On the hypoellipticity of differential operators

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0. Introduction.

0.0. In this paper we consider the regularity property of differential operators. A linear differential operator P (of order m) with C^∞ coefficients defined in a domain Ω in the n-dimensional Euclidean space R_n, in short, a differential operator P (of order m), is written as follows:

$$P=P(x, D)=\sum_{|\alpha|\leq m} a_\alpha(x)D^\alpha, \qquad a_\alpha(x)\in C^\infty(\Omega).$$

Here $x=(x^1, \cdots, x^n)$ is a generic point of Ω; $D_j=-i\partial/\partial x^j$, $j=1, \cdots, n$, $i^2=-1$; $D^\alpha=D_1^{\alpha_1}\cdots D_n^{\alpha_n}$ for $\alpha=(\alpha_1, \cdots, \alpha_n)\in N^n$ with $N=\{0, 1, 2, \cdots\}$; and $|\alpha|=\alpha_1+\cdots+\alpha_n$. $P_m(x, \xi)=\sum_{|\alpha|=m} a_\alpha(x)\xi^\alpha$ is the principal symbol of P, where $\xi=(\xi_1, \cdots, \xi_n)\in R_n$ and $\xi^\alpha=\xi_1^{\alpha_1}\cdots\xi_n^{\alpha_n}$. Function spaces $C^\infty(\Omega)$, $C_0^\infty(\Omega)$, $\mathscr{H}_{(s)}^{\mathrm{loc}}(\Omega)$, and $\mathscr{D}'(\Omega)$ are as usual[1]. $C_0^\infty(K)$ and $\mathscr{H}_{(s)}(K)$ are subspaces of $C_0^\infty(\Omega)$ and of $\mathscr{H}_{(s)}^{\mathrm{loc}}(\Omega)$ respectively, consisting of those elements with support in K where K is any compact subset of Ω with interior points. Note that the topology of $\mathscr{H}_{(s)}^{\mathrm{loc}}(\Omega)$ can be defined by the seminorm system[2] $\{||\chi_j\varphi||_{(s)}\}$ $(j\in N, \varphi\in\mathscr{H}_{(s)}^{\mathrm{loc}}(\Omega))$ where $\chi_j\in C_0^\infty(\Omega)$ with $\chi_j=1$ on $\bar{\Omega}_j$ and $\chi_j=0$ on $\complement\Omega_{j+1}$.[3] Here the Ω_j are subdomains of Ω with $\bar{\Omega}_j\subset\Omega_{j+1}$, the closures $\bar{\Omega}_j$ being compact and $\bigcup_j\Omega_j=\Omega$. A differential operator P is called *hypoelliptic* in Ω if we have

$$\text{sing supp } P\varphi=\text{sing supp } \varphi^{4)} \text{ for all } \varphi\in\mathscr{D}'(\Omega).$$

0.1. In § 1, we prove certain inequalities as necessary conditions of hypoellipticity. We also show that the adjoint operator of a hypoelliptic differential operator is locally solvable. In § 2, we show that hypoelliptic operators in $C_0^\infty(K)$ have finite dimensional nullspaces and closed ranges. Finally we give an example of non-elliptic differential operators with finite index on a compact C^∞ manifold.

1) As for definitions and notations we follow mainly those in Hörmander [2].

2) $||\varphi||_{(s)}=\left\{\int_{R_n}(1+|\xi|^2)^s|\varphi^\wedge(\xi)|^2 d\xi\right\}^{1/2}$, where $\varphi^\wedge$ is the Fourier transform of $\varphi\in\mathscr{H}_{(s)}(R_n)$.

3) $\complement M$ denotes the complement in Ω of the set $M\subset\Omega$. $\bar{M}$ denotes the closure in Ω of the set $M\subset\Omega$.

