On intermediate logics I

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We have been studying the intermediate logics between the classical and the intuitionistic, especially those of propositional logics. In this paper, we add some further results. One of them is, we report, that we can classify the intermediate logics somewhat neatly, so that we can recognize a structure, which we will call as *cone structure*, in the system of the intermediate logics. It seems to us that this cone structure will very distinctly show properties interconnecting the intermediate logics and it will provide us a useful tool for the study of intermediate logics, especially as we study the inclusion and non-inclusion relations of intermediate logics or monotonously descending sequences of intermediate logics. The classification will be defined in § 4, and in the succeeding sections, the cone structure will be studied in detail. In § 2, Boolean operations on logics are defined and some theorems on them are prepared. These operations prove to be a useful technique when we try to prove theorems in §§ 5-7. The section 3 is rather out of flow of this paper. That is a remark to § 2 dealing with non-intermediate logics. Hence, § 3 is dispensable for the understanding of the cone structure. But the fact dealt in it will, we think, show a certain characteristic of the intermediate logics.

The existence of the logics between the classical and the intuitionistic was first notified by Gödel [5]. And those logics were given the name *intermediate logic* by Umezawa [30] and [31]. Ever since many authors have contributed to this field of study, with or without consciousness of the notion of the intermediate logics in mind. In the bibliography placed at the conclusion of this paper, we collect those works as far as we are aware of. This bibliography will be supplemented in the papers planned to succeed this as we obtain further informations concerning it.

The classical and the intuitionistic logics have many devotees, and attempts have been made to construct *the* mathematics on them, but we do not think it reasonable to work on the intermediate logics with the same principle as we hold when dealing with the classical or the intuitionistic. So we shall not discuss such problems as *the* mathematics built on an intermediate logic or the application of the intermediate logics, though we do not deny the usefulness of them, especially, that of many valued logics. In a word, we regard the intermediate logics merely as *objects* of mathematical study. If we take those logics as axiomatic systems, we can think that the logics, as a whole, give a classification of the classically valid formulas by the interdeducibility in the intuitionistic logic. This point of view is more interesting to us than to regard them as logics in the usual sense.

To make this paper self-contained, we will transcribe those definitions and results already mentioned in other papers whenever it is necessary, though often without proof.

By *logic*, we mean a set of well formed formulas (wffs) which is closed for substitution and modus ponens. Since we are mainly dealing with the intermediate propositional logics, we always mean those logics simply by *logic*, unless mentioned otherwise. Our study does not restrict its range only to the logics defined by the axiomatic method. But it also covers such logics as defined by (characteristic) models or by other methods if any, as far as they fit to our definition abovementioned. By *model*, we always mean a characteristic model, since it is not necessary for us to use the notion of model in the usual sense. And unless mentioned otherwise, a model is always for an intermediate propositional logic. Moreover, we deal with only such models as have only *one* designated value, since we know (cf. [14]) that any models can be reduced to models with only one designated value. Further, we take it for granted that *any logic has a model*, since the Lindenbaum algebra of the logic is one.

§ 1. Preliminaries.

The logical connectives we use are $\supset$, &, $\vee$ and $\neg$. We use, possibly with indices, the lower case letters $a, b, c, \cdots$ for propositional variables and the upper case letters $A, B, C, \cdots$ for wffs. By $\bigwedge_{1\leq i\leq n} A_i$ (or $\bigvee_{1\leq i\leq n} A_i$), we mean the formula $A_1 \& A_2 \& \cdots \& A_n$ (or $A_1 \vee A_2 \vee \cdots \vee A_n$). By $A \equiv B$, we mean the formula $(A \supset B) \& (B \supset A)$.

DEFINITION 1.1. $K = ((a_1 \supset a_0) \supset a_1) \supset a_1$,

$$Z = ((a \supset b) \supset c) \supset (((b \supset a) \supset c) \supset c),$$

$$X_n = \bigvee_{1\leq i<j\leq n+1} (a_i \equiv a_j),$$

$$\begin{cases} P_1 = K, \\ P_{i+1} = ((a_{i+1} \supset P_i) \supset a_{i+1}) \supset a_{i+1} \qquad (i \geq 1), \end{cases}$$

$$R_n = a_1 \vee (a_1 \supset a_2) \vee (a_2 \supset a_3) \vee \cdots \vee (a_{n-1} \supset a_n) \vee \neg a_n.$$

We call the intuitionistic propositional logic as $\boldsymbol{L}$. By $\boldsymbol{M} + A_1 + A_2 + \cdots + A_n$ where $\boldsymbol{M}$ is a logic and $A_1, A_2, \cdots, A_n$ are wffs, we mean a logic obtained by adding new axiom schemes $A_1, A_2, \cdots, A_n$ to $\boldsymbol{M}$. The logic of the form $\boldsymbol{L} + A$ is often abbreviated as $\boldsymbol{LA}$. Hence the classical propositional logic can be expressed as $\boldsymbol{L}+K$ or $\boldsymbol{LK}$. By $\boldsymbol{M}\supset\boldsymbol{N}$ or by $\boldsymbol{N}\subset\boldsymbol{M}$, we mean that the logic $\boldsymbol{M}$, as a set of wffs, includes the logic $\boldsymbol{N}$, as a set of wffs. $\boldsymbol{M}\supset\subset\boldsymbol{N}$ means that the two logics $\boldsymbol{M}$ and $\boldsymbol{N}$ are equivalent, that is, define the same set of wffs, and $\boldsymbol{M}\supsetneqq\boldsymbol{N}$ means that $\boldsymbol{M}\supset\boldsymbol{N}$ but not $\boldsymbol{M}\supset\subset\boldsymbol{N}$. The logics $\boldsymbol{S}_n (n=1, 2, \cdots)$, which will play an important rôle in § 4, are defined by models as follows.

DEFINITION 1.2. *The values of $\boldsymbol{S}_n$ are $1, 2, \cdots, n$ and ω, where 1 is the sole designated value and ω is regarded as greater than any positive integers. The logical operations are defined as follows:*

$$v_1 \supset v_2 = \begin{cases} 1 & \text{if } v_1 \geq v_2, \\ v_2 & \text{otherwise}, \end{cases}$$

$$v_1 \,\&\, v_2 = \max(v_1, v_2),$$

$$v_1 \vee v_2 = \min(v_1, v_2),$$

$$\neg v = v \supset \omega.$$

DEFINITION 1.3. *$\boldsymbol{S}_\omega$ is an extension of $\boldsymbol{S}_n$ by taking the values to be all the positive integers and ω.*

Theorems as follow are known.

THEOREM 1.4. $\boldsymbol{LK}\supset\subset\boldsymbol{S}_1\supsetneqq\boldsymbol{S}_2\supsetneqq\cdots\supsetneqq\boldsymbol{S}_n\supsetneqq\cdots\supsetneqq\boldsymbol{S}_\omega\supsetneqq\boldsymbol{L}$.

THEOREM 1.5. $\boldsymbol{S}_n\supset\subset\boldsymbol{L}+Z+P_n\supset\subset\boldsymbol{L}+R_n$,

$\boldsymbol{S}_\omega\supset\subset\boldsymbol{L}+Z$.

THEOREM 1.6. $\boldsymbol{L}+Z+A\supset\subset\boldsymbol{S}_n$ *if and only if* $\boldsymbol{S}_n \ni A$ *and* $\boldsymbol{S}_{n+1} \not\ni A$.

The proof of 1.4 is immediate. The first relation of 1.5 is proved in [11] and [12], and the latter is in [3]. The theorem 1.6 is proved in [13].

Let $\boldsymbol{M}$ be a model. We mean by $\mathbf{V}_M$ the set of the truth values of $\boldsymbol{M}$ and by $\mathbf{D}_M$ the set of the designated values of $\boldsymbol{M}$ (the subscript $\boldsymbol{M}$ is often omitted). $\mathbf{D}_M$ is usually a set with only one element as mentioned in the beginning, whose element will be, then, denoted as 1 (or 1_M) if there occurs no confusion.

