}<\infty$$

where $\langle s\rangle=s-1$. If $r=\infty$, then the integral norms in (4.2′) and (4.3′) should be interpreted as *ess. sup.* in the variables.

THEOREM 4.1. *Let* $1\leq p<q\leq\infty$. *Then we have, for* $s>\frac{1}{p}-\frac{1}{q}$, $\left(\frac{1}{\infty}=0\right)$,

$$B_p^{s,r}(R)\subset B_q^{t,r}(R), \quad 1\leq r\leq\infty, \quad t=s-\frac{1}{p}+\frac{1}{q},$$

with the continuous injection.

PROOF. This is an immediate consequence of Example 2.1, and of our Theorem 2.1.

REMARK 4.1. From our argument in Example 2.1, we can easily extend the previous theorem to the functions with values in a Banach space.

Consider the operator $A=\frac{d}{dx}$ in the space $L^p(T)$, $1\leq p<\infty$, with the domain $D(A)=\left\{f\in L^p;\ \frac{d}{dx}f \text{ in the sense of distribution belongs to } L^p(T)\right\}$. Then as we have seen in Example 2.1*, A is a closed linear operator of type (h) in $L^p(T)$.

DEFINITION 4.1*. For $1\leq p<\infty$, $1\leq r\leq\infty$, and for $s>0$,

$$B_p^{s,r}(T)=(L^p(T),\ D((A)^m)))_{s/m,r}$$

where m is any fixed integer$>s$.

As seen from Example 2.1* and from the argument in Remark 2.2, $-A$ generates the translation semigroup in $L^p(T)$. Thus, as in the case of $B_p^{s,r}(R)$, we can give a more concrete characterization of the space $B_p^{s,r}(R)$, but we do not repeat the procedure. Also we can give the following definition, considering the operator $A=\frac{d}{dx}$ in $C(T)$ (cf. Example 2.1*, and Definition 4.1′).

DEFINITION 4.1*′. For $1\leq r<\infty$ and $s>0$,

$$B_\infty^{s,r}(T)=(C(T),\ C^m(T))_{s/m,r}$$

where m is any fixed integer s, and $C^m(T)=D((A)^m)$.

From our Theorems 2.1 and 2.1*, and from our Example 2.1*, it follows immediately.

THEOREM 4.1*. *Let* $1\leq p<q\leq\infty$. *Then we have, for* $s>\frac{1}{p}-\frac{1}{q}$,

$$B_p^{s+\delta,r}(T)\subset B_q^{t,r}(T),\quad 1\leqslant r\leqslant\infty,\quad t=s-\frac{1}{p}+\frac{1}{q},\ \delta\geqslant 0$$

with the continuous injection if $\delta=0$, *and the compact injection if* $\delta>0$ *and if* $r<\infty$.

4.2. In the space $L^p(R^n)$, $1\leqslant p<\infty$, consider the operator $A=\Delta=\left(\frac{\partial}{\partial x^1}\right)^2+\cdots+\left(\frac{\partial}{\partial x^n}\right)^2$ with the domain $D(A)=\{f\in L^p(R^n);\ \Delta f$ in the sense of distribution belongs to $L^p(R^n)\}$. Then as seen in Example 2.2, $-A$ generates a semigroup $G(t)$ of class (C_0):

$$G(t)f(x)=\begin{cases} f(x) & \text{if } t=0\,, \\ \left(\frac{1}{\sqrt{2\pi t}}\right)^n\int_{R^n}e^{-(|x-y|^2)/2t}f(y)dy & \text{if } t>0 \end{cases}\tag{4.4}$$

for $f\in L^p(R^n)$.

DEFINITION 4.2. For $1\leqslant p<\infty$, $1\leqslant r\leqslant\infty$, and for $s>0$, we define

$$\Lambda(s,p,r;R^n)=(L^p(R^n),D((A)^m))_{s/2m,r}$$

where m is any fixed integer $>s/2$.

Consider the operator $A=\Delta=\left(\frac{\partial}{\partial x^1}\right)^2+\cdots+\left(\frac{\partial}{\partial x^n}\right)^2$ in the space $C(R^n)$. Then $-A$ generates a semigroup $G(t)$ of class (C_0) in $C(R^n)$, and $G(t)f(x)$ is given by the formula (4.4) $(f\in C(R^n))$.

DEFINITION 4.2.′ For $1\leqslant r\leqslant\infty$, and for $s>0$, we define

$$\Lambda(s,\infty,r;R^n)=(C(R^n),D((A)^m))_{s/2m,r}$$

where m is any integer $>s/2$.

