Generation of local integral unitary groups over an unramified dyadic local field

Dedicated to Prof. Shôkichi Iyanaga on his 60th birthday

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In a paper of O. T. O'Meara and B. Pollak [3] they extended the theory of generation by symmetries of orthogonal groups over a field to orthogonal groups over a ring of integers, especially in the case of a dyadic local field. The purpose of this paper is to consider a similar problem for unitary groups.

In case of orthogonal groups the following results are known ([3]). Let L be a lattice on a regular quadratic space over a local field F. $O(L)$ denotes the group of units of L. Let $S(L)$ be the subgroup of $O(L)$ generated by symmetries in $O(L)$. If F is not a dyadic field then $O(L)=S(L)$. If F is an unramified dyadic field which is different from Q_2 then $O(L)=S(L)$. If $F=Q_2$ then there are cases where $O(L)\neq S(L)$. In these cases it is necessary to add additional generators $E_w^\cdot$ in order to generate $O(L)$. Moreover complete list of (nine) exceptional lattices (in modular cases) are given in [3]. Similar problems are studied in [1], §7 for unitary groups, especially in non-dyadic local cases.

In this paper we shall consider following cases. Let E be a ramified extension of degree two over an unramified dyadic local field F. Let L be a lattice on a non-degenerate hermitian space V with respect to the non-trivial automorphism of the extension E/F. $U(L)$ denotes the group of units of L. We shall prove the following theorem.

THEOREM. *Let $S(L)$ be the subgroup generated by symmetries in $U(L)$. If $F\neq Q_2$ then we have $U(L)=S(L)$. If $F=Q_2$ we have two 4-dimensional modular lattices for which $U(L)\neq S(L)$. We have to add the generators $T_{w,\mu}$ of [1] in order to generate $U(L)$.*

As an application we calculate in §7 the number of proper genera in a genus of a lattice in our cases.

§ 1. Preliminaries.

1.1. We shall explain here necessary notations and terminologies which are used in the following sections. For the complete definitions we refer to the paper [2].

Let F be an unramified local field over Q_2. Let E be a ramified quadratic extension of F. The operation of the nontrivial automorphism of E over F is denoted by $-$. Let $\mathfrak{O}, \mathfrak{o}$; $U, \mathfrak{u}$; p, π denote respectively the ring of integers; the group of units; a prime element in E and F. We have two cases.

a) $E=F(\sqrt{\pi})$, b) $E=F(\sqrt{1+\pi})$.

Case a) is called 'ramified prime' and case b) is called 'ramified unit'. We abbreviate them respectively to R. P. case and R. U. case.

1.2. Let V be a finite dimensional hermitian vector space over E with respect to the involution $-$. Let (x, y) with x, y in V be the associated inner product. $U(V)$ denotes the unitary group of V. Let $L=\sum \mathfrak{O}x_i$ be a $\mathfrak{O}$-lattice on V. By this base we associate the matrix $((x_i, x_j))$ to L. For example to hyperbolic plane L we associate $H(i)=\begin{pmatrix} 0 & p^i \\ \bar{p}^i & 0 \end{pmatrix}$: We write $L\cong H(i)$ in $\mathfrak{O}u+\mathfrak{O}v$. $\mathfrak{s}L, \mathfrak{n}L, dL$ denote respectively scale, norm, discriminant of L. Scaling by $\alpha\in F$ means the change of inner product from (x, y) to $\alpha(x, y)$. A vector x in L which is not contained in pL is called a maximal vector in L. By a p^i-modular lattice L we mean $(x, L)=p^i\mathfrak{O}$ for every maximal vector x in L. By suitable scaling it suffices to consider the modular lattice when $i=0$ or 1.

For a vector s in V and an element σ in E with $\sigma+\bar{\sigma}=(s, s)$, put $Sx=x-(x, s)\sigma^{-1}s$ for $x\in V$. Then we have $S\in U(V)$. S is called symmetry and written by (s, σ).

$U(L)$ denotes the unit group of L and $S(L)$ denotes its subgroup generated by symmetries in $U(L)$. Sp means Spur$_{E/F}$. ω is used always as a skew-symmetric element in E. $\oplus$ means orthogonal sum. Let $L=K\oplus M$ be a splitting of L. We denote by $U(L, K)$ the group of elements of $U(L)$ which are identity on K. Then we have $U(L, K)\cong U(M)$.

§ 2. 2-dimensional modular lattices.

By the results in [2] we can write 2-dimensional modular lattices L in canonical forms. This expression is not unique but we collect the results in [2] in tables for our later use.

