Determination of some Frobenius types I

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1. Introduction. Let us denote by Ω the set $\{1, 2, \cdots, n\}$. Let G be a subset of $\Omega\times\Omega$ of Frobenius type in the sense of [1]. Following the notations in [1], we denote by Δ_G the set of all column vectors u in $\boldsymbol{R}^n$ satisfying the following two conditions:

(α) $u\cdot P=u$ for some stochastic matrix P which has G as its support,

(β) $\sum_{i=1}^{n} u_i=1$, for $u=(u_1, u_2, \cdots, u_n)$.

In [1] we have proved that the closure $\bar{\Delta}_G$ of Δ_G in $\boldsymbol{R}^n$ is a compact convex set in $\boldsymbol{R}^n$, and in fact $\bar{\Delta}_G$ is the convex hull of the finite subset $E_G^* = \{V_C; C\in\langle G\rangle\}$, where $\langle G\rangle$ denotes the set of all G-cycles (see [1]) and for every C in $\langle G\rangle$ there is associated a vector V_C (which we call henceforth a G-cycle vector) as follows: let $C=\langle i_1, i_2, \cdots, i_p\rangle$, then

$$V_C=(u_1, u_2, \cdots, u_n) ,$$

where

$$u_k=\begin{cases} 1/p & \text{if } C \text{ passes through the } k\text{-th vertex,} \\ 0 & \text{otherwise.}\end{cases}$$

The purpose of this note is to determine all subsets G of $\Omega\times\Omega$ satisfying the following conditions.

(I) G is of Frobenius type.

(II) There exists a G-cycle C_0 with support Ω. Thus C_0 has the associated G-cycle vector $V_{C_0}=n^{-1}(1, 1, \cdots, 1)$.

(III) Let F be the mapping from the set $\langle G\rangle$ into the set 2^Ω of all subsets of Ω defined as follows:

$$F(C)=\mathrm{Supp}\ (C) .$$

Then the cardinality of the image $F(\langle G\rangle)$ is 2^{n-1}.

(IV) For every G-cycle C, the associated G-cycle vector V_C is an extreme point of $\bar{\Delta}_G$.

We shall show first that if G satisfies (I) and (II), the cardinality of $F(\langle G\rangle)$ is at most 2^{n-1}. Then under these conditions we prove the following:

THEOREM. *There exists one and only one subset G_0 of $\Omega\times\Omega$ satisfying (I),* *(II), (III), (IV) up to permutations of the set Ω. This subset G_0 is given by the following incidence-matrix;*

$$G_0=\begin{pmatrix} * & * & * & * & \cdots & * \\ * & 0 & * & * & \cdots & * \\ * & 0 & 0 & * & \cdots & * \\ \vdots & & \ddots & & & \\ \vdots & & & \ddots & & \\ \vdots & & & & \ddots & \\ * & 0 & 0 & 0 & \cdot & * \\ * & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

2. Determination of a class of graphs of Frobenius types.

Let G be a given subset of $\Omega\times\Omega$ of Frobenius type and let $\langle G\rangle$ be the set of all G-cycles. For any cycle C in $\langle G\rangle$, we denote its support by Supp (C), i.e.

$$\mathrm{Supp}\,(C)=\{i_1, i_2, \cdots, i_p\} \quad \text{for} \quad C=\langle i_1, i_2, \cdots, i_p\rangle .$$

Let C, C' be in $\langle G\rangle$. Then we say that C' is a *proper sub-cycle* of C if Supp (C') is a proper subset of Supp (C).

Now let $\mathfrak{S}=(C_1, C_2, \cdots, C_r)$ be a sequence of G-cycles. For each vertex $j\in\Omega$, we denote by $m(j,\mathfrak{S})$ the number of indices i in $[1, r]$ such that $j\in\mathrm{Supp}\,(C_i)$.

DEFINITION. Let k be a positive integer. Let C be a cycle in $\langle G\rangle$ and let $\mathfrak{S}=(C_1, C_2, \cdots, C_r)$ be a sequence of G-cycles. We say that $\mathfrak{S}$ is *a covering of C of constant multiplicity k* if the following conditions are all satisfied:

(i) $\mathrm{Supp}\,(C)=\bigcup_{i=1}^{r}\mathrm{Supp}\,(C_i)$,

(ii) each C_i is a proper sub-cycle of C,

(iii) $m(j,\mathfrak{S})=k$ for every j in Supp (C).