DEFINITION 1.7. *For $k\geq 2$, and for models $\boldsymbol{M}_i$ $(1\leq i\leq k)$, $(\boldsymbol{M}_1,\cdots,\boldsymbol{M}_k)$ is a model with the truth values $(v_1,\cdots,v_k)$, where $v_i\in\mathbf{V}_{M_i}$ $(1\leq i\leq k)$, and with the designated values $(d_1,\cdots,d_k)$, where $d_i\in\mathbf{D}_{M_i}$ $(1\leq i\leq k)$, and with the logical operations defined by*

$$(v_1, \cdots, v_k) * (w_1, \cdots, w_k) = (v_1 * w_1, \cdots, v_k * w_k),$$

$$\neg(v_1, \cdots, v_k) = (\neg v_1, \cdots, \neg v_k),$$

where $*$ is $\supset$, $\&$ or $\vee$. If each $\boldsymbol{M}_i$ is $\boldsymbol{M}$, $(\boldsymbol{M}_1, \cdots, \boldsymbol{M}_k)$ is abbreviated as $\boldsymbol{M}^k$.

Let $\boldsymbol{M}$ and $\boldsymbol{N}$ be models. Let be that $\mathbf{V}_M\cap\mathbf{V}_N=\emptyset$, and let $\mathbf{V}'_N$ be the set of the undesignated values of $\boldsymbol{N}$. By $\supset_M$ (or $\supset_N$), etc., we mean those logical operations of $\boldsymbol{M}$ (or $\boldsymbol{N}$). For any $v\in\mathbf{V}_N$, let $\alpha(v)$ be v or 1_M according as $v\in\mathbf{V}'_N$ or $v=1_N$, and let $\beta(v)$ be v or $\neg_M 1_M$ according as $v \in V'_N$ or $v = 1_N$.

DEFINITION 1.8. *$M \uparrow N$ is a model whose set of the values is $V_M \cup V'_N$ and whose designated value is 1_M and whose logical operations are defined as follows:*

$x \supset y$	$y \in V_M$	$y \in V'_N$
$x \in V_M$	$x \supset_M y$	y
$x \in V'_N$	1_M	$\alpha(x \supset_N y)$

$y \& y$	$y \in V_M$	$y \in V'_N$
$x \in V_M$	$x \&_M y$	y
$x \in V'_N$	x	$x \&_N y$

$x \vee y$	$y \in V_M$	$y \in V'_N$
$x \in V_M$	$x \vee_M y$	x
$x \in V'_N$	y	$\beta(x \vee_N y)$

	$\neg x$
$x \in V_M$	$\neg_N 1_N$
$x \in V'_N$	$\alpha(\neg_N x)$.

This operation $\uparrow$ on models is discussed in [14] and [29].

LEMMA 1.9. *Let be that $M = S_1$ and $A \in N$. Then, for any assignment f of $M \uparrow N$, $f(A)$ gets the value 1 or $\neg_M 1_M$ (which is often expressed as 2).*

PROOF. If we take $\mathbf{D}$ to be $\{1, 2\}$, then $M \uparrow N$ becomes equivalent to N.

When we deal with the intermediate models, the notion of regular model defined as follows is considerably important.

DEFINITION 1.10. *An intermediate model M is regular if it satisfies the following conditions.*

(i) *$\mathbf{D}$ has only one element.*

(ii) *If $\mathbf{D} \ni v$ and $\mathbf{D} \ni v \supset w$, then $\mathbf{D} \ni w$.*

(iii) *If $\mathbf{D} \ni v$, then $\mathbf{D} \ni w \supset v$ for any value w.*

(iv) *If $\mathbf{D} \ni v \supset w$ and $\mathbf{D} \ni w \supset v$, then $v = w$.*

COROLLARY 1.11. *If M is regular and 1 is its designated value, then $1 \supset v = 1$ if and only if $v = 1$.*

THEOREM 1.12. *Any intermediate logics have regular models.*

This theorem is proved in [14]. By this, we can always regard an intermediate model as regular.

§2. Boolean operations on logics.

Let M and N be two logics. They are two sets of wffs by definition. We want to introduce two Boolean operations on them, that is, *union* $\cup$ and *intersection* $\cap$. The operation of intersection for the axiomatic systems has been studied by Miura [23]. It is easily seen that the intersection $M \cap N$ as the set of wffs still keeps the property of logic, that is, closed for substitution and modus ponens and that it is intermediate. Miura's result is the following

THEOREM 2.1. *If the sets of the propositional variables contained in the axiom* *scheme A and in the axiom scheme B are mutually disjoint, then*

$$(\mathbf{L}+A)\cap(\mathbf{L}+B)\supset\subset\mathbf{L}+A\vee B.$$

PROOF. The $\supset$ part is obvious, since $\mathbf{L}\ni A\supset A\vee B$ and $\mathbf{L}\ni B\supset A\vee B$. The $\subset$ part is proved as follows. Let F be a provable formula both in $\mathbf{L}+A$ and $\mathbf{L}+B$. Then there are substituted cases $A_1, \cdots, A_m$ of A and $B_1, \cdots, B_n$ of B such that $(\bigwedge_{1\leq i\leq m} A_i)\supset F$ and $(\bigwedge_{1\leq j\leq n} B_j)\supset F$ are provable in $\mathbf{L}$. And so $((\bigwedge_{1\leq i\leq m} A_i)\vee(\bigwedge_{1\leq j\leq n} B_j))\supset F$, hence $(\bigwedge_{\substack{1\leq i\leq m\\ 1\leq j\leq n}} (A_i\vee B_j))\supset F$ is provable in $\mathbf{L}$. So F is provable in $\mathbf{L}+A\vee B$.

THEOREM 2.2. *If* $\mathbf{M}$ *and* $\mathbf{N}$ *are two models, then* $\mathbf{M}\cap\mathbf{N}\supset\subset(\mathbf{M}, \mathbf{N})$.

This is well known and easy to prove.

The Boolean union of two logics as sets does not necessarily constitute a logic. For example, let us take $\mathbf{LZ}$ and $\mathbf{LP}_2$. Since $\mathbf{LZ}+R_2\supset\subset\mathbf{LR}_2\supset\subset\mathbf{S}_2\supsetneqq\mathbf{LZ}$, $\mathbf{LZ}$ does not contain R_2. And likewise, R_2 is not contained in $\mathbf{LP}_2$ since

$$\mathbf{LP}_2+R_2\supset\subset\mathbf{LR}_2\supset\subset\mathbf{S}_2\supsetneqq\mathbf{LP}_2$$

(for the last relation, cf. §6). Suppose that $\mathbf{LZ}\cup\mathbf{LP}_2$ be a logic, then it is closed for substitution and modus ponens and it contains Z and P_2. So it must contain R_2 since $\mathbf{L}+Z+P_2\supset\subset\mathbf{LR}_2\ni R_2$. If we insist on the former definition of logic, we must define the operation $\cup$ differently.

DEFINITION 2.3. $\mathbf{L}_1\cup\mathbf{L}_2$ *is a logic whose elements are just such wffs C as, there exists a finite sequence of wffs* $C_1, \cdots, C_n=C$ *such that, for each* C_i, *at least one of the following three conditions holds*:

(i) $C_i\in\mathbf{L}_1$,

(ii) $C_i\in\mathbf{L}_2$,

(iii) *there exist* j, $k<i$ *such that* $C_k=C_j\supset C_i$.

It is easily seen that $\mathbf{L}_1\cup\mathbf{L}_2$ is an intermediate logic. This definition means that $\mathbf{L}_1\cup\mathbf{L}_2$ is the infimum of such logics as include both $\mathbf{L}_1$ and $\mathbf{L}_2$. It is also easily seen that, if $C\in\mathbf{L}_1\cup\mathbf{L}_2$, there exist $C_1\in\mathbf{L}_1$ and $C_2\in\mathbf{L}_2$ such that $(C_1\ \&\ C_2)\supset C$, or $C_1\supset(C_2\supset C)$, is provable in $\mathbf{L}$. The following two theorems are almost immediate from the definition.

THEOREM 2.4. $(\mathbf{L}+A)\cup(\mathbf{L}+B)\supset\subset\mathbf{L}+A+B\supset\subset\mathbf{L}+A\ \&\ B$.

In this case, it should be noted that A and B can have some propositional variables in common.