4) sing supp $\varphi=\complement\{x\in\Omega;\ \varphi$ is C^∞ near $x\}$ for $\varphi\in\mathscr{D}'(\Omega)$. It is well-known that we have, for any differential operator P,

$$\text{sing supp } P\varphi\subset\text{sing supp } \varphi$$

where $\varphi\in\mathscr{D}'(\Omega)$.

0.2. As an application of Theorem 1.2, the author had obtained a result concerning first order hypoelliptic differential operators. But recently the author was informed that Mr. Y. Kato independently obtained a more general result by a similar method. See [4]. So we deleted this part from the original manuscript. In the preparation of the present paper, the author owes very much to Prof. K. Yosida.

1. Some inequalities.

Let E and F be vector spaces and P a linear operator defined in E into F. Let E_1 and F_1 be subspaces of E and F respectively. Let us denote by $P^\wedge_{E_1,F_1}$, in short $P^\wedge$, the restriction of P to the domain $E_1\cap\{e;\ e$ in the domain of P with $Pe\in F_1\}$, and call $P^\wedge$ the operator induced in the pair (E_1, F_1) by P. We have the obvious

LEMMA 1.1. *Let P be a linear differential operator defined in Ω. Then P is a continuous operator in $\mathscr{D}'(\Omega)$, and the following operators $P^\wedge$ induced by P in each pair are closed operators*:

(1.1) $P^\wedge\colon \mathscr{H}_{(s)}^{\mathrm{loc}}(\Omega)\to C^\infty(\Omega)$ *for every fixed* $s\in R_1$;

(1.2) $P^\wedge\colon \mathscr{H}_{(s)}(K)\to \mathscr{H}_{(t)}(K)$ *for s and t in R_1,*

where K is any fixed compact subset of Ω with interior points. Thus $\mathscr{G}_P=\{\varphi\in\mathscr{H}_{(s)}^{\mathrm{loc}}(\Omega);\ P^\wedge\varphi\in C^\infty(\Omega)\}$ equipped with the graph topology becomes a Fréchet space.

THEOREM 1.1. *Let P be a hypoelliptic differential operator in Ω and Q be any fixed differential operator in Ω. Let s be a fixed real number. For any $t\in R_1$ and $j\in N$, we have*

$$||\chi_j Q\varphi||_{(t)}\leq c\{||\chi_{j_1}\varphi||_{(s)}+||\chi_{j_2}P\varphi||_{(t')}\}$$

for all $\varphi\in C^\infty(\Omega)$ with suitable $c>0, j_1, j_2\in N$ and $t'\in R_1$.

PROOF. Let $P^\wedge$ and $Q^\wedge$ be respectively the induced operators of P and Q in the pair $(\mathscr{H}_{(s)}^{\mathrm{loc}}(\Omega), C^\infty(\Omega))$. $P^\wedge$ and $Q^\wedge$ are closed operators. From the hypoellipticity of P, the domoin of $P^\wedge$ is contained in $C^\infty(\Omega)$, and hence in the domain of $Q^\wedge$. The mapping $J\colon[\varphi, P\varphi]\in\mathscr{G}_P\to Q\varphi\in C^\infty(\Omega)$ being closed, J is continuous by Banach's closed graph theorem. Recalling Sobolev's lemma $C^\infty(\Omega)=\bigcap_s\mathscr{H}_{(s)}^{\mathrm{loc}}(\Omega)$, we obtain the desired inequality by writing down explicitly the continuity of J in terms of the seminorm systems. q.e.d.

In the sequel, we consider only the case $Q=1=$ the identity.

REMARK 1.1. The author found that essentially the same inequality (in case

$Q=1$ and designed for pseudo-differential operators P) was obtained earlier by Hörmander [3].

By the local property of differential operators, we have the

LEMMA 1.2. *Let K be any compact subset of Ω with interior points. We have, for any $t\in R_1$,*

$$\|\varphi\|_{(t)}\leqq c\{\|\varphi\|_{(0)}+\|P\varphi\|_{(t')}\} \text{ for all } \varphi\in C_0^\infty(K) \tag{1.3}$$

with suitable $c>0$ and $t'\in R_1$.

