Determination of some Frobenius types II

Dedicated to Professor Shôkichi Iyanaga on his 60th birthday

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§1. Introduction.

This note is a continuation of our previous papers [1], [2]. In this paper we will describe how to determine all subsets G of $\Omega\times\Omega$, where $\Omega=\{1,2,\cdots,n\}$, satisfying the following conditions (cf. [1] and [2] for the definitions and notations):

I. There is a G-cycle with support Ω (consequently G is of Frobenius type).

II. For any G-cycle C, the associated G-cycle vector V_C is an extreme point of the convex set $\bar{\Delta}_G$, i. e. $E_G=E_G^*$ in the sense of [1].

III. The number of G-cycles is $\geq 2^{n-1}-n+2$.

IV. There exists a G-cycle of length 1.

In order to state our main result, let us recall a particular subset G_0 of $\Omega\times\Omega$ defined in [2]. G_0 consists of the following $\frac{n(n+1)}{2}$ elements of $\Omega\times\Omega$: $(1,1)$, $(1,i)$ $(2\leq i\leq n)$, $(i,1)$ $(2\leq i\leq n)$, (i,j) $(1<i<j\leq n)$. Our main results are the following theorems A and B.

THEOREM A. *Let G be a subset of $\Omega\times\Omega$ satisfying the above conditions I, II. Then the number of G-cycles is of the form $2^{n-1}-\delta$, where δ is a non-negative integer, called the defect of G. Suppose that G satisfies I, II, III and IV. Then there exists a permutation σ of Ω and a subset Σ of $\Omega\times\Omega$ such that $\sigma(G)=G_0-\Sigma$.*

The proof of Theorem A is divided into several steps. In §2 we will prove that if G satisfies I and II, then the mapping φ from the set $\langle G\rangle$ of all G-cycles into the set 2^Ω of all subsets of Ω, defined by $\varphi(C)=\mathrm{Supp}(C)$, is injective. Thus we see that there is no confusion when we identify a G-cycle with its support. In §3, we will compute the number of G-cycles when G is of the form $G=G_0-\Sigma$. In §4, we give a criterion that G is isomorphic with a subset of the form $G_0-\Sigma$. In §5, we will complete the proof of Theorem A. In §6, we will give as an application of Theorem A the determination of all subsets G of $\Omega\times\Omega$ satisfying I, II, III and of defect≤ 3.

The precise result is the following:

THEOREM B. *Let G be a subset of $\Omega\times\Omega$ satifying I, II, III. Let δ be the* *defect of G. Suppose that $\delta \leq 3$. Then there is a permutation σ of Ω and a subset Σ of G_0 such that $\sigma(G)=G_0-\Sigma$. Furthermore*

(i) *if $\delta=0$, then $\Sigma=\phi$,*

(ii) *if $\delta=1$,* then $\Sigma=\{(i, j)\}$ where $(i, j) \in \{(1,1),(2,1),(1, n),(2, n)\}$,

(iii) *if $\delta=2$, then Σ is one of the following sets:*
$\{(1,1),(2,1)\}$, $\{(1,1),(2, n)\}$, $\{(1,1),(1, n)\}$, $\{(2,1),(2, n)\}$, $\{(2,1),(1, n)\}$, $\{(1, n),(2, n)\}$, $\{(3,1)\}$, $\{(1, n-1)\}$, $\{(3, n)\}$, $\{(2, n-1)\}$.

(vi) *if $\delta=3$, then Σ is one of the following sets:*
$\{(1,1),(3,1)\}$, $\{(1,1),(1, n-1)\}$, $\{(1,1),(2, n-1)\}$, $\{(1,1),(3, n)\}$, $\{(1,1),(2,1),(1, n)\}$, $\{(1,1),(2,1),(2, n)\}$, $\{(1,1),(1, n),(2, n)\}$, $\{(2,1),(1, n),(2, n)\}$, $\{(2,1),(1, n-1)\}$, $\{(2,1),(2, n-1)\}$, $\{(2,1),(3,1)\}$, $\{(2,1),(3, n)\}$, $\{(1, n),(3,1)\}$, $\{(1, n),(1, n-1)\}$, $\{(1, n),(2, n-1)\}$, $\{(1, n),(3, n)\}$, $\{(2, n),(3,1)\}$, $\{(2, n),(2, n-1)\}$, $\{(2, n),(1, n-1)\}$, $\{(2, n),(3, n)\}$.

§2. The uniqueness of a G-cycle with a given support.

LEMMA 2.1. *Let G be a subset of $\Omega \times \Omega$ satisfying I and II. Then the mapping φ from the set $\langle G\rangle$ of all G-cycles into the set 2^{Ω} of all subsets of Ω defined by $\varphi(C)=\operatorname{Supp}(C)$ is injective.*

This Lemma is obviously a corollary of the following:

LEMMA 2.2. *Let G be a subset of $\Omega \times \Omega$ of Frobenius type. Suppose that there are two distinct G-cycles C, C′ such that* $\operatorname{Supp}(C)=\operatorname{Supp}(C')$. *Then the associated G-cycle vector V_C is not an extreme point of $\bar{\Delta}_G$.*

PROOF. We may assume that $\Omega=\operatorname{Supp}(C)=\operatorname{Supp}(C')$ and that

$$C=\langle 1,2, \cdots, n\rangle, \quad C'=\langle i_1, i_2, \cdots, i_n\rangle .$$

Define two permutations σ, τ of Ω by

$$\sigma=\begin{pmatrix}1, & 2, & \cdots, & n-1, & n \\ 2, & 3, & \cdots, & n, & 1\end{pmatrix} \quad i.e. \quad \sigma=(1,2, \cdots, n) \text{ and}$$
$$\tau=\begin{pmatrix}i_1, & i_2, & \cdots, & i_{n-1}, & i_n \\ i_2, & i_3, & \cdots, & i_n, & i_1\end{pmatrix} \quad i.e. \quad \tau=(i_1, i_2, \cdots, i_n) .$$

Denote by Ω_0 the subset of Ω consisting of the fixed points of $\tau^{-1}\sigma$. Denote by Ω_1 the complement of Ω_0 in Ω: $\Omega_1=\Omega-\Omega_0$. Then obviously both Ω_0 and Ω_1 are stable under $\tau^{-1}\sigma$.

Now for each i in Ω_1, we associate a G-cycle $C^{(i)}$ as follows:

$$C^{(i)}=\langle i, \tau(i), \sigma\tau(i), \cdots, \sigma^{k-1}\tau(i)\rangle$$

where k is the smallest positive integer such that $\sigma^k\tau(i)=i$. We note that the length $k+1$ of the G-cycle C is less than n. In fact, we have $k+1\leq n$. If we had $k+1=n$, then $i=\sigma^k\tau(i)=\sigma^{-1}\tau(i)$; but this is impossible. Thus each $C^{(i)}$ $(i\in\Omega_1)$ is a proper sub-cycle of C. Furthermore it is easy to see that $\mathrm{Supp}\,(C^{(i)})\neq\mathrm{Supp}\,(C^{(j)})$ if $i\neq j$.

We claim now that $\mathfrak{S}=\{C^{(i)};\ i\in\Omega_1\}$ is a covering of C of constant multiplicity. Denote by $\mathfrak{S}_j$ the subset of $\mathfrak{S}$ consisting of the $C^{(i)}$ with $j\in\mathrm{Supp}\,(C^{(i)})$. For our purpose it is enough to show that $|\mathfrak{S}_j|=|\mathfrak{S}_{\sigma(j)}|$ for all $j\in\Omega$.

Suppose first that $\sigma(j)=\tau(j)$. Then one has immediately

$$\mathfrak{S}_j=\mathfrak{S}_j\cap\mathfrak{S}_{\sigma(j)}=\mathfrak{S}_{\sigma(j)}.\quad \text{Hence } |\mathfrak{S}_j|=|\mathfrak{S}_{\sigma(j)}|.$$

Suppose that $\sigma(j)\neq\tau(j)$. Then one can check easily

$$\mathfrak{S}_j-\mathfrak{S}_j\cap\mathfrak{S}_{\sigma(j)}=\{C^{(j)}\} \text{ and}$$
$$\mathfrak{S}_{\sigma(j)}-\mathfrak{S}_j\cap\mathfrak{S}_{\sigma(j)}=\{C^{(\tau^{-1}\sigma(j))}\}\ .$$

Thus we get $|\mathfrak{S}_j|=|\mathfrak{S}_j\cap\mathfrak{S}_{\sigma(j)}|+1=|\mathfrak{S}_{\sigma(j)}|$, q. e. d.

For our later discussion, we introduce the following notation: Let G, φ: $\langle G\rangle\to 2^{\Omega}$ be as in Lemma 2.1. Then we denote by $\{G\}$ the image of $\langle G\rangle$ under φ.

§3. The cardinality of $\langle G_0-\Sigma\rangle$.

Let G_0 be the subset of $\Omega\times\Omega$ given in §1. Let Σ be a non-empty subset of $\Omega\times\Omega$ in G_0. We compute the cardinality of $\langle G_0-\Sigma\rangle$, where $G_0-\Sigma$ satisfies the conditions I, II.

To begin with, let us recall the main properties of G_0 (cf. [2]): G_0 satisfies I, II and the number of G_0-cycles is 2^{n-1}. Furthermore, for every subset $X=\{i_1, i_2, \cdots, i_r\}$ $(i_1<i_2<\cdots<i_r)$ of $\Omega-\{1\}$, $\langle 1, i_1, \cdots, i_r\rangle$ is a G_0-cycle and every G_0-cycle is obtained in this manner.

LEMMA 3.1. *Let Σ be a subset of G_0 and put $G=G_0-\Sigma$. Then G satisfies the conditions I, II, if and only if G contains* $\{(1, 2), (2, 3), (3, 4), \cdots, (n-1, n), (n, 1)\}$.

PROOF. Suppose that Σ contains $(i, i+1)$ for some i. Then $\langle G\rangle$ does not contain the cycle $\langle 1, 2, \cdots, n\rangle$. Now since this is the only cycle in $\langle G_0\rangle$ with support $\{1, 2, \cdots, n\}$, G does not satisfy the condition I.

Suppose now that G contains $\{(1, 2), (2, 3), \cdots, (n-1, n), (n, 1)\}$. Then $\langle G\rangle$ contains the cycle $\langle 1, 2, \cdots, n\rangle$. Hence G satisfies I. By Lemma 3 of [2], G satisfies also II, since every G-cycle passes through the point 1, q. e. d.