In particular, if $k=1$ in this definition, we say that $\mathfrak{S}$ is a *disjoint covering* of C.

LEMMA 1. *Let $\mathfrak{S}=(C_1, C_2, \cdots, C_r)$ be a covering of a G-cycle C of constant multiplicity k. Then*

$$V_C=\sum_{i=1}^{r}\frac{l(C_i)}{l(C)\cdot k}V_{C_i} \tag{1}$$

In particular, V_C is not an extreme point of $\bar{\Delta}_G$.

PROOF. Fix a vertex $j\in\mathrm{Supp}\,(C)$. Let $\{j_1, j_2, \cdots, j_k\}$ be the set of indices i in $[1, r]$ such that $j\in\mathrm{Supp}\,(C_i)$. Then the j-th entry of $\sum_{i=1}\frac{l(C_i)}{l(C)\cdot k}V_{C_i}$ is given by

$$\sum_{\nu=1}^{k}\frac{l(C_{j_\nu})}{l(C)\cdot k}\frac{1}{l(C_{j_\nu})}=\frac{1}{l(C)} .$$

But this is equal to the j-th entry of V_C. On the other hand, if $j\in\Omega-\mathrm{Supp}(C)$, the j-th entry of V_C and $\sum\frac{l(C_i)}{l(C)\cdot k}V_{C_i}$ are both zero by condition (i). Hence we proved (1). Now, since the sum of the entries of V_C, $V_{C_1},\cdots,V_{C_r}$ are all equal to 1, one has

$$1=\sum_{i=1}^{r}\frac{l(C_i)}{l(C)\cdot k}.$$

On the other hand, at least two V_{C_i}, V_{C_j} are distinct by the condition (ii). Thus, V_C is not an extreme point of $\bar{\Delta}_G$, q.e.d.

LEMMA 2. *Let C be a G-cycle of length m and let p be a given integer with $1\leq p<m$. Suppose that there exist exactly $\binom{m}{p}$ subsets of Ω each of which is a support of a proper sub-cycle C' of C of length p. Then C has a covering $\mathfrak{S}$ of constant multiplicity $\binom{m-1}{p-1}$.*

PROOF. Let $\mathfrak{S}=(C_1, C_2,\cdots,C_M)$, $M=\binom{m}{p}$, be the sequence of all proper sub-cycles of length p of C (arranged in any order) such that $\mathrm{Supp}(C_i)\neq\mathrm{Supp}(C_j)$ for $i\neq j$. Then the conditions (i), (ii) above are easily verified. Furthermore, the condition (iii) is also verified by taking $k=\binom{m-1}{p-1}$, q.e.d.

LEMMA 3. *Let C be in $\langle G\rangle$. If there exists a vertex j in* Supp (C) *such that $j\in\mathrm{Supp}(C')$ for any proper sub-cycle C' of C. Then V_C is an extreme point of $\bar{\Delta}_G$. Furtheremore, for every proper sub-cycle C' of C, $V_{C'}$ is an extreme point of $\bar{\Delta}_G$.*

PROOF. Suppose V_C is not an extreme point of $\bar{\Delta}_G$. Then, since $\bar{\Delta}_G$ is the convex hull of the set $\{V_{C'};\ C'\in\langle G\rangle\}$, there exist $C_1, C_2,\cdots,C_r$ in $\langle G\rangle$ and $\alpha_1>0,\cdots\alpha_r>0$ such that

$$V_C=\sum_{i=1}^{r}\alpha_i V_{C_i},\qquad \sum_{i=1}^{r}\alpha_i=1,\qquad V_C\neq V_{C_i}\qquad \text{for } i=1,\cdots,r. \tag{2}$$

Then, one has immediately that

$$\mathrm{Supp}(C)\supsetneqq\mathrm{Supp}(C_i)\qquad \text{for } i=1,2,\cdots r.$$

Now comparing the j-th entry of (2), one has

$$\frac{1}{l(C)}=\sum_{i=1}^{r}\frac{\alpha_i}{l(C_i)}. \tag{3}$$

Let k be any vertex in $\mathrm{Supp}(C)-\mathrm{Supp}(C_1)$. Then comparing k-th entry of (2), one has

$$\frac{1}{l(C)}=\sum_{k\in \mathrm{Supp}(C_i)}\frac{\alpha_i}{l(C_i)}<\sum_{i=2}^{r}\frac{\alpha_i}{l(C_i)}.$$

But this contradicts with (3), q.e.d.