From [2] p. 454 we have Table I

	①	②	③	④
	$i=0$, R. P.	$i=1$, R. P.	$i=0$, R. U.	$i=1$, R. U.
$\mathfrak{n}H(i)$	$2\mathfrak{O}$	$4\mathfrak{O}$	$2\mathfrak{O}$	$2\mathfrak{O}$

From [2] Props. 9.1, 9.2 and 10.4 we have Table II

		I $\mathfrak{n}L=\mathfrak{n}H(i)$		II $\mathfrak{n}L=2^{-1}\mathfrak{n}H(i)$	
		I_1 isotropic	I_2 anisotropic	II_1 isotropic	II_2 anisotropic
R. P.	①	$H(0)$	$\begin{pmatrix} 2 & 1 \\ 1 & 2\varepsilon \end{pmatrix}$	$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 \\ 1 & 4\varepsilon \end{pmatrix}$
	②	$H(1)$		$\begin{pmatrix} 2 & p \\ \bar{p} & 0 \end{pmatrix}$	$\begin{pmatrix} 2 & p \\ \bar{p} & 4\varepsilon \end{pmatrix}$
R. U.	③	$H(0)$		$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 \\ 1 & 2\varepsilon \end{pmatrix}$
	④	$H(1)$	$\begin{pmatrix} 2 & p \\ \bar{p} & 2\varepsilon \end{pmatrix}$		

REMARK.

1. ε is a unit except the case ③II_2.
2. In the case $\mathfrak{n}L=\mathfrak{O}$ (i.e. ①II, ③II cases) L splits.
3. In ①I_2 and ④I_2 cases expressions are modulo scaling.
4. In the cases ①I_2 and ④I_2, $(x, x)\in 2\mathfrak{u}$ for every maximal vector x in L. See [2] 9.2.

§ 3. Generation theorem, low dimensional cases.

3.0. For given two vectors x, x' in L which satisfy $(x, x)=(x', x')$, the symmetry which transposes x to x' is given by $St=t-(t, x-x')(x, x-x')^{-1}(x-x')$ for $t\in V$ if $(x, x-x')\neq 0$. S is in $U(L)$ if and only if $(L, x-x')(x, x-x')^{-1}(x-x')\subseteq L$.

3.1. If $\dim V=1$ then every element of $U(V)$ is symmetry, so we have $U(L)=S(L)$.

3.2. If $L\cong H(i)$ in $\mathfrak{O}u+\mathfrak{O}v$ then $U(L)=S(L)$.

PROOF. Take $\varphi\in U(L)$ and write $\varphi u=\alpha u+\beta v$, $\varphi v=\gamma u+\delta v$. Assume either β or γ to be unit, for example $\beta\in U$. By 3.0 there exists S in $S(L)$ such that $Su=\varphi u$. Put $\varphi'=S^{-1}\varphi$ and write $\varphi' u=u$, $\varphi' v=\gamma u+v$. Put $\omega=\bar{p}^i\gamma^{-1}$. Then ω is a skew-symmetric element in E and we have $\varphi'=(u, \omega)\in S(L)$. Next we consider φ for which $p\mid\beta$, $p\mid\gamma$. All such $\varphi\in U(L)$ make a subgroup H of $U(L)$ and $S(L)\supseteq U(L)-H$.

So we are through if we can prove $H\neq U(L)$. Take for example, φ_1 such that $\varphi_1 u=v$, $\varphi_1 v=\bar{p}^i p^i u$ then $\varphi_1\notin H$ and $\varphi\in U(L)$.

3.3. Let $L=\mathfrak{O}x\oplus M$ be a decomposition of L, where $(x, x)=1$ and $\mathfrak{s}M\subseteq\mathfrak{O}$. For $\varphi\in U(L)$ put $\varphi x=\alpha x+t$ with t in M. If $\alpha-1\in U$ then by 3.0 there exists

$S\in S(L)$ such that $\varphi x=Sx$. If $p\mid\alpha-1$ we replace φ by $\left(x, \frac{1+\omega}{2}\right)\varphi$ and by appropriate choice of a skew symmetric element ω in E we can assume in R. P. case $|\alpha-1|\geqq|2|$ and in R. U. case $|\alpha-1|\geqq|p|$. So if t is not maximal then there exists S in $S(L)$ such that $\varphi x=Sx$.

3.4. Suppose $L=\mathfrak{O}x\oplus\mathfrak{O}y$ in $\begin{pmatrix}1 & 0\\ 0 & \varepsilon\end{pmatrix}$. Then we have $U(L)=S(L)$.

PROOF. Take $\varphi\in U(L)$ and write $\varphi x=\alpha x+\beta y$. If $p^i\mid\alpha-1$ with $i=1,2$ then $p^i\mid\beta$. We can apply 3.3 and the problem is reduced to 1-dimensional case.

