On a special class of pseudo-differential operators

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Introduction. The theory of pseudo-differential operators on $\boldsymbol{R}^n$ was first discussed in Kohn-Nirenberg [4] and in Unterberger and Bakobza [5]. Then Hörmander [6] treated pseudo-differential operators on differentiable manifold M in more intrinsic manner.

In this note a special subclass of pseudo-differential operators on $M\times\boldsymbol{R}^1$ is treated, where M is a differentiable σ-compact manifold and $\boldsymbol{R}^1$ is the real line. Operators in this class are called β-pseudo-differential operators for the time being.

Briefly speaking, a β-pseudo-differential operator is a pseudo-differential operator on $M\times\boldsymbol{R}^1$ which has constant coefficients in the direction of $\boldsymbol{R}^1$. Some global properties of β-pseudo-differential operators are required by the theory of elliptic operators on M.

The aim of this note is to establish these properties by slightly modifying the discussions in Hörmander [6]. Applications of the theory of β-pseudo-differential operators will appear in the immediately following paper in this volume.

§ 1. β-pseudo-differential operators.

Let M be a σ-compact differentiable manifold of dimension n. We denote by $\mathscr{D}(M)$ the function space of complex valued C^∞ functions on M with compact support. The space of C^∞ functions on M is denoted by $\mathscr{E}(M)$. If M is an open set in $\boldsymbol{R}^n$ and $x=(x_1, \cdots, x_n)$ is in M, we denote by D^α with multi-index $\alpha=(\alpha_1, \cdots, \alpha_n)$ of non negative integers and $D_j=\frac{1}{i}\frac{\partial}{\partial x_j}$ the differential operator $D_1^{\alpha_1}D_2^{\alpha_2}\cdots D_n^{\alpha_n}$. For the other notations for various distribution spaces, we follow the notations in L. Schwartz [1] and A. Grothendieck [2].

DEFINITION 1. A continuous linear mapping P from $\mathscr{D}(M)\hat{\otimes}\mathscr{S}'(\boldsymbol{R}^1)$ into $\mathscr{E}(M)\hat{\otimes}\mathscr{S}'(\boldsymbol{R}^1)$ will be called a β-pseudo-differential operator if there exists a sequence $\{z_j=s_j+it_j\}_{j=0,1,2,3,\cdots}$ of complex numbers with decreasing real parts $s_0>s_1>s_2>\rightarrow -\infty$ such that, for all $f\in\mathscr{D}(M)$ and $g\in\mathscr{E}(M)$ which is real valued with $dg\neq 0$ on supp$f$, there is an asymptotic expansion

$$(1)\qquad e^{-i\lambda(\rho g+s\sigma)}P(fe^{i\lambda(\rho g+s\sigma)})\sim\sum_{j=0}^{N-1}p_j(f;\rho g, x, \sigma)\lambda^{z_j})$$

which has the following property:

$e^{-i\lambda(\rho g+\sigma s)}P(fe^{i\lambda(\rho g+\sigma s)})$ is independent of s and, for any integer $N>0$ and compact set $\mathscr{K}$ of real functions g in $\mathscr{E}(M)$ with $dg\neq 0$ on supp$f$,

$$\lambda^{-\varepsilon N}\Big(e^{-i\lambda(\rho g+s\sigma)}P(fe^{i\lambda(\rho g+s\sigma)})-\sum_{j=0}^{N-1}p_j(f;\,\rho g,\,x,\,\sigma)\lambda^{\varepsilon j}\Big) \tag{2}$$

remains bounded in $\mathscr{E}(M\times S)$ with $S=\{(\rho,\sigma)\in \boldsymbol{R}^2\,|\,1/2\leq\rho^2+\sigma^2\leq 2\}$. We call the formal sum

$$\sigma_P(f,g)=\sum_{j=0}^{\infty}p_j(f;\,\rho g,\,x,\,\sigma)\lambda^{\varepsilon j}$$

the symbol of P.

PROPOSITION 2. *Put*

$$p(f;\,\rho g;\,x,\,\sigma,\,\lambda)\otimes 1_s=e^{-i\lambda(\rho g+\sigma s)}P(fe^{i\lambda(\rho g+\sigma s)})\ .$$

Then, $P(f,\rho g;\,x,\sigma,\lambda)$ is an $\mathscr{E}(M\times S)$ valued C^∞ function in λ.

PROOF. Since $fe^{i\lambda(\rho g+\sigma s)}$ is a $\mathscr{D}(M)\hat{\otimes}\mathscr{S}'(\boldsymbol{R}^1)$ valued C^∞ function of λ,ρ and σ in $\boldsymbol{R}^1\times S$, $P(fe^{i\lambda(\rho g+\sigma s)})$ is an $\mathscr{E}(M)\hat{\otimes}\mathscr{S}'(\boldsymbol{R}^1)$ valued C^∞ function of λ,ρ and σ. $e^{-i\lambda(\rho g+\sigma s)}$ is also an $\mathscr{E}(M)\hat{\otimes}\mathscr{O}_M$ valued C^∞ function of λ, ρ, and σ. The multiplication of functions is a hypocontinuous bilinear mapping from $(\mathscr{E}(M)\hat{\otimes}\mathscr{O}_M)\times(\mathscr{E}(M)\hat{\otimes}\mathscr{S}'(\boldsymbol{R}^1))$ to $\mathscr{E}(M)\hat{\otimes}\mathscr{S}'(\boldsymbol{R}^1)$. Thus $p(f,\rho g;\,x,\sigma,\lambda)\otimes 1_s=e^{-i\lambda(\rho g+\sigma s)}P(fe^{i\lambda(\rho g+\sigma s)})$ is an $\mathscr{E}(M)\hat{\otimes}\mathscr{S}'(\boldsymbol{R}^1)$ valued C^∞ function of λ,ρ and σ. For any $\varphi\in\mathscr{S}(\boldsymbol{R}^1)$ with $\int_{\boldsymbol{R}^1}\varphi(s)ds=1$,

$$p(f;\,\rho g,\,x,\,\sigma,\,\lambda)=\langle e^{-i\lambda(\rho g+\sigma s)}P(fe^{i\lambda(\rho g+\sigma s)}),\,\varphi\rangle$$

is an $\mathscr{E}(M)$ valued C^∞ function of λ,ρ and σ. This proves Proposition 2.

REMARK 3. If f runs in a bounded set of $\mathscr{D}(M)$, the asymptotic expansion is uniform in f. In fact, the mappings $\Psi_{N,\rho,g,\sigma,\lambda}$ defined by

$$\Psi_{N,\rho,g,\sigma,\lambda}(f)=\lambda^{-\varepsilon N}\Big(e^{-i\lambda(\rho g+\sigma s)}P(fe^{i\lambda(g\rho+s\sigma)})-\sum_{j=0}^{N-1}p_j(f;\,\rho g,\,x,\,\sigma)\lambda^{\varepsilon j}\Big)$$

are continuous linear mappings from $\mathscr{D}(M)$ into $\mathscr{E}(M\times S)$. These mappings constitute a bounded set in $L_s(\mathscr{D}(M),\ \mathscr{E}(M\times S))$. Since $\mathscr{D}(M)$ is a barrelled space, this set is equi-continuous.

In the following we shall treat the case that M is an open subset Ω (not necessarily connected) of $\boldsymbol{R}^n$. In this case, for any f in $\mathscr{D}(\Omega)$, we shall define

$$p(f,\,x,\,\xi,\,\sigma)=e^{-i(x\cdot\xi+s\sigma)}P(fe^{i(x\cdot\xi+s\sigma)})$$

and

$$p_j(f, x, \xi, \sigma) = p_j(f, x\cdot\xi;\, x, \sigma)\,, \qquad j=0, 1, 2, \cdots.$$

PROPOSITION 4.

$p(f, x, \xi, \sigma)$ is in $\mathscr{E}(\Omega)\hat{\otimes}\mathscr{O}_M(\boldsymbol{R}^{n+1})$.

$p_j(f, x, \xi, \sigma)\in\mathscr{E}(\Omega)\hat{\otimes}\mathscr{E}(\boldsymbol{R}^{n+1}-\{0\})$ and p_j is homogeneous in (ξ, σ) of degree z_j. Moreover, for arbitrary multi-index α_1 and any integer α_2, the set

$$(3)\qquad (|\xi|+|\sigma|)^{-s_N+|\alpha_1|+\alpha_2} D_\xi^{\alpha_1} D_\sigma^{\alpha_2}\Big(p(f, x, \xi, \sigma)-\sum_{j=0}^{N-1} p_j(f, x, \xi, \sigma)\Big)$$

is bounded in $\mathscr{E}(\Omega)$, when $|\xi|+|\sigma|\to\infty$.

PROOF. The fact that $p(f, x, \xi, \sigma)$ is in $\mathscr{E}(\Omega)\hat{\otimes}\mathscr{E}(\boldsymbol{R}^{n+1})$ can be proved in the same way as in Proposition 1. Let ξ satisfy $1/2\le|\xi|\le 2$, then

$$\lambda^{-s_N}\Big[p(f, x, \lambda\rho\xi, \lambda\sigma)-\sum_{j=0}^{N-1} p_j(f, x, \rho\xi, \sigma)\lambda^{z_j}\Big]$$

remains bounded in $\mathscr{E}(\Omega\times S)$ uniformly in λ and ξ. Therefore for any $\alpha\ge 0$, the set

$$\lambda^{-s_N}\Big[D_\sigma^\alpha p(f, x, \lambda\rho\xi, \lambda\sigma)-\sum_{j=0}^{N-1} D_\sigma^\alpha p_j(f, x, \rho\xi, \sigma)\lambda^{z_j}\Big]$$

is bounded in $\mathscr{E}(\Omega\times S)$ uniformly in λ and ξ. Thus $D_\sigma^\alpha p_j(f, x, \rho\xi, \sigma)$ is an $\mathscr{E}(\Omega\times S)$-valued continuous function in ξ.

Differentiability of $p(f, x, \rho\xi, \sigma)$ gives

$$(4)\qquad -i\frac{\partial p(f, x, \rho\xi, \sigma)}{\partial\rho\xi_j} = -x_j p(f, x, \rho\xi, \sigma)+p(x_j f, x, \rho\xi, \sigma)\,.$$

Both $x_j p(f, x, \lambda\rho\xi, \lambda\sigma)$ and $p(x_j f, x, \lambda\rho\xi, \lambda\sigma)$ admit asymptotic expansion in λ in $\mathscr{E}(\Omega\times S)$, uniformly in ξ. So that,

$$(5)\qquad \lambda^{-s_N}\Big(D_{\rho\xi}^\beta p(f, \lambda\rho\xi, \lambda\sigma)-\sum_{0}^{N-1} p_j(f, x, \lambda\rho\xi, \lambda\sigma)\Big)$$

also admits asymptotic expansion in $\mathscr{E}(M\times S)$ uniformly in ξ. We show that the expansion of (5) does not contain any term of positive power in λ. If $\beta=0$, this is assumed. So, considered as a $\mathscr{D}'(M\times S\times\{1/2\le|\xi|^2\le 1\})$-valued function in λ, (5) is bounded. Therefore, it does not contain any term of positive power in λ. This implies that (5) is bounded in $\mathscr{E}(M\times S)$ uniformly in ξ. Thus $D_{\rho\xi}^\beta p_N(f, x, \rho\xi, \sigma)$ is a $\mathscr{E}(M\times S)$ valued continuous function in ξ. This implies that $p_j(f, x, \rho\xi, \sigma)$ is in $\mathscr{E}(M\times S\times\{1/2\le|\xi|^2\le 2\})$ thus $p_j(f, x, \xi, \rho)$ is in $\mathscr{E}(M\times\{|\xi|^2+|\sigma|^2=1\})$. Since (5) is bounded in $\mathscr{E}(M\times S)$ uniformly in ξ, λ, f,

$$\lambda^{-sN}\Big(D_\sigma^{\alpha_2}D_{\rho\xi}^{\alpha_1}p(f,\lambda\xi\rho,\lambda\sigma)-\sum_0^{N-1}D_{\rho\xi}^{\alpha_1}D_\sigma^{\alpha_2}p_j(f,x,\lambda\rho\xi,\lambda\sigma)\Big)$$

is bounded uniformly in ξ, λ.

Introducing $\lambda\xi\rho$ and $\lambda\sigma$ as new variables instead of $\rho\xi,\sigma$, we have the desired estimate.