Now, we assume that there is a G-cycle C_0 of length n. Let F be the mapping from $\langle G\rangle$ into the set of 2^{Ω} of all subsets of Ω, defined as follows:

$$F(C)=\mathrm{Supp}\,(C)\,.$$

We denote by $\mu(G)$ the cardinality of the set $F(\langle G\rangle)$. Then we have the following:

THEOREM 1. *Suppose that V_{C_0} is an extreme point of $\bar{\Delta}_G$, then $\mu(G)\leqq 2^{n-1}$. Moreover, if $\mu(G)=2^{n-1}$, then*

$$N_p+N_{n-p}=\binom{n}{p}\qquad \textit{for any } p\ (1\leqq p\leqq n)\,, \tag{4}$$

where N_p is the number of all subsets of Ω which are supports of G-cycles of length p.

PROOF. Let (A, B) be a partition of Ω (including the trivial one such that one component is empty), and denote by $\mathfrak{P}$ the set of all the partitions (A, B), where we identifiy (A, B) with (B, A). Now let us consider the mapping f from E_G^* (the set of all G-cycle vectors, see [1]) into $\mathfrak{P}$ such that $f(V_C)=(F(C), \Omega-F(C))$. We claim that f is injective. In fact, if $V_{C_1}\neq V_{C_2}$, then we have $\mathrm{Supp}\,(C_1)\neq \mathrm{Supp}\,(C_2)$, hence $F(C_1)\neq F(C_2)$. Let us note that $F(C_1)\neq\Omega-F(C_2)$ and $F(C_2)\neq\Omega-F(C_1)$. Indeed, since V_{C_0} is an extreme point of $\bar{\Delta}_G$, for any (A, B) in $\mathfrak{P}$, at least one of A or B can not be the support of a G-cycle, otherwise we would get a disjoint covering of C_0. Thus f is injective and $\mu(G)\leqq|\mathfrak{P}|=2^{n-1}$. Suppose the cardinality of E_G^* is equal to 2^{n-1}, then f is bijective. Thus for any (A, B) in $\mathfrak{P}$, at least one of A or B should be the support of a G-cycle. So we have $N_p+N_{n-p}=\binom{n}{p}$ for any p $(1\leqq p\leqq n)$.

LEMMA 5. *If V_{C_0} is an extreme point of $\bar{\Delta}_G$ and the number of all extreme points of $\bar{\Delta}_G$ is 2^{n-1}, then for any G-cycle C_p of length p, there exist*

1) *a G-cycle C_{p-1} of length $p-1$, such that* $\mathrm{Supp}\,(C_{p-1})\subset\mathrm{Supp}\,(C_p)$, *for any p $(1<p\leqq n)$, and*
2) *a G-cycle C_{p+1} of length $p+1$, such that* $\mathrm{Supp}\,(C_{p+1})\supset\mathrm{Supp}\,(C_p)$, *for any p $(1\leqq p<n)$ respectively.*

PROOF. 1). Note that we have $\mu(G)=2^{n-1}$, so we may apply Theorem 1. Suppose for some p $(1<p\leqq n)$, C_p has no proper sub-cycles of length $p-1$. Then, putting $C_p=\langle i_1, i_2, \cdots, i_p\rangle$, any subset of $\mathrm{Supp}\,(C_p)$ consisting of $p-1$ element in $\mathrm{Supp}\,(C_p)$ is not a support of a G-cycle. Denote these subsets by $D_1, D_2, \cdots, D_p$ where $D_s=\{i_1, i_2, \cdots, \widehat{i_s}, \cdots, i_p\}$. Then each $\Omega-D_s$ is a support of a G-cycle of