$\Lambda(s,p,r;R^n)$ is Taibleson's Lipschitz space [10]. A more concrete characterization can be obtained by Theorem 1.7. Taibleson obtained other concrete characterizations, employing the fact that $G(t)$ is a holomorphic semigroup, or considering the square root of A. Most of his characterizations are thus explained by the theory of mean space (for this, consult Komatsu [6]). For $1<p<\infty$, $D((A)^m)=W_p^m(R^n)$ by definition, and $\Lambda(s,p,r;R^n)$ is usualy written as $B_p^{s,r}(R^n)$. From our Theorem 2.2 and considerations in Example 2.2, we obtain immediately

THEOREM 4.2. *For* $1\leqslant p<q\leqslant\infty$, $1\leqslant r\leqslant\infty$, *and* $s>\frac{n}{p}-\frac{n}{q}>0$, *we have*

$$\Lambda(s,p,r;R^n)\subset\Lambda\left(s-\frac{n}{p}+\frac{n}{q},q,r;R^n\right)$$

with the continuous injection.

REMARK 4.3. The deduction of the above theorem is essentially in the same spirit as the proof of Lemma 11 of Taibleson [10], but he used indispensably the fact that the above semigroup $G(t)$ is a bounded holomorphic semigroup in each space (see our Introduction). He went farther however. In fact, he showed that the restriction mapping $f \longrightarrow f|_{R^m}$ $(m<n)$ from $\Lambda(s, p, r; R^n)$ into $\lambda\left(s-\frac{n}{p}+\frac{m}{q}, q, r; R^m\right)$ is well-defined and is continuous, if $s>\frac{n}{p}-\frac{m}{q}>0$, $1\leqslant p\leqslant q\leqslant\infty$, $1\leqslant r\leqslant\infty$. However, if $1<p\leqslant q\leqslant\infty$, we can show the above relation by the method of the mean space (or rather the trace method of Lions. See Lions-Peetre [7]) as the combination of our Theorem 4.1 and Grisvard's Theorem 5.1 [3].

REMARK 4.4. From our considerations in Example 2.2, we can easily extend the previous theorem to functions with values in a Banach space.

Consider the operator $A=\Delta=\left(\frac{\partial}{\partial x^1}\right)^2+\cdots+\left(\frac{\partial}{\partial x^n}\right)^2$ in the space $L^p(T^n)$ with the domain $D(A)=\{f\in L^p(T^n);\ \Delta f$ in the sense of distribution belongs to $L^p(T^n)\}$. Then as seen in Example 2.2*, $-A$ generates a semigroup $G(t)$ of class (C_0) in $L^p(T^n)$:

$$G(t)f(x)=\begin{cases} f(x) & \text{if } t=0\,, \\ \int_{T^n} g_t(x-y)f(y)dy & \text{if } t>0 \end{cases} \tag{4.5}$$

for $f\in L^p(T^n)$. Here

$$g_t=\left(\frac{1}{\sqrt{2\pi t}}\right)^n \sum_{k\in Z^n} e^{-(|x-k|^2)/2t},\ t>0\,. \tag{4.5}$$

DEFINITION 4.2*. For $1\leqslant p<\infty$, $1\leqslant r\leqslant\infty$, and for $s>0$, we define

$$\Lambda(s, p, r;\ T^n)=(L^p(T^n),\ D((A)^m))_{s/2m, r}$$

where m is any integer $>s/2$.

Similarly, considering the operator $-A=-\left\{\left(\frac{\partial}{\partial x^1}\right)^2+\cdots+\left(\frac{\partial}{\partial x^n}\right)^2\right\}$ in the space $C(T^n)$, which generates a semigroup $G(t)$ of class (C_0) given by the formula (4.5) and (4.6) (for $f\in C(T^n)$), we have the following definition (see Example 2.2*).

DEFINITION 4.2*′. For $1\leqslant r\leqslant\infty$ and for $s>0$, we define

$$\Lambda(s, \infty, r;\ T^n)=(C(T^n),\ D((A)^m))_{s/2m, r}$$

where m is any fixed integer $>s/2$.

For more concrete characterizations of the above spaces, consult Taibleson [10] (as before a characterization can be obtained by Theorem 1.7). From our Theorems 2.2 and 2.2*, and from our Example 2.2*, we immediately obtain

THEOREM 4.2*. *For* $1\leq p<q\leq\infty$, $1\leq r\leq\infty$, *and for* $s>\frac{n}{p}-\frac{n}{q}>0$ *we have*

$$\Lambda(s+\delta, p, r;\ T^n)\subset\Lambda\left(s-\frac{n}{p}+\frac{n}{q}, q, r;\ T^n\right), \qquad \delta\geq 0\,,$$

with the continuous injection if $\delta=0$, *and with the compact injection if* $\delta>0$ *and if* $r<\infty$.

