On the automorphism groups of homogeneous bounded domains

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Introduction

Let $\mathscr{D}(V, F)$ be a Siegel domain of the second kind, associated to the convex cone V and the V-hermitian form F (See §1 for the definition). This is originally defined by Piatetskij-Šapiro [11]. As is proved in [11], a $\mathscr{D}(V, F)$ is a domain in a certain $\boldsymbol{C}^n$, which is not bounded but holomorphically equivalent to a bounded domain in $\boldsymbol{C}^n$. Vinberg, Gindikin and Piatetskij-Šapiro [17] proved by using their theory of j-algebras the following realization theorem: every homogeneous bounded domain in $\boldsymbol{C}^n$ is holomorphically equivalent to some $\mathscr{D}(V, F) \subset \boldsymbol{C}^n$ which is affine homogeneous, *i.e.* an orbit of some affine transformation group of $\boldsymbol{C}^n$.

The first aim of the present paper is to prove the uniqueness of this $\mathscr{D}(V, F)$ (See Theorem 5.1): or more precisely, if a homogeneous bounded domain in $\boldsymbol{C}^n$ is holomorphically equivalent to affine homogeneous Siegel domains $\mathscr{D}(V, F)$ and $\mathscr{D}(V', F')$ in $\boldsymbol{C}^n$, then $\mathscr{D}(V, F)$ is carried to $\mathscr{D}(V', F')$ by some linear transformation of $\boldsymbol{C}^n$, as stated by Piatetskij-Šapiro [10] without proof. Since the uniqueness theorem of realizations is closely related to the classification of homogeneous bounded domains, it is desirable to give a rigorous proof of it. As a corollary (Corollary 5.2), it follows that if a symmetric domain is an affine homogeneous Siegel domain of the first kind $\mathscr{D}(V)$ (See §1 for the definition), then the cone V is self-adjoint (Rothaus [14]). The converse is known (Rothaus [13]). The second aim is to study the irreducible decomposition of homogeneous bounded domains, which is concerned with the classification of homogeneous bounded domains (See §6). A homogeneous bounded domain $\mathscr{D}$ is called *irreducible* if it is not holomorphically equivalent to the direct product of two homogeneous bounded domains. We shall prove in Theorem 6.1 that every homogeneous bounded domain is uniquely decomposed into the direct product of irreducible homogeneous bounded domains, which contains as a special case the well-known theorem for the case of symmetric domains. We shall also give a criterion of irreducibility of $\mathscr{D}$ (Theorem 6.3).

After some preperations in §1, we shall prove in §2 and §3 several facts which are implicitly used in Vinberg, Gindikin, Piatetskij-Šapiro [17] without proof. We give the explicit construction of a certain solvable group $\mathscr{T}$ (See §2). The group $\mathscr{T}$ acts simply transitively on an affine homogeneous Siegel domain $\mathscr{D}(V, F)$. In the case that the $\mathscr{D}(V, F)$ is a symmetric domain, the group $\mathscr{T}$ coincides with the Iwasawa subgroup of the holomorphic automorphism group of the $\mathscr{D}(V, F)$. In §3, we shall prove that the identity component $\mathscr{G}_h$ of the holomorphic automorphism group of an affine homogeneous $\mathscr{D}(V, F)$ has the trivial centre and that $\mathscr{G}_h$ is isomorphic to the identity component of a real algebraic group (Theorem 3.1 and Theorem 3.2). In §4, it is proved that the conjugate class of $\mathscr{T}$ in $\mathscr{G}_h$ is uniquely determined by the given homogeneous bounded domain $\mathscr{D}$ and is independent of realizations of $\mathscr{D}$ as Siegel domains of the second kind. This is essential in proving the uniqueness theorem.

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§1. Preliminaries

Throughout this paper we will use the following notations.