REMARK 1.2. If we denote by $\alpha(t)$ the infinimum of t' such that the inequality (1.3) holds with $c<+\infty$. Then we easily see that $\alpha(t)$ is non-decreasing. In fact, since $\|\varphi\|_{(s)}\leqq\|\varphi\|_{(t)}$ for $s\leqq t$, we have for any fixed $t'>\alpha(t)$,

$$\|\varphi\|_{(s)}\leqq c(t')\{\|\varphi\|_{(0)}+\|P\varphi\|_{(t')}\}\,.$$

Hence $\alpha(s)\leqq\alpha(t)$ if $s\leqq t$.

By Poincaré's inequality we have, from Lemma 1.2, the following

LEMMA 1.3. *For every $x\in\Omega$ there exists a sufficiently small neighborhood U of x such that*

$$\|\varphi\|_{(t)}\leqq c\|P\varphi\|_{(t')} \text{ for all } \varphi\in C_0^\infty(\bar U)\,.$$

THEOREM 1.2. *If a differential operator P (of order m) is hypoelliptic in Ω, then its formal adjoint P^* is locally solvable in Ω, that is, at every $x\in\Omega$ there exists a neighborhood U of x such that we can find, for every $\varphi\in C_0^\infty(U)$, a distribution u satisfying $P^*u=\varphi$ in U.*

PROOF. By Lemma 1.3, $P\varphi=0$ with $\varphi\in C_0^\infty(\bar U)$ implies $\varphi=0$. Thus for any $\psi\in C_0^\infty(U)$ we can define a linear functional L on $P(C_0^\infty(U))$ by $\int\psi\varphi\,dx=L(P\varphi)$, where φ is an arbitrary element in $C_0^\infty(U)$. Since $|L(P\varphi)|=\left|\int\psi\varphi\,dx\right|\leqq\|\psi\|_{(-t)}\|\varphi\|_{(t)}\leqq c\|\psi\|_{(t)}\|P\varphi\|_{(t')}$, there exists a u in $\mathscr{H}_{(-t')}$ such that $\langle u, P\varphi\rangle=\int\psi\varphi\,dx$ for all $\varphi\in C_0^\infty(U)$ (the Hahn-Banach theorem). This proves the theorem.

COROLLARY 1.1.

$$C_{2m-1}(x,\xi)=2\,Re\,i\sum_{j=1}^n\partial P_m(x,\xi)/\partial\xi_j\,\overline{\partial P_m(x,\xi)/\partial x^j}=0$$

if $P_m(x,\xi)=0$ at $x\in\Omega$ and $\xi\in R_n$. Here $\overline{\cdot}$ denotes the complex conjugate.

PROOF. This is an immediate consequence of the solvability of P^*. See Chapter 6 of Hörmander [2].

REMARK 1.3. Hörmander [3] proved the above Corollary 1.2 directly from Lemma 1.3 by substituting a suitable function into φ. We note that for any differential operators of the form P^n, where $n \geqq 2$ and P is an arbitrary differential operator, the above relation in Corollary 1.1 trivially holds. However we have the

THEOREM 1.3. *A differential operator P is hypoelliptic if and only if its n-th power P^n is hypoelliptic, where n is any natural number $\geqq 2$.*

PROOF. This follows immediately from the following

LEMMA 1.4. *Let P and Q be differential operators.*

(1.4) *If P and Q are hypoelliptic, then PQ is hypoelliptic;*

(1.5) *If PQ is hypoelliptic, then Q is hypoelliptic.*

PROOF OF LEMMA. Let φ be any element in $\mathscr{D}'(\Omega)$. Then we have

$$\text{sing supp } PQ\varphi = \text{sing supp } Q\varphi = \text{sing supp } \varphi ,$$

if P and Q are hypoelliptic. This proves (1.4). Next, if PQ is hypoelliptic,

$$\text{sing supp } \varphi = \text{sing supp } PQ\varphi \subset \text{sing supp } Q\varphi \subset \text{sing supp } \varphi,$$

that is,

$$\text{sing supp } Q\varphi = \text{sing supp } \varphi .$$

Hence Q is hypoelliptic.