3.5. In the cases ①I_2, ②II_2, ④I_2 in the Table II, write $L=\mathfrak{O}x+\mathfrak{O}y$. Then we have $U(L)=S(L)$.

PROOF. Let $\varphi\in U(L)$ and write $\varphi x=\alpha x+\beta y$. If $\beta\in U$ then by 3.0 the problem is reduced to 1-dimensional case. If $p\mid\beta$ than $\alpha\in U$. Put $S'=(y,\sigma)$ with $\sigma=\varepsilon$ in ①I_2, $\sigma=2\eta+p$ with $(y,y)=4\eta$, $\eta\in\mathfrak{o}$ in ②II, and $\sigma=\varepsilon+\varepsilon\sqrt{1+\pi}$ in ④I_2 respectively. Then the coefficient of y in $S'\varphi x$ is a unit and the problem is reduced to the above case.

3.6. Let L be a 3-dimensional unimodular lattice. Then $U(L)=S(L)$.

PROOF. By [2] Prop. 10.3 we may assume that L has a splitting $L=\mathfrak{O}x+H(0)$ with $(x,x)=1$. By 3.0 and 3.3 we are through if we can prove that (i) $\varphi\in U(L)$, (ii) $(x, x-\varphi x)\in p\mathfrak{O}$ and (iii) $\varphi x-x$ is maximal in L, can not happen simultaneously. Write $\varphi x=\alpha x+t$ with $t\in H(0)$ maximal. Then there exists an isotropic vector $u\in H(0)$ such that $(u,t)=1$. Put $w=u-\varphi x$. We have $(w,\varphi x)=0$. Thus w is in $\varphi H(0)$ which is improper. Since $(w,w)=1$, we have a contradiction.

§4. $U(L)=X(L)$.

4.1. Let L be a lattice of scale $p^i\mathfrak{O}$ with $i=0,1$ and $\dim L\geqq 3$. We say u, v to be hyperbolic pair (in L) if $\mathfrak{O}u+\mathfrak{O}v\cong H(i)$. For splitting $L=H(i)\oplus M$, $\mu\in\mathfrak{O}$ and $w\in M$ such that

$$\mathrm{Sp}\,(\mu(u,v))=-(w,w) \quad\cdots\cdots(1)$$

we put as in [1] p. 102.

$$T_{w,\mu}x=x+(x,u)(v,u)^{-1}w+(\mu(x,u)(v,u)^{-1}-(x,w)(u,v)^{-1})u\,.$$

Then, $T_{w,\mu}\in U(L)$ and $\det T_{w,\mu}=1$. We have

$$T_{w,\mu}u=u,\quad T_{w,\mu}v=\mu u+v+w,\quad \text{and}$$

$$T_{w,\mu}w=-(w,w)(u,v)^{-1}u+w\ .$$

$X(L)$ is the subgroup of $U(L)$ generated by $T_{w,\mu}$ and $S(L)$ where $T_{w,\mu}$ is taken for all p^i-modular hyperbolic splitting of L and for all pairs w,μ which satisfy (1). When we fix $H(i)$ we write $X(L)$ by $X_h(L)$.

4.2. Let u,v,j be isotropic vectors in L such that $(u,v)=(j,v)=p^i$ and $L=(\mathfrak{O}u+\mathfrak{O}v)\oplus M$. Then there exists $T_{w,\mu}$ in $X(L)$ such that $T_{w,\mu}u=j$.

PROOF. We can write $j=u+\beta v+w$, $w\in M$, and we have $\mathrm{Sp}(\beta(v,u))=-(w,w)$. Then 4.2 follows from 4.1. We remark $T_{w,\mu}v=v$.

4.3. Let $u,v;\ j,l$ be hyperbolic pairs in L. Then there exists $\varphi\in X(L)$ such that $\varphi u=\varepsilon j$, with $\varepsilon\in U$.

PROOF. We have splitting $L=(\mathfrak{O}u+\mathfrak{O}v)\oplus M$. Write $j=\alpha u+\beta v+w$, $l=\gamma u+\delta v+w'$, with $w,w'\in M$ and $\alpha,\beta,\gamma,\delta\in\mathfrak{O}$. If β (or δ) $\in U$ then there exists S in $S(L)$ such that $Su=j$ (or l) by 3.0. If either α or γ is a unit, say $\alpha\in U$, then $\alpha^{-1}j=u+\beta'v+w''$. By 4.2 there exists $T\in X(L)$ such that $Tu=\alpha^{-1}j$. So we have to consider the cases where all $\alpha,\beta,\gamma,\delta$ are divisible by p. Thus w,w' are maximal in M and we have $(w',w')\in 4\mathfrak{O}$, $(w,w)\in 4\mathfrak{O}$ and $(w,w')\in p^iU$. So $\mathfrak{O}w+\mathfrak{O}w'$ is a hyperbolic plane which splits M. So there exists an isotropic vector $z\in M$ such that $(w,z)=p^i$. Then we have $(u,v+z)=((\alpha+1)^{-1}j,v+z)$. Here $v+z$ is isotropic and $\alpha+1\in U$. We can apply 4.1. When γ (or δ) is a unit, we have $T\in X(L)$ such that $Tu=\varepsilon l$. Put $T':j\to p\bar{p}^{-1}l,\ l\to j$. Then $T'\in S(L)$ with $T'T=\varepsilon j$.