DEFINITION. Let (i, j) be an element of G_0. We denote by $I(i, j)$ the subset of Ω defined as follows:

$$I(i, j)=\begin{cases}\{k \in \Omega;\ i<k<j\} & if\ j \neq 1,\\ \{k \in \Omega;\ i<k \leq n\} & if\ j=1.\end{cases}$$

We also denote by $J(i, j)$, $L(i, j)$ the subsets of Ω defined as follows:

$$J(i, j)=I(i, j) \cup \{i, j\}\ ;\ L(i, j)=\Omega-\{1\}-J(i, j).$$

LEMMA 3.2. *Let (i, j) be an element of G_0. Denote by $\mathfrak{X}_{ij}$ the subset of $\langle G_0\rangle$ consisting of all G-cycles of the form $\langle * \cdots *, i, j, * \cdots *\rangle$. Then we have $|\mathfrak{X}_{ij}|=2^{|L(i, j)|}$. Furthermore, we have*

$$\begin{array}{ll} |L(i, j)|=(i-2)+(n-j) & if\ 2 \leq i<j \leq n, \\ |L(1, j)|=n-j & if\ 2<j \leq n, \\ |L(i, 1)|=i-2 & if\ 2 \leq i<n, \\ |L(1,1)|=0. & \end{array}$$

PROOF. We distinguish several cases.
Case (i). $2 \leq i<j \leq n$.

There is a bijection from $2^{L(i, j)}$ onto $\mathfrak{X}_{ij}$ as follows: Let $\{i_1, i_2, \cdots, i_t, i_{t+1}, \cdots, i_s\}$ be a subset of $L(i, j)$ such that $1<i_1<\cdots<i_t<i$; $j<i_{t+1}<i_{t+2}<\cdots<i_s \leq n$. The association

$$\{i_1, \cdots i_t, i_{t+1}, \cdots, i_s\} \rightarrow\langle 1, i_1, \cdots, i_t, i, j, i_{t+1}, \cdots, i_s\rangle$$

gives the desired bijection.
Case (ii). $i=1$, $2<j \leq n$.

There is a bijection from $2^{L(1, j)}$ onto $\mathfrak{X}_{1j}$ as follows: Let $\{i_1, i_2, \cdots, i_s\}$ be a subset of $L(1, j)$ such that $j<i_1<\cdots<i_s \leq n$. The association

$$\{i_1, i_2, \cdots, i_s\} \rightarrow\langle 1, j, i_1, \cdots, i_s\rangle$$

gives the desired bijection.
Case (iii). $2 \leq i<n$, $j=1$.

There is a bijection from $2^{L(i, 1)}$ onto $\mathfrak{X}_{i1}$ as follows: Let $\{i_1, i_2, \cdots, i_s\}$ be a subset of $L(i, 1)$ such that $1<i_1<i_2<\cdots<i_s<i$. The association

$$\{i_1, i_2, \cdots, i_s\} \rightarrow\langle 1, i_1, \cdots, i_s, i\rangle$$

gives the desired bijection.
Case (iv). $i=1$, $j=1$.

$L(1,1)$ is empty and $\mathfrak{X}_{11}$ consists of $(1,1)$ alone. Henc there is a bijection from $2^{L(1,1)}$ onto $\mathfrak{X}_{11}$, q. e. d.

LEMMA 3.3. *Let Σ be a subset of G_0 such that*

$$\Sigma \cap \{(1, 2), (2, 3), \cdots, (n-1, n), (n, 1)\} = \phi .$$

Then $G = G_0 - \Sigma$ *satisfies the condition I, II and we have*

$$\langle G \rangle = \langle G_0 \rangle - \bigcup_{(i, j) \in \Sigma} \mathfrak{X}_{ij} .$$

PROOF. Obvious from the definition and Lemma 3.1.

LEMMA 3.4. *Let* (i, j) *and* (p, q) *be two elements of* G_0. *Then we have*

$$\mathfrak{X}_{ij} \cap \mathfrak{X}_{pq} \neq \phi \text{ if and only if } I(i, j) \cap J(p, q) = J(i, j) \cap I(p, q) = \phi .$$

Furthermore, when this is the case, we have

$$|\mathfrak{X}_{ij} \cap \mathfrak{X}_{pq}| = 2^{|\Omega - \{1\} - J(i, j) - J(p, q)|} .$$

PROOF. Let us consider the case where $2 \leq i < j \leq n$, $2 \leq p < q \leq n$. If $\mathfrak{X}_{ij} \cap \mathfrak{X}_{pq} \neq \phi$, then we must have either $j \leq p$ or $q \leq i$.

Conversely, when this is the case, we have $\mathfrak{X}_{ij} \cap \mathfrak{X}_{pq} \neq \phi$ since

$$\langle 1 * \cdots * i, j, * \cdots * p, q * \cdots * \rangle \text{ or } \langle 1 * \cdots * p, q * \cdots * i, j * \cdots * \rangle$$

is in $\mathfrak{X}_{ij} \cap \mathfrak{X}_{pq}$, according to $j \leq p$ or $q \leq i$. Furthermore the cardinality of $\mathfrak{X}_{ij} \cap \mathfrak{X}_{pq}$ is easily verified to be equal to the one given in the Lemma. The other cases are treated similarly, q. e. d.

Example. Let $G = G_0 - \{(i, j)\}$ $(j \not\equiv i+1 \text{ mod. } n)$. Then

$$|\langle G \rangle| = 2^{n-1} - 1 \Longleftrightarrow (i, j) \in \{(1, 1), (2, 1), (1, n), (2, n)\}$$
$$|\langle G \rangle| = 2^{n-1} - 2 \Longleftrightarrow (i, j) \in \{(3, 1), (1, n-1), (2, n-1), (3, n)\} .$$

§4. Frobenius sets with fixed points.

In this section we give a criterion for a subset G of Frobenius type to be isomorphic with a subset of G_0.

LEMMA 4.1. *Let G be a subset of $\Omega \times \Omega$ satisfying I and II. Suppose that there is an element p of Ω such that no subset of $\Omega - \{p\}$ is the support of a G-cycle. Then there exists a permutation σ of Ω such that $\sigma(G) \subset G_0$. The converse is also true.*

PROOF. Let us define a binary relation in $\Omega - \{p\}$. Let $x, y \in \Omega - \{p\}$. We write $x \succ y$ either if $x = y$ or there exists distinct elements $z_1, \cdots, z_r$ of $\Omega - \{p\}$ such that $x = z_1$, $y = z_r$, $(z_j, z_{j+1}) \in G$ $(j = 1, \cdots, r-1)$. This relation is clearly reflexive and transitive. It is also symmetric: $x \succ y$ and $y \succ x$ imply $x = y$. In fact, if $x \neq y$, we have elements $z_1, \cdots, z_r, u_1, \cdots, u_s$ in $\Omega - \{p\}$ such that $x = z_1$, $y = z_r$,

$u_1=y$, $u_s=x$, $z_i\neq z_j$, $u_i\neq u_j$ for $i\neq j$ $(z_i, z_{i+1})\in G$ $(1\leq i\leq r-1)$, $(u_i, u_{i+1})\in G$ $(1\leq i\leq s-1)$.

If $\{z_2, \cdots, z_r\}\cap\{u_2, \cdots, u_{s-1}\}=\phi$, put $z_k=z_r$, $u_l=u_2$. Now suppose that $\{z_2, \cdots, z_r\}\cap\{u_2, \cdots, u_{s-1}\}\neq\phi$. Let k be the smallest integer in $[2, r]$ such that $z_k\in\{u_2, \cdots, u_{s-1}\}$ and put $z_k=u_l$. Then we have a G-cycle $\langle z_1, \cdots, z_k, u_{l+1}, \cdots, u_{s-1}\rangle$ not passing through p, contrary to our hypothesis. Thus $\succ$ defines a structure of a partially ordered set in $\Omega-\{p\}$.

Now let $q_1, \cdots, q_r$ be the totality of maximal elements of the partially ordered set $\Omega-\{p\}$ thus defined. Next let $q_{r+1}, \cdots, q_s$ be the totality of maximal elements of $\Omega-\{p\}-\{q_1, \cdots, q_r\}$. Then let $q_{s+1}, \cdots, q_t$ be the totality of maximal elements of $\Omega-\{p\}-\{q_1, \cdots, q_s\}$. Keep going in this manner, then finally we get a sequence $q_1, \cdots, q_{n-1}$ consisting of elements of $\Omega-\{q\}$. This sequence has the following property: if $(q_i, q_j)\in G$, then $i<j$. Define a permutation σ of Ω by

$$\sigma(p)=1, \sigma(q_1)=2, \cdots, \sigma(q_{n-1})=n .$$

It is then immediate to see that $\sigma(G)\subset G_0$. Conversely, if $G\subset G_0$, every G-cycle passes through 1, q. e. d.

DEFINITION. Let (G, p) be a pair satisfying the hypotheses of Lemma 4.1. Then we say that p is a *fixed point* of G.

§5. The case where $N_1>0$.

Let G be a subset of $\Omega\times\Omega$ satisfying I, II. We denote by N_i the number of G-cycles of length i. Then as we know by Theorem 1 of [2], $|\langle G\rangle|\leq 2^{n-1}$. We put $\delta=2^{n-1}-|\langle G\rangle|$ and call δ the defect of G.

We assume furthermore that $N_1>0$, i. e. $\langle G\rangle$ contains at least one G-cycle of length 1. Fix a G-cycle $\langle p\rangle$ of length 1. We denote by Z_p the set of all G-cycles passing through the vertex p. We put $Z_p'=\langle G\rangle-Z_p$.

Now denote by $\mathfrak{D}_p$ the subset of $2^{\Omega-\{p\}}$ consisting of all subsets D of $\Omega-\{p\}$ such that $D\notin\{G\}$, i. e. there is no G-cycle with support D.

Let us define a mapping ϕ_p: $Z_p\to\mathfrak{D}_p$ by $\phi_p(C)=\mathrm{Supp}\,(C)-\{p\}$ for $C\in Z_p$. Note that $\phi_p(C)\notin\{G\}$ for every $C\in Z_p$. In fact, if $\phi_p(C)\in\{G\}$, then we would get a disjoint covering of Ω by $\langle p\rangle$ and $\phi_p(C)$ in the sense of [2], however this is impossible because of the condition II.

