Local solutions of stochastic differential equations associated with certain quasilinear parabolic equations

By Hiroshi TANAKA

1. Introduction. Let $I=[0, T]$ for some fixed $T>0$ and $\{\beta_t, t\in I\}$ be the d-dimensional Brownian motion starting at 0. Given functions $a^{ij}(t, x, v)$, $b^i(t, x, v)$ $(1\leq i, j\leq d)$ and $c(t, x, v)$ on $I\times R^d\times R^1$, we consider the stochastic differential equation

(1.1a) $$d\xi^{(s,x)}(t)=a(t, \xi^{(s,x)}, u)d\beta_t+b(t, \xi^{(s,x)}, u)dt,\ \ \xi^{(s,x)}(s)=x,\ t\in I,$$

(1.1b) $$u(s, x)=\mathrm{E}[f(\xi^{(s,x)}(T))\exp\int_s^T c(t, \xi^{(s,x)}, u)dt],\quad 0\leq s\leq T$$

for a given data f on R^d. In the above pair of equations, $\xi^{(s,x)}$ and u mean $\xi^{(s,x)}(t)$ and $u(t, \xi^{(s,x)}(t))$ respectively. Other notational meanings will be explained in § 2. When $c=0$, stochastic differential equations of this kind were considered by Yu. N. Blagoveščenskii [1] in the investigation of local solutions of Cauchy's problems for degenerated quasi-linear parabolic equations. But, his stochastic differential equations needed a slight modification. In this paper, we extend a part of Blagoveščenskii's results to the case $c\neq 0$, and then treat similar equations on a compact manifold. In Theorem I (§ 3), we construct a local solution of (1.1) by successive approximation under the assumption of Lipschitz continuity of a^{ij}, b^i, c and f. It will be remarked that $u(s, x)$ satisfies a backward quasi-linear diffusion equation *if* it is *smooth* enough. § 4 is devoted to the case of compact manifold M (Theorem II). In this case, a similar method of successive approximation as in § 3 seems to be too complicated to carry out, and so we take another way; that is, we first imbed M into the Euclidean N-space R^N for some N, and then extend all coefficients and data to the whole of R^N by a suitable method to the effect that the resulting stochastic differential equation in R^N has a solution which can be converted onto M.

2. Notations and preliminaries

Let $\{\beta_t, t\in I\}=\{(\beta_t^1(\omega), \cdots, \beta_t^d(\omega)), t\in I\}$ be the d-dimensional Brownian motion with $\beta_0=0$, built on a probability space $(\Omega, \mathbf{B}, \mathbf{P})$. We may and do assume that the paths $\beta_t(\omega)$ are continuous. For $0\leq s\leq t\leq T$ denote by $\mathbf{B}_t^s$ the smallest σ-field on Ω that makes $\{\beta_\tau-\beta_s : s\leq\tau\leq t\}$ measurable. Choosing arbitrary sub σ-field $\mathbf{B}_0$

of $\mathbf{B}$ such that $\mathbf{B}_0$ and $\mathbf{B}_T^0$ are independent, we set $\mathbf{B}_t=\mathbf{B}_0\vee\mathbf{B}_t^0$. By an integral $\int_0^t a(s)d\beta_s^i$, we mean the stochastic integral of K. Itô [3]; this is defined for a real valued function $a(t, \omega)$ on $I\times\Omega$ such that i) $a(t, \omega)$ is (t, ω)-measurable, ii) $a(t, \omega)$ is $\mathbf{B}_t$-measurable for each $t\in I$, and iii) $\int_0^T a(t, \omega)^2dt<\infty$ with probability 1. We shall often write $\int a(s)d\beta_s$ for the n-vector $\left(\sum_j\int a^{1j}(s)d\beta_s^j, \cdots, \sum_j\int a^{nj}(s)d\beta_s^j\right)$ when $a(s)=\{a^{ij}(s)\}$ is an $n\times d$-matrix with each $a^{ij}(s)$ satisfying i), ii), and iii). The precise meaning of the stochastic differential equation (1.1a) is as follows: for fixed $s\leq T$ and $x\in R^d$, $\{\xi^{(s,x)}(t), s\leq t\leq T\}=\{(\xi^{(s,x),1}(t), \cdots, \xi^{(s,x),d}(t)), s\leq t\leq T\}$ is a stochastic process on R^d with continuous paths such that $\xi^{(s,x)}(t)$ is $\mathbf{B}_t^s$-measurable for each $t\in[s, T]$ and

$$\xi^{(s,x),i}(t)=x^i+\int_s^t a^i(\tau, \xi^{(s,x)}(\tau), u(\tau, \xi^{(s,x)}(\tau)))d\xi_\tau + \int_s^t b^i(\tau, \xi^{(s,x)}(\tau), u(\tau, \xi^{(s,x)}(\tau)))d\tau \qquad s\leq t\leq T,\ i=1, \cdots, d, \tag{2.1}$$

where $a^i(t, x, v)=(a^{i1}(t, x, v), \cdots, a^{id}(t, x, v))$ and x^i is the i-th component of $x\in R^d$.

In this section we consider the case in which a^{ij} and b^i do not depend on v (so that the equation (1.1) reduces to (2.3) below) and prepare, for the need of the next section, a simple estimate (Lemma 2.2) concerning the dependence of the solution upon the initial position x under the assumption 1.

For real valued functions $a^{ij}(t, x)$ $(1\leq i, j\leq d)$ and $b^i(t, x)$ $(1\leq i\leq d)$ on $I\times R^d$, set

$$A(t)=\sup_{\substack{x\neq y\\ 0\leq s\leq t}}\sum_{i,j}|a^{ij}(s, x)-a^{ij}(s, y)|^2/|x-y|^2 \tag{2.2a}$$

$$B(t)=\sup_{\substack{x\neq y\\ 0\leq s\leq t}}\sum_{i}|b^i(s, x)-b^i(s, y)|^2/|x-y|^2 \tag{2.2b}$$

and make the following.

Assumption 1. a^{ij} and b^i are bounded, and $A\equiv A(T)<\infty$, $B\equiv B(T)<\infty$.

Under this assumption, it is well known that the stochastic differential equation

$$d\xi(t)=a(t, \xi(t))d\beta_t+b(t, \xi(t))dt,\ t\in I,\ \xi(0)=x \tag{2.3}$$

has a unique solution, which is denoted by $\xi(t, x)$ to stress the initial position x. First we list a simple lemma without proof.