4.4. Let $L=H_\nu\oplus M_\nu$ with $H_\nu=\mathfrak{O}u_\nu+\mathfrak{O}v_\nu\cong H(i)$ for $\nu=1,2$. Then there exists $T\in X(L)$ such that $Tu_1=u_2$ and $Tv_1=v_2$.

PROOF. By 4.3 there exist a unit ε and $\varphi\in X(L)$ such that $\varphi u_1=\varepsilon u_2$. Put $\varphi v_1=\bar{\varepsilon}^{-1}l$. Then $\varphi H_1=\mathfrak{O}u_1+\mathfrak{O}l$ and $(u_2,l)=(u_2,v_2)$. By 4.2 there exists $T'\in X(L)$ such that $T'u_2=u_2$, $T'l=v_2$. Thus we have $T'\varphi H_1=H_2$. Put $T''=T'\varphi$ and put $\sigma:T''u_1\to u_2$ and $T''v_1\to v_2$. Then $\sigma\in S(H_2)$ and we can take $\sigma T''$ for T.

4.5. Let $L=H(i)\oplus M$. Then $U(L)=X(L)U(L,H(i))$.

PROOF. Take $\varphi\in U(L)$. Then $\varphi H(i)=\mathfrak{O}\varphi u+\mathfrak{O}\varphi v$. By 4.4 there exists T in $X(L)$ such that $Tu=\varphi u$ and $Tv=\varphi v$. Then $T^{-1}\varphi\in U(L,H(i))$.

4.6. Suppose $L=\mathfrak{O}x\oplus M$ where M is 2-dimensional p-modular lattice and $(x,x)\in\mathfrak{u}$. Then $L\cong\mathfrak{O}x'\oplus H(1)$ with $(x',x')\in\mathfrak{u}$.

PROOF. By a suitable scaling we can write L in R. P. case

$L\cong(1)\oplus\begin{pmatrix}\pi & p\\ \bar{p} & 4\eta\end{pmatrix}$ in $\mathfrak{O}x\oplus(\mathfrak{O}u+\mathfrak{O}v)$. Then
$L\cong(\varepsilon)\oplus\begin{pmatrix}0 & p\\ \bar{p} & 4\eta\end{pmatrix}$ in $\mathfrak{O}y+(\mathfrak{O}(px+u)+\mathfrak{O}v)$
where y can be chosen in L with $(y, y)=\varepsilon\in\mathfrak{u}$. Since $\begin{pmatrix}0 & p\\ \bar{p} & 4\eta\end{pmatrix}\cong H(1)$ we are through in R. U. case. We can do similarly in R. U. case.

4.7. Suppose $L=J\oplus M$ where J is proper unimodular with dim 1 or 2 and $\mathfrak{s}M\subseteq p\mathfrak{O}$. Then there exists $x\in L$ with $(x, x)\in\mathfrak{u}$ such that $U(L)=S(L)\,U(L, \mathfrak{O}x)$.

PROOF. (i) Suppose $J=\mathfrak{O}x$ with $(x, x)=1$. Take $\varphi\in U(L)$ and write $\varphi x=\alpha x+t$. By 3.3 we have to consider R. P. cases where $2\,||\,\alpha-1$. Let $M=M_1\oplus M_2\oplus M_3$ be a splitting of M where M_1 is p-modular, M_2 is proper $2\mathfrak{O}$-modular and M_3 is the remaining part of M.

(a) Case $M_2\neq\{0\}$. Take $\eta y\in M_2$ such that $(y, y)=2\varepsilon$ with $\varepsilon\in\mathfrak{u}$. Write $\pi=2\rho$ and put $S=(px+\eta y, -\rho+\varepsilon\eta^2+p)$. Choose $\eta\in\mathfrak{u}$ such that $\eta^2\varepsilon\equiv\rho \bmod 2\mathfrak{O}$. Then S is in $S(L)$ and $Sx=(1+p\alpha')x+\eta\alpha' y$ with $\alpha'\in U$. Considering $S\varphi x$ the problem is reduced to the case where $p\,||\,\alpha-1$.