LEMMA 5.1. ϕ_p: $Z_p\to\mathfrak{D}_p$ *is an injective mapping. Furthermore*

$$|\mathfrak{D}_p-\phi_p(Z_p)|=\delta .$$

PROOF. ϕ_p is injective by Lemma 2.1. Now we have

$$Z_p' = 2^{\Omega - \{p\}} - \mathfrak{D}_p \quad \text{and} \quad Z_p' = \langle G \rangle - Z_p \, .$$

Hence $|Z_p'| = 2^{n-1} - |\mathfrak{D}_p|$, $|Z_p'| = 2^{n-1} - \delta - |Z_p|$. Therefore $|\mathfrak{D}_p| = |Z_p| + \delta = |\phi_p(Z_p)| + \delta$ which completes the proof.

DEFINITION. A subset E of $\Omega - \{p\}$ is called *p-exceptional* if $E \in \mathfrak{D}_p - \phi_p(Z_p)$. A subset N of $\Omega - \{p\}$ is called *p-normal* if $N \in \phi_p(Z_p)$.

LEMMA 5.2. *Let G be a subset of $\Omega \times \Omega$ satisfying I, II, III, of defect δ. Then* $2^{N_1} - N_1 - 1 \leq \delta$.

PROOF. Let $\langle p_i \rangle$ $(i = 1, \cdots, N_1)$ be the totality of G-cycles of length 1. For every subset X of $\{p_2, \cdots, p_{N_1}\}$ such that $|X| > 1$ we have $X \notin \{G\}$ and $X \cup \{p_1\} \notin \{G\}$. Thus X is p_1-exceptional.

Now put $A = \Omega - \{p_1, \cdots, p_{N_1}\}$. Then for every subset Y of $\{p_2, \cdots, p_{N_1}\}$ such that $Y \neq \{p_2 \cdots, p_{N_1}\}$, we have $A \cup Y \notin \{G\}$ and $\{p_1\} \cup A \cup Y \notin \{G\}$. Hence $A \cup Y$ is p_1-exceptional.

We have thus obtained altogether $(2^{N_1 - 1} - N_1) + (2^{N_1 - 1} - 1)$ p_1-exceptional sets. Hence we get $2^{N_1} - N_1 - 1 \leq \delta$, q. e. d.

COROLLARY 5.3. *If $\delta \leq 3$, then $N_1 \leq 2$.*

LEMMA 5.4. *Let G be a subset of $\Omega \times \Omega$ satisfying I, II, III of defect δ. Suppose $n \geq 4$. Then $N_1 \leq 1$.*

PROOF. Let $\langle p_1 \rangle, \cdots, \langle p_{N_1} \rangle$ be the set of all G-cycles of length 1 and put $\{q_1, \cdots, q_{n-N_1}\} = \Omega - \{p_1, \cdots, p_{N_1}\}$.

We note first that $n - N_1 \geq 2$. In fact, by the conditions II and III we have $n > N_1$. Suppose for a moment that $n - N_1 = 1$. Then from $n - 2 \geq \delta \geq 2^{N_1} - N_1 - 1$ we get $n - 2 \geq 2^{n-1} - n$. Then we have $n \leq 3$ which is impossible by our assumption.

Now let us derive a contradiction by assuming $N_1 \geq 2$. Put $F = 2^{N_1} - N_1 - 1$. Then the following F subsets

$$\{q_1, \cdots, q_{n-N_1}\} \cup X, \, (X \subsetneqq \{p_2, \cdots, p_{N_1}\})$$
$$Y, \, (Y \subset \{p_2, \cdots, p_{N_1}\}, \, |Y| \geq 2)$$

are all p_1-exceptional by Lemma 4.2. Hence the number ε of p_1-exceptional subsets among $\{q_1\}, \cdots, \{q_{n-N_1}\}$ satisfies $\varepsilon + F \leq \delta$.

Put
$$\varepsilon = \delta - F - i \; (i \geq 0).$$

Then the number ν of p_1-normal subsets among $\{q_1\}, \cdots, \{q_{n-N_1}\}$ is given by

$$\nu = n - N_1 - \varepsilon.$$

Putting $n = 2 + \delta + \rho \; (\rho \geq 0)$, we have

$$\nu = \nu_0 + i, \text{ where } \nu_0 = F + 2 + \rho - N_1 \, .$$

We may assume that $\{q_1\}, \cdots, \{q_i\}$ are p_1-normal. Now let k be the number of p_1-exceptional subsets among $\{p_2, q_1\}, \cdots, \{p_2, q_\nu\}$. Then, since we have $F+\varepsilon+k\leq\delta$, k must satisfy $k\leq i$. Therefore among the sets $\{p_2, q_1\}, \cdots, \{p_2 q_\nu\}$, at least $\nu-k$ sets are not p_1-exceptional. So we may assume that $\{p_2, q_1\}, \cdots, \{p_2, q_{\nu-k}\}$ are either G-cycle of length 2 or p_1-normal.

We note that $\nu-k\geq 3$ if $N_1\geq 3$. In fact

$$\nu-k\geq\nu-i=\nu_0=(2^{N_1}-2N_1+1)+\rho\geq 3+\rho .$$

Also when $N_1=2$, we have $\nu-k\geq 2$ if $\rho>0$.

Let us consider now the case where $\nu-k\geq 2$. Then at least one of the sets $\{p_2, q_1\}, \cdots, \{p_2, q_{\nu-k}\}$ is p_1-normal. In fact, otherwise they are all G-cycles. Hence we have two distinct G-cycles $\langle p_1, q_1, p_2, q_2\rangle$ and $\langle p_1, q_2, p_2, q_1\rangle$ with the same support, which is impossible.

So we may assume that $\{p_2, q_1\}$ is p_1-normal. Then $\{p_1, p_2, q_1\}$ is the support of a G-cycle C. However this induces a disjoint covering of C by $\langle p_2\rangle$ and $\langle p_1, q_1\rangle$. This is a contradiction.

Thus we have only to consider the case where $\nu-k<2$. Then, as we have seen in the above, we have $n=\delta+2$, $F=1$. Also we have $\varepsilon=\delta-1-i$ $(i\geq 0)$ and $\nu=n-2-\varepsilon=\delta-\varepsilon$.

We distinguish several cases.

Case (i). $\varepsilon=\delta-1$.

We may assume that the δ p_1-exceptional subsets are given by

$$\{q_1, q_2, \cdots, q_\delta\},\ \{q_2\}, \cdots, \{q_\delta\} .$$

Suppose that $\{q_1, p_2\}\notin\{G\}$. Then $\{p_1, p_2, q_1\}$ is in $\{G\}$ since $\{q_1, p_2\}$ is p_1-normal. Then we get a disjoint covering of the G-cycle C with $\mathrm{Supp}(C)=\{p_1, p_2, q_1\}$, by $\langle p_2\rangle$ and $\langle p_1, q_1\rangle$. This is absurd.

Suppose that $\{p_2, q_1\}\in\{G\}$. Then, since $\{q_2, \cdots, q_\delta\}\notin\{G\}$ this is p_1-normal: $\{p_1, q_2, \cdots, q_\delta\}\in\{G\}$. Then, we get a disjoint covering of Ω by $\langle p_2, q_1\rangle$ and by a G-cycle C with $\mathrm{Supp}(C)=\{p_1, q_2, \cdots, q_\delta\}$. This is impossible.

Case (ii). $\varepsilon=\delta-1-i$ with $i>0$.

We may assume that $\{q_1\}, \cdots, \{q_{1+i}\}$ are p_1-normal and that $\{q_{i+2}\}, \cdots, \{q_\delta\}$ are p_1-exceptional. Now at least one of the sets $\{p_2, q_1\}, \cdots, \{p_2, q_{1+i}\}$ is not p_1-exceptional. So we may assume that $\{p_2, q_1\}$ is not p_1-exceptional. Thus $\{p_2, q_1\}$ is either a G-cycle or p_1-normal. If $\{p_2, q_1\}$ is p_1-normal, then we get a contradiction as in the above. Suppose that $\{p_2, q_1\}$ is in $\{G\}$. If there is an index j such that $2\leq j\leq 1+i$

and that $\{p_2, q_j\} \in \{G\}$, then we get a contradiction as in the above. So we may assume that $\{p_2\, q_j\} \notin \{G\}$ for $j=2, \cdots, 1+i$. If one of these is p_1-normal, then we get a contradiction. Thus we may assume that $\{p_2, q_2\}, \cdots, \{p_2, q_{1+i}\}$ are all p_1-exceptional. Then the δ p_1-exceptional sets are exhausted by $\{q_1 \cdots, q_\delta\}, \{q_{i+2}\}, \cdots, \{q_\delta\}, \{p_2, q_2\}, \cdots, \{p_2, q_{i+1}\}$.

Now suppose $\{q_1, q_2, p_2\}$ is not in $\{G\}$. Then $\{q_1, q_2, p_2\}$ must be p_1-normal. Therefore there is a G-cycle such that $\mathrm{Supp}(C)=\{p_1, p_1, q_2, q_2\}$. But then we get a disjoint covering of C by $\langle p_1, q_2\rangle$ and $\langle p_2, q_1\rangle$.

Finally suppose that $\{q_1, q_2, p_2\} \in \{G\}$. Then again we get a G-cycle C such that $\mathrm{Supp}(C)=\{p_1, p_2, q_1, q_2\}$. We get then a contradiction as in the above.

THEOREM 5.5. *Let G be a subset of $\Omega\times\Omega$ satisfying the conditions I, II, III and IV. Suppose $n\geq 4$. Then G has a fixed point.*

PROOF. Let $\langle p\rangle$ be the unique G-cycle of length 1. (Note that we have $N_1=1$ by Lemma 5.4.) Now put

$$Q=\{q\in\Omega-\{p\}\,;\ \{p, q\}\notin\{G\}\},\ r=|Q|,$$
$$X=\{x\in\Omega-\{p\}\,;\ \{p, x\}\in\{G\}\},\ m=|X|.$$

We have then $n=1+r+m\geq\delta+2$. Hence one has

$$m\geq(\delta-r)+1\geq 1.$$

Furthermore, since $\{q\}$ is p-exceptional for every q in Q, one has $\delta\geq r$. Thus if $m=1$, we have $\delta=r$.

We distinguish several cases.

Case (i). $m=1$.