LEMMA 2.1. *If $f(t)$ and $g(t)$ are nonnegative measurable functions on $[0, T]$ and if for some constant $A\geq 0$ the inequality*

$$f(t)\leq A+\int_0^t f(s)g(s)ds<\infty,\ 0\leq t\leq T$$

holds, then $f(t) \leq A \exp\int_0^t g(s)ds,\ 0 \leq t \leq T.$

LEMMA 2.2. *Set* $\xi_t(x, y) = \xi(t, y) - \xi(t, x)$ *for* $x, y \in R^d$. *Then, for* $0 \leq t \leq T$

$$\mathrm{E}\{|\xi_t(x, y)|^2\} \leq |x-y|^2 \exp\int_0^t (A(s) + 2\sqrt{B(s)})ds \tag{2.4}$$

PROOF. Set $\sigma_t^i = (\sigma_t^{i1}, \cdots, \sigma_t^{id})$, $i=1, \cdots d$, and $\tau_t = (\tau_t^1, \cdots, \tau_t^d)$ where

$$\sigma_t^{ij} = a^{ij}(t, \xi(t, y)) - a^{ij}(t, \xi(t, x))$$
$$\tau_t^i = b^i(t, \xi(t, y)) - b^i(t, \xi(t, x)).$$

Then, with the notation $|*|$ for the usual norm of d-vecter $*$, we have $\sum_i |\sigma_t^i|^2 \leq A(t)\,|\xi_t(x, y)|^2$ and $|\tau_t|^2 \leq B(t)\,|\xi_t(x, y)|^2$ by the assumption 1. Let $f(x) = |x|^2$, and $f_i = 2x^i, f_{ij} = 2\delta_{ij}$. We now apply the transformation formula concerning stochastic differentials ([4]) to the stochastic differential $df(\xi_t(x, y))$ where

$$d\xi_t(x, y) = \sigma_t d\beta_t + \tau_t dt,\ \xi_0(x, y) = y - x,$$

and then use the above estimates on σ_t^i and τ_t. Then

$$\begin{aligned} |\xi_t(x, y)|^2 &= |x-y|^2 + \int_0^t \sum_{i,j} f_i(\xi_s(x, y))\sigma_s^{ij} d\beta_s^j \\ &\quad + \int_0^t \sum_i f_i(\xi_s(x, y))\tau_s^i ds + \frac{1}{2}\int_0^t \sum_{i,j,k} f_{ij}(\xi_s(x, y))\sigma_s^{ik}\sigma_s^{jk} ds \\ &\leq |x-y|^2 + \int_0^t \sum_{i,j} 2(\xi_s^i(x, y))\sigma_s^{ij} d\beta_s^j \\ &\quad + \int_s^t 2\sqrt{B(s)}\,|\xi_s(x, y)|^2 ds + \int_s^t A(s)\,|\xi_s(x, y)|^2 ds\,. \end{aligned} \tag{2.5}$$

Noticing that $\mathrm{E}\{|\xi_t(x, y)|^2\} < \infty$ which follows immediately from (2.3), we take the expectation of both sides of the above inequality. Then, Lemma 2.1 applied to this resulting inequality implies (2.4).

3. Existence of local solutions of stochastic differential equations (1.1).

Suppose that we are given coefficients $a^{ij}(t, x, v)$, $b^i(t, x, v)$, $c(t, x, v)$ $((t, x, v) \in I \times R^d \times R^1)$ and a real valued function $f(x)$ $(x \in R^d)$ as in the introduction, and consider the stochastic differential equation (1.1). We will prove the existence and uniqueness of local solution under the assumption 2. Let $\rho^2 = |x-y|^2 + |u-v|^2$ and set for $s \in [0, T]$

$$\mathrm{A}(s) = \sup_{\substack{(x,u) \neq (y,v) \\ s \leq t \leq T}} \rho^{-2} \sum_{i,j} |a^{ij}(t, x, u) - a^{ij}(t, y, v)|^2,\ A = A(0) \tag{3.1a}$$

(3.1b) $$B(s)=\sup_{\substack{(x,u)\neq(y,v)\\ s\leq t\leq T}}\rho^{-2}\sum_i |b^i(t,x,b)-b^i(t,y,v)|^2,\ B=B(0)$$

(3.1c) $$C(s)=\sup_{\substack{(x,u)\neq(y,v)\\ s\leq t\leq T}}\rho^{-2}|c(t,x,u)-c(t,y,v)|^2,\ C=C(0)$$

(3.1f) $$F=\sup_{x\neq y}|f(x)-f(y)|^2/|x-y|^2$$

Assumption 2. A^{ij}, b^i, c^+ and f are bounded, and $A, B, C, F<\infty$, where c^+ is the positive part of c.

DEFINITION. Let $s_0\in[0,T)$. By a solution of (1.1) in $(s_0, T]$, we mean a family of stochastic processes $\{\xi^{(s,x)}(t), s\leq t\leq T\}$ $((s,x)\in(s_0,T]\times R^d)$ with continuous paths such that $\xi^{(s,x)}(t)$ is $\mathbf{B}_t^s$-measurable for each $t\in[s,T]$, and (2.1) and (1.1b) hold.

THEOREM I. *Under the assumption* 2, *there exists* $s_0\in[0,T)$ *such that* (1.1) *has a solution in* $(s_0,T]$ *and the corresponding function* $u(s,x)$ *satisfies*

(3.2) $$\sup_{\substack{x\neq y\\ s_1\leq t\leq T}}\frac{|u(s,x)-u(s,y)|}{|x-y|}<\infty,\ \textit{for any}\ s_1\in(s_0,T].$$

Furthermore, a solution for which (3.2) *holds is unique.*

The proof is based on successive approximation and will be completed after a series of lemmas. First we set $u_0(s,x)=f(x)$, and then for $n=1,2,\cdots$, define successively as follows:

$$a_n^{ij}(s,x)=a^{ij}(s,x,u_{n-1}(s,x)),\ b_n^i(s,x)=b^i(s,x,u_{n-1}(s,x)),$$

$$c_n(s,x)=c(s,x,u_{n-1}(s,x)),$$

(3.3) $$\xi_n^{(s,x)}(t)=x+\int_s^t a_n(\tau,\xi_n^{(s,x)}(\tau))d\beta_\tau+\int_s^t b_n(\tau,\xi_n^{(s,x)}(\tau))d\tau$$

(3.4) $$u_n(s,x)=\mathrm{E}[f(\xi_n^{(s,x)}(T))\exp\int_s^T c_n(t,\xi_n^{(s,x)}(t))dt].$$

We define $A_n(s)$ by

$$A_n(s)=\sup_{\substack{x\neq y\\ s\leq t\leq T}}\sum_{i,j}|a_n^{ij}(t,x)-a_n^{ij}(t,y)|^2/|x-y|^2,\ A_n=A_n(0)$$

and also $B_n(s)$, $C_n(s)$, $U_n(s)$ by a similar way. The following lemma shows that the coefficients in (3.3) satisfy Lipschitz condition, so that the above definitions make sense for all n.

LEMMA 3.1. (i) *If* $U_{n-1}<\infty$, *then*

$$A_n(s)\leq A(s)(1+U_{n-1}(s))\leq A(1+U_{n-1})<\infty,$$

and similar inequalities for $B_n(s)$ and $C_n(s)$ hold.