THEOREM 2.5. *If* $\mathbf{L}_2\supset\mathbf{L}_3$, *then*

$$\mathbf{L}_1\cup\mathbf{L}_2\supset\mathbf{L}_1\cup\mathbf{L}_3$$

and

$$\mathbf{L}_1\cap\mathbf{L}_2\supset\mathbf{L}_1\cap\mathbf{L}_3.$$

COROLLARY 2.6. *If* $L_2 \supset\subset L_3$, *then*

$$L_1 \cup L_2 \supset\subset L_1 \cup L_3$$

and

$$L_1 \cap L_2 \supset\subset L_1 \cap L_3.$$

THEOREM 2.7. *If* $A \in M$ *and* $B \in N$, *then* $A \,\&\, B \in M \cup N$ *and* $A \vee B \in M \cap N$.

PROOF. Since the formulas A, B and $A \supset (B \supset A \,\&\, B)$ belong to $M \cup N$, the formula $A \,\&\, B$ belongs to $M \cup N$. Next, let us suppose that M and N are models. Let f_M and f_N be assignments of M and N, respectively. Then the function f such that $f(a) = (f_M(a), f_N(a))$ is an assignment of $M \cap N$. Conversely, any assignment of $M \cap N$ can be thus decomposed into the product of those of M and N. Since, for any f_M and f_N, $f_M(A) = 1_M$ and $f_N(B) = 1_N$, $f(A \vee B) = (f_M(A) \vee f_M(B), f_N(A) \vee f_N(B))$ $= (1_M \vee f_M(B),\ f_N(A) \vee 1_N) = (1_M,\ 1_N)$. Hence $A \vee B \in M \cap N$.

THEOREM 2.8.

(1) $L_1 \cap L_2 \supset\subset L_2 \cap L_1$.

(2) $L_1 \cup L_2 \supset\subset L_2 \cup L_1$.

(3) $L_1 \cap (L_2 \cap L_3) \supset\subset (L_1 \cap L_2) \cap L_3$.

(4) $L_1 \cup (L_2 \cup L_3) \supset\subset (L_1 \cup L_2) \cup L_3$.

(5) $(L_1 \cup L_2) \cap L_3 \supset\subset (L_1 \cap L_3) \cup (L_2 \cap L_3)$.

(6) $(L_1 \cap L_2) \cup L_3 \supset\subset (L_1 \cup L_3) \cap (L_2 \cup L_3)$.

(7) $(L_1 \cap L_2) \cup L_2 \supset\subset L_2$.

(8) $L_1 \cap (L_1 \cup L_2) \supset\subset L_1$.

PROOF. We will prove only (5) and (6), since others are almost immediate from the definitions.

(5) Suppose that $A \in (L_1 \cup L_2) \cap L_3$. Then there exist formulas $A_1 \in L_1$, $A_2 \in L_2$ and $A_3 \in L_3$ such that both $(A_1 \,\&\, A_2) \supset A$ and $A_3 \supset A$ are provable in L. Hence $((A_1 \,\&\, A_2) \vee A_3) \supset A$, and so, $((A_1 \vee A_3) \,\&\, (A_2 \vee A_3)) \supset A$ is provable in L. This means that $(L_1 \cap L_3) \cup (L_2 \cap L_3) \ni A$, since $A_1 \vee A_3 \in L_1 \cap L_3$ and $A_2 \vee A_3 \in L_2 \cap L_3$. This reasoning can be pursued backward.

(6) Suppose that $A \in (L_1 \cup L_3) \cap (L_2 \cup L_3)$. Then there exist formulas $A_1 \in L_1$, $A_2 \in L_2$ and $A_3 \in L_3$ such that $A_1 \supset (A_3 \supset A)$ and $A_2 \supset (A_3 \supset A)$ are provable in L. Hence $(A_1 \vee A_2) \supset (A_3 \supset A)$ is provable in L. By this, $A \in (L_1 \cap L_2) \cup L_3$ since $A_1 \vee A_2 \in L_1 \cap L_2$ and $A_3 \in L_3$. The converse can be proved likewise.

REMARK 2.9. The above resasoning (2.1-8) can be easily extended to the predicate logics.

REMARK 2.10. The method of 2.4 used to obtain the union is axiomatic. We do not know yet a constructive method to obtain a model which is the union of two models. In case the operation $\cup$ is the set operation, a method has been discovered by Kalicki [16], which will be discussed in the following section.

It will be interesting if we can define for a logic $\boldsymbol{M}$ an operation $\boldsymbol{M}^c$ which makes (in a certain sense) the complement of $\boldsymbol{M}$. If we require $\boldsymbol{M}^c$ to be intermediate, both $\boldsymbol{M}$ and $\boldsymbol{M}^c$ must include $\boldsymbol{L}$. This is not desirable from the sense of the Boolean complement. So we should, perhaps, not insist that logics be intermediate.

Suppose that we define $\boldsymbol{M}^c$ to be $\{A;\ A \notin \boldsymbol{M}\}$. Then $\boldsymbol{L}^c$ would contain a formula a which is the propositional variable a itself. $\boldsymbol{L}^c$ must be closed for substitution in order that it might be a logic in our sense, and so it would contain any formulas since they are obtained as substituted cases of a. Hence $\boldsymbol{L}^c$ must consist of all the wffs. This property is not desirable again. So we must give up this definition for $\boldsymbol{M}^c$.

Next, suppose that we define $\boldsymbol{M}^c$ to be $\{A;\ A \notin \boldsymbol{M} \text{ and } A \in \boldsymbol{LK}\}$. $\boldsymbol{L}^c$ contains $(a \supset a) \vee \neg(a \supset a)$ if $\boldsymbol{L}^c$ is closed for substitution. This is contradictory.

We have not succeeded in defining $\boldsymbol{M}^c$ to our hearts' content. Our conjecture for the possibility of well defining $\boldsymbol{M}^c$ is rather negative by the above reasoning and others.

§ 3. Models with many designated values.

As mentioned in § 1, any intermediate models can be reduced to equivalent regular models with only one designated value. We think that this property is somewhat characteristic to the intermediate models from the reasoning as below.

DEFINITION 3.1. *$\boldsymbol{L}(A)$, where A is a formula, is a set of wffs determined by only one axiom scheme A and substitution.*

REMARK 3.2. By the definition, $\boldsymbol{L}(A)$ is closed for substitution. But $\boldsymbol{L}(A)$ is not necessarily closed for modus ponens. Let us take, for example, $a \supset (b \supset a)$ as A. If $\boldsymbol{L}(A)$ is closed for modus ponens, the formula $b \supset A$ must belong to $\boldsymbol{L}(A)$ since A and $A \supset (b \supset A)$ belong to $\boldsymbol{L}(A)$. But, in fact, $b \supset A$ cannot be obtained from A by any substitution.

Now, let us call $\boldsymbol{L}(a \supset a)$ as $\boldsymbol{N}$ and its characteristic model, if any, as $\boldsymbol{M}$. As is easily seen, $\boldsymbol{N}$ is closed for modus ponens, that is, a logic, though not intermediate, in our sense.

LEMMA 3.3. *If there exists such $\boldsymbol{M}$, then the number of its designated values is greater than one.*

PROOF. Suppose that 1 is the sole designated value of $\boldsymbol{M}$. Since $\boldsymbol{N}$ contains

$a \supset a$ and $b \supset b$, $f(a \supset a) = f(b \supset b) = 1$ for any assignment f of $\boldsymbol{M}$. On the other hand, $1 \supset 1$ must get the value 1. Hence, for any f, $f((a \supset a) \supset (b \supset b)) = 1$. So $(a \supset a) \supset (b \supset b) \in \boldsymbol{M}$. But this formula cannot belong to $\boldsymbol{N}$. This is contradictory.

We do not know yet the minimum number of the designated values for the existence of $\boldsymbol{M}$. But as is shown below, there exists an $\boldsymbol{M}$ with infinitely many designated values, which, we conject, would not be reduced to a model with finitely many designated values. We will define $\boldsymbol{M}$ by and by.

DEFINITION 3.4. *The set* $\mathbf{V}$ *of the truth values of* $\boldsymbol{M}$ *is that defined by the following two conditions.*

(i) *Every positive integer belongs to* $\mathbf{V}$.

(ii) *If* $\mathbf{V} \ni \alpha, \beta$, *then* $\mathbf{V} \ni (0_i, \alpha, \beta)$ *where* i *is* 0, 1, 2, 3 *or* 4.