2. Nullspaces and ranges.

Let K be any fixed compact subset of Ω with interior points. In this section we study the nullspace and range of a hypoelliptic differential operator in $C_0^\infty(K)$. Let T and T' be strictly increasing divergent sequences of positive numbers: $T = \{t_i\}_{i \in N}$, and $T' = \{t'_i\}_{i \in N}$ such that for every $t = t_i \in T$ we assign $t' = t'_i \in T'$ to satisfy the inequality in Lemma 1.2. Let P_i be the closure as an operator from $\mathscr{H}_{(t_i)}(K)$ into $\mathscr{H}_{(t'_i)}(K)$ of the differential operator P in $C_0^\infty(K)$. The preclosedness of the operator P follows from (1.2) in Lemma 1.1. Then by Lemma 1.2 and passing to the limit we have

$$\|\varphi\|_{(t_i)} \leqq c_i \{\|\varphi\|_{(0)} + \|P_i \varphi\|_{(t'_i)}\} \tag{2.1}$$

for all $\varphi \in D^i$ with a suitable constant $c_i > 0$. Here we denote by D^i the domain of P_i.

LEMMA 2.1.

(2.2) $D^{i+1} \subset D^i$, *and* $\varphi \in D^{i+1}$ *implies* $P_{i+1}\varphi = P_i \varphi$;

(2.3) $\bigcap_i D^i = C_0^\infty(K)$.

PROOF. Let $\varphi \in D^{i+1}$. Then there exist $\varphi_k \in C_0^\infty(K)$ such that $\varphi_k \to \varphi$ in $\mathscr{H}_{(t_{i+1})}(K)$ and $P\varphi_k \to P_{i+1}\varphi$ in $\mathscr{H}_{(t'_{i+1})}(K)$. Since $t_{i+1} > t_i$ and $t'_{i+1} > t'_i$, we have $\varphi_k \to \varphi$ in $\mathscr{H}_{(t_i)}(K)$ and $P\varphi_k \to P_{i+1}\varphi$ in $\mathscr{H}_{(t'_i)}(K)$. By the definition of P_i, we have $\varphi \in D^i$ and $P_{i+1}\varphi = P_i\varphi$. This shows (2.2). (2.3) follows from the inclusion relations $C_0^\infty(K) \subset D^i \subset \mathscr{H}_{(t_i)}(K)$ and $\bigcap_i \mathscr{H}_{(t_i)}(K) = C_0^\infty(K)$. q.e.d.

LEMMA 2.2.

(2.4) $\quad\mathscr{N}_i$, *the nullspace of* P_i, *is of finite dimension;*

(2.5) $\quad\mathscr{R}_i$, *the range of* P_i, *is closed.*

PROOF. For $\varphi \in \mathscr{N}_i$, we have, from (2.1),

$$\|\varphi\|_{(t_i)} \leqq c_i \|\varphi\|_{(0)} .$$

Since the injection from $\mathscr{H}_{(t_i)}(K)$ into $\mathscr{H}_{(0)}(K)$ is compact, we see that $\mathscr{N}_i$ is of finite dimension. Thus we can decompose $\mathscr{H}_{(t_i)}(K)$ into the topological direct sum of closed subspaces:

$$\mathscr{H}_{(t_i)}(K) = \mathscr{N}_i \dotplus \mathscr{F}_i .$$

If we show

$$\|\varphi\|_{(t_i)} \leqq c\|P_i\varphi\|_{(t'_i)} \quad \text{for} \quad \varphi \in \mathscr{F}_i \cap D^i$$