(ii) *If $A_n<\infty$,*

$$\mathrm{E}\{|\xi_n^{(s,x)}(t)-\xi_n^{(s,y)}(t)|^2\}\leq|x-y|^2\exp\{(A_n(s)+2\sqrt{B_n(s)})(t-s)\}.$$

(iii) *If $U_{n-1}<\infty$, then*

$$U_n(s)\leq 2\{F+(T-s)^2||f||^2C(1+U_{n-1}(s))\}$$
$$\times\exp[\{(A+2\sqrt{B})(1+U_{n-1}(s))+2||c^+||\}(T-s)]^{1)}.$$

(iv) *A_n, B_n, C_n, $U_n<\infty$ for all n.*

PROOF. Since (i) is obvious and (ii) is immediate from Lemma 2.2, we prove (iii). Noting (3.4) and then using Schwarz inequality, we have after simple calculations

$$\begin{aligned}&|u_n(s,x)-u_n(s,y)|^2\\ &\leq 2\mathrm{E}[|f(\xi_n^{(s,x)}(T))-f(\xi_n^{(s,y)}(T))|^2\exp 2\int_s^T c_n(t,\xi_n^{(s,x)}(t))dt]\\ &+2\mathrm{E}[|f(\xi_n^{(s,y)}(T))|^2|\exp\int_s^T c_n(t,\xi_n^{(s,x)}(t))dt-\int_s^T c_n(t,\xi_n^{(s,y)}(t))dt|^2]\\ &\leq 2e^{2(T-s)||c^+||}F\mathrm{E}[|\xi_n^{(s,x)}(T)-\xi_n^{(s,y)}(T)|^2]\\ &+2e^{2(T-s)||c^+||}||f||^2(T-s)C_n(s)\int_s^T\mathrm{E}[|\xi_n^{(s,x)}(t)-\xi_n^{(s,y)}(t)|^2]dt.\end{aligned}$$

Inserting the expression (ii) into the above and using (i), we obtain (iii) after a short calculation. (iv) follows from $U_0=F<\infty$, (iii) and (i).

LEMMA 3.2. $s_0\equiv\inf\{t\in[0,T];\ \sup_n U_n(t)<\infty\}<T$.

PROOF. It is enough to show that for some κ $(\geq U_0)$ and $t\in[0,T)$ the inequality $U_{n-1}(t)<\kappa$ implies $U_n(t)\leq\kappa$, and for this by (iii) of Lemma 3.1 it is also enough to prove the existence of $\kappa\geq U_0$ and $t\in[0,T)$ such that

$$2\{F+(T-t)^2||f||^2C(1+\kappa)\}\exp[\{(A+2\sqrt{B})(1+\kappa)+2||c^+||\}(T-t)]\leq\kappa.$$

But, the above inequality holds if $\kappa>2F$ and $T-t$ is small enough.

In the following lemma and in its proof, $K, K_0, K_1, \cdots$ denote suitably chosen constants independent of n and t. They may depend on s, but are monotone decreasing in s. Also, when we think of $u_n(t,x)$ as a function of x with t fixed, we denote it by $u_n(t)$.

LEMMA 3.3. *For fixed $s\in(s_0,T]$ we set*

$$\delta_n(s,t)=\sup_n\mathrm{E}\{|\xi_{n+1}^{(s,x)}(t)-\xi_n^{(s,x)}(t)|^2\},\quad s\leq t\leq T.$$

Then $\delta_n(s,t)$ and $||u_{n+1}(s)-u_n(s)||^2$ are dominated by $K(TK_0)^n(n!)^{-1}$.

1) $||\cdot||$ is the supremum norm.

PROOF. For $s\leq t\leq T$ set

$$\eta_n(t)=\xi_{n+1}^{(s,x)}(t)-\xi_n^{(s,x)}(t)$$
$$\alpha_n^{ij}(t)=a_{n+1}^{ij}(t,\xi_{n+1}^{(s,x)}(t))-a_n^{ij}(t,\xi_n^{(s,x)}(t))$$
$$\gamma_n^i(t)=b_{n+1}^i(t,\xi_n^{(s,x)}(t))-b_n^i(t,\xi_n^{(s,x)}(t))\ .$$

Using the expression $d\eta_n^i(t)=\sum_j\alpha_n^{ij}(t)d\beta_t^j(t)dt+\gamma_n^i(t)dt$ and then the assumption 2, we have

$$\mathbf{E}\{|\eta_n(t)|^2\}\leq 2\int_s^t\mathbf{E}\{\sum_{i,j}|\alpha_n^{ij}(\tau)|^2\}d\tau+2T\int_s^t\mathbf{E}\{\sum_i|\gamma_n^i(\tau)|^2\}d\tau$$
$$\leq 2(A+BT)\int_s^t(1+2U_n(\tau))\mathbf{E}\{|\eta_n(\tau)|^2\}d\tau+4(A+BT)\int_s^t||u_n(\tau)-u_{n-1}(\tau)||^2d\tau,$$

and hence

$$\delta_n(s,t)\leq K_1\int_s^t\delta_n(s,\tau)d\tau+K_2\int_s^t||u_n(\tau)-u_{n-1}(\tau)||^2d\tau \tag{3.5}$$

where $K_1=2(A+BT)(1+2\sup_n U_n(s))$, $K_2=4(A+BT)$. Applying Lemma 2.1 to (3.5) we have

$$\delta_n(s,t)\leq e^{K_1(t-s)}K_2\int_s^T||u_n(\tau)-u_{n-1}(\tau)||^2d\tau\ . \tag{3.6}$$

and hence

$$\int_s^T\delta_n(s,t)dt\leq K_3\int_s^T||u_n(t)-u_{n-1}(t)||^2dt\ , \tag{3.7}$$

for suitable K_3. On the other hand, by a similar method as in the proof of (iii) of Lemma 3.1,

$$||u_{n+1}(s)-u_n(s)||^2\leq K_4\delta_n(s,T)+K_5\int_s^T\delta_n(s,t)dt$$
$$+K_6\int_s^T||u_n(t)-u_{n-1}(t)||^2dt,$$

and inserting the expression (3.5) with $t=T$ into the above

$$||u_{n+1}(s)-u_n(s)||^2\leq K_7\int_s^T\delta_n(s,t)dt+K_8\int_s^T||u_n(t)-u_{n-1}(t)||^2dt\ . \tag{3.8}$$

From (3.7) and (3.8)

$$||u_{n+1}(s)-u_n(s)||^2\leq K_9\int_s^T||u_n(t)-u_{n-1}(t)||^2dt\ .$$

Since K_9 can be chosen to be monotone decreasing in s, the above inequality implies the conclusion of Lemma 3.3 for $\| u_{n+1}(s)-u_n(s) \|^2$ and hence the same for $\delta_n(s, t)$ by (3.6).