DEFINITION 3.5. *The set* $\mathbf{D}$ *of the designated values consists of all the values with the form* $(0_0, \alpha, \beta)$ *where* α *and* β *are any values.*

For the elements α and β of $\mathbf{V}$, the relation $\alpha = \beta$ is defined as usual, that is, $\alpha = \beta$ if and only if α and β have the same appearance.

DEFINITION 3.6. *The logical operations of* $\boldsymbol{M}$ *are as follow*:

$$\alpha \supset \beta = \begin{cases} (0_0, \alpha, \beta) & \text{if } \alpha = \beta, \\ (0_1, \alpha, \beta) & \text{otherwise.} \end{cases}$$

$$\alpha \,\&\, \beta = (0_2, \alpha, \beta).$$

$$\alpha \vee \beta = (0_3, \alpha, \beta).$$

$$\neg \alpha = (0_4, \alpha, \alpha).$$

THEOREM 3.7. *Let* A *and* B *be two formulas which contain at the most the propositional variables* $a_1, \cdots, a_n$. $A = B$ *if and only if* $f(A) = f(B)$ *for any assignment* f *of* $\boldsymbol{M}$.

PROOF. Let f be an assignment such that $f(a_i) = i$ $(i = 1, \cdots, n)$. $A = B$ is obvious from $f(A) = f(B)$ since $f(A)$ (and also $f(B)$) corresponds faithfully to the Polish expression of the formula. The converse part is immediate.

THEOREM 3.8. $\boldsymbol{M} \supset \boldsymbol{N}$.

PROOF. If $\boldsymbol{N} \ni A$, then there necessarily exists a formula B such that $A = B \supset B$. Let f be an assignment of $\boldsymbol{M}$. $f(A) = f(B) \supset f(B) = (0_0, f(B), f(B)) \in \mathbf{D}$.

THEOREM 3.9. $\boldsymbol{N} \supset \boldsymbol{M}$.

PROOF. Suppose that $\boldsymbol{M} \ni A$. By the hypothesis, $f(A) \in \mathbf{D}$ for any assignment f of $\boldsymbol{M}$. Hence $f(A) = (0_0, \alpha, \alpha)$ where α is some value. Hence A is of the form $B \supset C$. For any f, $f(B) = f(C)$. So $B = C$ by 3.7. Hence $A = B \supset B \in \boldsymbol{N}$.

The above reasoning also applies to others. Let us take $\boldsymbol{L}(a \vee \neg a)$ for example, which is also a logic in our sense. The formula $(a \vee \neg a) \vee \neg (b \vee \neg b)$ is the formula needed in the proof of the lemma 3.3. The logical operations of $\boldsymbol{M}$ must be changed as follow:

$$\alpha \supset \beta = (0_1, \alpha, \beta).$$
$$\alpha \,\&\, \beta = (0_2, \alpha, \beta).$$
$$\alpha \vee \beta = \begin{cases} (0_0, \alpha, \beta) & \textit{if } \beta = (0_4, \alpha, \alpha), \\ (0_3, \alpha, \beta) & \textit{otherwise}. \end{cases}$$
$$\neg\alpha = (0_4, \alpha, \alpha).$$

The above are logics with *one* axiom scheme. But the above construction of the model also applies to those with *more than one* axiom schemes only if they do not contain any inference rules.

The definition of logical operations for the logic with $a \supset a$ and $a \vee \neg a$ for the axiom schemes, for example, is as follows:

$$\alpha \supset \beta = \begin{cases} (0_0, \alpha, \beta) & \textit{if } \alpha = \beta, \\ (0_1, \alpha, \beta) & \textit{otherwise}. \end{cases}$$
$$\alpha \,\&\, \beta = (0_2, \alpha, \beta).$$
$$\alpha \vee \beta = \begin{cases} (0_0, \alpha, \beta) & \textit{if } \beta = (0_4, \alpha, \alpha), \\ (0_3, \alpha, \beta) & \textit{otherwise}. \end{cases}$$
$$\neg\alpha = (0_4, \alpha, \alpha).$$

REMARK 3.10. Even if A may be a contradiction in the usual sense, $\boldsymbol{L}(A)$ does not necessarily contain all the wffs.

REMARK 3.11. As is well known, $\boldsymbol{L}(a)$ is the logic which contains all the wffs.

We do not know yet if the lemma 3.3 holds generally for arbitrary $\boldsymbol{L}(A)$'s. Our conjecture is rather negative, since $\boldsymbol{L}(a)$ is characterized by a model with $\{1\}$ as its $\mathbf{V}$ and $\mathbf{D}$.

We give one more example of models with many designated values. This example is of Kalicki [16]. Let $\boldsymbol{M}_1$ and $\boldsymbol{M}_2$ be intermediate models. We can regard them as regular models. Let $\boldsymbol{M}$ be the union as sets of $\boldsymbol{M}_1$ and $\boldsymbol{M}_2$. Kalicki defines a model for $\boldsymbol{M}$ as follows:

DEFINITION 3.12. *$\mathbf{V}_{\boldsymbol{M}}$ is that of $(\boldsymbol{M}_1, \boldsymbol{M}_2)$, $\mathbf{D}_{\boldsymbol{M}} = \{(v_1, v_2);\ v_1 = 1_{M_1}$ or $v_2 = 1_{M_2}\}$ and the logical operations of $\boldsymbol{M}$ are those of $(\boldsymbol{M}_1, \boldsymbol{M}_2)$.*

THEOREM 3.13. *$\boldsymbol{M} \ni A$ if and only if $\boldsymbol{M}_1 \ni A$ or $\boldsymbol{M}_2 \ni A$.*

PROOF. Suppose that $\boldsymbol{M} \ni A$ but that $\boldsymbol{M}_1 \not\ni A$ and $\boldsymbol{M}_2 \not\ni A$. Then, for any assignment f of $\boldsymbol{M}$, $f(A) \in \mathbf{D}_{\boldsymbol{M}}$. There exists an assignment f_1 (f_2) of $\boldsymbol{M}_1$ $(\boldsymbol{M}_2)$ such that $f_1(A) \notin \mathbf{D}_{M_1}$ $(f_2(A) \notin \mathbf{D}_{M_2})$. Let f be an assignment of $\boldsymbol{M}$ such that $f(a) = (f_1(a), f_2(a))$ for any propositional variable a. Then $f(A) \notin \mathbf{D}_{\boldsymbol{M}}$, contrary to the hypothesis. The converse part is immediate.

THEOREM 3.14. *The model $\boldsymbol{M}$ cannot be reduced to a regular model with only one designated value if $\boldsymbol{M}$ is not closed for modus ponens.*

PROOF. Suppose that M which is not closed for modus ponens could be reduced to a regular model, which we would call as M again, with only one designated value. Since M is regular, $1 \supset v = 1$ if and only if $v = 1$. Since M is not closed for modus ponens, there exist formulas $A \in M_1$ and $B \in M_2$ such that $A \& B \notin M$ but $A \supset (B \supset A \& B) \in L \subset M$. Let f be an assignment of M such that $f(A \& B) = v \neq 1$. On the other hand, $f(A) = 1$, $f(B) = 1$ and $f(A \supset (B \supset A \& B)) = 1$. Hence $1 \supset (1 \supset v) = 1$, by 1.11 we get $1 \supset v = 1$, and so $v = 1$, contrary to the hypothesis.

§ 4. Definition of slices.

Hereafter we regard the constant ω to be an integer, which is regarded to be greater than any other integers, and when we want to mean other integers than ω, that is, integers in the usual sense, we will use the phrase *finite integers.*

LEMMA 4.1. *Let M be a logic, then there uniquely exists an integer n such that $M + Z \supset\subset S_n$.*

PROOF. Since $M \supset L$ and $L + Z \supset\subset S_\omega$, we obtain that $M + Z \supset S_\omega$. By 1.6, we know that there is no logic between S_i and S_{i+1} $(i = 1, 2, \cdots)$. And by 1.4, there is no logic which includes S_ω other than S_i's $(i = 1, 2, \cdots)$. Hence there must be some uniquely determined integer n such that $M + Z \supset\subset S_n$.

DEFINITION 4.2. $\mathscr{S}_n = \{M;\ M + Z \supset\subset S_n\}$.