with a suitable constant $c>0$, then the proof of this lemma will be complete. In fact, suppose that there exists a sequence $\{\varphi_k\}_{k\in N}$ such that $\varphi_k \in \mathscr{F}_i \cap D^i$ and

$$\|\varphi_k\|_{(t_i)} \geqq k\|P_i\varphi_k\|_{(t'_i)} . \tag{2.6}$$

We may assume $\|\varphi_k\|_{(t_i)} = 1$. Since the injection from $\mathscr{H}_{(t_i)}(K)$ into $\mathscr{H}_{(0)}(K)$ is compact, we may choose a subsequence $\{\varphi_{k'}\}$ from $\{\varphi_k\}$ such that $\varphi_{k'} \to \psi$ in $\mathscr{H}_{(0)}(K)$ where ψ is a certain element in $\mathscr{H}_{(0)}(K)$. Since $\|P_i\varphi_{k'}\|_{(t'_i)} \to 0$ by (2.6), we see that $\psi \in \mathscr{H}_{(t_i)}(K)$ and $\varphi_{k'} \to \psi$ in $\mathscr{H}_{(t_i)}(K)$ from (2.1). By the closedness of P_i, we have $\psi \in D^i$ and $P_i\psi = 0$. On the other hand, we observe that $\psi \in \mathscr{F}_i$ from the closedness of the subspace $\mathscr{F}_i$, and that $\psi \neq 0$ from $\|\varphi_{k'}\|_{(t_i)} = 1$. In conclusion, we have

$$\psi \in \mathscr{F}_i \cap D^i , \quad \psi \neq 0 \quad \text{and} \quad P_i\psi = 0 ,$$

but this is a contradiction. q.e.d.

THEOREM 2.1. *Let K be a compact subset of Ω with interior points. For a hypoelliptic differential operator considered as a linear operator in $C_0^\infty(K)$, we have*

i) *the dimension of the nullspace is finite;*

ii) *the range is closed.*

PROOF. Let $\mathscr{N}$ and $\mathscr{R}$ be respectively the nullspace and the range of the differential operator P acting in $C_0^\infty(K)$. P being hypoelliptic, $\mathscr{N}=\mathscr{N}_i$ and this proves the statement i) by (2.4). If we show $\mathscr{R}=\bigcap_i \mathscr{R}_i$, the proof will be complete. For any $\psi\in\bigcap_i \mathscr{R}_i$ there exists $\varphi_i\in D^i$ such that $P_i\varphi_i=\psi$. From the hypoellipticity of P and (2.2), we see $\varphi_i-\varphi_j\in C_0^\infty(K)$ for $j>i$. Thus we have $\varphi_i-\varphi_j+(\varphi_i-\varphi_j)\in\bigcap_{j>i} D^j\subset C_0^\infty(K)$. The inclusion $\mathscr{R}\subset\bigcap_i \mathscr{R}_i$ being obvious, we obtain $\mathscr{R}=\bigcap_i \mathscr{R}_i$. q.e.d.

REMARK 2.1. This proof is inspired by Grisvard [1].

LEMMA 2.3. *Let $x=(x^1,\cdots,x^n)$ be a local coordinate system of class C^∞ in an open set U. Let $y=(y^1,\cdots,y^n)$ be another local coordinate in U such that the Jacobian J of the coordinate transformation $x\to y$ is C^∞ and non-vanishing in U. Let P be a differential operator. We denote by P_x^* (resp. P_y^*) the adjoint operator with respect to the coordinate system x (resp. y), that is,*

$$\int_U P\varphi\cdot\psi dx=\int_U \varphi\cdot P_x^*\psi dx\left(resp.\ \int_U P\varphi\cdot\psi dy=\int_U \varphi\cdot P_y^*\psi dy\right)$$

for $\varphi,\psi\in C_0^\infty(U)$. If P_x^* is hypoelliptic in U, then so is P_y^*.