Now we complete the proof of the theorem. By the stochastic integral equation (3.3) that $\xi_n^{(s,x)}(t)$ satisfies, each component of $\xi_n^{(s,x)}(t)-x$ splits into a martingale (stochastic integral part based on the Brownian motion) and a process with absolutely continuous paths. We write $\xi_n^{(s,x)}(t)-x=X_n(t)+Y_n(t)$ for this decomposition. Then, by Doob's inequality on submartingales

$$\mathbf{P}\{\max_{s\le t\le T} |X_{n+1}(t)-X_n(t)|>2^{-n}\}\le 2^{2n}V_n,$$

$$V_n=\mathbf{E}\{|X_{n+1}(T)-X_n(T)|^2\}.$$

But, by the same way as we derived (3.5), V_n is dominated by the right hand side of (3.5), and hence $2^{2n}V_n$ is a general term of a convergent series. So, by Borel-Cantelli's lemma $X_n(t)$ converges uniformly in $t\in[s, T]$ as n tends to ∞ with probability 1. Since a similar reasoning based on Chebyschev's inequality can apply to $Y_n(t)$, the same conclusion holds for $Y_n(t)$ and hence for $\xi_n^{(s,x)}(t)$. Let $\xi^{(s,x)}(t)$ be the limit of $\xi_n^{(s,x)}(t)$ as n tends to ∞. Then, letting n tend to ∞ in (3.3) and (3.4), it is easily seen that $\{\xi^{(s,x)}(t)\}$ is a solution of (1.1) in $(s_0, T]$ satisfying (3.2). Finally, to prove the uniqueness, let $\{\xi^{(s,x)}(t)\}$ and $\{\xi_*^{(s,x)}(t)\}$ be solutions of (1.1) in $(s_0, T]$ both satisfying (3.2), and set

$$\delta(s, t)=\sup_x \mathbf{E}\{|\xi^{(s,x)}(t)-\xi_*^{(s,x)}(t)|^2\}, \quad s_0<s\le t\le T.$$

Then as in (3.5)

$$\delta(s, t)\le K_1'\int_s^t \delta(s, \tau)d\tau+K_2'\int_s^t \| u(\tau)-u_*(\tau)\|^2 d\tau$$

where $u_*(t)$ is defined from $\{\xi_*^{(s,x)}(t)\}$ as in (1.1b) and K_1', K_2' are suitable constants. Similar arguments after (3.5) are applicable, and we have $\delta(s, t)=0$ and hence ξ ξ_* are the same.

REMARK 1. Let $\{\xi^{(s,x)}\}$ be the solution constructed in Theorem I, and regard u as a given function in (1.1a). Then, the method of successive approximation for solving (1.1a) shows that for each $s\in(s_0, T]$ $\xi^{(s,\cdot)}(\cdot, \cdot)$ is measurable with respect to $\mathbf{F}\times\mathbf{F}_s\times\mathbf{B}_T^s$ where $\mathbf{F}$ (resp. $\mathbf{F}_s$) is the class of Borel set in R^1 (resp. $[s, T]$). Also by the uniqueness, we have $\xi^{(s,x)}(t)=\xi^{(s+h,z)}(t)$ $(z=\xi^{(s,x)}(s+h))$ for all $t\in[s+h, T]$ with probability 1 for each s, h $(s\le s+h\le T)$.

REMARK 2. We suppose, in addition to the assumption 2, that a^{ij}, b^i and c are continuous in t and c is bounded, and let $\{\xi^{(s,x)}\}$ be the solution constructed in Theorem 2. Further, suppose that $u(s, x)$ is of C^2 in x for each s and that $u(s, x)$, $u_i(s, x)$ and $u_{ij}(s, x)$ (partial derivatives with respect to the space variable) are bounded and continuous in s ([1] contains informations for proving smoothness of u). Then, by the transformation formula on stochastic differentials

$$\begin{aligned}
&u(s, x)-u(s+h, x)\\
&=\sum_i \mathbf{E}\left[\int_s^{s+h} u_i(s+h, \xi^{(s,x)}(t)) \exp\left(\int_s^t c d\tau\right) b^i dt\right]\\
&\quad+\int_s^{s+h} \mathbf{E}\left[u(s+h, \xi^{(s,x)}(t)) \exp\left(\int_s^t c d\tau\right) c dt\right]\\
&\quad+\frac{1}{2}\sum_{i,j,k} \mathbf{E}\left[\int_s^{s+h} u_{ij}(s+h, \xi^{(s,x)}(t)) \exp\left(\int_s^t c d\tau\right) a^{ik}a^{jk} dt\right]
\end{aligned}$$

for $s_0<s<s+h\leq T$, and hence

$$\begin{cases}
-u'(s, x)=\sum_{i,j} A^{ij}(s, x, u)u_{ij}+\sum_i b^i(s, x, u)u_i+c(s, x, u)u, & s_0<s<T\\
u(T, x)=f(x),
\end{cases}$$

where $u'(s, x)=\partial u(s, x)/\partial s$ and $A^{ij}(s, x, u)=\frac{1}{2}\sum_k a^{ik}(s, x, u)a^{jk}(s, x, u)$.

4. Stochastic differential equations on compact manifold

Let M be a compact C^∞-manifold of dimension d. Stochastic differential equations of the type (1.1) can be considered also on M. First we introduce a system of diffusion coefficients on M. Let $I=[0, T]$ as before, and suppose that to each local chart $\mathfrak{v}=(V, \phi)$ on M there corresponds a collection $\{a_{\mathfrak{v}}^{ij}, b_{\mathfrak{v}}^i, i, j=1, \cdots, d\}$ of functions from $I\times\phi(V)\times R^1$ into R^1. We say that a system $\{a^{ij}, b^i\}$ of diffusion coefficients is given on M, if these collections for different local charts are connected by the following transformation rule: for each pair $\mathfrak{v}=(V, \phi)$ and $\tilde{\mathfrak{v}}=(\tilde{V}, \tilde{\phi})$ of local charts on M

$$a_{\tilde{\mathfrak{v}}}^{ij}(t, \tilde{x}, v)=\sum_{k=1}^d \frac{\partial \tilde{x}^i}{\partial x^k} a_{\mathfrak{v}}^{kj}(t, x, v), \quad x\in\phi(V\cap\tilde{V}),\ v\in R^1 \tag{4.1a}$$

$$\begin{aligned}
b_{\tilde{\mathfrak{v}}}^i(t, \tilde{x}, v)=&\sum_{k=1}^d \frac{\partial\tilde{x}^i}{\partial x^k} b_{\mathfrak{v}}^k(t, x, v)\\
&+\frac{1}{2}\sum_{j,k,l}\frac{\partial^2\tilde{x}^i}{\partial x^k\partial x^l} a_{\mathfrak{v}}^{kj}(t, x, v)a_{\mathfrak{v}}^{lj}(t, x, v),\ x\in\phi(V\cap\tilde{V}),\ v\in R^1
\end{aligned} \tag{4.1b}$$

where $x=(x^1,\cdots,x^d)=\phi(q)$, $q\in V$, and $\tilde{x}=(\tilde{x}^1,\cdots,\tilde{x}^d)=\tilde{\phi}(q)$, $q\in\tilde{V}$. In addition to $\{a^{ij}, b^i\}$, suppose we are given a function $c(t,p,v)$ on $I\times M\times R^1$ and a function f on M. Our problem is to find, for some $s_0\in[0,T)$, a family $\Pi=\{\pi^{(s,p)}, (s,p)\in(s_0,T]\times M\}$ of stochastic processes $\pi^{(s,p)}=\{\pi^{(s,p)}(t), s\leq t\leq T\}$ on M such that