By 4.1, every logic belongs to some uniquely determined $\mathscr{S}_n$. $\mathscr{S}_n$ is a set of logics, which we call the *n-th slice* (of the system of the intermediate logics). That $\mathscr{S}_n \neq \emptyset$ $(n = 1, 2, \cdots)$ is obvious from the fact that every $\mathscr{S}_n$ contains S_n at the least.

To prove the theorem 4.6, we need some preparations.

LEMMA 4.3. *In L, the following formulas are provable.*

(1) $(a_0 \supset P_1) \equiv (a_1 \supset a_1)$.

(2) $(P_1 \supset a_0) \equiv a_0$.

(3) $(P_i \supset P_{i+1}) \equiv (a_1 \supset a_1) \quad (1 \leq i)$.

(4) $(P_{i+1} \supset P_i) \equiv P_i \quad (1 \leq i)$.

PROOF. We prove the formulas in the intuitionistic system LJ of Gentzen Since the inferences used are only structural or implicational, we will not show the names of the inferences used. Moreover, we will allow as beginning sequents such sequents as easily seen to be provable in LJ.

(1)
$$\dfrac{\dfrac{\dfrac{a_0 \rightarrow a_1 \supset a_0 \quad a_1 \rightarrow a_1}{a_0,\ (a_1 \supset a_0) \supset a_1 \rightarrow a_1}}{a_0 \rightarrow P_1}}{\rightarrow a_0 \supset P_1.}$$

Since $a_0 \supset P_1$ is thus provable in $\boldsymbol{L}$, (1) is immediate.

$$
(2)\quad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{a_1 \to P_1 \quad a_0 \to a_0}{a_1,\ P_1 \supset a_0 \to a_0}}{P_1 \supset a_0 \to a_1 \supset a_0 \quad a_1 \to a_1}}{P_1 \supset a_0,\ (a_1 \supset a_0) \supset a_1 \to a_1}}{P_1 \supset a_0 \to P_1 \qquad\qquad a_0 \to a_0}}{P_1 \supset a_0,\ P_1 \supset a_0 \to a_0}}{P_1 \supset a_0 \to a_0}}{\to (P_1 \supset a_0) \supset a_0.}
$$

The provability of $a_0 \supset (P_1 \supset a_0)$ is obvious.

(3) This is immediate from (1), if we take a_0 to be P_i, P_1 to be P_{i+1} and a_1 to be a_{i+1} in the sketch of the proof of (1).

(4) If we interpret (2) as we did (1) in (3), (4) is immediate from (2).

DEFINITION 4.4. $B_1 = a_1 \supset a_1$,

$B_i = P_{n+2-i} \qquad (2 \leq i \leq n+1)$,

$B_\omega = \neg(a_1 \supset a_1)$,

where n is an arbitrarily fixed positive and finite integer.

LEMMA 4.5. *In $\boldsymbol{L}$, the following formulas are provable.*

(1) $(B_i \supset B_j) \equiv B_1 \qquad (i \geq j)$.

(2) $(B_i \supset B_j) \equiv B_j \qquad (i < j)$.

(3) $(B_i \& B_j) \equiv B_{\max(i,j)}$.

(4) $(B_i \vee B_j) \equiv B_{\min(i,j)}$.

(5) $(\neg B_i) \equiv (B_i \supset B_\omega)$.

PROOF. Since B_ω is a contradiction, (5) is immediate. And if i or j is ω, or if $i=j$, the lemma is also immediate. We suppose otherwise.

(1) Suppose that $i = j+k$ $(k \geq 1)$. In $\boldsymbol{LJ}$,

$$\bigwedge_{0 \leq m \leq k-1} (P_{n+2-j-k+m} \supset P_{n+2-j-k+m+1}) \to P_{n+2-j-k} \supset P_{n+2-j}$$

is easily seen to be provable. Since each of the conjuncts in the left side of the arrow is provable in $\boldsymbol{LJ}$ by (3) of 4.3, $P_{n+2-j-k} \supset P_{n+2-j}$, which is $B_i \supset B_j$ itself, is provable, that is, equivalent to B_1.

(2) We suppose that $j = i+k$ $(k \geq 1)$. We will only show that $(B_i \supset B_j) \supset B_j$, that is, $(P_{n+2-i} \supset P_{n+2-i-k}) \supset P_{n+2-i-k}$, is provable in $\boldsymbol{LJ}$. For convenience' sake, we put $m = n+2-i-k$. It is obvious that both $a_{m+k} \to P_{m+k}$ and $P_m \to P_{m+k-1}$ are provable in $\boldsymbol{LJ}$.

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{a_{m+k} \to P_{m+k} \quad P_m \to P_{m+k-1}}{a_{m+k},\ P_{m+k} \supset P_m \to P_{m+k-1}}}{P_{m+k} \supset P_m \to a_{m+k} \supset P_{m+k-1}} \quad a_{m+k} \to a_{m+k}}{P_{m+k} \supset P_m,\ (a_{m+k} \supset P_{m+k-1}) \supset a_{m+k} \to a_{m+k}}}{P_{m+k} \supset P_m \to P_{m+k}} \quad P_m \to P_m}{P_{m+k} \supset P_m,\ P_{m+k} \supset P_m \to P_m}}{P_{m+k} \supset P_m \to P_m}}{\to (P_{m+k} \supset P_m) \supset P_m.}$$

For the proof of (3) and (4), suppose that $i>j$. Then they are immediate from the fact that $\boldsymbol{L} \ni B_i \supset B_j$.

The following theorem is due to S. Nagata. His proof will be published in near future. Our proof is along the line of his sketch proof delivered to us, only with slight changes.

THEOREM 4.6. *The following two conditions are equivalent* ($n<\omega$):

(i) $\boldsymbol{S}_n \ni A$ *and* $\boldsymbol{S}_{n+1} \not\ni A$.

(ii) $\boldsymbol{S}_n \supset \boldsymbol{L}+A \supset \boldsymbol{LP}_n$.

PROOF. By 1.6, (i) is implied from (ii). Suppose (i). Without loss of generality, we suppose that A contains only the propositional variables $b_1, \cdots, b_m$, which we write as $A(b_1, \cdots, b_m)$. By the hypothesis, there exists an assignment f of $\boldsymbol{S}_{n+1}$ such that $f(A)=2$ (cf. 1.9). For $1 \leq i \leq m$, we put $\varphi(b_i)=B_k$ where $k=f(b_i)$. We define A^* to be the formula $A(\varphi(b_1), \cdots, \varphi(b_m))$ which is a substituted case of A. Our aim is to prove $A^* \equiv B_2$ in $\boldsymbol{L}$, which means that B_2, that is, P_n, is provable in $\boldsymbol{L}+A$. We know that we can substitute equivalent formulas freely in the intuitionistic calculation. So, the calculation leading to $A^* \equiv B_2$ just goes as that of $f(A)=2$ by the assurance of the lemma 4.5.

COROLLARY 4.7. *Every slice has the minimum and the maximum elements.* $\boldsymbol{LP}_n$ *is the minimum and* $\boldsymbol{S}_n$ *is the maximum.*

It has been known that $\lim\limits_{n\to\infty} \boldsymbol{LP}_n \supset\subset \boldsymbol{L}$ and $\lim\limits_{n\to\infty} \boldsymbol{S}_n \supset\subset \boldsymbol{S}_\omega$. So 4.7 also applies to the case $n=\omega$.

COROLLARY 4.8. $\mathscr{S}_1$ *consists of only one element, which is the classical logic* $\boldsymbol{LK}$.

This is immediate from the fact that $\boldsymbol{S}_1 \supset\subset \boldsymbol{LP}_1$. As to the number of elements in other slices, we will discuss it in §6.

§5. The cone structure.

As shown in §4, the set of the intermediate logics can be devided into slices.

And here, we will show many structural relations interconnecting these slices. By these, the reason will become, we think, clear why we call it *cone structure*, likening the set of the intermediate logics as a cone, the slices as horizontal sections slicing the cone parallel to the bottom and the logics as dots scattered on slices.

DEFINITION 5.1. *Let* $\boldsymbol{M}$ *be a logic.* $\boldsymbol{M}(\omega)$ *is the logic* $\boldsymbol{M}\cap\boldsymbol{S}_\omega$, *which is called as the* ω*-projection of* $\boldsymbol{M}$.

COROLLARY 5.2.

(1) $\boldsymbol{M}\supset\boldsymbol{M}(\omega)$.