PROOF. Since the mapping $\varphi\to P\varphi$ is independent of coordinate system, we have

$$\int_U P\varphi\cdot\psi dx=\int_U P\varphi\cdot\psi J dy\ .$$

Hence

$$\int_U \varphi(P_x^*\psi)Jdy=\int_U \varphi P_y^*(\psi J)dy\ ,$$

that is,

$$JP_x^*\psi=P_y^*(J\psi)\ .$$

Since J is C^∞ and non-vanishing, the assertion holds. q.e.d.

Let Ω be a compact C^∞ manifold of dimension n, and P be a differential operator on Ω. The adjoint operator of P is determined depending on the choice of the partition of unity of Ω and the local coordinate systems. However, since the hypoellipticity of a differential operator is a local property and from the above lemma, it is well-defined that the adjoint operator of a differential operator is hypoelliptic on Ω or not. Therefore, from the proof of Theorem 2.1 and Banach's closed range theorem, we obtain the following

THEOREM 2.2. *Let Ω be a compact C^∞ manifold of dimension n. If a differential operator P and its adjoint P^* are both hypoelliptic in Ω, then P and P^* considered as linear operators in $C^\infty(\Omega)$ are of finite index, that is, their nullspaces are of finite dimension, and their ranges are closed and of finite codimension.*

It is well-known that any elliptic operator satisfies the hypothesis of Theorem 2.2. However we can give an example, slightly deviated from elliptic operators.

EXAMPLE 2.1. Let Ω be a compact C^∞ manifold of dimension 2, and P be a first order elliptic operator with C^∞ coefficients in Ω. In some neighborhood U_1 of a point $\omega_0 \in \Omega$ there exists a local coordinate (x, y) about ω_0 such that P is expressed, in this coordinate system, in the form

$$P=\frac{\partial}{\partial x}+i\,b(x, y)\frac{\partial}{\partial y}+c(x, y) \quad \text{in } U_1 ,$$

where $b(x, y)$ is a real-valued C^∞ function with $b(x, y)>0$ in U_1, and $c(x, y)$ is a C^∞ function in U_1. Take neighborhoods U_2 and U_3 of ω_0 with compact closure such that

$$U_3 \subset \bar{U}_3 \subset U_2 \subset \bar{U}_2 \subset U_1 .$$

Let $\varphi_j \in C_0^\infty(U_{j-1})$ $(j=2, 3)$ be such that $1 \geqq \varphi_j \geqq 0$ on U_{j-1} and $\varphi_j=1$ on $\bar{U}_j$. We shall construct a new differential operator $P^{(1)}$ from P: In U_1 we define $P^{(1)}$ by

$$P^{(1)}=\frac{\partial}{\partial x}+i\,b_1(x, y)\frac{\partial}{\partial y}+(1-\varphi_3(x, y))\,c(x, y)$$

where

$$b_1(x, y)=(1-\varphi_3(x, y))\,b(x, y)+x^{2m}\varphi_2(x, y) , \qquad m \in N ,$$

and outside U_1 we define $P^{(1)}$ as equal to P. Then we see that $P^{(1)}$ and its adjoint operator are both hypoelliptic in Ω. In fact, $P^{(1)}$ being elliptic in $\Omega-\{\omega_0\}$, so is its adjoint there. Since

$$P^{(1)}=\frac{\partial}{\partial x}+i\,x^{2m}\frac{\partial}{\partial y}$$

in U_3 in the coordinate system (x, y), its adjoint takes the form

$$P^{(1)*}=-\frac{\partial}{\partial x}-i\,x^{2m}\frac{\partial}{\partial y}=-P^{(1)}$$

in U_3 in this coordinate. The hypoellipticity of $P^{(1)}$ in U_3 was proved by Mizohata [5]. Therefore $P^{(1)*}$ is also hypoelliptic in U_3.

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