(π.1) $\pi^{(s,p)}(t)$ is continuous in t and $\pi^{(s,p)}(s)=p$ with probability 1,

(π.2) $\pi^{(s,p)}(t)$ is $\mathbf{B}_t^s$-measurable for each $t\in[s,T]$,

(π.3) for any $s_0<s\leq s_1<t<T$ and any local chart $\mathfrak{v}=(V,\phi)$,

$$\phi(\pi^{(s,p)}(t))=\phi(\pi^{(s,p)}(s_1))+\int_{s_1}^t a_{\mathfrak{v}}(\tau,\phi(\pi^{(s,p)}(\tau)), u(\tau,\pi^{(s,p)}(\tau)))d\beta_\tau \tag{4.2}$$

$$+\int_{s_1}^t b_{\mathfrak{v}}(\tau,\phi(\pi^{(s,p)}(\tau)), u(\tau,\pi^{(s,p)}(\tau)))d\tau$$

holds almost everywhere on $\{\pi^{(s,p)}(s_1)\in V$ and $t<\sigma\}$ where σ is the infimum of $\tau\geq s_1$ for which $\pi^{(s,p)}(\tau)\notin V$, and

(π.4) $u(s,p)$ satisfies

$$u(s,p)=\mathbf{E}\left[f(\pi^{(s,p)}(T))\exp\int_s^T c(t,\pi^{(s,p)}(t)), u(t,\pi^{(s,p)}(t)))dt\right]. \tag{4.3b}$$

Symbolically we write

$$d\pi^{(s,p)}(t)=a(t,\pi^{(s,p)},u)d\beta_t+b(t,\pi^{(s,p)},u)dt \tag{4.3a}$$

and call Π a solution of (4.3) in $(s_0,T]$. We make the following assumption:

Assumption 3. For every local chart $\mathfrak{v}=(V,\phi)$ and every compact subset K of $\phi(V)$, i) $a_{\mathfrak{v}}^{ij}(t,x,v)$, $b_{\mathfrak{v}}^i(t,x,v)$, $c^+(t,\phi^{-1}(x),v)$ are bounded on $I\times K\times R^1$, ii) $a_{\mathfrak{v}}^{ij}(t,x,v)$, $b_{\mathfrak{v}}^i(t,x,v)$, $c(t,\phi^{-1}(x),v)$ satisfy the Lipschitz condition as functions of $(x,v)\in K\times R^1$ uniformly in $t\in I$, and iii) $f(\phi^{-1}(x))$ satisfies the Lipschitz condition on K.

THEOREM II. *Under the above assumption, there exists an $s_0\in[0,T)$ such that (4.3) has a solution in $(s_0,T]$ and the corresponding function $u(s,p)$ satisfies*

$$\sup_{\substack{x,y\in K\\ x\neq y,\ s_1\leq s\leq T}} |u(s,\phi^{-1}(x))-u(s,\phi^{-1}(y))|/|x-y|<\infty \tag{4.4}$$

for each local chart (V,ϕ) and each compact subset K of $\phi(V)$, $s_1\in(s_0,T]$. Moreover, such a solution is unique.

We prove this theorem by reducing it to Theorem I by the method outlined below. We imbed M into the Euclidean space R^N of suitable dimension N, and then on the basis of the transformation rule (4.1) we introduce several functions $(a_\alpha^{ij}, b_\alpha^i, c_0, f_0)$ on $\varphi(M)$ (φ is the inbedding of M into R^N). These functions are extended to the whole of R^N (Lemma 4.1) to obtain a stochastic differential equation in R^N of the same type as (1.1), and we finally prove that the solution is confined on $\varphi(M)$ if the initial position is on $\varphi(M)$ to obtain a required solution on M by the mapping φ^{-1}.

The imbedding of M into R^N is well known, but in order to make the above procedures precise we first sketch the method of imbedding ([5]). For each $p\in M$ choose a local chart $\mathfrak{v}_p=(V_p, \psi_p)$ such that $p\in V_p$ and $x(p)=0$ where $x(q)=(x^1(q), \cdots, x^d(q))=\psi_p(q)$, $q\in V_p$. For fixed r_1, r_2 $(0<r_1<r_2)$ such that $[-r_2, r_2]^d\subset\psi_p(V_p)$, we set

$$Q_p=\{q\in V_p:\quad |x^i(q)|<r_1,\quad i=1, \cdots, d\}$$
$$R_p=\{q\in V_p:\quad |x^i(q)|<r_2,\quad i=1, \cdots, d\},$$

and let g be a C^∞-function in R^1 such that $g(t)=1$ for $|t|\leq r_1, g(t)=0$ for $|t|\geq r_2$ and $0<g(t)<1$ for $r_1<|t|<r_2$. We define a C^∞-function f_p on M by $f_p(q)=g(x^1(q))\cdots g(x^d(q))$ for $q\in R_p$ and $f_p(q)=0$ for $q\notin R_p$. Since $\{Q_p, p\in M\}$ is an open covering of M, there exist finite points $p_1, \cdots, p_{N_1}$ in M such that $\bigcup_{\alpha=1}^{N_1} Q_{p_\alpha}=M$. We set for simplicity

$$f_{p_\alpha}=f_\alpha,\ Q_{p_\alpha}=Q_\alpha,\ (V_{p_\alpha}, \psi_{p_\alpha})=(V_\alpha, \psi_\alpha),$$
$$a^{ij}_{\mathfrak{v}_{p_\alpha}}=a^{ij}_\alpha,\ b^i_{\mathfrak{v}_{p_\alpha}}=b^i_\alpha,$$

and denote by $x_\alpha=(x^1_\alpha, \cdots, x^d_\alpha)$ the local coordinates with respect to (V_α, ψ_α). We introduce $N(=N_1(d+1))$ C^∞-functions $\{f^i_\alpha, i=0, 1, \cdots, d, \alpha=1, \cdots, N_1\}$ on M as follows:

$$f^0_\alpha=f_\alpha, \alpha=1, \cdots, N_1$$

$$f^i_\alpha(q)=\begin{cases} f_\alpha(q)x^i_\alpha(q), & q\in V_\alpha \\ 0, & q\notin V_\alpha \end{cases}\quad \begin{pmatrix} i=1, \cdots, d \\ \alpha=1, \cdots, N_1 \end{pmatrix}.$$