length $n-(p-1)$ by Theorem 1. Now take a G-cycle $C^{(s)}$ with support $\Omega-D_s$, then we have a covering $\{C^{(1)}, C^{(2)}, \cdots, C^{(p)}, \underbrace{C_p, \cdots, C_p}_{p-1}\}$ of C_0 with constant multiplicity p. 2). Suppose for some p $(1 \leqq p < n)$, there is no G-cycle C' of length $p+1$ such that $\mathrm{Supp}(C') \supset \mathrm{Supp}(C)$. Put $\Omega - \mathrm{Supp}(C_p) = \{j_1, \cdots, j_{n-p}\}$ and $D_t = \Omega - \mathrm{Supp}(C_p) - \{j_t\}$ $(1 \leqq t \leqq n-p)$. Then each D_t is a support of a G-cycle $C^{(t)}$ of length $n-(p+1)$ by Theorem 1. Now we have a covering $\mathfrak{C} = \{C^{(1)}, \cdots, C^{(n-p)}, \underbrace{C_p, \cdots, C_p}_{n-(p+1)}\}$ of C_0 with constant multiplicity $n-(p+1)$.

THEOREM 2. *Suppose that G is a subset of $\Omega \times \Omega$ of Frobenius type such that* (i) *G has a cycle with support Ω,* (ii) *every G-cycle vector is an extreme point of $\bar{\Delta}_G$,* (iii) *the cardinality of E_G is 2^{n-1}. Then there is a permutation σ of Ω such that $\sigma(G) = G_0$, where G_0 is the subset of $\Omega \times \Omega$ given by the following incidence-matrix*

$$G_0 = \begin{pmatrix} * & * & * & * & \cdots & * \\ * & 0 & * & * & \cdots & * \\ * & 0 & 0 & * & \cdots & * \\ \cdot & \cdot & \cdot & & & \cdot \\ \cdot & \cdot & & \cdot & & \cdot \\ \cdot & \cdot & & & \cdot & \cdot \\ * & 0 & 0 & 0 & \cdots & * \\ * & 0 & 0 & 0 & \cdots & * \end{pmatrix}$$

The converse is also true.

PROOF. We show first that any G-cycle includes one and only one G-cycle of length 1. The existence of a G-cycle of length 1 is immediate by repeated use of Lemma 5. Let us show its uniqueness. Suppose there is a G-cycle C_p of length p such that which includes two different G-cycles $\langle p_1 \rangle$, $\langle p_2 \rangle$ of length 1. By Lemma 5, there exists a proper sub-cycle C_{p-1} of C_p of length $p-1$. We claim $p_1 \in \mathrm{Supp}(C_{p-1})$ and $p_2 \in \mathrm{Supp}(C_{p-1})$. Because if $\langle p_1 \rangle$ is not a proper sub-cycle of C_{p-1}, then $\{C_{p-1}, \langle p_1 \rangle\}$ is a disjoint covering of C_p. This is impossible, since V_{C_p} is an extreme point of $\bar{\Delta}_G$ by our hypothesis (ii). By repeated use of this argument, we get finally a G-cycle C_2 of length 2 such that $p_1 \in \mathrm{Supp}(C_2)$ and $p_2 \in \mathrm{Supp}(C_2)$. Then $\{\langle p_1 \rangle, \langle p_2 \rangle\}$ is a disjoint covering of C_2. Hence V_{C_2} is not an extreme point of $\bar{\Delta}_G$, but this is impossible. Therefore we have seen that there exists only one G-cycle $\langle i \rangle$ of length 1 and every G-cycle C satisfies $i \in \mathrm{Supp}(C)$. By a suitable permutation of Ω, we may assume $i=1$.

Next let us prove that $N_p = \binom{n-1}{p-1}$ for any $p = 1, 2, \cdots, n$. Since $1 \in \mathrm{Supp}(C_p)$ for any G-cycle C_p of length p, we have $N_p = \binom{n-1}{p-1}$. On the other hand using (4) in Theorem 1, we have

$$\binom{n}{p}=N_p+N_{n-p}\leqq\binom{n-1}{p-1}+\binom{n-1}{n-p-1}=\binom{n}{p},$$

therefore one has $N_p=\binom{n-1}{p-1}$ for $p=1, 2, \cdots, n$. Denoting (i, j)-th entry of incidence-matrix of G by p_{ij}, by $\langle 1\rangle\in\langle G\rangle$ and $N_1=1$ we have $p_{11}\neq 0$, $p_{ii}=0$ for all $i\neq 1$. By $N_2=\binom{n-1}{1}$, we have (noting that every G-cycle C satisfies $1\in\mathrm{Supp}(C)$) $p_{12}\neq 0$, $p_{13}\neq 0, \cdots, P_{1n}\neq 0$ and $p_{21}\neq 0$, $p_{31}\neq 0, \cdots, p_{n1}\neq 0$. Now let us consider the following cases, for any $i, j\in\Omega$, $i\neq 1$, $j\neq 1$, $i\neq j$.