(b) Case $M_2=\{0\}$. By 4.6 we may assume $M_1=\oplus H_i(1)=\oplus(\mathfrak{O}u_i+\mathfrak{O}v_i)$ by suitable choice of x. Put $\varphi x=\alpha x+\oplus(\beta_i u_i+\gamma_i v_i)+y'$, $y'\in M_3$. We are through if we can prove $p\,|\,\beta_i$, $p\,|\,\gamma_i$. If not, we may assume for example $p\nmid\gamma_i$. Put $w=-\bar{\gamma}_i p\bar{\alpha}^{-1}x+u_i$. Then $(\varphi x, w)=0$. Hence $w\in\varphi M_1\oplus\varphi M_3$. On the other hand $(w, w)\in 2\mathfrak{u}$, which is a contradiction.

(ii) Suppose $J=\mathfrak{O}x+\mathfrak{O}y$ with $(x, x)=1$. Take $\varphi\in U(L)$ and write $\varphi x=\alpha x+\beta y+z$ with $z\in M$.

(a) Suppose $\alpha=1+\alpha'$ with $p^2\nmid\alpha'$. Then we can apply 3.0, 3.3.

(b) Suppose $\alpha=1+2\alpha'$ with $\alpha'\in U$. Then by 3.3 this is an R. P. case and we can proceed as in (i).

4.8. Let $L=P\oplus M$ be a decomposition of L. Where P is of type ①I_2, ④I_2 and ②II. If $\mathfrak{s}M\subseteq p\mathfrak{s}P$. Then $U(L)=S(L)\,U(L, P)$.

PROOF. Take φ in $U(L)$ and write $\varphi x=\alpha x+\beta y+z$ with z in M.

(i) If $\beta\in U$ then by 3.0 there exists S in $S(L)$ such that $Sx=\varphi x$. If β is not a unit. Then put $S'=(y, \sigma)$ with σ as in the proof of 3.5. Consider $S'\varphi$ instead of φ. Then we are reduced to the case where β is a unit.

(ii) Let $\varphi x=x$ and $\varphi y=\gamma x+\delta y+z$ with z in M. Since $(\varphi x, \varphi y)=p^i$, we have $\bar{p}^i(1-\delta)=2\gamma$. This shows $(x, \varphi y-y)=0$. Since $(y, \varphi y-y)=\bar{\gamma}\,\bar{p}^i+(\bar{\delta}-1)(y, y)$, we are through if $\gamma \in U$. If $p \mid \gamma$. We put $S''=(x, 1)$. Then $S''\varphi x=-x$ and $S''\varphi y=\gamma' x+\delta y+z$. Since $\delta \in U$ we can assume $p^i \| \gamma'$. Write $S''\varphi=\varphi'$. Then we have $\varphi' x=-x$ and $\varphi' y=\gamma' x+\delta y+z$. Put $S'''t=t-(t, y+\varphi' y)(y, y+\varphi' y)^{-1}(y+\varphi' y)$. By the remark about γ' and $(\varphi' x, \varphi' y)=p^i$ we have $S''' \in S(L)$ and $S'''x=x$. We have $S'''y=-\varphi' y$. Thus $S'''^{-1}\varphi'$ induces an isometry on J and M. We have $U(L)=S(L)U(L, P)$.

§ 5. Relation between $X(L)$ and $S(L)$.

Let u, v, w, μ be as in 4.1. Put $S_1=(\mu u+w, -\bar{\mu}(v, u))$ and $S_2=(w, -\mu(u, v))$. We have $S_i \in U(V)$ and $T_{w,\mu}=S_1 S_2$. Since $T_{w,\mu} \in U(L)$ we have $S_1 \in U(L)$ if and only if $S_2 \in U(L)$. We have

$$(u, \omega) T_{w,\mu} v=\left(\mu-\frac{(v, u)}{\omega}\right)u+v+w \quad \cdots\cdots (2)$$

5.1. In cases ②, ③ we have $T_{w,\mu} \in S(L)$ by (2) and 5.2 below. So $X(L)=S(L)$. (See also [1] Hilfssatz 7, p. 103.)

5.2. If one of the following conditions (i), (ii), (iii) is satisfied, then $T_{w,\mu} \in S(L)$. (i) $\mu \in U$, (ii) w is not maximal, (iii) $(w, w) \in 2\mathfrak{u}$.

PROOF. If $\mu \in U$, then $S_1 \in U(L)$. So we have $T_{w,\mu} \in S(L)$. If (ii) is satisfied, by (2) we can assume $p^2 \nmid \mu$. Then we have $S_1 \in U(L)$ by 3.0. If (iii) is satisfied, by 5.1, we have to consider the cases ①, ④. Suppose $p \mid \mu$. Then $(\mu u+u, \mu u+v) \in 4\mathfrak{O}$. This contradicts the hypothesis. So in this case (i) is satisfied.