Then the mapping $\varphi: q\in M\to(f^i_\alpha(q), 0\leq i\leq d, 1\leq\alpha\leq N_1)\in R^N$ gives an imbedding of M into R^N. Next, we introduce the functions a^{ij}_α, b^i_α $(0\leq i\leq d,\ 1\leq j\leq d,\ 1\leq\alpha\leq N_1)$ on $I\times\varphi(M)\times R^1$ on the basis of the transformation rule (4.1). Each point y in the image $\varphi(M)$ has the coordinate $\varphi(p)=(f^i_\alpha(p),\ 0\leq i\leq d,\ 1\leq\alpha\leq N_1)$. Setting $x_\alpha=(x^1_\alpha, \cdots, x^d_\alpha)=\psi_\alpha(p)$ and choosing α_0 such that $p\in Q_{\alpha_0}$ we define for $0\leq i\leq d$, $1\leq j\leq d$, $1\leq\alpha\leq N_1$

$$\alpha^{ij}_\alpha(t, y, v)=\sum_{k=1}^d\frac{\partial(f^i_\alpha\circ\psi_{\alpha_0}{}^{-1})}{\partial x^k_{\alpha_0}}a^{kj}_{\alpha_0}(t, x_{\alpha_0}, v)$$

$$b_\alpha^i(t, y, v) = \sum_{k=1}^{d} \frac{\partial (f_\alpha^i \circ \phi_{\alpha_0}^{-1})}{\partial x_{\alpha_0}^k} b_{\alpha_0}^k(t, x_{\alpha_0}, v)$$
$$+ \frac{1}{2} \sum_{j,k,l=1}^{d} \frac{\partial^2 (f_\alpha^i \circ \phi_{\alpha_0}^{-1})}{\partial x_{\alpha_0}^k \partial x_{\alpha_0}^l} a_{\alpha_0}^{kj}(t, x_{\alpha_0}, v) a_{\alpha_0}^{lj}(t, x_{\alpha_0}, v) .$$

There may be many α_0's for which $p \in Q_{\alpha_0}$, but it is only a matter of applying the transformation rule to prove that the above definitions are independent of the choice of α_0. Also we set $c_0(t, y, v) = c(t, \phi^{-1}(y), v)$ and $f_0(y) = f(\phi^{-1}(y))$ for $y \in \varphi(M)$. Then, from our constructions and the assumption 3, we see easily that (i) a_α^{ij}, b_α^i, c_0^+ are bounded on $I \times \varphi(M) \times R^1$, (ii) each of a_α^{ij}, b_α^i, c_0 satisfies the Lipschitz condition as a function of $(y, v) \in \varphi(M) \times R^1$ uniformly in $t \in I$, and (iii) f_0 satisfies the Lipschitz condition on $\varphi(M)$.

Our next task is to extend a_α^{ij}, b_α^i and c_0 (resp. f_0) to the whole of $I \times R^N \times R^1$ (resp. R^N). We first consider the function f_0. For each $\alpha = 1, \cdots, N_1$, we choose three neighborhoods $Q_{\alpha,i}$ $(i = 1, 2, 3)$ of p such that the closure $\bar{Q}_{\alpha,i}$ is contained in $Q_{\alpha,i-1}$ for $i = 1, 2, 3$ $(Q_{\alpha,0} = Q_\alpha)$ and $\bigcup_{\alpha=1}^{N_1} Q_{\alpha,3} = M$. In the following, we write $y = (y_\alpha^i,$ $0 \le i \le d,\ 1 \le \alpha \le N_1)$ for points in R^N and sat $y_\alpha = (y_\alpha^i, 1 \le i \le d) \in R^d$ for $\alpha = 1, \cdots, N_1$. For each $\varepsilon > 0$ and $\alpha = 1, \cdots, N_1$ set

$$W_{\alpha,1}^\varepsilon = \{y \in R^N : y_\alpha \in \phi_\alpha(Q_{\alpha,1}) \textit{ and } |y^j - f_\beta^j \circ \phi_\alpha^{-1}(y_\alpha)| < \varepsilon \textit{ for } j = 0 \textit{ or } \beta \ne \alpha.\} \tag{4.5}$$

Then, we can easily show that $W_{\alpha,1}^\varepsilon \cap \varphi(M) = \varphi(Q_{\alpha,1})$ for sufficiently small ε. We fix an ε so that the above equality holds for all α, and suppress ε from $W_{\alpha,1}^\varepsilon$. Then for each $W_{\alpha,1}$ we can correspond a unique $\bar{y}$ in $\varphi(M)$ for which $\bar{y}_\alpha^i = y_\alpha^i$, $i = 1, \cdots, d$. Obviously $\bar{y} \in \varphi(Q_{\alpha,1})$. Now we define $f_\alpha(y) = f_0(\bar{y})$ for $y \in W_{\alpha,1}$. The Lipschitz continuity of f_0 on $\varphi(M)$ and the inequalities $|\bar{y} - \bar{z}|^2 \le |\bar{y}_\alpha - \bar{z}_\alpha|^2 + \text{constant} \cdot |y_\alpha - z_\alpha|^2 \le \text{constant} \cdot |y - z|^2$ for $y, z \in W_{\alpha,1}$ imply the Lipschitz continuity of f_α on $W_{\alpha,1}$. We next piece together f_α, $\alpha = 1, \cdots, N_1$, to get a nice function $\tilde{f}$ which coincides with f_0 on $\varphi(M)$. To do this, we define $W_{\alpha,2}(W_{\alpha,3})$ by the right hand side of (4.5) with $Q_{\alpha,1}$ and ε replaced by $Q_{\alpha,2}(Q_{\alpha,3})$ and $\varepsilon/2(\varepsilon/3)$ respectively. Since $\overline{W}_{\alpha,3} \subset W_{\alpha,2}$, we can choose a C^∞-function g on R^N such that $g = 1$ on $\overline{W}_{\alpha,3}$, $g = 0$ outside $W_{\alpha,2}$ and $0 < g < 1$ in $W_{\alpha,2} - \overline{W}_{\alpha,3}$. Since $\sum_{\alpha=1}^{N_1} g_\alpha$ is strictly positive on $\bigcup_{\alpha=1}^{N_1} W_{\alpha,3}$, there exists a strictly positive C^∞-function h which coincides with $\sum_{\alpha=1}^{N_1} g_\alpha$ on $\bigcup_{\alpha=1}^{N_1} W_{\alpha,3}$. We set $h_\alpha = g_\alpha / h$ and define $\tilde{f}$ by $\tilde{f}(y) = \sum_{\alpha=1}^{N_1} \tilde{f}_\alpha(y)$, where $\tilde{f}_\alpha = f_\alpha \cdot h_\alpha$ on $W_{\alpha,2}$ and $\tilde{f}_\alpha = 0$ outside $W_{\alpha,2}$. Then $\tilde{f}$ is bounded on the whole of R^N, satisfies the Lipschitz condition, and coincides with f_0 on $\varphi(M)$. As for a_α^{ij}, b_α^i and c_0, we think of them as functions of $y \in \varphi(M)$ with (t, v) fixed, and apply the above argument with *fixed*

$Q_{\alpha,i}$, $W_{\alpha,i}$, g_α and h. As a consequence, we have the following lemma.