$\mathscr{G}\mathscr{L}(L)$:	The group of all non-singular real (resp. complex) linear transformations on a real (resp. complex) vector space L.
$\mathscr{A}(L)$:	The group of all complex affine transformations of a complex vector space L.
$GL(L)$:	The Lie algebra of $\mathscr{G}\mathscr{L}(L)$.
$A(L)$:	The Lie algebra of $\mathscr{A}(L)$.
$\mathscr{G}, \mathscr{K}, \mathscr{H}, \cdots$:	Lie groups (denoted by Roman script letters).
$G, K, H, \cdots$:	The Lie algebras of $\mathscr{G}, \mathscr{K}, \mathscr{H}, \cdots$ respectively (denoted by the Roman capital letters)
$\mathscr{G}^0$:	The identity component of $\mathscr{G}$.
$\boldsymbol{R}, \boldsymbol{C}$:	The real and the complex number fields respectively.

We shall often denote by 1 the identity element of a group.

Let L be a complex vector space of n dimension and $\tilde{L}$ be the complex vector space of $n+1$ dimension in which L is imbedded as a hyperplane not through the origin. Then the action of each element of $\mathscr{A}(L)$ is naturally extended to the non-singular linear transformation on $\tilde{L}$. $\mathscr{G}\mathscr{L}(\tilde{L})$ is considered as a subgroup of $\mathscr{G}\mathscr{L}(\tilde{L}_R)$, where $\tilde{L}_R$ is the underlying real vector space of $\tilde{L}$. Therefore $\mathscr{A}(L)$ is considered as a closed subgroup of $\mathscr{G}\mathscr{L}(\tilde{L}_R)$. A subgroup $\mathscr{G}$ of $\mathscr{A}(L)$ is called a "*real algebraic group*" of $\mathscr{A}(L)$ if $\mathscr{G}$ is a real algebraic subgroup of $\mathscr{G}\mathscr{L}(\tilde{L}_R)$ via the above imbedding. A subgroup $\mathscr{G}$ is called a ***R**-triangular subgroup* of $\mathscr{A}(L)$ if $\mathscr{G}$ is represented as upper triangular matrices with respect to some base of $\tilde{L}_R$. It is easily seen that $\mathscr{G}$ is $\boldsymbol{R}$-triangular subgroup of $\mathscr{A}(L)$ if and only if the linear part of $\mathscr{G}$ is $\boldsymbol{R}$-triangular.

DEFINITION 1.1. ([16]) Let R be a real vector space of finite dimension and V be a domain of R. V is called a *convex cone* in R if the following conditions are satisfied:

1) For every $x \in V$ and every $\lambda > 0$, $\lambda x \in V$.
2) V is a convex set in R.
3) V contains no entire straight line.

DEFINITION 1.2. ([11]) Let V be a convex cone in R and W be a finite dimensional complex vector space. The mapping F of $W \times W$ into the complexification R_C of R is called a *V-hermitian form* if the following conditions are satisfied:

1) $F(u, v)$ is complex-linear in u. $u, v \in W$.
2) $\overline{F(u, v)} = F(v, u)$. (The bar means the conjugation of R_C with respect to R.)
3) $F(u, u) \in \bar{V}$. ($\bar{V}$ means the closure of V.)
4) $F(u, u) = 0$ if and only if $u = 0$.

The set

$$\mathscr{D}(V, F) = \{(x+iy, u) \in R_C \times W: \quad y - F(u, u) \in V\}$$

is a domain in $R_C \times W$, which is called a *Siegel domain of the second kind* associated to the cone V and the V-hermitian form F. This was originally defined by Piatetskij-Šapiro [11]. If $W = (0)$, then $\mathscr{D}(V, F)$ is reduced to

$$\mathscr{D}(V) = \{x+iy \in R_C: \quad x \in R, \ y \in V\},$$

which is called a *Siegel domain of the first kind*. For simplicity, we shall often call the Siegel domain for the Siegel domain of the second kind and often write $\mathscr{D}$ for $\mathscr{D}(V, F)$. It is known [11] that the Siegel domain $\mathscr{D}$ is holomorphically equivalent to a bounded domain in $R_C \times W$. Let V be a convex cone in R. Then