Now we can prove that

$$p(f,x,\xi,\sigma)\in\mathscr{E}(\Omega)\hat{\otimes}\mathscr{O}_M(\boldsymbol{R}^{n+1}).$$

Let $\varphi\in\mathscr{E}(\boldsymbol{R}^{n+1})$ and $\varphi\equiv1$ if $|\xi|^2+|\sigma|^2>1$, $\varphi\equiv0$ in some neighbourhood of 0. Then

$$\varphi(\xi,\sigma)p_0(f,x,\xi,\sigma)\in\mathscr{E}(\Omega)\hat{\otimes}\mathscr{O}_M(\boldsymbol{R}^{n+1}).$$

On the other hand the estimate proved above implies that

$$p(f,x,\xi,\sigma)-\varphi(\xi,\sigma)p_0(f,x,\xi,\sigma)\in\mathscr{E}(\Omega)\hat{\otimes}\mathscr{O}_M(\boldsymbol{R}^{n+1}).$$

Thus we have

$$p(f,x,\xi,\sigma)\in\mathscr{E}(\Omega)\hat{\otimes}\mathscr{O}_M(\boldsymbol{R}^{n+1}).$$

LEMMA 5. *If φ is in $\mathscr{D}_{\mathscr{F}}(\Omega)\hat{\otimes}\mathscr{S}'(\boldsymbol{R}^1)$ and $\mathscr{F}$ is compact in Ω, we have*

$$P(\varphi)=(2\pi)^{-n-1}\int_{R^{n+1}}p(f,x,\xi,\sigma)\hat{\varphi}(\xi,\sigma)e^{i(x\xi+s\sigma)}d\xi d\sigma\,, \tag{6}$$

where $f\in\mathscr{D}(\Omega)$ with $f\equiv1$ on $\mathscr{F}$. Here the integral is only the symbolic expression of the following fact:

$$\varphi(x,s)\in\mathscr{S}(\boldsymbol{R}^n)\hat{\otimes}\mathscr{S}'(\boldsymbol{R}^1)\ \text{so that}\ \hat{\varphi}(\xi,\sigma)\in\mathscr{S}(\boldsymbol{R}^n)\hat{\otimes}\mathscr{S}'(\boldsymbol{R}^1).$$

Since $p(f,x,\xi,\sigma)\in\mathscr{E}(\Omega)\hat{\otimes}\mathscr{O}_M(\boldsymbol{R}^{n+1})$, (see Proposition 4) *and*

$$\mathscr{O}_M(\boldsymbol{R}^{n+1})=\mathscr{O}_M(\boldsymbol{R}^n)\hat{\otimes}\mathscr{O}_M(\boldsymbol{R}^1)\,,$$

we have

$$p(f,x,\xi,\sigma)\hat{\varphi}(\xi,\sigma)\in\mathscr{E}(\Omega)\hat{\otimes}\mathscr{S}(\boldsymbol{R}^n)\hat{\otimes}\mathscr{S}'(\boldsymbol{R}^1).$$

In fact, the multiplication mapping

$$\begin{array}{c}(\mathscr{S}(\boldsymbol{R}^n)\otimes\mathscr{S}'(\boldsymbol{R}^1))\times(\mathscr{O}_M(\boldsymbol{R}^n)\otimes\mathscr{O}_M(\boldsymbol{R}^1))\\ \downarrow\\ \mathscr{S}(\boldsymbol{R}^n)\otimes\mathscr{S}'(\boldsymbol{R}^1)\end{array}$$

is a separately continuous bilinear mapping, where topologies in tensor products are projective topologies. Since $\mathscr{S}(\boldsymbol{R}^n)\otimes\mathscr{S}'(\boldsymbol{R}^1)$ and $\mathscr{O}_M(\boldsymbol{R}^n)\otimes\mathscr{O}_M(\boldsymbol{R}^1)$ are barrelled spaces (A. Grothendieck [2], Chap. I p. 44 cor.) *this bilinear mapping is hypocontinuous. Therefore, we can prolong this and obtain a hypocontinuous bilinear* *mapping,*

$$(\mathscr{S}(\boldsymbol{R}^n)\hat{\otimes}\mathscr{S}'(\boldsymbol{R}^1))\times(\mathscr{O}_M(\boldsymbol{R}^n)\hat{\otimes}\mathscr{O}_M(\boldsymbol{R}^1))\to\mathscr{S}(\boldsymbol{R}^n)\hat{\otimes}\mathscr{S}'(\boldsymbol{R}^1)\ .$$

This proves

$$p(f, x, \xi, \sigma)\hat{\varphi}(\xi, \sigma)\in\mathscr{E}(\Omega)\hat{\otimes}\mathscr{S}(\boldsymbol{R}^n)\hat{\otimes}\mathscr{S}'(\boldsymbol{R}^1)\ .$$

Expressing symbolically the Fourier inverse transform by integration, we have

$$(7)\qquad (2\pi)^{-n-1}\int_{\boldsymbol{R}^{n+1}} e^{i(x\cdot\xi+s\sigma)}p(f, x, \xi, \sigma)\hat{\varphi}(\xi, \sigma)d\xi d\sigma$$

in $\mathscr{E}(\Omega)\hat{\otimes}\mathscr{S}(\boldsymbol{R}^n)\hat{\otimes}\mathscr{S}'(\boldsymbol{R}^1)$.

The bilinear mapping $\mathscr{E}(\Omega)\times\mathscr{S}(\boldsymbol{R}^n)\to\mathscr{E}(\Omega)$ *defined by multiplication of functions induces a linear map from* $\mathscr{E}(\Omega)\hat{\otimes}\mathscr{S}(\boldsymbol{R}^n)\hat{\otimes}\mathscr{S}'(\boldsymbol{R}^1)$ *into* $\mathscr{E}(\Omega)\hat{\otimes}\mathscr{S}'(\boldsymbol{R}^1)$. *The image of* (7) *by this map is what we express by the symbol*

$$(2\pi)^{-n-1}\int_{\boldsymbol{R}^{n+1}} p(f, x, \xi, \sigma)\hat{\varphi}(\xi, \sigma)e^{i(x\cdot\xi+s\sigma)}d\xi d\sigma\ .$$

PROOF OF THE LEMMA 5. We have only to prove the formula (6) when $\varphi=\phi_1(x)\otimes\phi_2(s)$ with $\phi_1\in\mathscr{D}(\Omega)$, $\phi_2\in\mathscr{S}(\boldsymbol{R}^1)$. In this case,

$$\varphi=f\cdot\varphi=(2\pi)^{-n-1}\int_{\boldsymbol{R}^{n+1}} f(x)e^{i(x\cdot\xi+s\sigma)}\hat{\phi}_1(\xi)\otimes\hat{\phi}_2(\sigma)d\xi d\sigma\ .$$

This integral converges in $\mathscr{D}(\Omega)\hat{\otimes}\mathscr{S}'(\boldsymbol{R}^1)$, because this converges even in $\mathscr{D}(\Omega)\hat{\otimes}L^\infty(\boldsymbol{R}^1)$. Thus

$$P(\varphi)=P(f\varphi)=(2\pi)^{-n-1}\int_{\boldsymbol{R}^{n+1}} P(fe^{i(x\cdot\xi+s\sigma)})\hat{\phi}_1(\xi)\otimes\hat{\phi}_2(\sigma)d\xi d\sigma\ .$$

This, together with the definition of $p(f, x, \xi, \sigma)$, gives (6).

COROLLARY. *If* $\varphi=\phi_1\otimes\phi_2$, $\phi_1\in\mathscr{D}(\Omega)$, $\phi_2\in\mathscr{O}_M(\boldsymbol{R}^1)$, *then we have*

$$(8)\qquad P(\varphi)=(2\pi)^{-n-1}\int_{\boldsymbol{R}^n}\hat{\phi}_1(\xi)e^{ix\cdot\xi}d\xi\int_{\boldsymbol{R}^1} p(f, x, \xi, \sigma)\hat{\phi}_2(\sigma)e^{is\sigma}d\sigma\ ,$$

where the integral over $\boldsymbol{R}^1$ *means the coupling of functions* $(1+\sigma^2)^{-l}e^{i\sigma s}$ *in the space* $\mathscr{D}_{L^1}$ *and distribution*

$$p(f, x, \xi, \sigma)\hat{\phi}_2(\sigma)(1+\sigma^2)^l \text{ in the space } \mathscr{B}'\ .$$

Especially,

$$(9)\qquad P(\phi\otimes e^{is\sigma})=(2\pi)^{-n}\int_{R^n}\hat{\phi}_1 e^{ix\cdot\xi}d\xi\int_{R^1}p(f,x,\xi,\tau)\hat{\delta}_\sigma e^{is\tau}d\tau$$
$$=(2\pi)^{-n}e^{is\sigma}\int_{R^n}\hat{\phi}_1(\xi)p(f,x,\xi,\sigma)e^{ix\cdot\xi}d\xi\ .$$

COROLLARY. *A β-pseudo-differential operator P maps $\mathscr{D}(M)\hat{\otimes}\mathscr{S}'(\boldsymbol{R}^1)$ into $\mathscr{E}(M)\hat{\otimes}\mathscr{S}'(\boldsymbol{R}^1)$ continuously.*

§2. Fourier integral operators

Let $\Omega\subset\boldsymbol{R}^n$ be an open set. We denote by K a function in $\mathscr{E}(\Omega)\hat{\otimes}\mathscr{O}_M(\boldsymbol{R}^{n+1})$ such that there are functions $K_j(x,\xi,\sigma)$ in $\mathscr{E}(\Omega)\hat{\otimes}\mathscr{E}(\boldsymbol{R}^{n+1}-\{0\})$ which are positively homogeneous of degree $z_j=s_j+it_j$ in ξ,σ, with $s_j\to-\infty$ and that have the following property: for any multi-index α_1 and non negative integers α_2 and N,

$$(10)\qquad (|\xi|+|\sigma|)^{-s_N+|\alpha_1|+\alpha_2}D_\xi^{\alpha_1}D_\sigma^{\alpha_2}\Big(K(x,\xi,\sigma)-\sum_0^{N-1}K_j(x,\xi,\sigma)\Big)$$

is bounded in $\mathscr{E}(\Omega)$, when $|\xi|+|\sigma|\to\infty$.

If $\varphi\in\mathscr{D}(\Omega)\hat{\otimes}\mathscr{S}'(\boldsymbol{R}^1)$, we define

$$(11)\qquad (K\varphi)=(2\pi)^{-n-1}\int_{R^{n+1}}K(x,\xi,\sigma)\hat{\varphi}(\xi,\sigma)e^{i(x\cdot\xi+s\sigma)}d\xi d\sigma\ ,$$

where the integral has the same symbolic meaning as in Lemma 5. As stated in Lemma 5, K maps $\mathscr{D}(\Omega)\hat{\otimes}\mathscr{S}'(\boldsymbol{R}^1)$ continuously into $\mathscr{E}(\Omega)\hat{\otimes}\mathscr{S}'(\boldsymbol{R}^1)$.

LEMMA 6. *For any $f\in\mathscr{D}(\Omega)$, we have,*

$$e^{-i\lambda(x\cdot\xi+s\sigma)}K(f(x)e^{i\lambda(x\cdot\xi+s\sigma)})\sim\sum_{\alpha,j}\frac{1}{\alpha!}D_{\xi\lambda}^\alpha K_j(x,\xi\cdot\lambda,\sigma\cdot\lambda)D_x^\alpha f\ .$$

More precisely

$$(12)\qquad (|\lambda|^{-s_0+N}+|\lambda|^{-s_J})\Big(e^{-i\lambda(x\cdot\xi+s\sigma)}K(fe^{i\lambda(x\cdot\xi+s\sigma)})$$
$$-\sum_{|\alpha|<N}\sum_{j\in J}\frac{1}{\alpha!}D_{\xi\lambda}^\alpha K_j(x,\lambda\xi,\lambda\sigma)D^\alpha f\Big)$$

is bounded in $\mathscr{E}(\Omega\times S_2)$ with $S_2=\{(\xi,\sigma)\in\boldsymbol{R}^{n+1},1/2\le|\xi|^2+\sigma^2\le2\}$. Moreover, the expansion is uniform if f remains bounded in $\mathscr{D}(\Omega)$.

PROOF. Since $\widehat{e^{is\sigma\lambda}}=2\pi\delta(\tau-\lambda\sigma)$, we have

$$(13)\qquad e^{-i\lambda(x\cdot\xi+s\sigma)}K(fe^{i\lambda(x\cdot\xi+s\sigma)})$$
$$=e^{-i\lambda x\cdot\xi}\int_{R^n}K(x,\eta,\lambda\sigma)\hat{f}(\eta-\lambda\xi)e^{ix\cdot\eta}d\eta$$
$$=\int_{R^n}K(x,\eta+\lambda\xi,\lambda\sigma)e^{ix\cdot\eta}\hat{f}(\eta)d\eta\ .$$

Let $x \in \Omega$ remain in a fixed compact set F. Then we have for any multi-index β,

$$\left| D_x^\beta K(x, \eta+\lambda\xi, \lambda\sigma) - \sum_{|\alpha|<N} \frac{1}{\alpha!} D_{\xi\lambda}^\alpha D_x^\beta K(x, \lambda\xi, \lambda\sigma)\eta^\alpha \right|$$
$$\leq C|\eta|^N(1+\lambda)^{s_0-N}, \text{ if } |\eta|<\frac{\lambda}{4},$$

and $\leq C|\eta|^N$, for any η. Thus,

$$\left| D_x^\beta(e^{-i\lambda(x\cdot\xi+s\sigma)}K(fe^{i\lambda(x\cdot\xi+s\sigma)})) - \sum_{|\alpha|<N} \frac{1}{\alpha!} D_{\lambda\xi}^\alpha D_x^\beta K(x, \lambda\xi, \lambda\sigma) D^\alpha f \right|$$
$$\leq C(1+\lambda)^{s_0-N}\int_{|\eta|<\lambda/4} |\hat{f}(\eta)||\eta|^N d\eta + C\int_{|\eta|>\lambda/4} |\eta|^N \hat{f}(\eta) d\eta$$
$$\leq C(1+\lambda)^{s_0-N}\int_{R^n} |\hat{f}(\eta)||\eta|^N d\eta + C\int_{|\eta|>\lambda/4} (1+|\eta|)^{s_0-2N-n}|\eta|^N d\eta$$
$$\leq C(1+\lambda)^{s_0-N}.$$

On the other hand, we have

$$\text{(15)} \quad \left| \sum_{|\alpha|<N} \frac{1}{\alpha!} D_x^\beta D_{\lambda\xi}^\alpha K(x, \lambda\xi, \lambda\sigma) D^\alpha f - \sum_{j<J} \sum_\alpha \frac{1}{\alpha!} D_{\lambda\xi}^\alpha D_x^\beta K_j(x, \lambda\xi, \lambda\sigma) D^\alpha f \right|$$
$$\leq C(1+\lambda)^{s_J}.$$