The p-exceptional subsets are exhausted by $\{q\}$, $q\in Q$. Therefore, for every subset D of Q with $|D|\geq 2$, either D is in $\{G\}$ or D is p-normal. But D cannot be in $\{G\}$. In fact, if $D\in\{G\}$, then $\Omega-\{p\}-D$ is p-exceptional. However this is impossible, since $\Omega-\{p\}-D$ contains X. Thus we see that every subset D of Q containing more than one element is p-normal.

Suppose now that $\delta\geq 3$. Take 3 elements q_1, q_2, q_3 in Q. Then $\{q_1, q_2\}, \{q_2, q_3\}, \{q_3, q_1\}$ are all p-normal, so we may assume that $\langle p, q_1, q_2\rangle\in\langle G\rangle$ and $\langle p, q_3, q_2\rangle\in\langle G\rangle$. Now since $\{p, q_1, q_3\}\in\{G\}$, we have either $\langle p, q_1, q_3\rangle\in\langle G\rangle$ or $\langle p, q_3, q_1\rangle\in\langle G\rangle$.

Suppose $\langle p, q_1, q_3\rangle\in\langle G\rangle$. Then we have $(q_3, p)\in G$. On the other hand, since $\langle p, q_3, q_2\rangle\in\langle G\rangle$, we have $(p, q_3)\in G$. Thus we get $\langle p, q_3\rangle\in\langle G\rangle$, which is impossible. Similarly $\langle p, q_3, q\rangle\in\langle G\rangle$ is also impossible. Hence we see that $\delta\leq 2$. Consequently $n\leq 4$.

Now suppose that $\delta=1$. Then $n=3$. We may put

$$Q=\{q\},\ X=\{x\},\ \Omega=\{p, q, x\}.$$

We have $\{q, x\}\not\in\{G\}$, because otherwise we get a disjoint covering of Ω by $\langle p\rangle$ and $\langle q, x\rangle$, then it is immediate to see that p is a fixed point of G.

Suppose that $\delta=2$. Then $n=4$. We may put

$$Q=\{q_1, q_2\},\ X=\{x\},\ \Omega=\{p, q_1, q_2, x\}.$$

Let us show that p is a fixed point of G. Assume that there is a G-cycle C with $\mathrm{Supp}(C)\subset\Omega-\{p\}$. We have $\mathrm{Supp}(C)\subsetneqq\Omega-\{p\}$, because otherwise we would get a disjoint covering of Ω by C and $\langle p\rangle$. Thus C is of length 2, and $\mathrm{Supp}(C)\neq\{q_1, q_2\}$. So we may assume that $C=\langle q_1, x\rangle$. Since Q is p-normal, we have either $(p, q_1)\in G$ or $(q_1, p)\in G$. In any case, we have a G-cycle C' such that $\mathrm{Supp}(C')=\{p, x, q_1\}$. But this is impossible since we get a disjoint covering of C' by $\langle p\rangle$ and $\langle q_1, x\rangle$.
Case (ii). $m\geq 2$.

We claim that p is a fixed point of G. Let us derive a contradiction by assuming that there is a G-cycle C with $\mathrm{Supp}(C)\subset\Omega-\{p\}$. Put $C=\langle y_1, y_2, \cdots, y_r\rangle$. We claim that no two consequent vertices of C can belong to X. In fact, if $y_i\in X$, $y_{i+1}\in X$, then we get a G-cycle $C'=\langle y_i, p, y_{i+1}, y_{i+2}, \cdots, y_r, y_1, \cdots, y_{i-1}\rangle$ admitting a disjoint covering by $\langle p\rangle$ and C. We claim next that $\mathrm{Supp}(C)\not\supset X$. In fact, assume that $\mathrm{Supp}(C)\supset X$. Take any two indices i, j in $[1, r]$ such that $y_i\in X$, $y_j\in X$, $j\not\equiv i+1\ (\mathrm{mod.}\ r)$, $i<j$. Let us show that $\{y_i, y_j\}$ is p-exceptional. In fact, it is easy to see that $\{y_i, y_j\}\notin\{G\}$. Suppose that $\{y_i, y_j\}$ is p-normal. Then we get either $(y_i, y_j)\in G$ or $(y_j, y_i)\in G$. If $(y_j, y_i)\in G$, we get a G-cycle $C'=\langle p, y_i, y_{i+1}, \cdots, y_j\rangle$ admitting a disjoint covering by $\langle p\rangle$ and $\langle y_i, y_{i+1}, \cdots, y_j\rangle$. Similarly $(y_i, y_j)\in G$ is also impossible. Thus we have obtained $\binom{m}{2}$ p-exceptional subsets $\{x, x'\}$ $(x\in X, x'\in X)$. Moreover, other than these, we have $r+1$ (resp. r) p-exceptional subsets:

$$\Omega-\{p\}-\mathrm{Supp}(C),\ \{q\}\ (q\in Q),\ \text{if } l(C)\neq n-2\ (\text{resp. } l(C)=n-2).$$

Thus we have $\binom{m}{2}+r+1\leq\delta$, if $l(C)\neq n-2$,

$\binom{m}{2}+r\leq\delta$, if $l(C)=n-2$.

Hence we have $\binom{m}{2}+n-m\leq\delta\leq n-2$, i.e. $\binom{m}{2}+2\leq m$, if $l(C)\neq n-2$. But this is impossible, since the quadratic polynomial $\frac{1}{2}t(t-1)+2-t$ is everywhere positive on the real line. Hence $l(C)=n-2$ and we have $\binom{m}{2}+1\leq m$. It is then immediate to get $1\leq m\leq 2$. So we have $m=2$, $\delta=r+1$. The p-exceptional sets are exhausted by X and $\{q\}$ $(q\in Q)$. Put

$$\{q_0\}=\Omega-\{p\}-\mathrm{Supp}\,(C),\ X=\{x, x'\}.$$

Because of $(x, x')\notin G$, $(x', x)\notin G$, we have $\{q_0, x, x'\}\notin\{G\}$, so $\{q_0, x, x'\}$ is p-normal. Hence $\{p, q_0, x, x'\}\in\{G\}$ and there is a G-cycle C' with $\mathrm{Supp}\,(C')=\{p, q_0, x, x'\}$. Since x and x' cannot be two consequent vertices in C', we have either $C'=\langle p, x, q_0, y'\rangle$, or $C'=\langle p, x', q_0, x\rangle$. Suppose $C'=\langle p, x, q_0, x'\rangle$. Put $x=y_1$, $x'=y_s$. Then we get a G-cycle $\langle q_0, x', y_{s+1}, \cdots, y_r, x\rangle$. This G-cycle must be of length $n-2$ by what we have shown above. Thus $s=3$. Now $\{q_0, y_2\}$ is not in $\{G\}$, since otherwise we get a disjoint covering of Ω by $\langle q_0, y_2\rangle$ and $\langle p, x', y_4, \cdots, y_r, x\rangle$. Therefore $\{q_0, y_2\}$ is p-normal. Hence we have either $(p,q_0)\in G$ or $(q_0, p)\in G$. In any case we get a G-cycle C^* with $\mathrm{Supp}\,(C^*)=\Omega-\{y_2\}$ admitting a disjoint covering by $\langle p\rangle$ and $\langle x, q_0, x', y_4, \cdots, y_r\rangle$. This is impossible. Similarly $C'=\langle p, x', q_0, x\rangle$ is also impossible.

We have thus proved that $\mathrm{Supp}\,(C)\not\supset X$ for every G-cycle C contained in $\Omega-\{p\}$.

We claim next that $\mathrm{Supp}\,(C)\cap X=\phi$. In fact, assume that $\mathrm{Supp}\,(C)\cap X\neq\phi$. Since $\mathrm{Supp}\,(C)\not\supset X$, there is an element x in $X-\mathrm{Supp}\,(C)$. Let y_i be a vertex of the G-cycle $C=\langle y_1, \cdots, y_t\rangle$ such that $y_i\in X$.

Let us show that $\{x, y_{i-1}\}$ is p-exceptional. First we claim that $\{x,y_{i-1}\}\notin\{G\}$. In fact, if $\{x, y_{i-1}\}$ is in $\{G\}$, there is a G-cycle C^* with $\mathrm{Supp}\,(C^*)=\{p, x, y_{i-1}, y_i\}$ admitting a disjoint covering by $\langle x, y_{i-1}\rangle$ and $\langle p, y_i\rangle$, Next suppose that $\{x, y_{i-1}\}$ is p-normal. Then either $(y_{i-1}, x)\in G$ or $(x, y_{i-1})\in G$.

If $(y_{i-1}, x)\in G$, then we get a G-cycle $\langle p, y_i, y_{i+1}, \cdots, y_t, y_1, \cdots, y_{i-1}, x\rangle$ admitting a disjoint covering by C and $\langle p, x\rangle$. If $(x, y_{i-1})\in G$, then using $(y_{i-1}, p)\in G$, we get a G-cycle $\langle p, y_i, \cdots, y_{i-1}\rangle$ admitting a disjoint covering by $\langle p\rangle$ and C.

Thus we have shown that $\{x, y_{i-1}\}$ is p-exceptional for every x in $X-\mathrm{Supp}\,(C)$ and for every y_{i-1} which is adjacent to a vertex y_i of C belonging to X. Put $s=|\mathrm{Supp}\,(C)\cap X|$. Then we get in this manner at least $s(m-s)$ p-exceptional sets. Moreover, we have p-exceptional sets $\Omega-\{p\}-\mathrm{Supp}\,(C)$, $\{q\}$ $(q\in Q)$. These are mutually distinct since $\Omega-\{p\}-\mathrm{Supp}\,(C)\ni x$. So we have

$$s(m-s)+r+1\leq\delta\leq n-2\,.$$

Hence we have $s(m-s)+n-m\leq n-2$, i.e. $s(m-s)\leq m-2$. On the other hand, since $1 \leq s \leq m-1$, we have $s(m-s) \geq 1 \cdot (m-1)$. Hence $m-1 \leq m-2$ which is impossible. We have thus proved that $\mathrm{Supp}(C) \cap X = \phi$ for every G-cycle C contained in $\Omega - \{p\}$.

Hence $\mathrm{Supp}(C)$ is contained in Q. Put $Z = \mathrm{Supp}(C)$. Then we have that $Z \cup \{x\} \notin \{G\}$ for every x in X. In fact, $Z \cup \{x\}$ is contained in $\Omega - \{p\}$ and $x \in X$, which is impossible by what we have shown above. Now $Z \cup \{p, x\}$ is not in $\{G\}$. In fact, if $Z \cup \{p, x\} \in \{G\}$, then we get a G-cycle C^* with $\mathrm{Supp}(C^*) = Z \cup \{p, x\}$ admitting a disjoint covering by $\langle p, x \rangle$ and C.