$$\mathscr{A}ut\,V=\{g\in\mathscr{GL}(R):\quad gV=V\}$$

is a closed subgroup of $\mathscr{GL}(R)$, which is called the *automorphism group* of V. If $\mathscr{A}ut\,V$ is transitive on V, then V is said to be a *homogeneous convex cone*. For a Siegel domain $\mathscr{D}=\mathscr{D}(V, F)$ we define the group $\mathscr{G}_a$ as follows,

$$\mathscr{G}_a=\{g\in\mathscr{A}(R_c\times W):\quad g\mathscr{D}=\mathscr{D}\}\,.$$

This is a closed subgroup of $\mathscr{A}(R_c\times W)$ and is called the *affine automorphism group* of $\mathscr{D}$. If $\mathscr{G}_a$ is transitive on $\mathscr{D}$, then $\mathscr{D}$ is called an *affine homogeneous* Siegel domain of the second kind. We shall define the j-algebra due to Piatetskij-Šapiro ([17]).

DEFINITION 1.3. Let G be a Lie algebra over $\boldsymbol{R}$ and K be a subalgebra of G. If the collection (j) of linear endomorphisms j and the linear form ω are defined on G and if the following axioms are satisfied, then the system $\{G, K, (j), \omega\}$ is called a *j-algebra*.

0) $jK\subset K$ for every $j\in(j)$. For arbitrary two $j, j'\in(j)$, $jx\equiv j'x$ (mod. K) $x\in G$.
1) $j^2\equiv-1$ (mod. K)
2) $j[k, x]\equiv[k, jx]$ (mod. K) $k\in K, x\in G$.
3) $[jx, jy]\equiv j[jx, y]+j[x, jy]+[x, y]$ (mod. K) $x, y\in G$.
4) $\omega([k, x])=0$. $k\in K, x\in G$.
5) $\omega([jx, jy])=\omega([x, y])$. $x, y\in G$.
6) $\omega([jx, x])>0$. $x\neq0, x\notin K$.

We remark that if the axioms 1)—3) are satisfied for some j, then they are satisfied for every $j\in(j)$. For simplicity we will often write $\{G, K, (j)\}$, $\{G, K\}$ or G for $\{G, K, (j), \omega\}$. Then j-algebra is called "*effective*" if K contains no non-trivial ideal of G.

DEFINITION 1.4. ([17]) Let $\{G, K, (j)\}$ and $\{G', K', (j')\}$ be j-algebras. $\{G, K, (j)\}$ is called a *j-subalgebra* of $\{G', K', (j')\}$ if the following conditions are satisfied:

1) G is a subalgebra of G' and $K=G\cap K'$.
2) $jx\equiv j'x$ (mod. K') $x\in G$, $j\in(j)$, $j'\in(j')$.

In particular if G is an ideal of G', then $\{G, K, (j)\}$ is called a *j-ideal* of $\{G', K', (j')\}$.

DEFINITION 1.5. Let $\{G, K, (j)\}$ and $\{G', K', (j')\}$ be j-algebras. The homomorphism φ of G into G' is called a *j-homomorphism* if the following conditions are satisfied:

1) $\varphi j \equiv j' \varphi$ (mod. K') $j' \in (j')$, $j \in (j)$.

2) $\varphi(K) = \varphi(G) \cap K'$.

We should remark that $\varphi(G)$ becomes a j-subalgebra of G' by taking the collection of $j' \in (j')$ by which $\varphi(G)$ is stable.