So that we have the expansion in the topology of $\mathscr{E}(\Omega)\hat{\otimes}\mathscr{C}(S_2)$. To obtain expansion in $\mathscr{E}(\Omega)\hat{\otimes}\mathscr{E}(S_2)$, note that

$$D_{\xi_j} e^{-i\lambda(x\cdot\xi+s\sigma)}K(fe^{i\lambda(x\cdot\xi+s\sigma)})$$
$$=\lambda\int_{R^n} D_{\eta_j}K(x, \eta+\lambda\xi, \lambda\sigma)e^{ix\cdot\eta}\hat{f}(\eta)d\eta$$
$$=\lambda e^{-\lambda(x\cdot\xi+s\sigma)}(D_{\xi_j}K)(fe^{i\lambda(x\cdot\xi+s\sigma)}).$$

The kernel $D_{\xi_j}K$ has the similar property to that of K. Therefore repeating these processes, we see that

$$(\lambda^{-s_0+N}+\lambda^{-s_J})\Big[D_\xi^{\alpha_1}D_\sigma^{\alpha_2}e^{-i\lambda(x\cdot\xi+s\sigma)}K(fe^{i\lambda(x\cdot\xi+s\sigma)})$$
$$-\sum_{|\alpha|<N}\sum_{j\in J}\frac{1}{\alpha!}D_\xi^{\alpha_1}D_\sigma^{\alpha_2}D_{\lambda\xi}^\alpha K_j(x, \lambda\xi, \lambda\sigma)D^\alpha f\Big]$$

admits an asymptotic expansion in $\mathscr{E}(\Omega)$ which is uniform in $\xi, \sigma \in S_2$. We can prove that this doesn't contain any term of positive power in λ by the same method used in the proof of Proposition 4. Therefore, we can prove that

$$(\lambda^{-s_0+N}+\lambda^{-s_J})\Big[D_\xi^{\alpha_1}D_\sigma^{\alpha_2}(e^{-i\lambda(x\cdot\xi+s\sigma)}K(fe^{i\lambda(x\cdot\xi+s\sigma)})$$

$$-\sum_{|\alpha|<N}\sum_{j\in J}\frac{1}{\alpha!}D_{\lambda\xi}^{\alpha}K_j(x,\lambda\xi,\lambda\sigma)D^{\alpha}f)\Big]$$

is uniformly bounded in $\mathscr{E}(\Omega)$. This completes the proof.

The following lemma is due to Hörmander [3].

LEMMA 7. *Let B_0 be a bounded subset of $\mathscr{D}(\Omega)$, and let B be a bounded set of $\mathscr{E}(\Omega)$ with only real elements. If c is an upper bound for $|\mathrm{grad}\, h|$ in $\mathrm{supp} f$ when $f\in B_0$ and $h\in B$, then, for every positive integers N and k, there is a constant C such that*

$$\left|\int_{R^n} f\Big(e^{i\lambda h}-\sum_0^{k-1}\frac{1}{j!}(i\lambda h)^j\Big)(i\lambda h)^{-k}e^{-ix\cdot\xi}dx\right|\le C|\xi|^{-N}, \tag{16}$$

if $|\xi|>2\lambda c$. When $N=0$, the estimate holds for all $\xi\in\boldsymbol{R}^n$.

Using this, we can prove

THEOREM 8. *If K satisfies (10), the operator K defined by (11) is a β-pseudo-differential operator of order s_0. The symbol of K is given by*

$$\sigma_K(f,g)=\sum_{\alpha,j}\frac{1}{\alpha!}D_{\lambda\rho\xi}^{\alpha}K_j(x,\lambda\rho\xi_x,\lambda\sigma)D^{\alpha}(fe^{ih_x\rho\lambda}), \tag{17}$$

where

$$\xi_x=\mathrm{grad}\, g(x).\quad h_x(y)=g(y)-g(x)-\langle y-x,\xi_x\rangle .$$

PROOF. Let $F\subset\Omega$ be compact, B and B_0 be bounded sets of $\mathscr{E}(\Omega)$ and $\mathscr{D}(\Omega)$, respectively, such that, when $g\in B$ and $f\in B_0$, we have $|dg|\ge c>0$ which is independent of f and g. We wish to study

$$e^{-i\lambda(\rho g+s\sigma)}K(fe^{i\lambda(\rho g+s\sigma)}),\qquad (\rho,\sigma)\in S$$

$$S=\Big\{(\rho,\sigma)\in\boldsymbol{R}^2:\ \frac{1}{2}\le\rho^2+\sigma^2\le 2\Big\} .$$

We may assume that for any f in B_0 support f is contained in F.

At first we also require that

$$|\mathrm{grad}\, g(x)-\mathrm{grad}\, g(y)|\le\frac{1}{4}|\mathrm{grad}\, g(x)| \tag{18}$$

for any $x,y\in F$, $g\in B$. This hypothesis will be removed at the end of the proof.

Let $x\in F$, $\xi_x=\mathrm{grad}\, g(x)$ and $h_x(y)=g(y)-g(x)-\langle y-x,\xi_x\rangle$, then $h_x(y)$ vanishes to the second order at x. The function $u_\lambda=fe^{i\lambda(\rho g+t\sigma)}$ can be written as

$$u_\lambda(y,t,\rho,\sigma)=\Big[f(y)e^{i\lambda(\rho g(x)+\langle y-x,\rho\xi_x\rangle)}\sum_0^{k-1}\frac{1}{j!}(i\lambda\rho h_x(y))^j + e^{i\lambda\rho g(x)}R_\lambda(y)\Big]e^{i\lambda\sigma t}\,. \tag{19}$$

Where $R_\lambda(y)=R_\lambda(y,\rho)$ is the remainder term which we shall study later. Hence

$$\begin{aligned} e^{-i\lambda(g\rho+s\sigma)}K(u_\lambda)(x,\rho,\sigma,\lambda) &= \sum_{j=0}^{k-1}\frac{1}{j!}e^{-i\lambda(\rho x\cdot\xi+s\sigma)}K(f\cdot(i\lambda\rho h_x(y))^j e^{i(y\rho\xi+\sigma t)}) \\ &\quad + e^{-i\lambda\sigma s}K(R_\lambda(y,\rho))e^{i\lambda\sigma t}\,, \end{aligned} \tag{20}$$

$(1/2)C\lambda\le\lambda(|\rho\xi|+|\sigma|)\le 2(C+1)\lambda$, $f(\rho h_x)^j$ remains bounded uniformly in ρ. So, by Lemma 6, the sum in (20) admits an asymptotic expansion in $\mathscr{E}(\Omega\times S)$ which is given by

$$\sum_{l=0}^{k-1}\frac{1}{l!}\sum_{j,\alpha}\frac{1}{\alpha!}D^\alpha_{\lambda\rho\xi}K_j(x,\lambda\rho\xi,\lambda\sigma)D^\alpha(f(i\lambda\rho h_x)^l)\,.$$

Since $D^\alpha(f(i\lambda\rho h_x))$ vanishes for $\alpha<2l$, if k is sufficiently large the terms in this sum involving λ to a power larger than any given number will be the same as those in the formal sum

$$\sum_{\alpha,\nu}\frac{1}{\alpha!}D^\alpha_{\lambda\rho\xi}K_j(x,\lambda\rho\xi,\lambda\sigma)D^\alpha(fe^{i\lambda\rho h_x})\,.$$

Hence to prove the theorem we have only to estimate the error term $e^{-i\lambda\sigma s}K(R_\lambda e^{i\lambda\sigma t})$ with

$$\begin{aligned} R_\lambda(y) &= e^{i\lambda(\rho g(y)-\rho g(x))}f(y)-f(y)e^{i\lambda\langle y-x,\rho\xi_x\rangle}\sum_{j=0}^{k-1}\frac{1}{j!}(i\lambda\rho h_x(y))^j \\ &= e^{i\lambda\rho\langle y-x,\xi\rangle}f(y)F_\lambda(y,\rho)(i\lambda\rho h_x)^k\,, \end{aligned} \tag{21}$$

where

$$F_\lambda(y,\rho)=\Big[e^{i\lambda\rho h_x}-\sum_{j=0}^{k-1}\frac{1}{j!}(i\lambda\rho h_x)^j\Big](i\lambda\rho h_x)^{-k}\,. \tag{22}$$

Note that $(h_x)^k$ vanishes to the order $2k$ at x. So we have

$$f(y)(ih_x)^k=\sum_{|\alpha|=2k}H_\alpha(y)(y-x)^\alpha$$

for suitable H_α which can be chosen in a bounded set in $\mathscr{D}(\Omega)$ for all x in F and all $f\in B_0$ and $g\in B$.

Let $G_\alpha(y;\lambda,\rho)=F_\lambda(y,\rho)H_\alpha(y)$, then

(23) $$|\hat{G}_\alpha(\xi;\lambda,\rho)|\le C|\xi|^{-N},\ \text{if } |\xi|\ge\frac{1}{2}\lambda\rho|\xi_x|.$$

In fact, if $|\xi|\ge(1/2)\lambda\rho|\xi_x|$, then by (18)

$$|\operatorname{grad} h_x|=|\operatorname{grad} g(y)-\xi_x|\le\frac{1}{4}|\xi_x|.$$

By Lemma 7 we have (23). When $N=0$, (23) holds for any ξ.

$$\hat{R}_\lambda(\xi,\rho)=(2\pi)^n(\lambda\rho)^k e^{-i\lambda\rho(x\cdot\xi)}\sum_{|\alpha|=2k}(-D_\xi-x)^\alpha\hat{G}_\alpha(\xi-\lambda\rho\xi_x;\lambda,\rho).$$

Therefore we have

(24) $$\begin{aligned}&e^{-i\lambda\sigma s}K(R_\lambda(y)e^{i\lambda\sigma s})\\&=e^{-i\lambda\sigma s}(\lambda\rho)^k\int_{R^{n+1}}K(x,\xi,\tau)e^{-i\lambda\rho x\cdot\xi}\sum_{|\alpha|=2k}(-D_\xi-x)^\alpha\hat{G}_\alpha(\xi-\lambda\xi_x\rho;\lambda,\rho)\\&\qquad\otimes\delta(\tau-\lambda\sigma)e^{i(x\cdot\xi+s\tau)}d\xi d\tau\\&=(\lambda\rho)^k\int_{R^n}K(x,\xi,\lambda\sigma)e^{ix\cdot(\xi-\lambda\rho\xi_x)}\sum_{|\alpha|=2k}(-D_\xi-x)^\alpha\hat{G}_\alpha(\xi-\lambda\xi_x\rho;\lambda,\rho)d\xi\\&=(\lambda\rho)^k\int_{R^n}K(x,\xi+\lambda\rho\xi_x,\lambda\sigma)e^{ix\cdot\xi}\sum_{|\alpha|=2k}(-D_\xi-x)^\alpha\hat{G}_\alpha(\xi;\lambda,\rho)d\xi\\&=(\lambda\rho)^k\int_{R^n}\sum_{|\alpha|=2k}(D_\xi^\alpha K(x,\xi+\lambda\rho\xi_x,\lambda\sigma))e^{ix\cdot\xi}\hat{G}_\alpha(\xi;\lambda,\rho)d\xi.\end{aligned}$$

If $\rho\ge1/4$, using (23) and (24), we have

(25) $$\begin{aligned}&e^{-i\lambda\sigma s}K(R_\lambda(y,\rho)e^{i\lambda\sigma s})(x,\lambda,\rho,\sigma)\\&\le C(\lambda\rho)^k\int_{|\xi|\ge(1/2)\lambda\rho|\xi_x|}|\xi|^{-N}d\xi+C(\lambda\rho)^k\int_{|\xi|<(1/2)\lambda\rho|\xi_x|}(|\xi_x\lambda\rho|+|\lambda\sigma|)^{s_0-2k}d\xi\\&\le C(\lambda\rho)^k[(\lambda\rho|\xi_x|)^{-N+n}+(|\xi_x|\lambda\rho)^{s_0+n-2k}]\\&\le C\left(\left(\frac{\lambda c}{4}\right)^{-N+n+k}+\left(\frac{\lambda c}{4}\right)^{n-k}\right)\qquad\text{uniformly in } \rho,\sigma.\end{aligned}$$

And if $\rho<1/4$, then $\sigma>1/4$ and we have for large k,

$$\begin{aligned}\sum_{|\alpha|=2k}|D_\xi^\alpha K(x,\xi+\lambda\rho\xi_x,\lambda\sigma)|&\le C(1+|\xi+\lambda\rho\xi_x|+|\lambda\sigma|)^{s_0-2k}\\&\le C(|\xi+\lambda\rho\xi_x|+|\lambda\sigma|)^{s_0-2k}.\end{aligned}$$

So that

(26)
$$e^{-i\lambda\sigma s}K(R_\lambda(y,\rho)e^{i\lambda\sigma t})$$
$$\leq C(\lambda\rho)^k\int_{R^n}(|\xi+\lambda\rho\xi_x|+|\lambda\sigma|)^{s_0-2k}d\xi$$
$$\leq C(\lambda\rho)^k\int_{R^n}(|\xi|+|\lambda\sigma|)^{s_0-2k}d\xi$$
$$\leq C(\lambda\rho)^k(\lambda\sigma)^{s_0-2k+n}\int_{R^n}(1+|\xi|)^{s_0-2k}d\xi$$
$$\leq C(\lambda\sigma)^{s_0-k+n}$$
$$\leq C(2\lambda)^{s_0-k+n} \qquad \text{uniformly in } \rho, \sigma .$$

(25) and (26) imply that the asymptotic expansion holds in $\mathscr{C}(\Omega\times S)$ topology.

Next we shall prove that the expansion holds in $\mathscr{E}(\Omega\times S)$ topology.