Thus $Z \cup \{x\}$ is p-exceptional for every x in X. In this manner we get m p-exceptional sets. Moreover there are $r+1$ other p-exceptional sets: $\Omega - \{p\} - \mathrm{Supp}(C)$, $\{q\}$ $(q \in Q)$. Hence we get $m+r+1 \leq \delta$, i.e. $n \leq \delta$. But this is impossible because of our assumption $\delta + 2 \leq n$, q.e.d.

§6. The case where the defect is ≤ 3.

The purpose of this section is to determine all subsets G of $\Omega \times \Omega$, satisfying I, II and of defect ≤ 3. Therefore throughout in this section, let G be such a subset.

If G is of defect 0, then by [2] there exists a permutation σ of Ω such that $\sigma(G) = G_0$. Hence we may assume that the defect δ of G satisfies $1 \leq \delta \leq 3$. We denote by N_p the number of G-cycles of length p. Denote by $\mathfrak{P}$ the set of all partitions (A, B) of Ω. Note that we identify the partition (A, B) with the partition (B, A). Thus the cadinality $|\mathfrak{P}|$ of $\mathfrak{P}$ is 2^{n-1}.

We denote by f the mapping from $\langle G \rangle$ into $\mathfrak{P}$ defined by

$$f(C) = (\mathrm{Supp}(C), \Omega - \mathrm{Supp}(C)) \quad \text{for} \quad C \in \langle G \rangle .$$

DEFINITION. A partition (A, B) of Ω is called *G-regular* (or simply regular when there is no confusion) if (A, B) is in $f(\langle G \rangle)$. A partition (A, B) of Ω is called *G-singular* (or simply singular) if (A, B) is in $\mathfrak{P} - f(\langle G \rangle)$.

LEMMA 6.1. *The mapping f: $\langle G \rangle \to \mathfrak{P}$ defined above is injective. Furthermore $\delta = |\mathfrak{P} - f(\langle G \rangle)|$ is equal to the defect of G. A partition (A, B) of Ω is G-singular if $A \notin \{G\}$ and $B \notin \{G\}$. A partition (A, B) of Ω is G-regular if and only if exactly one of A, B is in $\{G\}$.*

PROOF. Let $C \in \langle G \rangle$ and put $A = \mathrm{Supp}(C)$, $B = \Omega - A$. Then $A \in \{G\}$, $B \notin \{G\}$. (If $B \in \{G\}$, then we would get a disjoint covering of Ω by G-cycles which is impossible by the conditions II, III.) Conversely, if (A, B) is a partition of Ω such that $A \in \{G\}$, $B \notin \{G\}$, then there exists a G-cycle C such that $A = \mathrm{Supp}(C)$.

Furthermore such a G-cycle C is unique by Lemma 2.2. Now

$$\begin{aligned}|\mathfrak{P}-f(\langle G\rangle)| &= |\mathfrak{P}|-|\langle G\rangle| \\ &= 2^{n-1}-|\langle G\rangle|\end{aligned}$$

is equal to the defect of G, q. e. d.

Let p be an integer such that $0\leq p\leq n$. We denote by $\langle G\rangle_p$ the subset of $\langle G\rangle$ consisting of G-cycles of length p. We denote by $\mathfrak{P}_p$ the subset of $\mathfrak{P}$ consisting of partitions (A, B) of Ω such that $|A|=p$ or $|B|=p$. Then the mapping f: $\langle G\rangle \rightarrow \mathfrak{P}$ considered above satisfies

$$f(\langle G\rangle_p)\cup f(\langle G\rangle_{n-p})\subset\mathfrak{P}_p\,.$$

LEMMA 6.2. *Suppose* $0<p<n$. *If all the partitions in* $\mathfrak{P}_p$ *are* G-*regular, then* $N_p>0$, $N_{n-p}>0$. *Furthermore* $N_p+N_{n-p}=\binom{n}{p}$.

PROOF. By our assumption we have $f(\langle G\rangle_p)\cup f(\langle G\rangle_{n-p})=\mathfrak{P}_p$ and this is a partition of $\mathfrak{P}_p$, if $p\neq n-p$. Hence we heve $N_p+N_{n-p}=|\mathfrak{P}_p|$ (if $p\neq n-p$), $N_p=|\mathfrak{P}_p|$ (if $p=n-p$). Now $|\mathfrak{P}_p|=\frac{1}{2}\binom{n}{p}$ (if $p\neq n-p$) and $|\mathfrak{P}_p|=\binom{n}{p}$ (if $p=n-p$). Thus we have always $N_p+N_{n-p}=\binom{n}{p}$.

We claim now $N_p>0$ and $N_{n-p}>0$. This is obvious if $p=n-p$. Assume $p\neq n-p$. If one of N_p or N_{n-p}, say N_p, is zero, then we have $N_{n-p}=\binom{n}{p}=\binom{n}{n-p}$. But then Ω has a covering by G-cycles of constant multiplicity by Lemma 2 of [2], q. e. d.

THEOREM 6.3. *Suppose that* $\delta=1$ *and* $N_1=0$. *Then the unique* G-*singular partition of* Ω *is of the form* $(\{p\}, \Omega-\{p\})$. *Furthermore* p *is a fixed point of* G. *Finally there is a permutation* σ *of* Ω *such that* $\sigma(G)$ *coincides with* $G_0-\{(1, 1)\}$.

PROOF. Since G is of defect 1, there exists exactly one G-singular partition (A, B) by Lemma 6.1. This partition (A, B) is in $\mathfrak{P}_1$ by Lemma 6.2. Hence we may put $A=\{p\}$, $B=\Omega-\{p\}$. Now the mapping f: $\langle G\rangle\rightarrow\mathfrak{P}$ defined above satisfies $f(\langle G\rangle_1)\cup f(\langle G\rangle_{n-1})=\mathfrak{P}-\{(A, B)\}$. Hence by $N_1=0$, we have $f(\langle G\rangle_{n-1})=\mathfrak{P}_1-\{(A, B)\}$. Therefore $N_{n-1}=\binom{n}{1}-1=n-1$. Thus we have $n-1$ G-cycles $C_1, \cdots, C_{n-1}$ in $\langle G\rangle_{n-1}$; obviously one has $p\in \mathrm{Supp}(C_i)$ for $i=1, \cdots, n-1$.

We claim now $p\in \mathrm{Supp}(C)$ for every G-cycle C. In fact, put $C=\langle i_1, \cdots, i_r\rangle$. Suppose $p\not\ni \mathrm{Supp}(C)$. Then we may assume that $\mathrm{Supp}(C_i)=\Omega-\{i\}$, $i=1, \cdots, r$. We can then easily verify that $(C_1, \cdots, C_r, C)$ forms a covering of Ω of constant multiplicity r which is impossible.

Thus p is a fixed point of G. Hence by Lemma 4.1, there is a permutation σ of Ω such that $\sigma(G)\subset G_0$. Hence we may assume that $G\subset G_0$. Put $\Sigma=G_0-G$. Then $\langle G\rangle=\langle G_0\rangle-\bigcup_{(i,j)\in\Sigma}\mathfrak{X}_{ij}$. By §3, we have then $\Sigma=\{(1,1)\}$ because of our assumption $N_1=0$, q. e. d.

THEOREM 6.4. *Suppose that* $n\geq 4$, $\delta=1$ *and* $N_1>0$. *Then we have* $N_1=1$. *There is a permutation* σ *of* Ω *such that* $\sigma(G)$ *is of the form* $G_0-\Sigma$, *where* Σ *is one of the following sets:*

$$\{(2,1)\},\ \{(1,n)\},\ \{(2,n)\}.$$

PROOF. G satisfies all the conditions I, II, III, IV. Hence by Theorem 5.5, G has a fixed point. Thus, by Lemm 4.1, there exists a permutation σ of Ω such that $\sigma(G)$ is of the form $\sigma(G)=G_0-\Sigma$. Now by §3 and $N_1=1$, Σ must be of the form $\Sigma=\{(2,1)\}$ or $\Sigma=\{(1,n)\}$ or $\Sigma=\{(2,n)\}$, q. e. d.

LEMMA 6.5. *Let G be a subset of $\Omega\times\Omega$ satisfying I, II and let* $(\{p_i\},\Omega-\{p_i\})$ $(i=1,\cdots,r)$ *be the totality of the G-singular partitions in* $\mathfrak{P}_1$. *Then for every G-cycle C, we have*

$$\mathrm{Supp}\,(C)\cap\{p_1,\cdots,p_r\}\neq\phi\,.$$

PROOF. Suppose G has a G-cycle C not passing through any point in $\{p_1,\cdots,p_r\}$. Put $C=\langle q_1,\cdots,q_s\rangle$ and $A_i=\Omega-\{q_i\}$ $(1\leq i\leq s)$. Then the A_i are all in $\{G\}$ and we get a covering $(A_1,\cdots,A_s,C)$ of Ω of constant multiplicity s which is impossible, q. e. d.

THEOREM 6.6. *Suppose that* $n\geq 4$, $\delta=2$ *and* $N_1>0$. *Then we have* $N_1=1$. *Furthermore G has a fixed point and there is a permutation σ of Ω such that $\sigma(G)$ is of the form $G_0-\Sigma$, where Σ is one of the following sets:*

$$\{(2,1),(1,n)\},\ \{(2,1),(2,n)\},\ \{(1,n),(2,n)\},\ \{(3,1)\}$$
$$\{(1,n-1)\},\ \{(2,n-1)\},\ \{(3,n)\}.\ \text{(The last two sets are valid for } n\geqq 5.)$$

PROOF. By Theorem 5.5, $N_1=1$ and G has a fixed point. Hence by Lemma 4.1, there exists a permutation σ of Ω such that $\sigma(G)$ is of the form $G_0-\Sigma$. By §3, we see that Σ must be one of the sets mentioned above, q. e. d.