5.3. Let $F \neq Q_2$ and let L be a 4-dimensional proper unimodular lattice of type ①. Then $U(L)=S(L)$.

PROOF. Suppose $L=\mathfrak{O}x+K$ with $(x, x)=1$ and write $\varphi x=\alpha x+t$ with $t \in K$ for φ in $U(L)$. By 3.3 we may assume $\alpha=1+p\alpha'$ with $\alpha' \in \mathfrak{O}$ and t is maximal. Then there exists an isotropic vector u in K such that $(t, u)=1$ and we have splitting $K=(\mathfrak{O}t+\mathfrak{O}u) \oplus \mathfrak{O}v$ with $(v, v)=\varepsilon \in \mathfrak{u}$. Since $F \neq Q_2$ we can take λ in $\mathfrak{u}$ such that $\lambda^2\varepsilon-1$ is a unit. Put $J=\mathfrak{O}x \oplus (\mathfrak{O}t+\mathfrak{O}(u+\lambda v))$. Then $\varphi x \in J$. The orthogonal complement of φx in J contains $u+\lambda v-\varphi x$ whose norm is a unit by the choice of λ. So it is isometric to $\mathfrak{O}t+\mathfrak{O}(u+\lambda v)$ by [2] Prop. 10.1. Hence the problem is reduced to 3-dimensional cases.

5.4. For a fixed hyperbolic pair u, v we have the relation

$$T_{y_1,\mu_1} T_{y_2,\mu_2}=T_{y_1+y_2,\mu_1\mu_2-(y_2,y_1)(u,v)^{-1}} \quad \cdots\cdots (3)$$

Similarly, we consider $T_{w,\mu}T_{w,\mu'}$ for the same vector w. By (1) we have $\mu=\mu'+\omega(u,v)^{-1}$. Then by (2) we have $T_{w,\mu'}\in S(L)T_{w,\mu}$. Since we consider $T_{w,\mu}$ modulo $S(L)$ we may write T_w instead of $T_{w,\mu}$. By this notation (3) is written by $T_{v_1}T_{v_2}=T_{v_1+v_2}$.

5.5. Let L be p^i-modular lattice with $\dim L\geqq 5$. Then $X(L)=S(L)$.

PROOF. Take $T_{w,\mu}$ with splitting $L=H(i)\oplus M$. By 5.1 we may assume the case ① or ④, moreover, by 5.2 w is maximal with $(w,w)\in 4\mathfrak{O}$. Take $t\in M$ such that $(w,t)=p^i$ and split $M=(\mathfrak{O}w+\mathfrak{O}t)\oplus N$. Since $N\neq\{0\}$ there exists $w_1\in N$ such that $(w_1,w_1)\in 2\mathfrak{u}$. Then $T_w\in S(L)$ follows from 5.2 and 5.4.

5.6. Let $F\neq Q_2$ and let L be a 4-dimensional improper unimodular lattice. Then $X(L)=S(L)$.

PROOF. As in 5.5 we split $L=H(i)\oplus(\mathfrak{O}w+\mathfrak{O}t)$ with $(w,w)\in 4\mathfrak{O}$. Then since we are considering cases ①, ④, $\mathfrak{O}w+\mathfrak{O}t$ is a hyperbolic plane. Write it by $\mathfrak{O}u+\mathfrak{O}v$ and $w=u+\lambda v$. Then we have $p\mid\lambda$. Take $\mu\in\mathfrak{u}$ such that $\mu-1$ is a unit and put $w_1=\mu u+v$. Then we have $(w_1,w_1)\in 2\mathfrak{u}$ and $(w-w_1,w-w_1)\in 2\mathfrak{u}$. Thus we have $T_w\in S(L)$ by 5.4.

5.7. Let $L=H(i)\oplus M$ with $\mathfrak{s}M\subseteq p^{i+1}\mathfrak{O}$. Then $X(L)=S(L)X(M)$.

PROOF. Take $T_{w,\mu}\in X_h(L)$. We may assume that μ is divisible at most by p. Thus we have $S_2\in U(L)$.

5.8. Let L be a lattice whose modular components consist of modular lattices to which 5.3, 5.5 and 5.6 are applicable. Then $X(L)=S(L)$.

PROOF. Let $L=H(i)\oplus M$ and let M have a splitting $M=M_1\oplus\cdots\oplus M_t$ where M_i is a p^{r_i}-modular component such that $r_1<r_2<\cdots<r_t$. For $r_j>i$ we have $T_{w_j}\in S(L)$ for w_j in M_j by 5.7. If $r_1=i$ we can apply 5.3, 5.5 or 5.6 and we have $T_{w_1}\in S(L)$.