From (11) we have

(27)
$$D_jK(\varphi)=K_{(j)}(\varphi)+K(D_j\varphi) ,$$

where $K_{(j)}$ is the operator with the kernel $D_jK(x,\xi,\sigma)$. Hence we have

(28)
$$D_j(e^{-i\lambda(\rho g+s\sigma)}K(fe^{i\lambda(g\rho+s\sigma)}))$$
$$=-\lambda\rho\frac{\partial g}{\partial x_j}e^{-i\lambda(g\rho+s\sigma)}K(fe^{i\lambda(g\rho+s\sigma)})$$
$$+e^{-i\lambda(g\rho+s\sigma)}\Big[(K(D_jf)e^{i\lambda(g\rho+s\sigma)})+\lambda\rho K\Big(f\frac{\partial g}{\partial x_j}e^{i\lambda(g\rho+s\sigma)}\Big)$$
$$+K_{(j)}(fe^{i\lambda(g\rho+s\sigma)})\Big].$$

Therefore, $D_j(e^{-i\lambda(g\rho+s\sigma)}K(fe^{i\lambda(g\rho+s\sigma)}))$ admits an asymptotic expansion in $\mathscr{C}(\Omega\times S)$ topology. Since the differentiation is continuous in $\mathscr{D}'(\Omega\times S)$ this is the formal differentiation of $e^{-i\lambda(g\rho+s\sigma)}K(fe^{i\lambda(g\sigma+s\sigma)})$.

(29)
$$D_\rho e^{-i\lambda(g\rho+s\sigma)}K(fe^{i\lambda(g\rho+t\sigma)})=-\lambda ge^{-i\lambda(g\rho+s\sigma)}K(fe^{i(\rho g+s\sigma)})$$
$$+\lambda e^{i\lambda(g\rho+s\sigma)}K(fge^{i\lambda(\rho g+s\sigma)}) .$$

Therefore similar argument is valid for

$$D_\rho e^{-i\lambda(g\rho+s\sigma)}K(fe^{i\lambda(g\rho+s\sigma)}) .$$

If $\varphi\in\mathscr{D}(\Omega)$, there holds

(30)
$$K(\varphi\otimes e^{is\sigma})=(2\pi)^{-n-1}\int_{R^{n+1}}e^{i(x\cdot\xi+s\tau)}K(x,\xi,\tau)\hat{\varphi}(\xi)2\pi\delta(\tau-\sigma)d\tau d\xi$$
$$=(2\pi)^{-n}e^{is\sigma}\int_{R^n}e^{ix\cdot\xi}K(x,\xi,\sigma)\hat{\varphi}(\xi)d\xi .$$

So we have

$$D_\sigma[e^{-is\sigma}K(\varphi\otimes e^{is\sigma})]=(2\pi)^{-n}\int_{R^n}e^{ix\xi}D_\sigma K(x,\xi,\sigma)\hat{\varphi}(\xi)d\xi$$
$$=e^{-is\sigma}(D_\sigma K)(\varphi\otimes e^{is\sigma}) .$$

Here $(D_\sigma K)$ is the operator with the Kernel $D_\sigma K(x,\xi,\sigma)$. Hence we have

$$D_\sigma e^{-i\lambda(g\rho+s\sigma)}K(fe^{i\lambda(g\rho+s\sigma)})=e^{-i\lambda(g\rho+s\sigma)}(D_\sigma K)(fe^{i\lambda(g\rho+s\sigma)}) .$$

Therefore we can apply to this case the argument used in treating (28) and (29).

Repeating this process, we can prove Theorem 8 under the hypothesis (18).

Finally we must remove the hypothesis (18). At any point x_0 in F, there exists a relatively compact open neighbourhood U_x, with the following properties:

(31) For any $g\in B$, there is a function $g_{x_0}\in C^\infty(\Omega)$ such that

(32) $g_{x0}=g+\text{const. on } U_{x_0},\ \ |\text{grad}\, g_{x_0}(x)-\text{grad}\, g_{x_0}(y)|<\frac{1}{4}|\text{grad}\, g_{x_0}(x)|,$

for any x, y in F.

To prove this, we shall introduce a function $\varphi_n(x)=\varphi(n(x-x_0))$ where $\varphi(z)$ is a C_0^∞ function which is identically one in some neighbourhood of the origin and has its support in the unit ball. We have the estimate

(33) $$|\text{grad}\,\varphi_n(x)|\leq C\cdot n .$$

Now define a function

$$g_n(x)=\varphi_n(x)(g(x)-g(x_0))+(1-\varphi_n(x))\langle\xi_{x_0}, x-x_0\rangle$$

where $\xi_{x_0}=\text{grad}\, g(x_0)$. Then

$$\text{grad}\, g_n(x)=\varphi_n(x)\,\text{grad}\, g(x)+(1-\varphi_n(x))\xi_{x_0}$$
$$+(g(x)-g(x_0))\,\text{grad}\,\varphi_n(x)-\langle\xi_{x_0}, x-x_0\rangle\,\text{grad}\,\varphi_n(x)$$

$$|\xi_{x_0}-\text{grad}\, g_n(x)|\leq|\varphi_n(x)(\text{grad}\, g(x)-\xi_{x0})$$
$$+(g(x)-g(x_0)-\langle\xi_{x_0}, x-x_0\rangle)\,\text{grad}\,\varphi_n(x)| .$$

Since the support of φ_n is contained in the sphere $|x-x_0|\leq 1/n$, we have

$$|\xi_{x_0}-\text{grad}\, g_n(x)|\leq 0\left(\frac{1}{n}\right) \text{ uniformly for } g\in B \text{ and for } x\in F .$$

Therefore for our purpose we have only to choose large n and define $g_{x_0}=g_n$. Because F is compact, we can choose a finite open covering $\{U_j\}_{j=1}^J$, functions $g_j\in\mathcal{E}(\Omega)$ and constants a_j such that $|dg_j|>C$ and $g_j=g+a_j$ on U_j and g_j satisfies hypothesis (18), that is, (18)′ $$|\operatorname{grad} g_j(x)-\operatorname{grad} g_j(y)|\le \frac{1}{4}|\operatorname{grad} g_j(x)|, \qquad \forall x, y\in F.$$

Let $\{\phi_j\}$ be a C_0^∞ partition of unity subordinate to the covering $\{V_j\}$ which is a star refinement of $\{U_j\}$. Then,

(34) $$e^{-i\lambda(g\rho+s\sigma)}K(fe^{i\lambda(g\rho+s\sigma)})=\sum_{j,k}^{J}e^{-i\lambda(g\rho+s\sigma)}\phi_kK(\phi_jfe^{i\lambda(g\rho+s\sigma)}).$$

If $\operatorname{supp}\phi_j\cap\operatorname{supp}\phi_k\neq\phi$, then both $\operatorname{supp}\phi_j$ and $\operatorname{supp}\phi_k$ are contained in an open set U_l. Therefore

(35) $$e^{-i\lambda(g\rho+s\sigma)}\phi_kK(\phi_jfe^{i\lambda(g\rho+s\sigma)})=e^{-i\lambda(g_l\rho+s\sigma)}\phi_kK(\phi_jfe^{i\lambda(g_l\rho+s\sigma)}).$$

Since g_l satisfies (18)′, (35) admits an asymptotic expansion in λ in $\mathscr{E}(\Omega\times S)$.

If $\operatorname{supp}\phi_j\cap\operatorname{supp}\phi_k=\phi$, then we have

(36) $$e^{-i\lambda(\rho g+s\sigma)}\phi_kK(\phi_jfe^{i\lambda(\rho g+s\sigma)})=e^{-i\lambda\rho(g-g_j+a_j)}\phi_ke^{-i\lambda(g_j\rho+s\sigma)}K(\phi_jfe^{i\lambda(\rho g_j+s\sigma)}).$$

Since g_l satisfies (18)

$$\phi_ke^{-i\lambda(g_j\rho_j+s\sigma)}K(\phi_jfe^{i\lambda(g\rho+s\sigma)})$$

admits the asymptotic expansion (17) with $K_j=\phi_kK_j$ and $f=\phi_jf$. Therefore for any integer $N\ge 0$,

(37) $$\lambda^{-N}(e^{-i\lambda\rho(g-g_j+a_j)}\phi_ke^{-i\lambda(g_j\rho+s\rho)}K(\phi_jfe^{i\lambda(g_j\rho+s\sigma)})$$

is bounded in $\mathscr{E}(\Omega\times S)$. This, together with (34), (35) and (36) completes our proof.

If $u\in\mathscr{S}(\boldsymbol{R}^k)$, we introduce for any real s, the norm

(38) $$\|u\|_s=\|u\|_{H^s(\boldsymbol{R}^k)}=(2\pi)^{-n/2}\left[\int|\hat{u}(\eta)|^2(1+|\eta|)^{2s}d\eta\right]^{1/2}.$$

THEOREM 9. *If K is the operator given by* (11), *then for any fixed $\varphi\in\mathscr{D}(\Omega)$ and $a\in\boldsymbol{R}$, there exists a constant C such that for any $u\in\mathscr{D}(\Omega)\hat{\otimes}\mathscr{S}(\boldsymbol{R}^1)$,*

(39) $$\|e^{-i(x\cdot\xi+s\sigma)}\varphi K(e^{i(x\cdot\xi+s\sigma)}u)\|_{H^{a-s_0}(\boldsymbol{R}^{n+1})}\le C(1+|\xi|+|\sigma|)^{|s_0|}\|u\|_{H^a(\boldsymbol{R}^{n+1})}.$$

Besides this, if $s_0\le 0$, there holds for $\forall b\in[s_0,-s_0]$

(40) $$\|e^{-i(x\cdot\xi+s\sigma)}\varphi K(e^{i(x\cdot\xi+s\sigma)}u)\|_{H^{a+b}(\boldsymbol{R}^{n+1})}\le C(1+|\xi|+|\sigma|)^{b}\|u\|_{H^a(\boldsymbol{R}^{n+1})}.$$

PROOF. First we shall prove with $a_1+b_1=s_0$ the inequality

(41) $$|\langle e^{-i(x\cdot\xi+s\sigma)}K(e^{i(x\cdot\xi+s\sigma)}u),\varphi v\rangle|\le C(1+|\xi|+|\sigma|)^s\|u\|_{H^a(\boldsymbol{R}^{n+1})}\|v\|_{H^b(\boldsymbol{R}^{n+1})}$$

holds for any $v\in\mathscr{S}(\boldsymbol{R}^{n+1})$. We have

(42) $$\langle e^{-i(x\cdot\xi+s\sigma)}K(e^{i(x\cdot\xi+s\sigma)}u),\ \varphi v\rangle$$
$$=(2\pi)^{-n-1}\int_{R^{n+1}} v(x,s)\varphi(x)dxds\int_{R^{n+1}} e^{i(x\cdot\zeta+s\rho)}K(x,\zeta+\xi,\rho+\sigma)\hat{u}(\zeta,\rho)d\zeta d\rho\ .$$

To compute this we introduce the Fourier transform

$$h(\eta,\tau:\zeta,\rho:\xi,\sigma)=\int_{R^{n+1}} e^{-i(x\cdot\eta+s\tau)}\varphi(x)K(x,\zeta+\xi,\rho+\sigma)dxds$$
$$=h_1(\eta:\zeta,\rho:\xi,\sigma)\otimes\delta(\tau)\ .$$

By partial integration, from (10), we have for any N,

(43) $$|h_1(\eta:\zeta,\rho:\xi,\sigma)(1+|\eta|)^N(1+|\zeta+\xi|+|\rho+\sigma|)^{-s_0}|<c\ .$$

Now we have

(44) $$\langle e^{-i(x\cdot\xi+s\sigma)}K(e^{i(x\cdot\xi+s\sigma)}u),\ \varphi v\rangle$$
$$=\int_{R^{n+1}}\int_{R^{n+1}}\hat{v}(\eta,\tau)h(-\eta-\zeta,-\tau-\rho:\zeta,\rho:\xi,\sigma)\hat{u}(\zeta,\rho)d\zeta d\rho d\eta d\tau$$
$$=\int_{R^{n+1}}\int_{R^{n}}\hat{v}(\eta,-\rho)h_1(-\eta-\zeta;\zeta,\rho:\xi,\sigma)\hat{u}(\zeta,-\rho)d\zeta d\rho d\eta\ .$$

Setting

$$V(\eta,\rho)=\hat{v}(\eta,-\rho)(1+|\eta|^2+\rho^2)^{b_1/2}$$
$$U(\zeta,\rho)=\hat{u}(\zeta,-\rho)(1+|\zeta|^2+\rho^2)^{a_1/2}$$
$$H(\eta:\zeta,\rho:\xi,\sigma)=(1+|\eta|^2+\rho^2)^{-b_1/2}(1+|\zeta|^2+\rho^2)^{-a_1/2}\times h_1(-\eta-\zeta:\zeta,l:\xi,\sigma)\ ,$$

we write (44) as

(45) $$\langle e^{-i(x\cdot\xi+s\sigma)}K(e^{i(x\cdot\xi+s\sigma)}u),\ \varphi v\rangle$$
$$=\int_{R^{n+1}}\int_{R^{n}}H(\eta:\zeta,\rho:\xi,\sigma)V(\eta,\rho)\cdot U(\zeta,\rho)d\zeta d\rho d\eta\ .$$

From (43) follows

(46) $$|H(\eta:\zeta,\rho:\xi,\sigma)|\leq C(1+|-\eta-\zeta|)^{-N}(1+|\zeta+\xi|+|\rho+\sigma|)^{s_0}$$
$$\times(1+|\eta|^2+\rho^2)^{-b_1/2}(1+|\zeta|^2+\rho^2)^{-a_1/2}\ .$$

For any $b_1\in\boldsymbol{R}$, there holds

(47) $$(1+|\eta|^2+\rho^2)^{-b_1/2}\leq(1+|-\eta-\zeta|)^{|b_1|}(1+|\zeta|^2+\rho^2)^{-b_1/2}\ ,$$

because $(1+|\eta|+|\rho|)\leq(1+|-\eta-\zeta|)(1+|\zeta|+|\rho|)$ and $(1+|-\eta-\zeta|)^{-1}(1+|\eta|+|\rho|)^{-1}\leq(1+|\zeta|+|\rho|)^{-1}$ hold. Thus we have