Case 1) $p_{ij}\neq 0$ and $p_{ji}\neq 0$.

Case 2) $p_{ij}=0$ and $p_{ji}=0$.

Case 3) $p_{ij}\neq 0$ and $p_{ji}=0$.

Case 1) is impossible, since $N_2=\binom{n-1}{1}$ and all G-cycles contains 1 in its support. We can see that Case 2) is also impossible. Suppose in fact for some i, j we have $p_{ij}=0$ and $p_{ji}=0$, then there is no G-cycle C_3 of length 3 such that $\mathrm{Supp}(C_3)=\{1, i, j\}$. It contradicts to the fact $N_3=\binom{n-1}{2}$. Hence only Case 3) is possible. Let us introduce an ordering in $\Omega-\{1\}$. For i, j in $\Omega-\{1\}$, we denote $i\ll j$ if $p_{ij}\neq 0$. We prove next that this ordering satisfies the transitive law: $i\ll j$ and $j\ll k$, then $i\ll k$. In fact, suppose $p_{ij}\neq 0$ and $p_{jk}\neq 0$, then we have $p_{ik}\neq 0$. Because, if $p_{ik}=0$, then we have $p_{ki}\neq 0$, therefore we have a G-cycle $C=\langle i, j, k\rangle$ whose support does not include 1. But this is impossible. It is then immediate to see that $\Omega-\{1\}$ is a totally ordered set. Set $\Omega-\{1\}=\{i_2, i_3, \cdots, i_n\}$ with $i_2\ll i_3\ll\cdots\ll i_n$. Let us define a permutation σ of Ω as follows:

$$\sigma(1)=1,\ \sigma(i_2)=2, \cdots, \sigma(i_n)=n\ .$$

We have then $(i, j)\in G$ if and only if $(\sigma(i), \sigma(j))\in G_0$ by our construction of σ. Hence one has $\sigma(G)=G_0$ and the proof of the first part of Theorem 2 is complete.

Conversely, let us show that G_0 satisfies the conditions (i), (ii) and (iii). We start from the following observation: if $C=\langle i_1, i_2, \cdots, i_p\rangle$ is a G_0-cycle, then $1\in\mathrm{Supp}(C)$. So we may assume that $i_1=1$. Then by the definition of G_0, it is obvious that $1=i_1<i_2<\cdots<i_p$. Conversely, for any subset $\{i_2, i_3 \cdots, i_p\}$ of $\Omega-\{1\}$ with $i_2<i_3<\cdots<i_p$, it is immediate to see that the cycle $\langle 1, i_2, \cdots, i_p\rangle$ is a G_0-cycle. Thus we have a bijection β between $\langle G_0\rangle$ and $2^{\Omega-\{1\}}$ given by

$$C\in\langle G_0\rangle\longleftrightarrow\beta(C)=\mathrm{Supp}\,(C)-\{1\}.$$

Especially, $\langle 1, 2, \cdots, n\rangle\in\langle G_0\rangle$. This proves (i). Let $C\in\langle G_0\rangle$, then $1\in\mathrm{Supp}\,(C)$ as we have seen above. Hence by Lemma 4, V_C is an extreme point of $\bar{\Delta}_{G_0}$. This proves (ii). The number $|E_{G_0}|$ of the extreme points of $\overline{\Delta}_{G_0}$ is equal to the number of G_0-cycles by the observation above and by (ii). Hence $|E_{G_0}|=|2^{\Omega-\{1\}}|=2^{n-1}$. This proves (iii), q.e.d.

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Reference

[1] Nobuko Iwahori, On some stochastic matrices of a given Frobenius type, J. Fac. Sci., Univ. of Tokyo, **13** (1966), 139-161.

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