Let $\mathscr{M} = \mathscr{G}/\mathscr{K}$ is a homogeneous complex manifold of dimension n which is holomorphically equivalent to a bounded domain in C^n. Then the pair $\{G, K\}$ has the structure of a j-algebra. The linear endomorphisms j satisfying the axioms 0)—3) are naturally induced by the homogeneous complex structure of $\mathscr{M}$. According to Koszul [9], the linear form ω of the j-algebra is uniquely determined by the Bergman metric of $\mathscr{M}$ as follows. Let π be the projection of $\mathscr{G}$ onto $\mathscr{G}/\mathscr{K}$ and ψ be the Kähler form associated to the Bergman metric. The left invariant 2-form $\pi^*\psi$ on $\mathscr{G}$ is considered as a 2-cocycle on G. Moreover it is a coboundary, as was proved in [9]. Therefore there exists a linear form ω on G satisfying the condition $\pi^*\psi = -\delta\omega$, where δ is the coboundary operator in the Lie algebra cohomology. It is easy to see that ω satisfies the axioms 4)—6).

We denote by $\mathscr{G}_h$ the group of all holomorphic automorphisms of a Siegel domain $\mathscr{D}(V, F)$. Then it can be seen that the affine automorphism group $\mathscr{G}_a$ is a closed subgroup of $\mathscr{G}_h$.

§ 2. The affine automorphism groups of Siegel domains

Let $\mathscr{M}$ be a connected differentiable manifold, $\mathscr{U}$ be a domain in $\mathscr{M}$ and $\mathscr{G}$ be a Lie transformation group on $\mathscr{M}$. We consider the subgroup

$$\mathscr{G}(\mathscr{U}) = \{g \in \mathscr{G}: \ g\mathscr{U} = \mathscr{U}\},$$

and denote by $\mathscr{N}$ the normalizer of $\mathscr{G}^0(\mathscr{U})$ in $\mathscr{G}$. The following two lemmas are essentially due to Vinberg [16]. But we shall prove them for completeness.

LEMMA 2.1. *If $\mathscr{G}(\mathscr{U})$ is transitive on $\mathscr{U}$, then $\mathscr{N}^0 = \mathscr{G}^0(\mathscr{U})$, where $\mathscr{N}^0$ and $\mathscr{G}^0(\mathscr{U})$ are the identity components of $\mathscr{N}$ and $\mathscr{G}(\mathscr{U})$ respectively.*

PROOF. Let x_0 be a point of $\mathscr{U}$. We choose an open neighbourhood U_1 of x_0 in $\mathscr{M}$ such that $U_1 \subset \mathscr{U}$. The set $U = \{g \in \mathscr{G}: \ gx_0 \in U_1\}$ is an open neighbourhood of the identity of $\mathscr{G}$. Since $\mathscr{G}^0(\mathscr{U})$ is transitive on $\mathscr{U}$, we have, for every $g \in U \cap \mathscr{N}$

$$g\mathscr{U} = g\,\mathscr{G}^0(\mathscr{U})x_0 = \mathscr{G}^0(\mathscr{U})gx_0 = \mathscr{U}.$$

Therefore $U \cap \mathscr{N} = U \cap \mathscr{G}(\mathscr{U})$. Since $\mathscr{G}(\mathscr{U})$ and $\mathscr{N}$ are closed subgroups of $\mathscr{G}$, the sets $U\cap\mathscr{N}$ and $U\cap\mathscr{G}(\mathscr{U})$ are neighbourhoods of the identities of $\mathscr{N}$ and $\mathscr{G}(\mathscr{U})$ respectively, which shows $\mathscr{N}^0=\mathscr{G}^0(\mathscr{U})$.

LEMMA 2.2. *Let L be a vector space over $\boldsymbol{C}$ and G be a real subspace of the Lie algebra $A(L)$ of the complex affine transformation group $\mathscr{A}(L)$. Then $\mathscr{N}=\{g\in\mathscr{A}(L):\ (\mathrm{Ad}\ g)G\subset G\}$ is a real algebraic group of $\mathscr{A}(L)$.*

PROOF. We choose a real base of $A(L)$ such that the first several vectors of it span G. Then $g\in\mathscr{N}$ if and only if Ad g is represented by a matrix of the form

$$\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}.$$

On the other hand the adjoint representation of $\mathscr{A}(L)$ is a rational representation defined over $\boldsymbol{R}$. Therefore $\mathscr{N}$ is defined by the polynomials on End $\tilde{L}_R$ with real coefficients, where $\tilde{L}$ is a complex vector space containing L as a hyperplane not through the origin. It follows that $\mathscr{N}$ is an algebraic subgroup of $\mathscr{GL}(\tilde{L}_R)$.