(48) $$|H(\eta:\zeta,\rho,\xi,\sigma)|\leq C(1+|-\zeta-\eta|)^{-N+b_1}(1+|\zeta|^2+\rho^2)^{-(a_1+b_1)/2}$$
$$\times(1+|\zeta+\xi|+|\rho+\sigma|)^{s_0}\ .$$

If $a_1+b_1=s_0$, then the estimate

$$(1+|\zeta+\xi|+|\rho+\sigma|)^{s_0}\leq(1+|\zeta|^2+\rho^2)^{s/2}(1+|\xi|+|\sigma|)^{|s_0|}$$

holds, so that

$$|H(\eta:\zeta,\rho,\xi,\sigma)|\leq C(1+|-\zeta-\eta|)^{-N+|b_1|}(1+|\xi|+|\sigma|)^{|s_0|}.$$

Since $\int_{R^n}(1+|\zeta|)^{-N+b_1}d\zeta<\infty$, from (45) we have

$$\begin{aligned}&|\langle e^{-i(x\cdot\xi+s\sigma)}K(e^{i(x\cdot\xi+s\sigma)}u),\varphi v\rangle|\\&\leq C(1+|\xi|+|\sigma|)^{|s_0|}\int_{R^1}d\rho\int_{R^n}\int_{R^n}(1+|-\zeta-\eta|)^{-N+|b_1|}|V(\eta,\rho)U(\zeta,\rho)|\,d\zeta d\eta\\&\leq C(1+|\xi|+|\sigma|)^{|s_0|}\int_{R^1}\Big[\int_{R^n}|U(\zeta,\rho)|^2d\zeta\Big]^{1/2}\Big[\int_{R^n}|V(\eta,\rho)|^2d\eta\Big]^{1/2}d\rho\\&\leq C(1+|\xi|+|\sigma|)^{|s_0|}\Big[\int|U(\zeta,\rho)|^2d\zeta d\rho\Big]^{1/2}\Big[\int|V(\eta,\rho)|^2d\eta d\rho\Big]^{1/2}\\&\leq C(1+|\xi|+|\sigma|)^{|s_0|}\|u\|_{a_1}\|v\|_{b_1}.\end{aligned}$$

Thus we have proved (41). From (41), (39) follows easily. If $s_0\leq 0$, we shall prove, with $a_1+b_1=-s_0$

(49) $$|\langle e^{-i(x\cdot\xi+s\sigma)}K(e^{i(x\cdot\xi+s\sigma)}u),\varphi v\rangle|\leq C(1+|\xi|+|\sigma|)^{s_0}\|u\|_{a_1}\|v\|_{b_1}.$$

In fact, then we have from (48),

(50) $$\begin{aligned}&|(1+|\xi|+|\sigma|)^{-s_0}H(\eta:\zeta,\rho:\xi,\sigma)|\\&\leq C(1+|-\zeta-\eta|)^{-N+|b_1|}(1+|\zeta|^2+\rho^2)^{s_0/2}(1+|\zeta+\xi|+|\rho+\sigma|)^{s_0}(1+|\xi|+|\sigma|)^{-s_0}\\&\leq C(1+|-\zeta-\eta|)^{-N+|b_1|}(1+|\xi|+|\sigma|)^{s_0}(1+|\xi|+|\sigma|)^{-s_0}\\&\leq C(1+|-\zeta-\eta|)^{-N+|b_1|}.\end{aligned}$$

Since $\int(1+|-\zeta-\eta|)^{-N+|b_1|}d\zeta=\int(1+|-\xi-\eta|)^{-N+|b_1|}d\eta<\infty$, we have, from (45) and (50),

$$\begin{aligned}&|\langle e^{-i(x\cdot\xi+s\sigma)}K(e^{i(x\cdot\xi+s\sigma)}u),\varphi v\rangle|(1+|\xi|+|\sigma|)^{-s_0}\\&\leq\int_{R^1}d\rho\int_{R^n}\int_{R^n}(1+|-\eta-\zeta|)^{-N+|b_1|}|V(\eta,\rho)U(\zeta,\rho)|d\zeta d\eta\leq C\|u\|_{a_1}\|v\|_{b_1}.\end{aligned}$$

This proves (49), the estimate (40) is a direct consequence of (49), if $b=s_0$.

By the theory of interpolation (see J.L. Lions and J. Peetre [3] and E. Magenes [7]), (40), for general b, follows easily from (49) and (50).

REMARK. Similar estimate holds for usual Fourier integral operator K. Namely, let Ω be an open set in $\boldsymbol{R}^n$, and K be a function in $\mathscr{E}(\Omega\times\boldsymbol{R}^n)$ such that, there exist functions $K_j(x, \xi) \in \mathscr{E}(\Omega \times (\boldsymbol{R}^n - \{0\}))$ which are positively homogeneous of degree $z_j = s_j + it_j$ with $s_j \to -\infty$ and for all N, α, β,

$$(1+|\xi|)^{-(\beta+s_N)} D_\xi^\beta \Big(K(x, \xi) - \sum_0^{N-1} K_j(x, \xi) \Big) \tag{51}$$

remains bounded in $\mathscr{E}(\Omega)$, when $|\xi| \to \infty$. Then for any fixed $\varphi \in \mathscr{D}(\Omega)$, $a \in \boldsymbol{R}$, there is a constant C such that for any $u \in \mathscr{D}(\Omega)$

$$\| \varphi e^{-ix\cdot\xi} K(e^{ix\cdot\xi} u) \|_{a-s_0} \leq C(1+|\xi|)^{|s_0|} \| u \|_a . \tag{52}$$

If $s_0 \leq 0$, we have, for any $b \in [s_0, -s_0]$,

$$\| \varphi e^{-ix\cdot\xi} K(e^{ix\cdot\xi} u) \|_{a+b} \leq C(1+|\xi|)^b \| u \|_a . \tag{53}$$

The proof is quite similar to the proof of Theorem 9.

THEOREM 10. *If K is the operator given by* (11) *with $s_0 \leq 0$, then for any fixed $\varphi \in \mathscr{D}(\Omega)$ and $a \in \boldsymbol{R}$ there exists a constant C such that for any b, $|b| \leq |s_0|$ and for any $u \in \mathscr{D}(\Omega)$, we have*

$$\| e^{-i(x\cdot\xi + s\sigma)} K(u e^{i(x\cdot\xi + s\sigma)}) \|_{H^{a+b}(\boldsymbol{R}^n)} \leq C(1+|\xi|+|\sigma|)^b \| u \|_{H^a(\boldsymbol{R}^n)} . \tag{54}$$

PROOF.

$$\begin{aligned} &e^{-i(x\cdot\xi+s\sigma)} K(u e^{i(x\cdot\xi+s\sigma)}) \\ &\quad = (2\pi)^{-n} \int_{\boldsymbol{R}^n} \varphi(x) K(x, \zeta, \sigma) \hat{u}(\zeta - \xi) e^{-ix\cdot(\eta-\xi)} d\zeta \\ &\quad = (2\pi)^{-n} \int_{\boldsymbol{R}^n} \varphi(x) K(x, \zeta+\xi, \sigma) \hat{u}(\zeta) e^{ix\cdot\zeta} d\zeta . \end{aligned}$$

For any $v \in \mathscr{D}(M)$, we have

$$\begin{aligned} &\langle \varphi e^{-i(x\cdot\xi+s\sigma)} K(u e^{i(x\cdot\xi+s\sigma)}), v \rangle \\ &\quad = (2\pi)^{-n} \int_{\boldsymbol{R}^n} v(x) \varphi(x) dx \cdot \int_{\boldsymbol{R}^n} K(x, \zeta+\xi, \sigma) \hat{u}(\zeta) e^{ix\cdot\zeta} d\zeta . \end{aligned} \tag{55}$$

Introducing Fourier transform

$$h(\eta, \zeta, \xi, \sigma) = \int_{\boldsymbol{R}^n} e^{-ix\cdot\eta} \varphi(x) K(x, \zeta+\xi, \sigma) dx .$$

We can write this as

$$\begin{aligned} &\langle \varphi e^{-i(x\cdot\xi+s\sigma)} K(u e^{i(x\cdot\xi+s\sigma)}), v \rangle \\ &\quad = \int_{\boldsymbol{R}^n} \int_{\boldsymbol{R}^n} \hat{v}(\eta) h(-\eta-\zeta, \zeta : \xi, \sigma) \hat{u}(\zeta) d\zeta d\eta . \end{aligned} \tag{56}$$

By partial integration we obtain from (10), for any $N > 0$,

(57) $$|h(\eta, \zeta: \xi, \sigma)|(1+|\eta|)^N(1+|\zeta+\xi|+|\sigma|)^{-s_0}|<C .$$

Setting

$$V(\eta)=(1+|\eta|^2)^{b_1/2}\hat{v}(\eta)$$
$$U(\zeta)=(1+|\zeta|^2)^{a_1/2}\hat{u}(\zeta)$$
$$H(\eta, \zeta: \xi, \sigma)=h(-\eta-\zeta, \zeta, \xi, \sigma)(1+|\eta|^2)^{-b_1/2}(1+|\zeta|^2)^{-a_1/2} ,$$

we have

(58) $$|H(\eta, \zeta: \xi, \sigma)|\le C(1+|\eta+\zeta|)^{-N}(1+|\zeta|^2)^{-a_1/2}(1+|\zeta+\xi|+|\sigma|)^{s_0}(1+|\eta|^2)^{-b_1/2}$$

and

(59) $$\langle\varphi e^{-i(x\cdot\xi+s\sigma)}K(ue^{i(x\cdot\xi+s\sigma)}), v\rangle=\int_{R^n}\int_{R^n}H(\eta, \zeta, \xi, \sigma)V(\eta)U(\zeta)d\eta d\zeta .$$

If $a_1+b_1=s_0<0$

$$(1+|\zeta+\xi|+|\sigma|)^{s_1}\le(1+|\zeta|)^{s_0}(1+|\xi|+|\sigma|)^{-s_0}$$
$$(1+|\eta|)^{-b_1/2}\le(1+|-\eta-\zeta|)^{b_1/2}(1+|\zeta|^2)^{-b_1/2} .$$

Thus

$$|H(\eta, \zeta: \xi, \sigma)|\le C(1+|\eta+\zeta|)^{-N+|b_1|}(1+|\zeta|^2)^{-(a_1+b_1)/2+s_0}(1+|\xi|+|\sigma|)^{-s_0} ,$$

so that

$$\int|H(\eta, \zeta: \xi, \sigma)|d\eta\le C(1+|\xi|+|\sigma|)^{-s_0} ,$$
$$\int|H(\eta, \zeta: \xi, \sigma)|d\zeta\le C(1+|\xi|+|\sigma|)^{-s_0} .$$

Therefore from (59),

$$\begin{aligned}|\langle\varphi e^{-i(x\cdot\xi+s\sigma)}K(e^{i(x\cdot\xi+s\sigma)}u), v\rangle|&\le C\|V(\eta)\|_{L^2}\|U(\zeta)\|_{L^2}(1+|\xi|+|\sigma|)^{-s_0}\\ &=C\|u\|_a\|v\|_b(1+|\xi|+|\sigma|)^{-s_0} ,\end{aligned}$$

so that,

(60) $$\|\varphi e^{-i(x\cdot\xi+s\sigma)}K(e^{i(x\cdot\xi+s\sigma)}u)\|_{H^{a-s_0}(M)}\le C\|u\|_{H^a(M)}(1+|\xi|+|\sigma|)^{-s_0} .$$

Next, $a_1+b_1=-s_0\ge0$, then by (58)

$$\begin{aligned}&|(1+|\xi|+|\sigma|)^{-s_0}H(\eta, \zeta, \xi, \sigma)|\\ &\le C(1+|\xi|+|\sigma|)^{-s_0}(1+|\eta+\zeta|)^{-N}(1+|\zeta|^2)^{-a_1/2}(1+|\zeta+\xi|+|\eta|)^{s_0}(1+|\eta|^2)^{-b_1/2}\\ &\le C(1+|\xi|+|\sigma|)^{-s_0}(1+|\eta+\zeta|)^{-N}(1+|\zeta|^2)^{-a_1/2}(1+|\zeta|)^{-s_0}(1+|\xi|+|\sigma|)^{s_0}(1+|\eta|^2)^{-b_1/2}\\ &\le C(1+|\eta+\zeta|)^{-N}(1+|\zeta|^2)^{-a_1/2}(1+|\zeta|)^{-s_0}(1+|\eta|^2)^{-b_1/2}\\ &\le C(1+|\eta+\zeta|)^{-N}(1+|\zeta|^2)^{-a_1/2-s_0-b_1/2}(1+|\zeta+\eta|)^{b_1/2}\le C(1+|\eta+\zeta|)^{-N+|b_1|/2} .\end{aligned}$$

Thus

$$\int |H(\eta, \zeta, \xi, \sigma)|d\eta \leq C(1+|\xi|+|\sigma|)^{s_0},$$
$$\int |H(\eta, \zeta, \xi, \sigma)|d\zeta \leq C(1+|\xi|+|\rho|)^{s_0},$$

Hence, from (59),

(61) $$|\langle \varphi e^{-i(x\cdot\xi+s\sigma)}K(e^{i(x\cdot\xi+s\sigma)}u), v\rangle| \leq C\|u\|_{H^a(R^n)}\|v\|_{H^b(R^n)}(1+|\xi|+|\sigma|)^{s_0}.$$

This implies

(62) $$\|\varphi e^{-i(x\cdot\xi+s\sigma)}K(ue^{i(x\cdot\xi+s\sigma)})\|_{H^{a+s}(R^n)} \leq C(1+|\xi|+|\sigma|)^{s_0}\|u\|_{H^a(M)}.$$

Interpolating (60) and (62), we have

$$\|\varphi e^{-i(x\cdot\xi+s\sigma)}K(ue^{i(x\cdot\xi+s\sigma)})\|_{H^{a+b}(R^n)} \leq C(1+|\xi|+|\sigma|)^{b}\|u\|_{H^a(M)}$$

for $s_0 \leq b \leq -s_0$.