(2) $\boldsymbol{M}(\omega)\in\mathscr{S}_\omega$, *that is,* $\boldsymbol{S}_\omega\supset\boldsymbol{M}(\omega)$.

(3) $\boldsymbol{M}(\omega)\supset\subset\boldsymbol{M}$ *if and only if* $\boldsymbol{M}\in\mathscr{S}_\omega$.

(4) $\boldsymbol{M}(\omega)\supset\subset\boldsymbol{S}_\omega$ *if and only if there exists an integer* n *such that* $\boldsymbol{M}\supset\subset\boldsymbol{S}_n$.

THEOREM 5.3. *Let be that* $\boldsymbol{M}$, $\boldsymbol{N}\in\mathscr{S}_n$. *Then* $\boldsymbol{M}\supset\subset\boldsymbol{N}$ *if and only if* $\boldsymbol{M}(\omega)\supset\subset\boldsymbol{N}(\omega)$.

PROOF. The case $n=\omega$ is trivial. Suppose that $n\neq\omega$. If $\boldsymbol{M}\supset\subset\boldsymbol{N}$, $\boldsymbol{M}(\omega)\supset\subset\boldsymbol{N}(\omega)$ is obvious from the definition. To prove the converse, suppose that $A\in\boldsymbol{M}$ but $A\notin\boldsymbol{N}$. It will be sufficient if we prove that there is a formula which belongs to $\boldsymbol{M}(\omega)$ but does not belong to $\boldsymbol{N}(\omega)$. The formula $A\vee Z$, where, we suppose, the sets of the propositional variables contained in A and Z are disjoint, belongs to $\boldsymbol{M}(\omega)$ by 2.7. We regard $\boldsymbol{N}$ to be a model. If $\boldsymbol{N}\ni Z$, $\boldsymbol{N}\supset\subset\boldsymbol{S}_n\supset\boldsymbol{M}\ni A$, contrary to the hypothesis. Hence, $\boldsymbol{N}\not\ni A\vee Z$ by the hypothesis that A and Z have no propositional variable in common. So $\boldsymbol{N}(\omega)\not\ni A\vee Z$ by (1) of 5.2.

DEFINITION 5.4. *Let* $\boldsymbol{M}$ *be a logic.* $\boldsymbol{M}(i)$ $(i=1,2,\cdots)$ *is the logic* $\boldsymbol{M}(\omega)\cup\boldsymbol{LP}_i$, *which is called as the* i*-projection of* $\boldsymbol{M}$.

COROLLARY 5.5. *If* $\boldsymbol{M}\in\mathscr{S}_n$, *then* $\boldsymbol{M}(i)\supset\subset\boldsymbol{M}\cap\boldsymbol{S}_i$ *for* $i\geq n$.

PROOF. $\boldsymbol{M}(i)=\boldsymbol{M}(\omega)\cup\boldsymbol{LP}_i\supset\subset(\boldsymbol{M}\cap\boldsymbol{S}_\omega)\cup\boldsymbol{LP}_i\supset\subset(\boldsymbol{M}\cup\boldsymbol{LP}_i)\cap(\boldsymbol{S}_\omega\cup\boldsymbol{LP}_i)\supset\subset\boldsymbol{M}\cap\boldsymbol{S}_i$, since $\boldsymbol{M}\supset\boldsymbol{LP}_i$ and $\boldsymbol{S}_\omega\cup\boldsymbol{LP}_i\supset\subset\boldsymbol{S}_i$. For $i<n$, see 5.13.

THEOREM 5.6. *Let be that* $\boldsymbol{M}$, $\boldsymbol{N}\in\mathscr{S}_n$. *If* $i\geq n$, *then* $\boldsymbol{M}\supset\subset\boldsymbol{N}$ *if and only if* $\boldsymbol{M}(i)\supset\subset\boldsymbol{N}(i)$.

PROOF. This can be proved just as the proof of 5.3. We only need to use the formula R_i instead of Z in 5.3 and use the corollary 5.5 as the definition of $\boldsymbol{M}(i)$.

THEOREM 5.7. *The sequence of logics* $\{\boldsymbol{M}(i)\}_{i=1,2,\dots}$ *is strictly descending from* $\boldsymbol{LK}$ *to* $\boldsymbol{M}(\omega)$ *and contains* $\boldsymbol{M}$ *in it. If* $\boldsymbol{M}\in\mathscr{S}_n$ $(n<\omega)$, *there is no intermediate logic between* $\boldsymbol{M}(i)$ *and* $\boldsymbol{M}(i+1)$ *for* $n\leq i<\omega$.

PROOF. It is obvious that $\boldsymbol{M}(1)\supset\subset\boldsymbol{LK}$ and $\lim_{i\to\infty}\boldsymbol{M}(i)\supset\subset\boldsymbol{M}(\omega)$, and that the sequence contains M. Since $LP_i \supset LP_{i+1}$, $M(i) \supset M(i+1)$. Since $M(i) \in \mathscr{S}_i$ and $M(i+1) \in \mathscr{S}_{i+1}$, $M(i) \supsetneqq M(i+1)$. Suppose that $n \leq i < \omega$ and that there exists a logic M' such that $M(i) \supsetneqq M' \supsetneqq M(i+1)$. M' cannot belong to $\mathscr{S}_{i+1}$ since, if it does, $M(i+1) \supset\subset M(i) \cap S_{i+1} \supset M'$, which is contrary to the hypothesis. Suppose that $M' \in \mathscr{S}_i$. Since $M(i) \supsetneqq M'$, $M(i+1) \supsetneqq M'(i+1)$ by 5.6. Hence $M' \supsetneqq M(i+1) \supsetneqq M'(i+1)$. And both $M(i+1)$ and $M'(i+1)$ belong to $\mathscr{S}_{i+1}$. So $M'(i+1) \supset\subset M' \cap S_{i+1} \supset M(i+1)$. This is contradictory.

THEOREM 5.8. *The sequence* $\{LP_n(\omega)\}_{n=1,2,\ldots}$ *is a strictly descending sequence on* ω*-th slice* $\mathscr{S}_\omega$. *That is,*

$$S_\omega \supset\subset LP_1(\omega) \supsetneqq LP_2(\omega) \supsetneqq \cdots \supsetneqq LP_n(\omega) \supsetneqq \cdots \supset L\,.$$

PROOF. That $S_\omega \supset\subset LP_1(\omega)$ and $LP_n(\omega) \supset LP_{n+1}(\omega)$ is immediate from the definition. It is necessary for us to prove only that $LP_n(\omega) \supsetneqq LP_{n+1}(\omega)$. By the definition, $LP_n(\omega) \supset\subset L + P_n \vee Z$. We will prove that $LP_{n+1}(\omega) \not\ni P_n \vee Z$. First we define a sequence of models N_n. N_1 is the model $S_1{}^2$. For $n \geq 2$, N_n is defined as follows: V is $\{1, 2, \cdots, n, \alpha, \beta, \omega\}$, where 1 is the designated value. The order relation $\geq$ for $1, 2, \cdots, n, \omega$ is as that of S_n, and α and β are regarded as greater than n but smaller than ω and uncomparable with each other. The logical operations are as follows:

$$v_1 \supset v_2 = \begin{cases} 1 & \textit{if } v_1 \geq v_2\,, \\ \alpha & \textit{if } v_1 = \beta \textit{ and } v_2 = \omega\,, \\ \beta & \textit{if } v_1 = \alpha \textit{ and } v_2 = \omega\,, \\ v_2 & \textit{otherwise}\,. \end{cases}$$

$$v_1 \,\&\, v_2 = \begin{cases} \omega & \textit{if } v_1 = \alpha \textit{ and } v_2 = \beta, \textit{ or if } v_1 = \beta \textit{ and } v_2 = \alpha\,, \\ \max(v_1, v_2) & \textit{otherwise}\,. \end{cases}$$

$$v_1 \vee v_2 = \begin{cases} n & \textit{if } v_1 = \alpha \textit{ and } v_2 = \beta, \textit{ or if } v_1 = \beta \textit{ and } v_2 = \alpha\,, \\ \min(v_1, v_2) & \textit{otherwise}\,. \end{cases}$$