PROPOSITION 2.1. *Let $\mathscr{D}$ be an affine homogeneous Siegel domain of the second kind and $\mathscr{G}_a$ be the affine automorphism group of $\mathscr{D}$. Then the identity component $\mathscr{G}_a^0$ of $\mathscr{G}_a$ coincides with that of a real algebraic group of $\mathscr{A}(R_C\times W)$.*

PROOF. Let $\mathscr{N}$ be the normalizer of $\mathscr{G}_a^0$ in $\mathscr{A}(R_C\times W)$. By Lemma 2.2, $\mathscr{N}$ is a real algebraic group of $\mathscr{A}(R_C\times W)$. And we see from Lemma 2.1 that $\mathscr{N}^0=\mathscr{G}_a^0$. *q.e.d.*

We shall study the structure of the affine automorphism group $\mathscr{G}_a$ of an affine homogeneous Siegel domain $\mathscr{D}$. We consider the set $E=\{(x+iy,\ u)\in R_C\times W:\ x=u=0\}$ and identify the cone V with $\mathscr{D}(V, F)\cap E=\{(iy, 0):\ y\in V\}$. We define the closed subgroup $\mathscr{H}$ of $\mathscr{G}_a$ by putting

$$\mathscr{H}=\{g\in\mathscr{G}_a:\ \ gV=V\}.$$

THEOREM 2.1. *Let $\mathscr{D}(V, F)$ be an affine homogeneous Siegel domain in $R_C\times W$. Then there exist the representation ρ (over $\boldsymbol{R}$) of $\mathscr{H}$ into $\mathscr{A}_{ut}V$ and the representation σ (over $\boldsymbol{C}$) of $\mathscr{H}$ into $\mathscr{GL}(W)$ satisfying the conditions*

1) $\rho(h)F(u, v)=F(\sigma(h)u, \sigma(h)v)\quad h\in\mathscr{H}\quad u, v\in W.$

2) *$\rho(\mathscr{H})$ is transitive on the cone V.*

Moreover the affine automorphism group $\mathscr{G}_a$ and its operation on $\mathscr{D}(V, F)$ are reconstructed by using the representations ρ and σ. More precisely let us consider the product manifold $\mathscr{G}'=\mathscr{H}\times R\times W$ and define the product of *elements of* $\mathscr{G}'$ *as follows,*

(2.1) $(h', a', c')\ (h, a, c)=(h'h,\ \rho(h')a+a'-2\,\mathrm{Im}\,F(\sigma(h')c,\ c'),\ \sigma(h')c+c')\ .$

With this multiplication, $\mathscr{G}'$ *becomes a Lie group and the subspace* $R\times W$ *is a closed (meta-abelian) subgroup of* $\mathscr{G}'$*, which is denoted by* RW*. We define the operation of* $\mathscr{G}'$ *on* $R_C\times W$ *by putting*

(2.2) $(h, a, c)(x+iy,\ u)=(\rho(h)x+a+i(\rho(h)y+2F(\sigma(h)u,\ c)+F(c,\ c)),\ \sigma(h)u+c)$

$\mathscr{D}(V, F)$ *is stable under the action of* $\mathscr{G}'$*. Then*

3) $\mathscr{G}_a$ *is isomorphic to* $\mathscr{G}'$ *and the operation of* $\mathscr{G}_a$ *on* $\mathscr{D}(V, F)$ *coincides with that of* $\mathscr{G}'$ *under this isomorphism. In particular we have*

$$\mathscr{G}_a=\mathscr{H}\cdot RW \qquad \text{(semi-direct)}$$

4) *If we denote by* $\mathscr{K}_a$ *the isotropy subgroup of* $\mathscr{G}_a$ *at* $(ir,\ 0)\in\mathscr{D}(V, F)$*, where* $r\in V$*, then* $\mathscr{K}_a\subset\mathscr{H}$*.*