THEOREM 11. *If K is the operator given by* (11) *with $s_0 \leq 0$, then for any fixed $\varphi \in \mathscr{D}(\Omega)$, there exists a constant C such that for any $u \in \mathscr{D}(\Omega)$, $0 \leq b \leq -s_0$,*

(63) $$\|\varphi e^{-is\sigma}K(ue^{is\sigma})\|_{H^{a+b}(R^n)} \leq C(1+|\sigma|)^{b+s_0}\|u\|_{H^a(R^n)}.$$

PROOF. For any $v \in \mathscr{S}'(\boldsymbol{R}^n)$, by (59),

(64) $$\langle \varphi e^{-is\sigma}K(ue^{is\sigma}), v\rangle = \int_{R^n}\int_{R^n} H(\eta, \zeta: 0, \sigma)V(\eta)U(\zeta)d\eta d\zeta.$$

And for $\forall N>0$,

(65) $$|H(\eta, \zeta: 0, \sigma)| \leq C(1+|\eta+\zeta|)^{-N}(1+|\zeta|^2)^{-a_1/2}(1+|\eta|^2)^{-b_1/2}(1+|\zeta|+|\sigma|)^{s_0}.$$

If $a_1+b_1=0$, then

$$\begin{aligned}|H(\eta, \zeta, 0, \sigma)| &\leq C(1+|\eta+\zeta|)^{-N+|b_1|}(1+|\zeta|)^{-(a_1+b_1)/2}(1+|\sigma|)^{s_0}\\ &= C(1+|\eta+\zeta|)^{-N+b_1}(1+|\sigma|)^{s_0}\end{aligned}$$

and

(66) $$\int_{R^n} H(\eta, \zeta, 0, \sigma)d\eta \leq C(1+|\sigma|)^{s_0}$$
$$\int_{R^n} H(\eta, \zeta: 0, \sigma)d\zeta \leq C(1+|\sigma|)^{s_0}.$$

By (64), (66), we have, if $a+b=0$,

$$|\langle \varphi e^{-is\sigma}K(ue^{is\sigma}), v\rangle| \leq C(1+|\sigma|)^{s_0}\|u\|_a\|v\|_b.$$

From this

(67) $$\|\varphi e^{-is\sigma}K(ue^{is\sigma})\|_a \leq C(1+|\sigma|)^{s_0}\|u\|_a .$$

If $a+b=s_0$, from (65) we have

(68) $$|H(\eta, \zeta, 0, \sigma)| \leq C(1+|\eta+\zeta|)^{-N+|b|}(1+|\zeta|+|\sigma|)^{s_0} \leq C(1+|\eta+\zeta|)^{-N+|b|} .$$

So that we have

(69) $$\int_{R^n} |H(\eta, \zeta, 0, \sigma)|d\eta < C. \quad \int_{R^n} |H(\eta, \zeta, 0, \sigma)|d\zeta < C .$$

From (64) and (69), we have

(70) $$|\langle \varphi e^{-is\sigma}K(ue^{is\sigma}), v\rangle| \leq C\|u\|_a\|v\|_b .$$

Therefore we have proved

(71) $$\|\varphi e^{-is\sigma}K(ue^{is\sigma})\|_{a-s_0} \leq C\|u\|_a .$$

Interpolating (67) and (71), we obtain the estimate (63).

THEOREM 12. *If K is the operator given by* (11) *satisfying* (10), *with* $s_0=-\infty$, *the mapping K has the unique continuous extension from* $\mathscr{E}'(\Omega)\hat{\otimes}\mathscr{S}'(\boldsymbol{R}^1)$ *to* $\mathscr{E}(\Omega)\hat{\otimes}\mathscr{O}_M(\boldsymbol{R}^1)$.

We call these operators K operators of order $-\infty$.

PROOF. Let $\varphi \in \mathscr{D}(\Omega)$, $\phi \in \mathscr{S}'(\boldsymbol{R}^1)$, then

(72) $$K(\varphi\otimes\phi)(x, s) = (2\pi)^{-n-1}\int_{R^{n+1}} K(x;\xi, \sigma)\hat{\varphi}(\xi)\otimes\hat{\phi}(\sigma)e^{i(x\cdot\xi+s\sigma)}d\xi d\sigma .$$

Since $s_0=-\infty$,

(73) $$K(x, \xi, \sigma) \in \mathscr{E}(\Omega)\hat{\otimes}\mathscr{S}(\boldsymbol{R}^{n+1}) = \mathscr{E}(\Omega)\hat{\otimes}\mathscr{S}(\boldsymbol{R}^n)\hat{\otimes}\mathscr{S}(\boldsymbol{R}^1) .$$

Therefore we can define the bilinear mapping

$$B: \quad \begin{array}{ccc} \mathscr{S}'(\boldsymbol{R}^n)\times\mathscr{S}'(\boldsymbol{R}^1) & \to & \mathscr{E}(\Omega)\hat{\otimes}\mathscr{O}_c'(\boldsymbol{R}^n)\hat{\otimes}\mathscr{O}_c'(\boldsymbol{R}^1) \\ \cup & & \cup \\ (\varphi, \phi) & \to & K(x, \xi, \sigma)\hat{\varphi}(\xi)\hat{\phi}(\sigma) , \end{array}$$

which is separately continuous. Since $\mathscr{S}'(\boldsymbol{R}^n)$, $\mathscr{S}'(\boldsymbol{R}^1)$ are barrelled $(\mathscr{D}\mathscr{F})$ spaces, this mapping B is continuous. The Fourier inverse transformation induces a continuous linear mapping from $\mathscr{E}(\Omega)\hat{\otimes}\mathscr{O}_c'(\boldsymbol{R}^n)\hat{\otimes}\mathscr{O}_c'(\boldsymbol{R}^1)$ to $\mathscr{E}(\Omega)\hat{\otimes}\mathscr{O}_M(\boldsymbol{R}^n)\hat{\otimes}\mathscr{O}_M(\boldsymbol{R}^1)$, defined by

$$\Psi: \quad g(x, \xi, \sigma) \to (2\pi)^{-n-1}\int_{R^{n+1}} g(x, \xi, \sigma)e^{i(x\cdot\xi+s\sigma)}d\xi d\sigma .$$

The following mapping Φ induced by multiplication of functions in $\mathscr{E}(\Omega)$ and in $\mathscr{O}_M(\boldsymbol{R}^n)$ is continuous

$$\Phi: \qquad \mathscr{E}(\Omega)\hat{\otimes}\mathscr{O}_M(\boldsymbol{R}^n)\hat{\otimes}\mathscr{O}_M(\boldsymbol{R}^1) \to \mathscr{E}(\Omega)\hat{\otimes}\mathscr{O}_M(\boldsymbol{R}^1)\ ,$$

because the mapping $\mathscr{E}(\Omega)\times\mathscr{O}_M(\boldsymbol{R}^n)\to\mathscr{E}(\Omega)$ is the composition of two continuous mappings $\mathscr{E}(\Omega)\times\mathscr{O}_M(\boldsymbol{R}^n)\to\mathscr{E}(\Omega)\otimes\mathscr{E}(\Omega)$ and $\mathscr{E}(\Omega)\times\mathscr{E}(\Omega)\to\mathscr{E}(\Omega)$. Composing B, Ψ and Φ, we obtain a bilinear continuous mapping from $\mathscr{S}'(\boldsymbol{R}^n)\times\mathscr{S}'(\boldsymbol{R}^1)$ to $\mathscr{E}(\Omega)\hat{\otimes}\mathscr{O}_M(\boldsymbol{R}^1)$ which can symbolically be written as

$$(\varphi, \phi) \to (2\pi)^{-1}\int_{R^{n+1}} K(x, \xi, \sigma)\hat{\varphi}(\xi)\hat{\phi}(\sigma)e^{i(x\cdot\xi+s\sigma)}d\xi d\sigma\ .$$

This last mapping induces the continuous linear mapping

$$\begin{array}{ccc} \mathscr{E}'(\Omega)\hat{\otimes}\mathscr{S}'(\boldsymbol{R}^1) & \to & \mathscr{E}(\Omega)\hat{\otimes}\mathscr{O}_M(\boldsymbol{R}^1) \\ \cup & & \cup \\ \varphi\otimes\phi & \to & (2\pi)^{-1}\int_{R^{n+1}} K(x, \xi, \sigma)\hat{\varphi}(\xi)\hat{\phi}(\sigma)e^{i(x\cdot\xi+s\sigma)}d\xi d\sigma\ . \end{array}$$

Since $\mathscr{D}(\Omega)\hat{\otimes}\mathscr{S}(\boldsymbol{R}^1)$ is dense in $\mathscr{E}'(\Omega)\hat{\otimes}\mathscr{S}'(\boldsymbol{R}^1)$ and the mapping $\tilde{K}$ that we have just defined is identical with the mapping K on $\mathscr{D}(\Omega)\hat{\otimes}\mathscr{S}(\boldsymbol{R}^1)$, $\tilde{K}$ is the unique extension of K to $\mathscr{E}'(\Omega)\hat{\otimes}\mathscr{S}'(\boldsymbol{R}^1)$.

§3. Calculus of β-pseudo-differential operators

We have proved in Lemma 5 that if P is a β-pseudo-differential operator on $\Omega\subset\boldsymbol{R}^n$ and if f is in $\mathscr{D}(\Omega)$, the map $\mathscr{D}(\Omega)\hat{\otimes}\mathscr{S}'(\boldsymbol{R}^1)\ni u\to P(f\cdot u)$ is of the form (11) with $K=p(f, x, \xi, \sigma)$. Therefore Lemma 6 implies

$$\sum_j p_j(uf, x, \xi, \sigma)\sim\sum_{\alpha,j}\frac{1}{\alpha!}D^\alpha p_j(f, x, \xi, \sigma)D_x^\alpha u(x)\ . \tag{74}$$

THEOREM 13. *Let P be a β-pseudo-differential operator on $\Omega\subset\boldsymbol{R}^n$ and $f_1, f_2\in\mathscr{D}(\Omega)$, $f_1=f_2$ in some neighbourhood of $x\in\Omega$, then*

$$p_j(f_1, x, \xi, \sigma)=p_j(f_2, x, \xi, \sigma)\ . \tag{75}$$

PROOF. If $u\in\mathscr{D}(\Omega)$ and $u\equiv1$ in some open neighbourhood of x where $f_1=f_2$, then $uf_1=uf_2$. So that we have

$$\begin{aligned}\sum_j p_j(f_1; x, \xi, \sigma) &= \sum_{j,\alpha}\frac{1}{\alpha!}D_\xi^\alpha p_j(f_1; x, \xi, \sigma)D_x^\alpha u(x)\\ &=\sum_j p_j(uf_1; x, \xi, \sigma)\\ &=\sum_j p_j(uf_2; x, \xi, \sigma)\\ &=\sum_{j,\alpha}\frac{1}{\alpha!}D_\xi^\alpha p_j(f_2, x, \xi, \sigma)D_x^\alpha u(x)\\ &=\sum p_j(f_2; x, \xi, \sigma)\ .\end{aligned}$$

From Theorem 13, it is possible to adopt the following

DEFINITION 14. If P is a β-pseudo-differential operator, we define $p_j(x, \xi, \sigma)$, $x \in \Omega$ as $p_j(f; x, \xi, \sigma)$, where $f \in \mathcal{D}(\Omega)$ with $f \equiv 1$ in some neighbourhood of x.

THEOREM 15. *If P is a β-pseudo-differential operator,*

$$\sum_j p_j(u; x, \xi, \sigma) = \sum_{\alpha, j} \frac{1}{\alpha!} D_\xi^\alpha p_j(x, \xi, \sigma) D_x^\alpha u(x) . \tag{76}$$

PROOF. With a function $f \in \mathcal{D}(\Omega), f \equiv 1$ in some neighbourhood of x, we have

$$\begin{aligned} \sum_j p_j(u, x, \xi, \sigma) &= \sum_j p_j(fu; x, \xi, \sigma) \\ &= \sum_{j, \alpha} \frac{1}{\alpha!} D_\xi^\alpha p_j(f; x, \xi, \sigma) D_x^\alpha u \\ &= \sum_{j, \alpha} \frac{1}{\alpha!} D_\xi^\alpha p_j(x, \xi, \sigma) D_x^\alpha u . \end{aligned}$$

THEOREM 16. *Let p be a continuous linear map from $\mathcal{D}(\Omega) \hat{\otimes} \mathcal{S}'(\boldsymbol{R}^1)$ to $\mathcal{E}(\Omega) \hat{\otimes} \mathcal{S}'(\boldsymbol{R}^1)$ such that $e^{-i\lambda(x \cdot \xi + s\sigma)} P(f e^{i(x \cdot \xi + s\sigma)})$ is independent of s and an asymptotic expansion*

$$e^{-i\lambda(x \cdot \xi + s\sigma)} P(f e^{i\lambda(x \cdot \xi + s\sigma)}) \sim \sum_0^\infty p_j(f; x, \xi, \sigma) \lambda^{2j}$$

holds in $\mathcal{E}(\Omega \times S_1)$, $S_1 = \{1/2 \leq |\xi|^2 + \sigma^2 \leq 2\}$. Then P is a β-pseudo-differential operator, and the symbol of P at the point x is given by

$$\sigma_p(f, g) = \sum \frac{1}{\alpha!} D_\xi^\alpha p_j(\xi_x, \sigma) D^\alpha (f e^{ih_x}) \tag{77}$$

where

$$\xi_x = \text{grad } g(x), \qquad h_x(y) = g(y) - g(x) - \langle y - x, \xi_x \rangle . \tag{78}$$

If $f \in \mathcal{D}(\Omega)$, the operator $u \to P(fu)$ is, by Lemma 5, of the form (11). Hence from Theorem 8, it follows that this is a β-pseudo-differential operator and that

$$\sum_j p_j(uf, g, x, \sigma) = \sum_{\alpha, j} \frac{1}{\alpha!} D_\xi^\alpha p_j(f, \xi_x, \sigma) D_x^\alpha (u e^{ih_x}) .$$

Taking $f \equiv 1$ in some neighbourhood of x, we obtain (77).