THEOREM 6.7. *Suppose that* $n\geq 4$, $\delta=2$ *and* $N_1=0$. *Then G has a fixed point. Furthermore there is a permutation σ of Ω such that $\sigma(G)$ is of the form $G_0-\Sigma$, where Σ is one of the following sets:*

$$\{(1,1),(2,1)\},\ \{(1,1),(1,n)\},\ \{(1,1),(2,n)\}.$$

PROOF. Let r be the number of G-singular partitions in $\mathfrak{P}_1$. Since $N_1=0$, we have $r>0$ by Lemma 6.2. Also by Lemma 6.1, we have $r\leq 2$. If $\mathfrak{P}_1$ contains only one single G-singular partition $(\{p\}, \Omega-\{p\})$, then p is a fixed point of G by Lemma 6.5. Thus we may assume that $r=2$. Let $(\{p\}, \Omega-\{p\})$ and $(\{q\}, \Omega-\{q\})$ be the G-singular partitions in $\mathfrak{P}_1$. We claim that either p or q is a fixed point of G. Then we get the Theorem by §3 and §4.

In order to prove this, let us denote by Z_p (resp. by Z_q) the set of all G-cycles passing through p (resp. q). Then by Lemma 6.5, we have $\langle G\rangle=Z_p\cup Z_q$. Therefore it is enough to show that either $Z_p\supset Z_q$ or $Z_q\supset Z_p$.

Suppose now $Z_p\not\subset Z_q$ and $Z_q\not\subset Z_p$. Then there exist G-cycles C, C' such that $C\in Z_p-Z_q$, $C'\in Z_q-Z_p$. Put $X=\mathrm{Supp}(C)\cup\mathrm{Supp}(C')$. Then X is in $\{G\}$. In fact, since $p\notin\Omega-X$, $q\not\ni\Omega-X$ we have $\Omega-X\notin\{G\}$ using the fact $\langle G\rangle=Z_p\cup Z_q$. Furthermore the partition $(X, \Omega-X)$ is distinct from the G-singular partitions $(\{p\}, \Omega-\{p\})$, $(\{q\}, \Omega-\{q\})$. Hence $(X, \Omega-X)$ is a G-regular partition. Therefore $X\in\{G\}$, i.e. there exists uniquely a G-cycle C_1 such that $\mathrm{Supp}(C_1)=X=\mathrm{Supp}(C)\cup\mathrm{Supp}(C')$. Now put $Y=\mathrm{Supp}(C)\cap\mathrm{Supp}(C')$. Then $p\notin Y$, $q\notin Y$. Put $Z=X-Y$. Then, as above $(Z, \Omega-Z)$ is a G-regular partition since $p\notin\Omega-Z$, $q\notin\Omega-Z$. Furthermore $Z\in\{G\}$. Therefore there exists uniquely a G-cycle C_2 such that $\mathrm{Supp}(C_2)=Z$. It is then easy to verify that (C, C', C_2) forms a covering of C_1 of constant multiplicity 2. This is absurd. Thus we have shown that $\langle G\rangle=Z_p$ or $\langle G\rangle=Z_q$, and the proof is complete.

THEOREM 6.8. *Suppose that $n\geq 5$, $\delta=3$ and $N_1>0$. Then there is a permutation σ of Ω such that $\sigma(G)$ is of the form $G_0-\Sigma$, where Σ is one of the following sets:*

$\{(2, 1), (1, n), (2, n)\}$, $\{(2, 1), (1, n-1)\}$, $\{(2, 1), (2, n-1)\}$, $\{(2, 1), (3, 1)\}$,
$\{(2, 1), (3, n)\}$, $\{(1, n), (3, 1)\}$, $\{(1, n), (1, n-1)\}$, $\{(1, n), (2, n-1)\}$,
$\{(1, n), (3, n)\}$, $\{(2, n), (3, 1)\}$, $\{(2, n), (1, n-1)\}$, $\{(2, n), (2, n-1)\}$,
$\{(2, n), (3, n)\}$.

PROOF. By Theorem 5.5, we have $N_1=1$ and G has a fixed point. Then by Lemma 4.1, there is a permutation σ of Ω such that $\sigma(G)=G_0-\Sigma$. Σ must be of the form mentioned above by §3, q. e. d.

LEMMA 6.9. *Suppose that $n\geq 5$, $\delta=3$, $N_1=0$. Then the number of G-singular partitions in $\mathfrak{P}_1$ is less than 3.*

PROOF. Let us derive a contradiction by assuming that there exist three distinct G-singular partition $(\{p_i\}, \Omega-\{p_i\})$ $(i=1, 2, 3)$.

Firstly we consider the case where G has a fixed point. Then we may assume that $G\subset G_0$. By Lemma 6.2, we heve $\binom{n}{2}=N_2+N_{n-2}$. On the other hand $G\subset G_0$ implies that

$$N_{n-2}\leq\binom{n-1}{1},\ N_{n-2}\leq\binom{n-1}{n-3}=\binom{n}{2}.$$

Hence $\binom{n}{2}=N_2+N_{n-2}\leq\binom{n-1}{1}+\binom{n-1}{2}=\binom{n}{2}$. Therefore we have $N_2=\binom{n-1}{1}$. Similarly from $\binom{n}{3}=N_3+N_{n-3}\leq\binom{n-1}{2}+\binom{n-1}{3}=\binom{n}{3}$, we have $N_3=\binom{n-1}{2}$. Then we have easily that $G=G_0-\{(1, 1)\}$. Therefore G is of defect 1 which is impossible. Thus we have shown that G has no fixed points.

Since $(\{p_i\}, \Omega-\{p_i\})$ $(i=1, 2, 3)$ exhaust all the G-singular partitions, we see that $(X, \Omega-X)$ is G-regular for every subset X of Ω such that $X\supset\{p_1, p_2, p_3\}$. Hence by Lemma 6.5, we have $\Omega-X\notin\{G\}$ and $X\in\{G\}$. In particular $\{p_1, p_2, p_3\}\in\{G\}$.

Put $D_i=\{p_1, p_2, p_3\}-\{p_i\}$ $(i=1, 2, 3)$. If all the D_i's are in $\{G\}$, we get a covering of $\{p_1, p_2, p_3\}$ of constant multiplicity by D_1, D_2, D_3, which is absurd. Thus we may assume that $D_3\notin\{G\}$. Then since $(D_3, \Omega-D_3)$ is G-regular, we have $\Omega-D_3\in\{G\}$.

We claim next that $\Omega-D_j\notin\{G\}$ for some j in $\{1, 2\}$. In fact, if we have $\Omega-D_j\in\{G\}$ for $j=1, 2, 3$, then Ω has a covering of constant multiplicity by $(\Omega-D_1, \Omega-D_2, \Omega-D_3, D, D)$, where $D=\{p_1, p_2, p_3\}\in\{G\}$. Thus we may assume that $\Omega-D_1\notin\{G\}$. Then $D_1\in\{G\}$.

We distinguish two cases:

(α) $D_2\in\{G\}$, (β) $D_2\notin\{G\}$.

Suppose that (α) is the case. Because of the non-existence of fixed points of G, there is a G-cycle C not passing through p_3. We have $\mathrm{Supp}(C)\cap\{p_1, p_2\}\neq\phi$, by Lemma 6.5.

Now suppose that $\{p_1, p_2\}\subset\mathrm{Supp}(C)$. Then Ω has a covering of constant multiplicity by C, $\Omega-D_3$, $\Omega-\{x\}$ $(x\in\mathrm{Supp}(C)-\{p_1, p_2\})$ which is impossible. Suppose $\mathrm{Supp}(C)\cap\{p_1, p_2\}=\{p_1\}$. Then $\mathrm{Supp}(C)\cup D_1$ is in $\{G\}$ and has a disjoint covering by C and D_1 which is absurd.

Suppose now $\mathrm{Supp}(C)\cap\{p_1, p_2\}=\{p_2\}$. Then $\mathrm{Supp}(C)\cup D_2$ is in $\{G\}$ and has a disjoint covering by C and D_2 which is impossible. Thus case (α) is impossible.

Suppose that (β) is the case, i.e. $D_1\in\{G\}$, $\Omega-D_2\in\{G\}$, $\Omega-D_3\in\{G\}$. Take any point i in $\Omega-\{p_1, p_2, p_3\}$. We claim now $\Omega-\{p_1, p_2, i\}\in\{G\}$. In fact, if $\Omega-\{p_1, p_2, i\}\notin\{G\}$, we have $\{p_1, p_2, i\}\in\{G\}$ since $(\{p_1, p_2, i\}, \Omega-\{p_1, p_2, i\})$ is

G-regular. Then we get a covering of Ω of constant multiplicity by $\{p_1, p_2, i\}$, $\Omega-D_3$, $\Omega-\{i\}$ which is impossible.

Thus there is a G-cycle C_1 such that $\mathrm{Supp}(C_1)=\Omega-\{p_1, p_2\ i\}$. Similarly there is a G-cycle C_2 such that $\mathrm{Supp}(C_2)=\Omega-\{p_1, p_3, i\}$.

We claim now, that in the G-regular partition $(\{p_1, i\}, \mathrm{Supp}(C_1)\cup \mathrm{Supp}(C_2))$, $\{p_1, i\}\notin\{G\}$. In fact, if $\{p_1, i\}\in\{G\}$ we get s disjoint covering of $\{p_1, p_2, p_3, i\}\in\{G\}$ by $\{p_1, i\}$ and D_1. Thus we have $\mathrm{Supp}(C_1)\cup \mathrm{Supp}(C_2)\in\{G\}$. But then $\mathrm{Supp}(C_1)\cup \mathrm{Supp}(C_2)$ has a covering of constant multiplicity by C_1, C_2 and D_1. Therefore case (β) is also impossible and this completes the proof of Lemma 6.9.

Remark. We can prove by a similar (but slightly involved) argument as in Lemma 6.9, the following proposition. But we omit its proof since we don't use it in this paper.

PROPOSITION. *Let G be a subset of $\Omega\times\Omega$ satisfying I, II, III. Suppose that $N_1=0$ and that the defect δ of G is ≥ 3. Then the number of G-singular partitions in $\mathfrak{P}_1$ is less than δ.*

LEMMA 6.10. *Suppose that $N_1=0$ and that there is a vertex p in Ω such that $\{p, i\}\in\{G\}$ for all $i\in\Omega-\{p\}$. Then p is a fixed point of G.*

PROOF. Suppose that there is a G-cycle C such that $\mathrm{Supp}(C)\not\ni p$. Put $C=\langle i_1, \cdots, i_r\rangle$. Then $r\geq 2$ since $N_1=0$. We have then two distinct G-cycles $\langle p, i_1, \cdots, i_r\rangle$, $\langle i_1, p, i_2, \cdots, i_r\rangle$ with the same support $\{p, i_1, \cdots, i_r\}$ which is impossible by the condition II and Lemma 2.2, q. e. d.