LEMMA 4.1. *There exist functions* $\tilde{a}_\alpha^{ij}$, $\tilde{b}_\alpha^i$ $(0\leq i\leq d,\ 1\leq j\leq d,\ 1\leq\alpha\leq N_1)$ *and* $\tilde{c}$ *on* $I\times R^N\times R^d$ *and a function* $\tilde{f}$ *on* R^N *with the following properties:* (i) $\tilde{a}_\alpha^{ij}$, $\tilde{b}_\alpha^i$, $\tilde{c}^+$ *and* $\tilde{f}$ *are bounded,* (ii) $\tilde{a}_\alpha^{ij}$, $\tilde{b}_\alpha^i$, $\tilde{c}$ *and* $\tilde{f}$ *are extensions of* a_α^{ij}, b_α^i, c_0 *and* f_0 *respectively, and hence if* $y=(y_\gamma^i,\ 0\leq i\leq d,\ 1\leq\gamma\leq N_1)\in\varphi(Q_\alpha)$ *and* $y_\alpha=(y_\alpha^i,\ 1\leq i\leq d)\in\psi_\alpha(Q_\alpha)$, *then*

$$\tilde{a}_\beta^{ij}(t,y,v)=\sum_{k=1}^{d}\frac{\partial(f_\beta^i\circ\psi_\alpha^{-1})}{\partial y_\alpha^k}\tilde{a}_\alpha^{kj}(t,y,v) \tag{4.6a}$$

$$\tilde{b}_\beta^i(t,y,v)=\sum_{k=1}^{d}\frac{\partial(f_\beta^i\circ\psi_\alpha^{-1})}{\partial y_\alpha^k}\tilde{b}_\alpha^k(t,y,v) + \frac{1}{2}\sum_{j,k,l=1}^{d}\frac{\partial^2(f_\beta^i\circ\psi_\alpha^{-1})}{\partial y_\alpha^k\partial y_\alpha^l}\tilde{a}_\alpha^{kj}(t,y,v)\tilde{a}_\alpha^{lj}(t,y,v) \tag{4.6b}$$

$$(0\leq i\leq d,\ 1\leq j\leq d,\ 1\leq\beta\leq N_1),$$

(iii) *each of* $\tilde{a}_\alpha^{ij}$, $\tilde{b}_\alpha^i$, $\tilde{c}$ *satisfies the Lipschitz condition as a function of* $(y,v)\in R^N\times R^1$ *uniformly in* $t\in I$, (iv) f *satisfies the Lipschitz condition.*

We next consider the stochastic differential equation:

$$d\xi_\alpha^{(s,y),i}(t)=\sum_{j=1}^{d}\tilde{a}_\alpha^{ij}(t,\xi^{(s,y)},u)d\beta_t^j+\tilde{b}_\alpha^i(t,\xi^{(s,y)},u)dt\ , \tag{4.7a}$$

$$\xi^{(s,y)}(s)=y;\ 0\leq s\leq t\leq T,\ 0\leq i\leq d,\ 1\leq\alpha\leq N_1,$$

$$u(s,y)=\mathrm{E}\left[\tilde{f}(\xi^{(s,y)}(T))\exp\int_s^T c(t,\xi^{(s,y)},u)dt\right] \tag{4.7b}$$

where $\xi^{(s,y)}(t)=(\xi_\alpha^{(s,y),i}(t),\ 0\leq i\leq d,\ 1\leq\alpha\leq N_1)$ and $y=(y_\alpha^i,\ 0\leq i\leq d,\ 1\leq\alpha\leq N_1)$. By Lemma 4.1 and Theorem I, this stochastic differential equation has a unique local solution $\{\xi^{(s,y)}(t),\ s\leq t\leq T\}$ $(s\in(s_0,T],\ y\in R^N)$.

LEMMA 4.2. *For each* $y\in\varphi(M)$ *and* $s\in(s_0,T]$, $\xi^{(s,y)}(t)$ *is on* $\varphi(M)$ *with probability* 1.

PROOF. Considering u as a given function, we set $\tilde{a}_\alpha^{ij}(t,y,u(t,y))=\tilde{a}_\alpha^{ij}(t,y)$, $\tilde{b}_\alpha^i(t,y,u(t,y))=\tilde{b}_\alpha^i(t,y)$. For each $z\in\varphi(M)$ and α with $z\in\varphi(Q_\alpha)$, denote by $\sigma_\alpha(t,z)$ the supremum of $t'\in[t,T]$ such that $\xi_\alpha^{(t,z)}(\tau)\in\psi_\alpha(Q_\alpha)$ for all $\tau\in[t,t']$. We first prove that for each $y\in\varphi(M)$ $\xi^{(s,y)}(t)$ is on $\varphi(M)$ up to σ with probability 1, where $\sigma=\sigma_\alpha(s,y)$, $y\in\varphi(Q_\alpha)$. For $x\in\psi_\alpha(Q_\alpha)\subset R^d$, we set

$$\tilde{a}_\alpha^{ij}(t,z)=a_\alpha^{ij}(t,x),\ \tilde{b}_\alpha^i(t,z)=b_\alpha^i(t,x)$$

where $z=(z_\beta^i,\ 0\leq i\leq d,\ 1\leq\beta\leq N_1)$, $z_\beta^i=(f_\beta^i\circ\psi_\alpha^{-1})(x)$. Then, $a_\alpha^{ij}(t,x)$ and $b_\alpha^i(t,x)$ are bounded and satisfy the Lipschitz condition as functions of $x\in\psi_\alpha(Q_\alpha)$ uniformly in

$t\in[s,T]$, and hence the following stochastic equation has a unique solution $\{\eta_\alpha(t),\ s\leq t\leq T\}=\{(\eta_\alpha^1(t),\cdots,\eta_\alpha^d(t)),\ s\leq t\leq T\}$ ([2]).