§6. Cases: $F=Q_2$ and 4-dimensional p^i-modular L.

We have only to consider the cases ① and ④.

6.1. Let $F=Q_2$. Suppose $L=H(0)\oplus M$ where M is of type ①II. Then we have $X(L)=S(L)$.

PROOF. Let $L=(\mathfrak{O}u+\mathfrak{O}v)\oplus(\mathfrak{O}x+\mathfrak{O}y)$ where $\mathfrak{O}u+\mathfrak{O}v\cong H(0)$ and the matrix of $\mathfrak{O}x+\mathfrak{O}y$ is type of ①II. Suppose $\mu\notin U$, write $\pi=2\rho$ and put $S=(u+px,-\rho)$. Considering $ST_{w,\mu}$ the problem is reduced to the case where $\mu\in U$.

6.2. In the cases ①I_2 and ④I_2 with $L\cong H(i)\oplus\begin{pmatrix} 2 & p^i \\ p^i & 2\varepsilon \end{pmatrix}$, $X(L)=S(L)$ follows from § 2 remark 5 and 5.2 ii) and iii).

6.3. Exceptional cases can be treated as in [3], § 10. In this paragraph $-$ is used as in [3] § 10 as the reduction mod $p\mathfrak{O}$ and is not used as the conjugate operation. Let $L\cong H(i)\oplus H(i)$ in $(\mathfrak{O}u+\mathfrak{O}v)\oplus(\mathfrak{O}x+\mathfrak{O}y)$. We have only to consider the cases ①, ④. Here [2] Props. 10.1 and 10.2 are true when considered modulo $p\mathfrak{O}$ and by choosing σ suitably. We remark in the case ④ that $\overline{(s, \sigma)}$ is written in the form $\overline{(s, \sigma)}\bar{y}=\bar{y}+\overline{p^{-1}(y, s)}\bar{s}$ and if $x-y\in pL$ then we have $\overline{(x, \sigma)}=\overline{(y, \sigma')}$ for our choice of σ, σ'. Let $f: U(L)\to GL(\bar{V})$ be the canonical homomorphism. For convenience' sake we consider R. P. case when $i=0$. $f(S(L))$ is generated from following six elements. The left side of semicolon is defined as the image of f of the right hand side.

$$\begin{array}{llcll} (u, v); & (u+v, 1) & , & (x, y); & (x+y, 1) \\ A_x; & (u+v+y, 1), & & A_y; & (u+v+x, 1) \\ A_u; & (v+x+y, 1), & & A_v; & (u+x+y, 1). \end{array}$$

Following identities are easily proved.

$$(u, v)(x, y)=(x, y)(u, v), \quad (u, v)^2=(x, y)^2=1$$

Let $\{u, v\}\ni\lambda$ and $\{x, y\}\ni\mu$ be generic elements of each set respectively. Then we have relations

$$\begin{array}{l} A_\lambda A_\mu=A_\mu A_\lambda\,, \quad A_x A_y=A_y(x, y)\,, \quad A_\lambda^2=A_\mu^2=1 \\ (u, v)A_u=A_v(u, v)\,, \quad A_\lambda(x, y)=(x, y)A_\lambda,\ A_\mu(u, v)=(u, v)A_\lambda\,. \end{array}$$

Considering these relations every element of $f(S(L))$ can be reduced to one of the following three forms: $\Pi, A\Pi, A_\lambda A_\mu\Pi$. Here Π is one of $(u, v), (x, y)$ or the product of them and A is either A_λ or A_μ. As we can see easily from this there is no $\varphi\in f(S(L))$ such that $\varphi u\equiv u, \varphi v\equiv v+y$ mod $p\mathfrak{O}$. Put $\varphi' u=u, \varphi' v=2u+v-2x+y$. By the cancellation law ([2] Prop. 9.3) φ' can be extended to an element of $U(L)$. Above consideration shows $\varphi'\notin S(L)$.

6.4. We repeat here the theorem which is stated in the introduction.

THEOREM. *Let E/F be a ramified quadratic extension over the unramified dyadic local field F. Let L be a lattice on a vector space over E with non degenerate hermitian form with respect to the non-trivial automorphism of E/F. Let $U(L), S(L)$ denote respectively the group of units of L and the subgroup generated* *by symmetries in* $U(L)$. *Then*

(i) $U(L)=X(L)$ (for $X(L)$ see 4.1).

(ii) *If* $F\neq Q_2$ *then* $X(L)=S(L)$.