THEOREM 6.11. *Suppose that $n\geq 5$, $\delta=3$, $N_1=0$. Then G has a fixed point and there is a permutation σ of Ω such that $\sigma(G)$ is of the form $G_0-\Sigma$ where Σ is one of the following sets:*

$$\{(1, 1), (3, 1)\},\ \{(1, 1), (1, n-1)\},\ \{(1, 1), (2, n-1)\},\ \{(1, 1,), (3, n)\},$$
$$\{(1, 1), (2, 1), (1, n)\},\ \{(1, 1), (2, 1), (2, n)\},\ \{(1, 1), (1, n), (2, n)\},$$

PROOF. If we can prove the existence of a fixed point of G, the remaining part of the Theorem is immediate by § 3. So let us prove the existence of a fixed point of G.

Since $N_1=0$, $\mathfrak{P}_1$ contains at least one G-singular partition by Lemma 6.2. Hence let r be the number of G-singular partitions in $\mathfrak{P}_1$. Then $r=2$ or $r=1$ by Lemma 6.9. Now suppose $r=1$ and let $(\{p\}, \Omega-\{p\})$ be the unique G-singular partition in $\mathfrak{P}_1$. Then p is a fixed point of G by Lemma 6.5.

Suppose now that $r=2$. Let $(\{p_1\}, \Omega-\{p_1\})$, $(\{p_2\}, \Omega-\{p_2\})$ be the two singular partitions in $\mathfrak{P}_1$. We distinguish several cases.

Case (I). The partition $(\{p_1, p_2\}, \Omega-\{p_1, p_2\})$ is G-singular.

The three G-singular partitions are exhausted by $(\{p_1\}, \Omega-\{p_1\})$, $(\{p_2\}, \Omega-\{p_2\})$, $(\{p_1, p_2\}, \Omega-\{p_1, p_2\})$. Hence all the other partitions are G-regular. In particular $(\{p_1, i\}, \Omega-\{p_1, i\})$ is G-regular for $i\in\Omega-\{p_1, p_2\}$. Put $\Omega-\{p_1, p_2\}=\{1, 2, \cdots, k\}$. Now if $\Omega\supset X\supsetneqq\{p_1, p_2\}$, then $(X, \Omega-X)$ is a G-regular and $\Omega-X\notin\{G\}$ by Lemma 6.5. Hence $X\in\{G\}$. Taking as X a subset of the form $\{p_1, p_2, i\}$, we have either $(p_1, p_2)\in G$ or $(p_2, p_1)\in G$. Note that we cannot have simultaneously $(p_1, p_2)\in G$, $(p_2, p_1)\in G$ since $\{p_1, p_2\}\notin\{G\}$. We may assume that $(p_2, p_1)\in G$, $(p_1, p_2)\notin G$. Then by $\{p_1, p_2, i\}\in\{G\}$ we have $(p_1, i)\in G$, $(i, p_2)\in G$ for $i=1, \cdots, k$.

Now we claim the following:

LEMMA 6.12. *Let G, p_1, p_2, $1, \cdots, k$ be as above. Then we have either*

$$\{p_1, 1\}\in\{G\},\ \{p_1, 2\}\in\{G\},\ \cdots,\ \{p_1, k\}\in\{G\}\ \text{ or}$$
$$\Omega-\{p_1, 1\}\in\{G\},\ \Omega-\{p_1, 2\}\in\{G\},\ \cdots,\ \Omega-\{p_1, k\}\in\{G\}.$$

Therefore, in the first case p_1 is a fixed point of G. In the second case, p_2 is a fixed point of G.

PROOF. Suppose $\{p_1, j\}\in\{G\}$, $\{p_1, i\}\notin\{G\}$ for some $i, j\in\{1, \cdots, k\}$. Because of the G-regularity of the partition $(\{p_1, i\}, \Omega-\{p_1, i\})$, we have then $\Omega-\{p_1, i\}\in\{G\}$. Thus there is a G-cycle $C=\langle x_1, \cdots, x_m\rangle$ such that $\mathrm{Supp}(C)=\Omega-\{p_1, i\}$. Since C passes throught the vertex p_2, we may put $x_1=p_2$.

Let us consider to begin with the case where $x_2\neq j$. Then, as we have noticed above, $\{p_1, p_2, j, x_2\}\in\{G\}$ which admits a disjoint covering by G-cycles $\langle p_1, j\rangle$, $\langle p_2, x\rangle$. This is impossible. Therefore we have $j=x_2$. Now consider the partition $(\{p_1, i, j\}, \Omega-\{p_1, i, j\})$. By what we have noticed above, this is G-regular. Hence either $\{p_1, i, j\}\in\{G\}$ or $\Omega-\{p_1, i, j\}\in\{G\}$. If $\Omega-\{p_1, i, j\}\in\{G\}$, then there is a vertex x_l with $2<l\leq m$ such that $(x_l, p_2)\in G$. Then we have $\langle p_2, x_l\rangle\in\langle G\rangle$. On the other hand $\{p_1, p_2, x_2, x_l\}\in\{G\}$ which has a disjoint covering by G-cycles $\langle p_1, x_2\rangle$, $\langle p_2, x_l\rangle$. This is impossible. Thus we have shown $\{p, i, j\}\in\{G\}$. Hence either $(i, j)\in G$ or $(j, i)\in G$.

Suppose $(i, j)\in G$. Then $\{p_1, p_2, i, j\}\in\{G\}$. Moreover there are two distinct G-cycles $\langle p_2, p_1, i, j\rangle$, $\langle p_1, i, p_2, j\rangle$ having the same support $\{p_1, p_2, i, j\}$. This is impossible by Lemma 2.2.

Suppose now $(j, i)\in G$. Then $\{p_1, p_2, i, j\}$ is in $\{G\}$ and there are two distinct G-cycles $\langle p_1, j, i, p_2\rangle$, $\langle p_2, j, p_1, i\rangle$ having the same support $\{p_1, p_2, i, j\}$. This is again impossible.

Thus we have shown that if $\{p_1, j\}\in\{G\}$ for some $j\in\{1, \cdots, k\}$, then $\{p_1, i\}$

$\in \{G\}$ for all i in $\{1, \cdots, k\}$.

Conversely, if $\Omega-\{p_1, i\} \in \{G\}$ for some $i \in \{1, \cdots, k\}$, then because of the G-regularity of the partition $(\{p_1, j\}, \Omega-\{p_1, j\})$, we have $\Omega-\{p_1, j\} \in \{G\}$ for all j in $\{1, \cdots, k\}$. This completes the proof of Lemma 6.12.

Now let us return to the proof of Theorem 6.11. Suppose that the first alternative is the case in Lemma 6.12. Then we claim that p_1 is a fixed point of G. In fact let C be a G-cycle not passing through p_1. Then C passes through p_2 by Lemma 6.5. So we may put $C=\langle y_1, \cdots, y_t\rangle$, $y_1=p_2$. Then $(p_2, y_2) \in G$. Moreover $(y_2, p_2) \in G$ as we have seen above. Hence $\langle p_2, y_2\rangle \in \langle G\rangle$. Since $n \geq 5$, there is a point $i \in \Omega-\{p_1, p_2, y_2\}$. Then $\{p_1, i\} \in \{G\}$ and also $\{p_1, p_2, i, y_2\} \in \{G\}$ which has a disjoint covering by G-cycles $\langle p_1, i\rangle$, $\langle p_2, y_2\rangle$. This is impossible. Thus p_1 is a fixed point of G.

Next suppose that the second alternative is the case, i. e. $\Omega-\{p_1, i\} \in \{G\}$ for all $i \in \{1, \cdots, k\}$. We claim then that p_2 is a fixed point of G.

In fact suppose that there is a G-cycle C not passing through p_2. Then C passes through p_1. Hence there is a point $i \in \Omega-\{p_1, p_2\}$ such that $(i, p_1) \in G$. Then, since $(p_1, i) \in G$, we have $\langle p_1, i\rangle \in \langle G\rangle$, i. e. $\{p_1, i\} \in \{G\}$. This is impossible.

Case (II). The partition $(\{p_1, p_2\}, \Omega-\{p_1, p_2\})$ is G-reguar.

We note first that $\{p_1, p_2\} \in \{G\}$ by Lemma 6.5. In this case we begin with the following:

LEMMA 6.13. *There exists a point i in $\Omega-\{p_1, p_2\}$ such that either $(\{p_1, i\}, \Omega-\{p_1, i\})$ or $(\{p_2, i\}, \Omega-\{p_2, i\})$ is G-singular.*

PROOF. Suppose that the partitions $(\{p_1, i\}, \Omega-\{p_1, i\})$, $(\{p_2, j\}, \Omega-\{p_2, j\})$ $(i, j \in \Omega-\{p_1, p_2\})$ are all G-regular and let us derive a contradiction.

We claim first that there does not exist a pair (i, j) of points i, j in $\Omega-\{p_1, p_2\}$ such that $\{p_1, i\} \in \{G\}$, $\{p_2, j\} \in \{G\}$.

In fact, suppose that such a pair (i, j) exists. If $i=j$, then $\{p_1, p_2, i\}$ is in $\{G\}$ and has a covering of constant multiplicity 2 by $\langle p_1, p_2\rangle$, $\langle p_1, i\rangle$, $\langle p_2, j\rangle$, which is impossible. Hence $i \neq j$.

If $(\{p_1, p_2, i, j\}, \Omega-\{p_1, p_2, i, j\})$ is G-regular, then $\{p_1, p_2, i, j\}$ is in $\{G\}$ and has a disjoint covering by $\langle p_1, i\rangle$ and $\langle p_2, j\rangle$ which is also impossible. Thus $(\{p_1, p_2, i, j\}, \Omega-\{p_1, p_2, i, j\})$ is G-singular. Hence the three partitions $(\{p_1\}, \Omega-\{p_1\})$, $(\{p_2\}, \Omega-\{p_2\})$, $(\{p_1\ p_2, i, j\}, \Omega-\{p_1, p_2, i, j\})$ exhaust all the G-singular partitions.

Now since $n \geq 5$, there is a point h in $\Omega-\{p_1, p_2, i, j\}$. The partition $(\{p_1, h\}, \Omega-\{p_1, h\})$ being G-regular, we have either $\{p_1, h\} \in \{G\}$ or $\Omega-\{p_1, h\} \in \{G\}$. If

$\{p_1, h\} \in \{G\}$, we have $\{p_1, p_2, h, j\} \in \{G\}$ becuase of the G-regularity of the partition $(\{p_1, p_2, h, j\}, \Omega-\{p_1, p_2, h, j\})$. Moreover $\{p_1, p_2, h, j\}$ has a disjoint covering by $\langle p_1, h\rangle, \langle p_2, j\rangle$ which is absurd.