$$\eta_\alpha^i(t)=y_\alpha^i+\sum_{j=1}^d\int_s^t a_\alpha^{ij}(\tau,\eta_\alpha(\tau))\chi(\tau<\tilde{\sigma})d\beta_\tau^j$$
$$+\int_s^t b_\alpha^i(\tau,\eta_\alpha(\tau))\chi(\tau<\tilde{\sigma})d\tau \qquad (1\leq i\leq d,\ s\leq t\leq T)$$

where $\tilde{\sigma}$ is the supremum of $\tau\in[s,T]$ for which $\eta_\alpha(\tau')\in\psi_\alpha(Q_\alpha)$ for all $\tau'\in[s,\tau]$, and $\chi(A)$ denotes the indicator function of A. Set $\eta_\beta^i(t)=(f_\beta^i\circ\psi_\alpha^{-1})(\eta_\alpha(t))$ for $0\leq i\leq d$ and $1\leq\beta\leq N_1$. Then, using the transformation formula on stochastic differentials and (4.6), we have

$$d\eta_\beta^i(t)=\sum_{j,k=1}^d\frac{\partial(f_\beta^i\circ\psi_\alpha^{-1})}{\partial x^k}(\eta_\alpha(t))a_\alpha^{kj}(t,\eta_\alpha(t))d\beta_t^j$$
$$+\sum_{k=1}^d\frac{\partial(f_\beta^i\circ\psi_\alpha^{-1})}{\partial x^k}(\eta_\alpha(t))b_\alpha^k(t,\eta_\alpha(t))dt$$
$$+\frac{1}{2}\sum_{j,k,l=1}^d\frac{\partial^2(f_\beta^i\circ\psi_\alpha^{-1})}{\partial x^k\partial x^l}a_\alpha^{kj}a_\alpha^{lj}dt$$
$$=\sum_{j=1}^d\tilde{a}_\beta^{ij}(t,\eta(t))d\beta_t^j+\tilde{b}_\beta^i(t,\eta(t))dt,\ t<\tilde{\sigma}$$

where $\eta(t)=(\eta_\beta^i(t),\ 0\leq i\leq d,\ 1\leq\beta\leq N_1)$. Since $\{\xi^{(s,y)}(t\wedge\sigma),\ s\leq t\leq T\}$ satisfies the same stochastic differential equation as above, we have $\xi^{(s,x)}(t\wedge\sigma)=\eta(t\wedge\tilde{\sigma})$, with probability 1. But, this means that $\xi^{(s,x)}(t)$ is on $\varphi(M)$ up to σ with probability 1. Next, let $Q_{\alpha,1}$ be the same as in the paragraph preceding to Lemma 4.1, and for each $z\in\varphi(M)$ denote by $\alpha(z)$ the first α such that $\varphi(Q_{\alpha,1})\ni z$. We set $\sigma_0=s$ and for $n\geq 1$ $\sigma_n=$the supremum of $t\in[\sigma_{n-1},T]$ such that $\xi_\alpha^{(s,y)}(t')\in\psi_\alpha(Q_\alpha)$ for all $t'\in[\sigma_{n-1},t]$ where $\alpha=\alpha(\xi^{(s,y)}(\sigma_{n-1}))$. We prove that for $n=1,2,\cdots$

$$P[\xi^{(s,y)}(t)\in\varphi(M)\ \textit{for all}\ t\in[s,\sigma_n]]=1\,, \tag{4.8}$$

by induction. This is true for $n=1$, as we have just proved. For $n\geq 2$, using the induction hypothesis, we have

$$\{\xi^{(s,y)}(t)\in\varphi(M)\ \textit{for all}\ t\in[s,\sigma_n]\} \tag{4.9}$$
$$=\bigcup_{\alpha,\beta=1}^{N_1}\{\alpha(\xi^{(s,y)}(\sigma_{n-2}))=\alpha,\ \alpha(\xi^{(s,y)}(\sigma_{n-1}))=\beta \text{ and }$$
$$\xi^{(s,y)}(t)\in\varphi(M)\ \textit{for all}\ t\in[\sigma_{n-1},\sigma_n]\}$$
$$\underset{\text{a.s.}}{=}\bigcup_{\alpha,\beta=1}^{N_1}\bigcup_{\gamma}\Lambda_{\alpha,\beta,\gamma}{}^{2)},\qquad \Lambda_{\alpha,\beta,\gamma}=\Lambda_{\alpha,\beta,\gamma}^1\cap\Lambda_{\alpha,\beta,\gamma}^2\,,$$

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2) $\Lambda\underset{\text{a.s.}}{=}\Lambda'$ means that $\mathbf{P}[(\Lambda-\Lambda')\cup(\Lambda'-\Lambda)]=0$.

$$\Lambda^1_{\alpha,\beta,r}=\{\sigma_{n-2}<r\leq\sigma_{n-1},\ \alpha(\xi^{(s,y)}(\sigma_{n-2}))=\alpha,\ \xi^{(s,y)}(r)\in\varphi(Q_\beta)\}$$

$$\Lambda^2_{\alpha,\beta,r}=\{\alpha(\xi^{(s,y)}(\sigma_{n-1}))=\beta,\ \xi^{(s,y)}(t)\in\varphi(M)\ \textit{for all}\ t\in[r,\sigma']\},$$

where the union in r is taken over all rationals r in $[s,T]$ and σ' is the supremum of $t\in[r,T]$ for which $\xi_\beta^{(s,y)}(t')\in\varphi_\beta(Q_\beta)$ for all $t'\in[r,t]$. Since $\xi^{(s,y)}(t)=\xi^{(r,z)}(t)$ $(z=\xi^{(s,y)}(r))$ for $t\geq r$ with probability 1 by Remark 1 (§3), we have

$$\begin{aligned}\mathbf{P}\{\Lambda_{\alpha,\beta,r}\}&=\mathbf{E}\{\mathbf{P}[\Lambda^2_{\alpha,\beta,r}\mid\mathbf{B}_r],\ \Lambda^1_{\alpha,\beta,r}\}^{3)}\\&=\mathbf{E}\{\mathbf{P}[\alpha(\xi^{(r,z)}(\sigma_\alpha(r,z)))=\beta,\ \xi^{(r,z)}(t)\in\varphi(M)\\&\qquad \textit{for all}\ t\in[r,\sigma_\beta(r,z)]]_{z=\xi^{(s,y)}(r)},\ \Lambda^1_{\alpha,\beta,r}\}\\&=\mathbf{P}\{\Lambda^1_{\alpha,\beta,r},\ \alpha(\xi^{(s,y)}(\sigma_{n-1}))=\beta\}.\end{aligned}$$

So that we have

$$(4.9)\underset{\text{a.s.}}{=}\bigcup_{\alpha,\beta=1}^{N_1}\bigcup\{\Lambda^1_{\alpha,\beta,r},\ \alpha(\xi^{(s,y)}(\sigma_{n-1}))=\beta\}\underset{\text{a.s.}}{=}\Omega$$

and hence (4.8). On the other hand, $\sigma_n=T$ for some $n=n(\omega)$ with probablity 1, because, if $\sigma_n(\omega)<T$ for all n, then $\xi^{(s,y)}(t)$ would be discontinuous at $t=\lim\sigma_n(\omega)\leq T$. This remark and (4.8) complete the proof of the lemma.

Lemma 4.2 enables us to define stochastic processes on M by $\pi^{(s,p)}(t)=\varphi^{-1}(\xi^{(s,y)}(t))$, $y=\varphi(p)$, and obviously the family $\{\pi^{(s,p)}(t)\}$ is a solution of (4.3) in $(s_0,T]$. Also the uniqueness is reduced to Theorem I by the mapping φ.

University of Tokyo

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3) $\mathbf{E}[X,A]=\int_A X d\mathbf{P}$.