(iii) *Let* $F=Q_2$ *and let* L *be a* p^i*-modular lattice. Then we have* $X(L)=S(L)$ *except the case* ① *with* $L\cong H(0)\oplus H(0)$ *and the case* ④ *with* $L\cong H(1)\oplus H(1)$.

PROOF. (i) follows from 4.5, 4.7, 4.8 and [2] Prop. 10.3.

(ii) follows from 5.5, 5.6 and 3.6.

(iii) follows from the consideration of § 5.

§ 7. An application.

7.1. Let E, F be as in 6.4. Let W be the group of elements of unit norm in E. Put $W_1=\{\bar{\delta}\delta^{-1}\,|\,\delta\in U\}$ and let W_2 be the subgroup of W generated by $-\sigma^{-1}\bar{\sigma}$ where σ is an element of E^* such that $(s,\sigma)\in S(L)$ for some maximal vector s in L. W_1 is a subgroup of W of index 2. From [1] p. 89, $U(L)=X(L)$ and $\det T_{w,\mu}=1$ it follows $W_2=\{\det\varphi\,|\,\varphi\in U(L)\}$. For $\sigma=a+bp$ in E with $a,b\in F$ we put $g(\sigma)=\delta=1+\varepsilon p$ or $2\varepsilon+p$ with $\varepsilon\in\mathfrak{o}$ according as $|a|\geqq|b|$ or $|a|<|b|$.

Let us consider the inclusion relations among W, W_1, W_2.

7.2. R. P. cases.

(i) Suppose $i=0,1$ and $\mathfrak{n}L\supseteq 2\mathfrak{O}$. Take $s\in L$ such that $(s,s)\in 2\mathfrak{n}$. Then every $\delta=1+\varepsilon p$ and $2\varepsilon+p$ can be the image of g for some σ with $(s,\sigma)\in S(L)$.

(ii) Suppose $i=1$ and $\mathfrak{n}L=4\mathfrak{O}$. In this case $\delta=g(\sigma)$ with $(s,\sigma)\in S(L)$ for some maximal vector s in L is of the form $\delta=2\eta'+p$ with $\eta'\in\mathfrak{o}$. As $p^{-1}\bar{p}=-1$ we have $W_2=W_1$.

7.3. In R. U. cases we remark $\omega=\sqrt{1+\pi}$, $p=1+\omega$ and $-1=\bar{\omega}\omega^{-1}$.

(i) Suppose $i=0$ and $\mathfrak{n}L=\mathfrak{O}$. Then L splits and we have $W=W_2$.

(ii) Suppose $\mathfrak{n}L=2\mathfrak{O}$. In $i=0$ case there is no element σ such that $(s,\sigma)\in S(L)$ for which $g(\sigma)=2\varepsilon+p$ with $\varepsilon\in\mathfrak{o}$. In this case we have $W_2=W_1$. In case $i=1$ we have $W=W_2$.

7.4. Let E be a quadratic extension of algebraic number field F. Let 2 be unramified in F/Q. By K_A we shall mean adelization of an algebraic set K. As in the proof of [1] Satz 30 the number of proper genera in a genus of a modular lattice L is the number of double cosets $U^+(V)_A\psi U(L)_A$ in $U(V)_A$ where ψ is an element of $U(V)_A$. By determinantial map this number is the index of $\det U(L)_A$

in W_A. Let r be the number of nondyadic prime spots $\mathfrak{p}$ in F which is ramified in E (i.e., $\mathfrak{p}=\mathfrak{P}^2$) such that $L_{\mathfrak{P}}$ is improper. For ramified dyadic $\mathfrak{p}$ put $s_{\mathfrak{p}}=1$ if in R. P. case $L_{\mathfrak{P}}$ has a splitting $L_{\mathfrak{P}}= \oplus H(1)$ and in R. U. case $L_{\mathfrak{P}}$ has a splitting $L_{\mathfrak{P}}= \oplus H(0)$. Put $s_{\mathfrak{p}}=0$ otherwise. Put $s=\sum s_{\mathfrak{p}}$ where $\mathfrak{p}$ is taken all ramified dyadic spots. Then from [1] Satz 30 and 7.3 we have:

7.5. THEOREM. *Every genus of L contains 2^{r+s} proper genera.*

References

[1] S. Böge geb. Becken; Schiefhermitesche Formen über Zahlkörpern und Quaternionschiefkörpern, J. reine angew. Math., **221** (1966), 85-112.

[2] R. Jacobowitz; Hermitian forms over local fields, Amer. J. Math., **84** (1962), 441-465.

[3] O'Meara, O. T. and B. Pollak; Generation of local integral orthogonal groups, Math. Zeit., **87** (1965), 385-400.