$$\neg v = v \supset \omega\,.$$

N_{n+1}, thus defined, can be regarded as $S_n \uparrow S_1{}^2$, or $S_1 \uparrow N_n$. $N_1 \ni P_1$, hence $N_1 \ni P_1 \vee Z$. But $N_2 \not\ni P_1 \vee Z$, since $P_1 \vee Z$ gets the value 2 if we assign the values ω, 2, α, β and 2 to the propositional variables a_0, a_1, a, b and c respectively. Next, let us prove $N_{n+1} \ni P_{n+1}$ by assuming $N_n \ni P_n$. By the hypothesis and 1.9, the value $f(P_n)$ is 1 or 2 for any assignment f of N_{n+1}. So, $f(P_{n+1}) = 1$ for any f. Hence $N_{n+1} \ni P_{n+1}$ and, by this, $N_{n+1} \ni P_{n+1} \vee Z$. But $N_{n+1} \not\ni P_n \vee Z$, since $P_n \vee Z$ gets the value 2 if we assign the values $\omega, n+1, \cdots, 2, \alpha, \beta$ and 2 to the propositional variables $a_0, a_1, \cdots, a_n, a, b$ and c respectively. Since $N_n \ni P_n \vee Z$, $N_n \supset LP_n(\omega)$. On the other hand, as $N_{n+1} \not\ni P_n \vee Z$, $LP_{n+1}(\omega) \not\ni P_n \vee Z$.

COROLLARY 5.9. $\lim_{n\to\infty} LP_n(\omega) \supset\subset L$.

COROLLARY 5.10. *If* $m \geq n$, *then* $LP_n \cap S_m \supset\subset LP_m \cup LP_n(\omega)$.

COROLLARY 5.11. *If* $p<q<m$, *then* $LP_m \cup LP_p(\omega) \supsetneqq LP_m \cup LP_q(\omega)$.

PROOF. This can be just proved as 5.8. Only we need to use the formula of the form $P_p \vee R_m$ instead of $P_n \vee Z$. By this corollary, we can recognize a strictly descending finite sequence from S_m to LP_m on the m-th slice.

THEOREM 5.12. *There exists* L_{n+1} *such that* $LP_n \supsetneqq L_{n+1} \supsetneqq LP_{n+1}$.

PROOF. It is easily seen that no logics on $\mathscr{S}_n$ can have the property mentioned in the theorem. We prove that $LP_n(n+1)$, which is on $\mathscr{S}_{n+1}$, has that property. The relation $LP_n \supsetneqq LP_n(n+1)$ holds obviously. Since $LP_n(n+1) \in \mathscr{S}_{n+1}$, $LP_n(n+1) \supset LP_{n+1}$. Suppose that $LP_{n+1} \supset LP_n(n+1)$. Then $LP_{n+1} \ni P_n \vee Z$, since $LP_n(n+1) \supset\subset L + P_{n+1} + P_n \vee Z$. But this cannot occur as shown in the proof of 5.8.

THEOREM 5.13. *If* $M \in \mathscr{S}_{n+1}$ *and* $M \supset LP_n \cap S_{n+1}$, *then there exists* $N \in \mathscr{S}_n$ *such that* $N(n+1) \supset\subset M$.

PROOF. Since $M \supset LP_n \cap S_{n+1}$, $M(\omega) \supset LP_n \cap S_\omega$. We define N to be $M(\omega) \cup LP_n$. Then,

$$\begin{aligned} N(n+1) &= ((M(\omega) \cup LP_n) \cap S_\omega) \cup LP_{n+1} \\ &\supset\subset ((M(\omega) \cap S_\omega) \cup (LP_n \cap S_\omega)) \cup LP_{n+1} \\ &\supset\subset (M(\omega) \cup (LP_n \cap S_\omega)) \cup LP_{n+1} \\ &\supset\subset M(\omega) \cup LP_{n+1} \\ &\supset\subset (M \cap S_\omega) \cup LP_{n+1} \\ &\supset\subset (M \cup LP_{n+1}) \cap (S_\omega \cup LP_{n+1}) \\ &\supset\subset M \cap S_{n+1} \\ &\supset\subset M\,. \end{aligned}$$

§6. The rank of logic.

As a preparation, we prove the following

LEMMA 6.1. *If* $n \geq j > i \geq 0$, *then* $a_j \supset a_i \to P_n$ *is provable in* $\boldsymbol{LJ}$.

PROOF. The following three sequents are provable in $\boldsymbol{LJ}$.

(1) $P_i \to P_{j-1}$.

(2) $a_i \to P_i$.

(3) $P_j \to P_n$.

Now,

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{a_j \to a_j \qquad \dfrac{\overset{(2)}{a_i \to P_i} \qquad \overset{(1)}{P_i \to P_{j-1}}}{a_i \to P_{j-1}}}{a_j,\ a_j \supset a_i \to P_{j-1}}}{a_j \supset a_i \to a_j \supset P_{j-1}} \qquad a_j \to a_j}{a_j \supset a_i,\ (a_j \supset P_{j-1}) \supset a_j \to a_j}}{a_j \supset a_i \to P_j} \qquad \overset{(3)}{P_j \to P_n}}{a_j \supset a_i \to P_n\ .}$$

THEOREM 6.2. *If* $M \in \mathscr{S}_m$ *and* $N \in \mathscr{S}_n$ $(m,\ n<\omega)$ *are two regular models, then* $M \uparrow N \in \mathscr{S}_{m+n}$.

PROOF. Let f be an assignment of $M \uparrow N$. By 6.1, $f(P_{m+n})=1$ if $f(a_j) \geq f(a_i)$ for some j and i such that $n \geq j > i \geq 0$. Hence, if $f(P_{m+n}) \neq 1$, it must be that $V'_N \supset \{f(a_i);\ 0 \leq i \leq k\}$ and $V_M \supset \{f(a_i);\ k+1 \leq i \leq m+n\}$ for some k. Suppose that $k \geq n$. Then, the value $f(P_n)$ must be 1 since $P_n \in N$. Hence, $f(P_{m+n})$ gets the value 1. Suppose that $k<n$. Then $f(P_k)$ is either 1 or a value in V'_N. If it is 1, $f(P_{m+n})$ is obviously 1. Otherwise, $f(((a_{k+1} \supset P_k) \supset a_{k+1}) \supset a_{k+1}) = f(a_{k+1})$, since $f(a_{k+1}) \supset f(P_k) = f(P_k)$ and $f(P_k) \supset f(a_{k+1}) = 1$. Hence we can regard the value $f(P_{m+n})$ to be the value $g(P_{m+n-k-1})$ by some assignment g of M, which is 1 since $m+n-k-1 \geq m$ and $P_m \in M$. So, again, $f(P_{m+n})$ must be 1. Hence, $M \uparrow N \ni P_{m+n}$. Next, by the hypothesis, we have an assignment f_M (f_N) of M (N) such that $f_M(P_{m-1}) > 1_M$ $(f_N(P_{n-1}) > 1_N)$. It is obvious that both of the assignments do not assign the designated values to any propositional variables in the formulas. Let f be an assignment of $M \uparrow N$ such that:

$$f(a_i) = \begin{cases} f_N(a_i) & if\ 0 \leq i \leq n-1\ . \\ f_M(a_{i-n}) & if\ n \leq i \leq m+n-1\ . \end{cases}$$

Then, it is obvious that $f(P_{m+n-1}) > 1$.

REMARK. Jaśkowski [15] defines a sequence $\{J_n\}_{n=1,2,\dots}$ which converges to L as follows:

$$J_1 = S_1\ ,$$

$$J_{i+1} = S_1 \uparrow J_i^i\ .$$

And McKay [20] simplifies it as follows:

$$J'_1 = S_1\ ,$$

$$J'_{i+1} = S_1 \uparrow J_i'^2\ .$$

By the above theorem, we know that both J_n and J'_n are on the n-th slice.