REMARK 17. It is obvious that a β-pseudo-differential operator on $\Omega \times \boldsymbol{R}^1$ is a pseudo-differential operator on $\Omega \times \boldsymbol{R}^1$ in the sense of Hörmander [3].

THEOREM 18. *Let P, Q be β-pseudo-differential operators on $\Omega \times \boldsymbol{R}^1$, Ω is open in $\boldsymbol{R}^n$ and let $f \in \mathcal{D}(\Omega)$. Then $R = QfP$ is also a β-pseudo-differential operator and we have*

$$\sum_i r_i(x, \xi, \sigma) = \sum_{\alpha, j, k} \frac{1}{\alpha!} D_\xi^\alpha q_k(x, \xi, \sigma) D_x^\alpha(fp_j(x, \xi, \sigma)) . \tag{79}$$

PROOF. Let $u \in \mathscr{D}(\Omega)$, and let $g \in \mathscr{E}(\Omega)$ be real valued and $dg \neq 0$ in supp u. Then for any ρ, σ in $S = \{(\rho, \sigma) \in \boldsymbol{R}^2, 1/2 \leq \rho^2 + \sigma^2 \leq 2\}$

$$e^{i\lambda(g\rho+s\sigma)} fP(e^{i\lambda(g\rho+s\sigma)}u) \sim \sum_0^\infty fp_j(u, g; x, \sigma, \rho)\lambda^{z_j} \tag{80}$$

in $\mathscr{E}(\Omega \times S_1)$. Thus

$$e^{-i\lambda(g\rho+s\sigma)} R(ue^{i\lambda(g\rho+s\sigma)}) = e^{-i\lambda(g\rho+s\sigma)} Q(e^{i\lambda(g\rho+s\sigma)} e^{-i\lambda(g\rho+s\sigma)} f P(ue^{i\lambda(g\rho+s\sigma)}))$$
$$\sim \sum_{j,k=0}^\infty Q_k(fp_j(u, g; x, \sigma, \rho))\lambda^{z_j+z'_k} \quad \text{in } \mathscr{E}(\Omega \times S_1) .$$

Therefore R is a β-pseudo-differential operator. Setting $g = \langle x, \xi \rangle$, we obtain (79).

THEOREM 19. *To every β-pseudo-differential operator P there is one and only one β-pseudo-differential operator ${}^tP$, called its formal adjoint, such that*

$$\langle Pu, v \rangle = \langle u, {}^tPv \rangle \tag{81}$$

for any $u \in \mathscr{D}(\Omega) \otimes \mathscr{S}(\boldsymbol{R}^1)$ and $v \in \mathscr{D}(\Omega) \otimes \mathscr{S}'(\boldsymbol{R}^1)$. The symbol of ${}^tP$ is given by

$$\sum_j {}^tp_j(x, \xi, \sigma) = \sum_{\alpha, j} \frac{1}{\alpha!} (-D_x)^\alpha p_j^{(\alpha)}(x, -\xi, -\sigma) . \tag{82}$$

PROOF. It is obvious that the operator ${}^tP$ is uniquely determined and maps $\mathscr{D}(\Omega) \hat{\otimes} \mathscr{S}(\boldsymbol{R}^1)$ into $\mathscr{D}'(\Omega \times \boldsymbol{R}^1)$. To prove the existence of ${}^tP$, it suffices to show that for every $f \in \mathscr{D}(\Omega)$ there is a β-pseudo-differential operator Q_f such that

$$\langle P(fu), v \rangle = \langle u, Q_f v \rangle \text{ if } u, v \in \mathscr{D}(\Omega) \hat{\otimes} \mathscr{S}(\boldsymbol{R}^1) .$$

For then, we obtain $gQ_f = fQ_g$ for all $f, g \in \mathscr{D}(\Omega)$. So that there is an operator ${}^tP$ satisfying

$$Q_f = f\,{}^tP, \text{ for all } f .$$

Obviously ${}^tP$ is a β-pseudo-differential operator if and only if all Q_f are, and ${}^tP$ satisfies (81).

Set $K(x, \xi, \sigma) = P(f; x, \xi, \sigma)$, which satisfies condition (10), then we have by Lemma 5

$$\langle P(uf), \varphi \otimes \phi \rangle = (2\pi)^{-n-1} \int_{R^{n+1}} \int_{R^{n+1}} K(x, \xi, \sigma) \hat{u}(\xi, \sigma) e^{i(x \cdot \xi + s\sigma)} \varphi(x) \phi(s) d\xi d\sigma dx ds$$
$$= \langle u, Q(\varphi \otimes \phi) \rangle ,$$

$Q(\varphi \otimes \phi)$ is the Fourier transform of the function of ξ, σ.

(83) $$q(\varphi\otimes\phi)(\xi,\sigma)=(2\pi)^{-n-1}\int_{\Omega}\int_{R^1}K(x,\xi,\sigma)\varphi(x)\phi(s)e^{i(x\cdot\xi+s\sigma)}dxds$$
$$=(2\pi)^{-n}\tilde{\phi}(\sigma)\int_{\Omega}K(x,\xi,\sigma)\varphi(x)e^{ix\cdot\xi}dx$$

where $\tilde{\phi}$ is the inverse Fourier transform of ϕ. By integration by part.

$$\xi^\alpha q(\varphi\otimes\phi)(\xi,\sigma)=(2\pi)^{-n}\hat{\phi}(\sigma)\int_{\Omega}(-D_x)^\alpha(K(x,\xi,\sigma)\varphi(x))e^{ix\cdot\xi}dx$$

so that by (10), we have

$$|D_x^\alpha K(x,\xi,\sigma)|\leq C(|\xi|+|\sigma|)^{s_0}.$$

Since α is arbitrary, $\int_{R^n}D_x^\alpha K(x,\xi,\sigma)\varphi(x)e^{ix\cdot\xi}dx$ belongs to $\mathscr{S}(R^n)\hat{\otimes}\mathscr{O}_M(R^1)$. The map $\phi\to q(\varphi\otimes\phi)$ can be extended continuously from $\mathscr{S}'(R^1)$ to $\mathscr{S}(R^n)\hat{\otimes}\mathscr{S}'(R^1)$.

Now we shall seek asymptotic expansion of

$$e^{-i(x\cdot\eta+s\sigma)}Q(\varphi e^{i(x\cdot\eta+s\tau)}),\qquad |\tau|+|\eta|\to\infty.$$

This is the Fourier transform of the function

$$(\xi,\sigma)\to q(\varphi e^{i(x\cdot\xi+s\sigma)})(\xi-\eta,\sigma-\tau).$$

By (83), this is equal to

$$(2\pi)^{-n}\delta(\sigma-\tau+\tau)\int_{\Omega}K(x,\xi-\eta,\sigma-\tau)\varphi(x)e^{i(x\cdot(\xi-\eta+\eta))}dx$$
$$=(2\pi)^{-n}\delta(\sigma)\otimes\int_{\Omega}K(x,\xi-\eta,-\tau)\varphi(x)e^{ix\cdot\xi}dx.$$

Therefore $e^{-i(x\cdot\eta+s\tau)}Q(\varphi e^{i(x\cdot\eta+s\tau)})$ is independent of s. We now study the Taylor expansion of $K(x,\xi-\eta,-\tau)$ at $(-\eta,-\tau)$. The partial sum is

$$(2\pi)^{-n}\sum_{|\alpha|<N}\int\frac{\xi^\alpha}{\alpha!}D_\xi^\alpha K(x,-\eta,-\tau)\varphi(x)e^{ix\cdot\xi}dx.$$

At the point x, the Fourier transform of this is

$$\sum_{|\alpha|<N}\frac{1}{\alpha!}(-D_x)^\alpha(D_\eta^\alpha K(x,-\eta,-\tau))\varphi(x).$$

This has the asymptotic expansion by Lemma 6.

The remainder term $R_\eta^N(\xi)$ can be written as

$$R_\eta^N(\xi)=(2\pi)^{-n}\int\Big(K(x,\xi-\eta,-\tau)-\sum_{|\alpha|<N}\frac{\xi^\alpha}{\alpha!}D_\eta^\alpha K(x,-\eta,-\tau)\Big)\varphi(x)e^{ix\cdot\xi}dx.$$

To estimate R_η^N we again integrate by parts, then, we have

$$(-\xi)^\beta R_{\eta,\tau}^N(\xi)=(2\pi)^{-n}\int_\Omega e^{ix\cdot\xi}D_x^\beta\Big(K(x,\xi-\eta,-\tau)-\sum_{|\alpha|<N}\frac{\xi^\alpha}{\alpha!}D_\xi^\alpha K(x,-\eta,-\tau)\Big)\varphi(x)e^{ix\cdot\xi}dx\ ,$$

and

$$(84)\qquad |(-\xi)^\beta R_{\eta,\tau}^N(\xi)|\le\begin{cases}C|\xi|^N(|\eta|+|\tau|)^{s_0-N}; & \text{if}\quad |\xi|<\frac{1}{4}(|\eta|+|\tau|)\\ C(|\xi|^N+|\xi|^{s_0}) & \text{if}\quad |\xi|\ge\frac{1}{2}(|\eta|+|\tau|)\ .\end{cases}$$

If $|\xi|<1/2(|\eta|+|\tau|)$, taking $\beta=N$, then we have

$$|R_{\eta,\tau}^N(\xi)|\le C(|\eta|+|\tau|)^{s_0-N}\ .$$

If $(|\tau|+|\eta|)\le 2|\xi|$ choosing $|\beta|$ large, we have

$$|R_{\eta,\tau}^N(\xi)|\le C(|\xi|+|\tau|)^M\le C(|\eta|+|\tau|)^{-M}\ .$$

Therefore

$$\hat{R}_{\eta,\tau}^N(x)\le\int|R_\eta^N(\xi)|\,d\xi=0(|\eta|+|\tau|)^{-s_0-N+n},\qquad |\eta|+|\tau|\to\infty$$

so that

$$e^{-i(x\cdot\eta+s\sigma)}Q(ve^{i(x\cdot\xi+s\sigma)})\sim\sum_{\alpha,j}\frac{1}{\alpha!}(-D_x)^\alpha K_j^{(\alpha)}(x,\eta)\ .$$

Where the series is asymptotic in $\mathscr{E}(\Omega)$ topology. By the same argument used in the proof of Theorem 7 we can prove that this expansion holds in $\mathscr{E}(\Omega)$ topology. It is easy to prove that the operator K is defined on $\mathscr{D}(\Omega)\hat{\otimes}\mathscr{S}'(\boldsymbol{R}^1)$ by the formula (11) with kernel of type (10), therefore K can easily be extended to a continuous mapping from $\mathscr{D}(\Omega)\hat{\otimes}\mathscr{S}'(\boldsymbol{R}^1)$ to $\mathscr{E}(\Omega)\hat{\otimes}\mathscr{S}'(\boldsymbol{R}^1)$. Thus Q is a β-pseudo-differential operator.

DEFINITION 20. A β-pseudo-differential operator P of order s_0 is called elliptic if the principal part $p_0(x,\xi)$ of degree s_0 of the symbol is $\neq 0$ for every real $\xi\neq 0$ and $x\in\Omega$.

THEOREM 21. *If P is an elliptic β-pseudo-differential operator of order s_0 on $\Omega\times\boldsymbol{R}^1$, then one can find a β-pseudo-differential operator E of order $-s_0$, such that for every $f\in C_0^\infty(\Omega)$, the symbols of the operators EfP and PfE are identically one on any open set when $f=1$. The symbol of E is uniquely determined.*

We omit the proof, but note that symbol $e=\sum_j e_j(x, \xi, \sigma)$ of E is determined uniquely by

$$\sum_{\alpha, j, k} \frac{1}{\alpha!} D_\xi^\alpha e_j(x, \xi, \sigma) D_x^\alpha e_j(x, \xi, \sigma)=1 \tag{85}$$

or

$$\sum_{\alpha, j, k} \frac{1}{\alpha!} D_\xi^\alpha p_k(x, \xi, \sigma) D_x^\alpha e_j(x, \xi, \sigma)=1 . \tag{86}$$

§4. The case M is a manifold.

In this section, we again assume that M is a σ-compact differentiable n-manifold. In this case we shall restate the results corresponding to those which were obtained in the preceding sections.