Suppose $\{p_1, h\} \notin \{G\}$. Then we have $\Omega-\{p_1, h\} \in \{G\}$. Moreover $\Omega-\{i, h\}$ is in $\{G\}$. Now $\Omega-\{h\}$ is in $\{G\}$ and has a covering of constant multiplicity by $\Omega-\{p_1, h\}$, $\{p_1, i\}$, $\Omega-\{i, h\}$. Thus we get a contradiction.

We claim next that there does not exist a pair (i, j) of points i, j in $\Omega-\{p_1, p_2\}$ such that $\Omega-\{p_1, i\} \in \{G\}$, $\Omega-\{p_2, j\} \in \{G\}$.

In fact, suppose that such a pair (i, j) exists. If $i=j$, we get a covering of constant multiplicity of $\Omega-\{i\}$ by $\Omega-\{p_1, i\}$, $\Omega-\{p_2, i\}$, $\{p_1, p_2\}$ which is absurd. Hence $i \neq j$. But then we get a covering of Ω of constant multiplicity by $\Omega-\{p_1, i\}$, $\Omega-\{p_2, j\}$ and $\Omega-\{x\}$ for $x \in \Omega-\{p_1, p_2, i, j\}$ which is impossible.

Thus we have shown that
(i) if $\{p_1, i\} \in \{G\}$ for some $i \in \Omega-\{p_1, p_2\}$, then $\{p_2, j\} \notin \{G\}$ for all $j \in \Omega-\{p_1, p_2\}$, hence $\Omega-\{p_2 j\} \in \{G\}$ by the G-regularity of $(\{p_2, j\}, \Omega-\{p_2, j\}$, (ii) if $\Omega-\{p_1, i\} \in \{G\}$ for some $i \in \Omega-\{p_1, p_2\}$, then $\{p_2, j\} \in \{G\}$ for all $j \in \Omega-\{p_1, p_2\}$.

Now let us introduce a structure of a partially ordered set in the set $S=\Omega-\{p_1, p_2\}$. Let $i, j \in S$. We write $i \prec j$ if either $i=j$ or there is a sequence $i_1, \cdots, i_s$ of points in S such that $i=i_1, j=i_s, (i_{t-1}, i_t) \in G$ $(2 \leq t \leq s)$. The relation $\prec$ is obviously reflexive and transitive. Moreover $\prec$ is symmetric since S contains no G-cycle.

Let us prove now that S is linearly ordered. In fact let i, j be distinct elements in S. By $n \geq 5$, there is an element $x \in \Omega-\{p_1, p_2, i, j\}$. Suppose $\{p_1, x\} \in \{G\}$. Then $\Omega-\{p_2, x\} \in \{G\}$. Hence there is a G-cycle containing p_1, i, j. Therefore we have $i \prec j$ or $j \prec i$. Suppose $\{p_1, x\} \notin \{G\}$. Then $\Omega-\{p_1, x\} \in \{G\}$ and we have a G-cycle containing p_2, i, j. Thus we have $i \prec j$ or $j \prec i$. Hence S is linearly ordered. Put $S=\{y_1, \cdots, y_m\}$, $y_1 \prec y_2 \prec \cdots \prec y_m$. Suppose $\{p_1, y_1\} \in \{G\}$. Then by what we have shown above, $\{p_1, y_t\} \in \{G\}$ for $1 \leq t \leq m$. Hence $\langle p_1, y_1, y_2, \cdots, y_m\rangle \in \langle G\rangle$. Therefore $\Omega-\{p_2\} \in \{G\}$ which is impossible since $(\{p_2\}, \Omega-\{p_2\})$ is G-singular.

Thus we have $\{p_1, y_1\} \notin \{G\}$. Then again by what we have shown above, $\{p_2, y_1\} \in \{G\}$. Hence we get a contradiction as above. This completes the proof of Lemma 6.13.

Let us now return to the proof of Theorem 6.11, Case (II). By Lemma 6.12, we may assume that the partitions $(\{p_1\}, \Omega-\{p_1\})$, $(\{p_2\}, \Omega-\{p_2\})$, $(\{p_1, 1\}, \Omega-\{p_1, 1\})$ exhaust all the G-singular partitions. Put $\Omega-\{p_1, p_2, 1\}=\{2, \cdots, k\}$. Then as in the proof of Lemma 6.13,. we have two alternatives:

(α) $\{p_1, i\} \in \{G\}$ $(2 \leq i \leq k)$, $\Omega - \{p_2, j\} \in \{G\}$ $(1 \leq j \leq k)$, or
(β) $\Omega - \{p_1, i\} \in \{G\}$ $(2 \leq i \leq k)$, $\{p_2, j\} \in \{G\}$ $(1 \leq j \leq k)$.

Suppose (α) is the case. Then as in the proof of Lemma 6.13, $S = \Omega - \{p_1\ p_2\}$ is endowed with the structure of a linearly ordered set. Put

$$S = \{y_1, \cdots, y_k\}, \quad y_1 \prec y_2 \prec \cdots \prec y_k .$$

If $y_1 \neq 1$, then $\langle p_1, y_1, \cdots, y_k \rangle \in \langle G \rangle$. Hence $\Omega - \{p_2\} \in \{G\}$ which is impossible. Hence $y_1 = 1$. Now if $(p_1, 1) \in G$, we get again $\langle p_1, 1, y_2, \cdots, y_k \rangle \in \langle G \rangle$ which is impossible. Thus $(p_1, 1) \notin G$. On the other hand, by the G-regularity of $(\{p_1, p_2, 1\}, \Omega - \{p_1, p_2, 1\})$, we have $\{p_1, p_2, 1\} \in \{G\}$ by Lemma 6.5. Thus we have $(1, p_1) \in G$. Therefore we have also $(p_2, 1) \in G$. We note also $(1, p_2) \notin G$. In fact, if $(1, p_2) \in G$, then $\{1, p_2\} \in \{G\}$. On the other hand we have $\langle p_1, y_2, \cdots, y_k \rangle \in \langle G \rangle$ since $\langle p_1, y_2 \rangle \in \langle G \rangle$ and $\langle p_1, y_2 \rangle \in \langle G \rangle$. Hence we get a disjoint covering of Ω by $\langle 1, p_2 \rangle$ and $\langle p_1, y_2, \cdots, y_k \rangle$, which is absurd.

Let us now show that $(p_2, y_s) \in G$, $(y_s, p_2) \notin G$ $(1 \leq s \leq k)$ by induction on s. This is true for $s = 1$ above. Suppose it is valid for s. Since $\{p_1, p_2, y_{s+1}\} \in \{G\}$ by the G-regularity of the partition $(\{p_1, p_2, y_{s+1}\}, \Omega - \{p_1, p_2, y_{s+1}\})$ and by Lemma 6.5, we have either $(p_2, y_{s+1}) \in G$ or $(y_{s+1}, p_2) \in G$. If $(y_{s+1}, p_2) \in G$, then $\{p_1, y_{s+1}, p_2, y_s\}$ is in $\{G\}$. Moreover there are two distinct G-cycles $\langle p_1, y_{s+1}, p_2, y_s \rangle$ and $\langle p_2, y_s, y_{s+1}, p_1 \rangle$, which is however impossible. Thus $(y_{s+1}, p_2) \notin G$. Thus $(p_2, y_{s+1}) \in G$ and our induction is complete.

It is easy then to see that every G-cycle C passes through the point p_1. In fact, otherwise C passes through p_2. But since we have $(x, p_2) \notin G$ for every $x \in \Omega - \{p_1, p_2\}$, such a G-cycle C cannot exist. Thus, in case (α), p_1 is a fixed point of G.

Next suppose that (β) is the case. Then p_2 is a fixed point by Lemma 6.10 since $\langle p_2, x \rangle \in \langle G \rangle$ for every $x \in \Omega - \{p_2\}$. This completes the proof of Theorem 6.11.

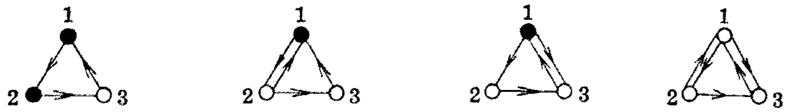
Appendix

We give here the classifications of subsets G of $\Omega \times \Omega$, $\Omega = \{1, 2, \cdots, n\}$, satisfying the conditions I, II and such that of defect $\delta \leq 3$. Since we gave the classification for the case $n - 2 \geq \delta$ already in §6, we consider here only the case where $n - 2 < \delta$. Thus the following cases will be settled: (i) $\delta = 1$, $n = 3$, (ii) $\delta = 2$, $n = 3$, (iii) $\delta = 3$, $3 \leq n \leq 4$.

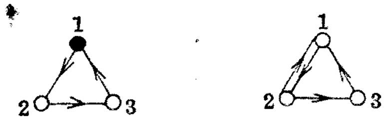
We describe G by means of the directed graph associated with G (cf. [1]).

We use below the symbol $\bullet i$ to indicate the G-cycle $\langle i\rangle$ of length 1.

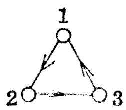
Case (i) $\delta=1$, $n=3$.



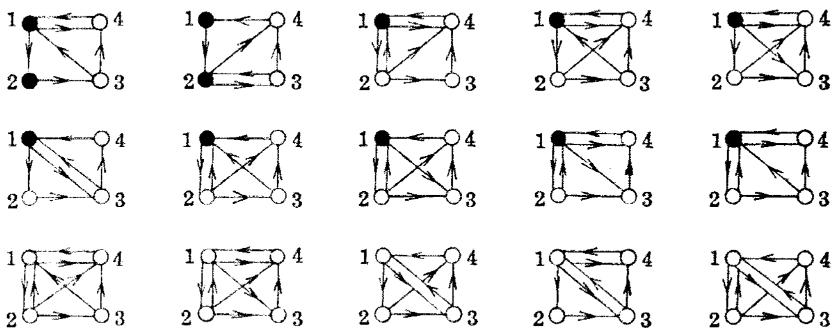
Case (ii) $\delta=2$, $n=3$.



Case (iii) $\delta=3$, $n=3$,



$\delta=3$, $n=4$.



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