DEFINITION 6.3. *The rank $r(M)$ of M is the minimum integer n such that $M \ni X_n$ if there are any, otherwise $r(M)=\omega$.*

COROLLARY 6.4. *$r(S_n)=n+1$ if n is finite, and $r(S_\omega)=\omega$.*

COROLLARY 6.5. *If M is an n-valued regular model, then $r(M)\leq n$.*

COROLLARY 6.6. *If $M \supset N$, then $r(M)\leq r(N)$.*

COROLLARY 6.7. *If $M \in \mathscr{S}_n$, then $r(M)\geq r(S_n)=n+1$.*

COROLLARY 6.8. *If $r(M)=\omega$, there is not a finite model for M.*

COROLLARY 6.9. *There is not a finite model for $M \in \mathscr{S}_\omega$.*

COROLLARY 6.10. *Let M be an n-valued regular model $(2\leq n<\omega)$. Then, $M=S_1$ or $M=S_1 \uparrow N$ for some N if and only if $r(M)=n$.*

These corollaries are almost immediate and some of them are treated in [14].

THEOREM 6.11. *The n-th slice $\mathscr{S}_n$ has infinitely many elements if $n\geq 2$.*

PROOF. The logics $S_1 \uparrow S^k_{n-1}$ $(k=1, 2, \cdots)$ all belong to $\mathscr{S}_n$ by 6.2, since $S^k_{n-1} \supset\subset S_{n-1} \in \mathscr{S}_{n-1}$. And, as $r(S_1 \uparrow S^k_{n-1})=n^k+1$ by 6.10, they are all distinct.

COROLLARY 6.12. $r(LP_n)=\omega$ $(n\geq 2)$.

PROOF. This is immediate from 6.6 and 6.11. By 6.8, we know that there is not a finite model for LP_n $(n\geq 2)$.

COROLLARY 6.13. *If $M \subset LP_2$, then M has not a finite model.*

COROLLARY 6.14. *The n-th slice $\mathscr{S}_n$ $(n\geq 3)$ has more than one logics such that they have not finite models.*

PROOF. LP_n and $LP_2 \cap S_n$ are these examples.

§ 7. Δ-projection.

DEFINITION 7.1. *Let A be a formula which does not contain the propositional variables $d_1, d_2, \cdots$. Then,*

$$\Delta(A)=\Delta^1(A)=((d_1 \supset A)\supset d_1)\supset d_1 \ ,$$

$$\Delta^{i+1}(A)=((d_{i+1} \supset \Delta^i(A))\supset d_{i+1})\supset d_{i+1} \qquad (i\geq 1) \ .$$

The logic $L+\Delta^n(A)$ is often written as $\Delta^n(LA)$.

THEOREM 7.2. *If $LA \in \mathscr{S}_n (n<\omega)$, then $\Delta(LA)\in \mathscr{S}_{n+1}$.*

PROOF. By the hypothesis, $S_n \ni A$ but $S_{n+1} \not\ni A$. So, for any assignment f of S_{n+1}, $f(A)$ is 1 or 2. Hence $f(\Delta(A))=1$. There exists an assignment g of S_{n+2} such that $g(A)=3$, hence $g(\Delta(A))=2$ if we put $g(d_1)=2$. Hence $S_{n+2} \not\ni \Delta(A)$.

COROLLARY 7.3. *Under the same hypothesis, $\Delta^m(LA)\in \mathscr{S}_{n+m}$.*

COROLLARY 7.4. $LA \supsetneqq \Delta(LA) \supsetneqq \Delta^2(LA) \supsetneqq \cdots \supsetneqq \Delta^n(LA) \supsetneqq \cdots$

THEOREM 7.5. *$LA \supset LB$ implies $\Delta^n(LA)\supset \Delta^n(LB)$.*

PROOF. It will be sufficient only if we prove the case $n=1$. By the hypothesis, there are some formulas $A_1, A_2, \cdots, A_k$ which are substituted cases of A, such that $A_1, A_2, \cdots, A_k \rightarrow B$ is provable in $\boldsymbol{LJ}$. From this, we can easily derive $\Delta(A_1), \Delta(A_2), \cdots, \Delta(A_k) \rightarrow \Delta(B)$.

COROLLARY 7.6. *$\boldsymbol{LA} \supset\subset \boldsymbol{LB}$ implies $\Delta^n(\boldsymbol{LA}) \supset\subset \Delta^n(\boldsymbol{LB})$.*

COROLLARY 7.7. *For any formula A, $\boldsymbol{LP}_n \ni \Delta^n(A)$ $(n \geq 1)$.*

THEOREM 7.8. *If $\boldsymbol{LA} \in \mathscr{S}_2$, $\Delta^n(\boldsymbol{LA}) \subset \boldsymbol{LP}_{n+1} \cap \boldsymbol{S}_{n+2}$.*

PROOF. As the proof of 7.5, $\Delta^n(R_2)$ is deducible from P_{n+1}. Hence $\boldsymbol{LP}_{n+1} \supset \Delta^n(\boldsymbol{S}_2)$. $\boldsymbol{S}_{n+2} \supset \Delta^n(\boldsymbol{S}_2)$ is obvious. So, $\boldsymbol{LP}_{n+1} \cap \boldsymbol{S}_{n+2} \supset \Delta^n(\boldsymbol{S}_2) \supset \Delta^n(\boldsymbol{LA})$.

THEOREM 7.9. *If $\boldsymbol{LA} \in \mathscr{S}_n$ $(n<\omega)$, then $\Delta(\boldsymbol{LA}) \subset \boldsymbol{LP}_2 \cap \boldsymbol{S}_{n+1}$.*

PROOF. $\boldsymbol{S}_{n+1} \supset \Delta(\boldsymbol{LA})$ is immediate from 7.2. $\boldsymbol{LP}_2 \supset \Delta(\boldsymbol{LA})$ is proved from the fact that $\boldsymbol{LP}_1 \supset \boldsymbol{LA}$ and $\boldsymbol{LP}_2 \supset\subset \Delta(\boldsymbol{LP}_1)$.

We have defined two kinds of mappings from $\mathscr{S}_n$ into $\mathscr{S}_{n+1}$. One is the ω-projection, which maps $\boldsymbol{M} \in \mathscr{S}_n$ to $\boldsymbol{M}(n+1) \in \mathscr{S}_{n+1}$. By the reasoning of §5, we know that $\boldsymbol{S}_n(n+1) \supset\subset \boldsymbol{S}_{n+1} \supset \boldsymbol{M}(n+1) \supset \boldsymbol{LP}_n(n+1) \supset\subset \boldsymbol{LP}_n \cap \boldsymbol{S}_{n+1}$. This projection, figuratively speaking, maps the n-th slice parallel with the sequence $\{\boldsymbol{S}_i\}_{i=1,2,\cdots}$ which can be regarded as a generating line. The other is the Δ-projection, which maps $\boldsymbol{LA} \in \mathscr{S}_n$ to $\Delta(\boldsymbol{LA}) \in \mathscr{S}_{n+1}$. And we know that $\boldsymbol{LP}_2 \cap \boldsymbol{S}_{n+1} \supset \Delta(\boldsymbol{LA}) \supset \Delta(\boldsymbol{LP}_n) \supset\subset \boldsymbol{LP}_{n+1}$. So this projection maps the n-th slice parallel with the generating line $\{\boldsymbol{LP}_i\}_{i=1,2,\cdots}$.

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On the axiomatic method and the algebraic method for dealing with propositional logics. J. Fac. Sci., Univ. Tokyo, Sec. I, **14** (1967), 131-169.

Page 131, line 10. For "probability", read "provability".
Page 132, line 4. For A_k, read A_n.
Page 133, line 19. For $\boldsymbol{N}$, read $\boldsymbol{M}$.
Page 134, line 10 from the bottom. For "completion of **V**", read "completion of **W**".
Page 138, line 2. For "$\Gamma(\boldsymbol{M})$'", read "$\Gamma(\boldsymbol{M})$.".
Page 138, line 11 from the bottom. For $\boldsymbol{S}_w$, read $\boldsymbol{S}_\omega$.
Page 139, line 6. For $\neg_M d_M$, read $\neg_N d_N$.
Page 139, line 9 from the bottom. For "eet", read "set".
Page 140, line 7 from the bottom. For $\Subset$, read $\Supset$.
Page 141, line 4 from the bottom. For "**logics,**", read "**logics.**".
Page 143, line 6 from the bottom. For "been", read "be".
Page 146, line 2. For $\boldsymbol{S}_2$, read $\boldsymbol{S}_3$.
Page 160, line 22. For $\boldsymbol{S}_{12}$, read $\boldsymbol{S}_1^2$.
Page 169, line 4. For 107, read 106.
Page 169, line 13. For "Jaskowski", read "Jaśkowski".