Let $\{\varphi_j\}_{j\in J}$ be a smooth partition of unity on M then an operator P is a β-pseudo-differential operator if and only if every $\varphi_j P\varphi_k$, $j, k\in J$, is a β-pseudo-differential operator. Therefore corresponding to Theorem 16, we have

THEOREM 22. *Let P be a continuous linear map from $\mathscr{D}(M)\hat{\otimes}\mathscr{S}'(\boldsymbol{R}^1)$ to $\mathscr{E}(M)\hat{\otimes}\mathscr{S}'(\boldsymbol{R}^1)$. P is a β-pseudo-differential operator if and only if for any $\varphi_1, \varphi_2\in\mathscr{D}(M)$ whose supports are both contained in a coordinate neighbourhood U (not necessarily connected) and for any linear function $x\cdot\xi$ of coordinate functions $x_1, \cdots, x_n$ in U, an asymptotic expansion*

$$e^{-i\lambda(x\cdot\xi+s\sigma)}\varphi_i P(\varphi_2 e^{i\lambda(x\cdot\xi+s\sigma)})\sim\sum_{j=0} p_j(x; \xi, \sigma)\lambda^{2j}$$

holds in $\mathscr{E}(M\times S_1)$, where $S_1=\{(\xi, \sigma)\in\boldsymbol{R}^{n+1}, 1/2\leq\sigma^2+|\xi|^2\leq 2\}$. Then the symbol of $\varphi_1 P\varphi_2$ is given by

$$\sigma_{\varphi_j P\varphi_k}(f, \rho g)=\sum_{\alpha, j} \frac{1}{\alpha!} D_{\lambda\xi\rho}^\alpha p_j(x, \lambda\rho\xi_x, \lambda\sigma) D_x^\alpha(f e^{i\lambda\rho h_x}) . \tag{87}$$

where $\xi_x=\operatorname{grad} g(x)$, $h_x=g(y)-g(x)-\langle y-x, \xi_x\rangle$.

COROLLARY 1. *If P is a β-pseudo-differential operator and if $\varphi_1, \varphi_2\in\mathscr{D}(M)$ with $\operatorname{supp}\varphi_1\cap\operatorname{supp}\varphi_2=\phi$ then, $\varphi_1 P\varphi_2$ is of order $-\infty$.*

PROOF OF COROLLARY 1. We may assume that there is a coordinate neighbourhood U (not necessarily connected) containing $\operatorname{supp}\varphi_1\cup\operatorname{supp}\varphi_2$. Let ψ_1 (resp. ψ_2) be in $\mathscr{D}(U)$ satisfying $\psi_1\equiv 1$ (resp. $\psi_2\equiv 1$) some neighbourhood of $\operatorname{supp}\varphi_1$ (resp. $\operatorname{supp}\varphi_2$). Using the asymptotic expansion

$$e^{-i\lambda(x\cdot\xi+s\sigma)}\psi_1 P(\psi_2 e^{i\lambda(x\cdot\xi+s\sigma)})\sim\sum_{j=0}^{\infty} p_j(x, \xi, \sigma)\lambda^{2j}$$

we can write as

$$e^{-i\lambda(g\rho+s\sigma)}\varphi_1 P(\varphi_2 e^{i\lambda(g\rho+s\sigma)})=e^{-i\lambda(g\rho+s\sigma)}\varphi_1\cdot\psi_1 P(\varphi_2\psi_2 e^{i\lambda(g\rho+s\sigma)})$$
$$\sim\varphi_1\sum_{\alpha,j}\frac{1}{\alpha!}D^{\alpha}_{\lambda\rho\xi}p_j(x,\lambda\rho\xi_x,\lambda,\sigma)D_x(\varphi_2 e^{i\lambda x\cdot\varphi}) .$$

Since $\operatorname{supp}\varphi_1\cap\operatorname{supp}\varphi_2=\phi$, the right hand side of (88) is equal to 0.

COROLLARY 2. (i) *If* $\varphi_3, \varphi_4\in\mathscr{D}(M)$ *with* $\varphi_3=\varphi_1, \varphi_4=\varphi_2$ *in a neighbourhood* V *of* x,

(89) $$\sigma_{\varphi_1 P\varphi_2}(f, g)=\sigma_{\varphi_3 P\varphi_4}(f, g) \text{ in } V\times \boldsymbol{R}^1\times S^1$$

(ii) *if* $\varphi_1=\varphi_2=1$ *in a neighbourhood* V *of* x, *then*

(90) $$\sigma_P(f, g)=\sigma_{\varphi_1 P\varphi_2}(f, g) \text{ in } V$$

(iii) *if* $f_1\equiv f_2\equiv 1$ *in some neighbourhood of* x,

(91) $$\sigma_P(f_1, g)=\sigma_P(f_2, g) \text{ in } x .$$

These are direct consequences of Theorem 21 and Corollary 1.

We can define $\sigma_P(g)(x, \rho, \sigma, \lambda)$ as $\sigma_P(f, g)(x; \rho, \sigma, \lambda)$, where $f\in\mathscr{D}(M)$ and $f\equiv 1$ in some neighbourhood of x. We don't use the following theorem, however it will not be of no use to state it here.

THEOREM 23. *Let* P *be a* β*-pseudo-differential operator. Let* $J^{\mu}(M)$ *be the* μ*-jet bundle of* M. *Then for any* k *there are integers* $l>0$ *and function* Φ_l *from* $J^{l_k}(M)\times S_1$ *to complex number field* $\boldsymbol{C}$ *such that*

$$p_j(g, x, \rho, \sigma)\lambda^{\varepsilon_j}=\Phi_l(\gamma_l(g), \rho, \sigma)\lambda^{\varepsilon_j}, \qquad 0\leq j\leq k .$$

where $\gamma_l(g)$ *is the section of* $J^l(M)$ *defined by* g. *For this it is sufficient to choose* $l\leq 2(k-s_0)$.

PROOF. $p_j(g, x, \rho, \sigma)$ has an intrinsic meaning by definition. On the other hand, (87) implies that for fixed k, $p_j(g, x, \rho, \sigma)$, $0\leq j\leq k$ are determined completely by $\gamma_l(g)$ with sufficiently large l. (It is sufficient to choose as $l\leq 2(k-s_0)$.) Since the fibre of $J^l(M)$ over x is generated by the Image $\gamma_l(g)(x)$, $g\in\mathscr{D}(M)$, this implies our theorem.

REMARK 24. It is possible to state corresponding result for symbols of usual pseudo-differential operator in the sense of Hörmander [3].

In the rest of this section we assume that M is compact. Let $\{U_j\}_{j\in J}$ be a finite coordinate covering of M and we denote the diffeomorphism from U_j to an open subset Ω_j in $\boldsymbol{R}^n$ by Φ_j. Φ_j^* is the corresponding isomorphism from $\mathscr{E}(\Omega_j)$ to $\mathscr{E}(U_j)$, $j\in J$. We can choose and fix a partition of unity $\{\varphi_j\}$ such that if

$\operatorname{supp}\varphi_j \cap \operatorname{supp}\varphi_k \neq \phi$, then there is an index $l(i,j) \in J$ satisfying $\operatorname{supp}\varphi_j \cup \operatorname{supp}\varphi_k \subset U_{l(i,j)}$.

DEFINITION 25. We say

(i) a distribution $T \in \mathscr{D}'(M)\hat{\otimes}\mathscr{S}'(\boldsymbol{R}^1)$ belongs to $H^a(M\times\boldsymbol{R}^1)$ $a \in \boldsymbol{R}$, if

(92) $$\|T\|^2_{H^a(M\times\boldsymbol{R}^1)} = \sum_{j\in J} \|\varphi_j T\circ\Phi_j^*\otimes I\|_{H^a(\boldsymbol{R}^n\times\boldsymbol{R}^1)} < \infty .$$

(ii) A distribution $S \in \mathscr{D}'(M)$ belongs to $H^a(M)$, $a \in \boldsymbol{R}$, if

(93) $$\|S\|^2_{H^a(M)} = \sum_{j\in J} \|\varphi_j S\Phi_j^*\|^2_{H^a(\boldsymbol{R}^n)} < \infty .$$

We can easily prove the following theorems.

THEOREM 28. *Let Q be an elliptic β-pseudo-differential operator of order s_0 on $M\times\boldsymbol{R}^1$. Then one can find a β-pseudo-differential operator F of order $-s_0$ such that symbols of $F\cdot Q$ and $Q\cdot F$ are identically* 1 *on $M\times\boldsymbol{R}^1$.*

PROOF of THEOREM 28. Choose a coordinate patches $\{U_l\}$ and partition of unity as above and consider the mapping

(95) $$P_l = (\Phi^{*-1}\otimes I)\circ Q\circ(\Phi_l^*\otimes I) : \mathscr{D}(\Omega_l)\hat{\otimes}\mathscr{S}'(\boldsymbol{R}^1) \to \mathscr{E}(\Omega_l)\hat{\otimes}\mathscr{S}'(\boldsymbol{R}^1) .$$

This is, by definition, an elliptic β-pseudo-differential operator of order s_0 on $\Omega_l\times\boldsymbol{R}^1$. Therefore, there is an β-pseudo-differential operator E_l of order $-s_0$ on $\Omega\times\boldsymbol{R}^1$ with the properties stated in Theorem 21.

With

(96) $$F_l = (\Phi_l^*\otimes I)\circ E_l\circ(\Phi_l^{*-1}\otimes I) ,$$

we define F as

(97) $$Fu = \sum_{i,j} \varphi_i\cdot F_{l(i,j)}\cdot\varphi_j u .$$

Now we shall prove the symbol $Q\cdot F$ is identically one. To do this, fix a point x in M. Let I be the subset of the index set J such that for any $i \in I, x \in \operatorname{supp}\varphi_i$ and let $U = \bigcap_{i,j\in I} U_{l(i,j)}$. We note that, if $i, j, i', j' \in I$, then the symbols of mappings $u \to F_{l(i,j)}(u)$ and $u \to F_{l(i',j')}(u)$ is the same. In fact for any $\phi \in D(U)$ the symbols of $F_{l(i,j)}\phi P$ and $F_{l(i',j')}\phi P$ are equal in some neighbourhood of x. This is a relation invariant by coordinate transformation. Therefore we can represent this relation in terms of coordinate function. Since R is elliptic, Theorem 21 implies that the symbols of $F_{l(i,j)}$ and $F_{l(i',j')}$ are the same on U. We call this $\sigma(F)$. Then for any $\phi \in \mathscr{D}(U)$ with $\phi=1$ in some neighbourhood of x, the symbol of F

at x=the symbol of $F\cdot\phi=\sum_{i,j}\varphi_j\cdot$(the symbol of $(F_{l(i,j)}\circ\varphi_j\cdot\phi)$ at x)

$=\sum_{i,j}\varphi_j\cdot$(the symbol of $(F_{l(i,j)}\phi)$ at x)

$=\sum_{i,j}\varphi_j\sigma(F)$

$=\sigma(F)$

=the symbol of $F_{l(i,j)}$ for $\forall i, j\in I$.

Therefore, the symbol of $P\cdot F$ at x=the symbol of $P\cdot F_{l(i,j)}$ at $x=1$. This completes the proof.

THEOREM 29. *Let Q be a β-pseudo-differential operator of order $s_0\leq 0$, then for any fixed $\varphi_1, \varphi_2\in\mathscr{D}(M)$, whose supports are both contained in a coordinate neighbourhood U (not necessarily connected) and for any $a\in\boldsymbol{R}$ and $b\in[s_0, -s_0]$, there is a constant C such that for any $\varphi\in\mathscr{D}(M)\hat{\otimes}\mathscr{S}(\boldsymbol{R}^1)$,*

$$\|e^{-i(x\cdot\xi+s\sigma)}\varphi_1 Q(\varphi_2\varphi e^{i(x\cdot\xi+s\sigma)})\|_{H^{a+b}(M\times\boldsymbol{R}^1)} \leq C(1+|\xi|+|\sigma|)^b\|\varphi\|_{H^a(M\times\boldsymbol{R}^1)}, \tag{98}$$

where $x\cdot\xi$ is a linear function of local coordinate function $x_1, \cdots, x_n$ in U.

This is a simple consequence of Theorem 9.

COROLLARY 1. *Under the same hypothesis of Theorem 29,*

$$\|Q(\varphi_2 e^{i(x\cdot\xi+s\sigma)}\varphi)\|_{L^2(M\times\boldsymbol{R}^1)}\leq C(1+|\xi|+|\sigma|)^b\|\varphi\|_{H^{-b}(M\times\boldsymbol{R}^1)}. \tag{99}$$

THEOREM 30. *Let Q, φ_1, φ_2, and $x\cdot\xi$ be as in Theorem 29, there is a constant $C>0$ such that for any $b\in[+s_0, -s_0]$ and $u\in\mathscr{D}(M)$, we have*

$$\|(e^{-i(x\cdot\xi+s\sigma)}\varphi_1 Q(\varphi_2 e^{i(x\cdot\xi+s\sigma)}u)\|_{H^{a+b}(M)}\leq C(1+|\xi|+|\sigma|)^b\|u\|_{H^a(M)}. \tag{100}$$

This follows from Theorem 10.

COROLLARY 1. *Under the same hypothesis as in Theorem 30*

$$\|Q(\varphi_2 e^{i(x\cdot\xi+s\sigma)}u)\|_{L^2(M)}\leq C(1+|\xi|+|\sigma|)^b\|u\|_{H^{-b}(M)}. \tag{101}$$

Finally, we have

THEOREM 31. *Let Q be a β-pseudo-differential operator of order $s_0\leq 0$ and let a be an arbitrary real number, then there exists a constant C such that for any b in $[0, -s_0]$ and u in $\mathscr{D}(M)$, we have*

$$\|e^{-is\sigma}Q(e^{is\sigma}u)\|_{H^{a+b}(M)}\leq C(1+|\sigma|)^{b+s_0}\|u\|_{H^a(M)}. \tag{102}$$

This follows from Theorem 11.

PROOF. Consider a finite smooth partition of unity $\{\varphi_j\}_{j\in J}$ such that for any $j, k\in J$, there exists a coordinate neighbourhood U (not necessarily connected) which contains both supp φ_j and supp φ_k. Then from Theorem 11,

$$\| e^{-is\sigma}\varphi_j Q\varphi_k(e^{is\sigma}u) \|^2_{H^{a+b}(M)} \leq C(1+|\sigma|)^{b+s_0}\| u \|^2_{H^a(M)} \text{.} \tag{103}$$

Summing these by j and k, we obtain (102).

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