4.3. In the space $L^p(R^n)$, $1\leq p\leq\infty$, consider the operators $A^j=\frac{\partial}{\partial x^j}$, $j=1,\cdots,n$, with the domain $D(A^j)=\left\{f\in L^p(R^n);\ \frac{\partial}{\partial x^j}f \text{ in the sense of distribution belongs to } L^p(R^n)\right\}$. Each A^j is a closed linear operator of type (h) in $L^p(R^n)$ as is seen in Example 4.3.

DEFINITION 4.3. For $1\leq p<\infty$, $1\leq r\leq\infty$, $\vec{s}=(s_1,\cdots,s_n)$, $s_j>0$, $j=1,\cdots,n$, we define

$$B_p^{\vec{s},r}(R^n)=\bigcap_{j=1}^{n}(L^p(R^n),\ D((A^j)^{m_j}))_{s_j/m_j,r}$$

where each m_j is any fixed number$>s_j$.

From our Theorems 1.6 and 1.7 and Remark 2.3, we can give a more concrete characterization of the above introduced space. Denoting by $\langle s_j\rangle$ the greatest integer$<s_j$ (so $\langle s_j\rangle=s_j-1$ if s_j is an integer), we have

(i) if none of s_j, $j=1,\cdots,n$ are integers, then $B_p^{\vec{s},r}(R^n)\ni f$ if and only if $\left(\frac{\partial}{\partial x^j}\right)^k f\in L^p(R^n)$, $k=0,\cdots,\langle s_j\rangle$, and

$$\left\{\int_0^\infty t^{-(s_j-\langle s_j\rangle)r-1}\left[\int_{R^n}\left|\left(\frac{\partial}{\partial x^j}\right)^{\langle s_j\rangle}f(x)-\left(\frac{\partial}{\partial x^j}\right)^{s_j}f(x+te_j)\right|^p dx\right]^{r/p}dt\right\}^{1/r}<\infty \tag{4.7}$$

for all $j=1,\cdots,n$;

(ii) if for some j, s_j is an integer, then for this j, (4.7) should be replaced by

$$\left\{\int_0^\infty t^{-(r/2)-1}\left[\int_{R^n}\left|\left(\frac{\partial}{\partial x^j}\right)^{\langle s_j\rangle}f(x)-2\left(\frac{\partial}{\partial x^j}\right)^{\langle s_j\rangle}f(x+te_j)+\right.\right.\right.$$
$$\left.\left.\left.+\left(\frac{\partial}{\partial x^j}\right)^{s_j}f(x+2te_j)\right|^p dx\right]^{r/p}dt\right\}^{1/r}<\infty\ ; \tag{4.8}$$

here $e_j=(0,\cdots,0,\overset{j}{\check{1}},0,\cdots,0)$, and if $r=\infty$, then the integral norms in t of (4.7) and (4.8) should be interpreted as *ess. sup.* in t. Especially if

$$\vec{s}=(s_1, \cdots, s_n)=(\theta m_1, \cdots, \theta m_n)=\theta(m_1, \cdots, m_n),\ 0<\theta<1\ , \tag{4.9}$$

with positive integers m_j, $j=1, \cdots, n$, then we have, from Theorem 1.9 and our considerations in Example 2.4, the following equality:

$$B_p^{\vec{s},r}(R^n)=(L^p(R^n), \bigcap_{j=1}^{n} D((A^j)^{m_j}))_{\theta,r} \tag{4.10}$$

and moreover from our Theorem 2.4, it follows immediately.

THEOREM 4.3. *Let* $1\leq p<q<\infty$, $1\leq r\leq\infty$, *and let* $\vec{s}=(s_1, \cdots, s_n)$ *satisfy* (4.9). *Then if*

$$\kappa=\left(\frac{1}{p}-\frac{1}{q}\right)\sum_{k=1}^{n} s_k^{-1}<1\ ,$$

we have

$$B_p^{\vec{s},r}(R^n)\subset B_p^{\vec{t},r}(R^n),\quad \vec{t}=(t_1, \cdots, t_n),\quad t_j=s_j-\kappa s_j,\quad j=1, \cdots, n$$

with the continuous injection.

REMARK 4.5. This is a partial result of the imbedding theorem of Nikol'skii-Besov type ([2], [8]). In fact, their results hold without the supplementary assumption (4.9). We note, however, if all $s_j<2$, then we can eliminate the assumption (4.9) by applying the theory of fractional powers of closed operators and our Theorem 2.3. For this, consult Komatsu [5], [6]. We can also eliminate the assumption (4.9) if $1<p\leq 2\leq q<\infty$, by the theory of fractional powers and Mihlin's theorem generalized to functions with values in a Hilbert space (J. Schwartz [9]). In this case, we must pay attention to the choice of the space E_{jk} in Definition 2.4 for the applicability of the Mihlin-Schwartz theorem (the actual choice of E_{jk} in Example 2.4 should be changed). The proofs of these remarks are not difficult, but quite lengthy, and since they do not give the complete answer, we do not enter in their proofs here.

University of Tokyo

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