PROOF. As is proved in [11], every affine automorphism of $\mathscr{D}(V, F)$ has the following form:

(2.3)
$$\begin{cases} z\longrightarrow Az+a+2iF(Bu,\ c)+iF(c,\ c) \\ u\longrightarrow Bu+c \end{cases} \qquad \text{for } (z,\ u)\in R_C\times W\,,$$

where $A\in\mathscr{A}ut\,V$, $B\in\mathscr{GL}(W)$, $a\in R$, $c\in W$ and A, B satisfy the condition

(2.4) $$A\cdot F(u,\ v)=F(Bu,\ Bv)\ .$$

Let the affine automorphism (2.3) belongs to $\mathscr{H}$. Then the image of $(iy,\ 0)\in V$ under the transformation (2.3) is $(iAy+a+iF(c,\ c),\ c)$. Since V is stable by the action of $\mathscr{H}$, we see $a=c=0$. Therefore every $h\in\mathscr{H}$ has the form

$$\begin{cases} z\longrightarrow Az \\ u\longrightarrow Bu \end{cases}$$

where A and B satisfy the above conditions. Conversely the automorphism (2.3) with $a=c=0$ belongs to $\mathscr{H}$. We see $h=(A, B)\in\mathscr{GL}(R)\times\mathscr{GL}(W)$. The representations ρ and σ are defined by putting $\rho(h)=A$, $\sigma(h)=B$. From (2.4), 1) is verified. To show 2), we choose two arbitrary points $(iy,\ 0)$, $(iy',\ 0)\in V$. Since $\mathscr{G}_a$ is transitive on V, there exists an automorphism $g\in\mathscr{G}_a$ having the form (2.3) which carries $(iy,\ 0)$ to $(iy',\ 0)$. Therefore we have $a=c=0$, which implies $g\in\mathscr{H}$. Using (2.1) and (2.3), we can verify by direct calculations that the correspondence $(h,\ a,\ c)\longrightarrow((\rho(h),\ \sigma(h)),\ a,\ c)$ gives an isomorphism of $\mathscr{G}'$ onto $\mathscr{G}_a$, which shows 3). 4) is easily verified.

PROPOSITION 2.3. *Let* $\mathscr{D}(V, F)$ *be an affine homogeneous Siegel domain* *and* $\mathscr{G}_a$ *be the affine automorphism group of* $\mathscr{D}(V, F)$. *Let* $\mathscr{K}_a$ *be the isotropy subgroup of* $\mathscr{G}_a$ *at* $(ir, 0) \in \mathscr{D}(V, F)$. *Then* $\mathscr{K}_a{}^0$ *is a maximal compact subgroup of* $\mathscr{G}_a{}^0$.

PROOF. As is mentioned in §1, $\mathscr{G}_a$ is a closed subgroup of the holomorphic automorphism group $\mathscr{G}_h$ of $\mathscr{D} = \mathscr{D}(V, F)$. The isotropy subgroup $\mathscr{K}_h$ of $\mathscr{G}_h$ at $(ir, 0) \in \mathscr{D}$ is compact subgroup, since $\mathscr{D}$ is holomorphically equivalent of a bounded domain. Therefore $\mathscr{K}_a$ is compact. The mapping

$$\mathscr{D} \ni (x+iy, u) \longrightarrow (x, y-F(u, u), u) \in R \times V \times W$$

is a homeomorphism of $\mathscr{D}$ onto $R \times V \times W$, which is an open cell. On the other hand $\mathscr{G}_a{}^0$ is transitive on $\mathscr{D}$ and the isotropy subgroup $\mathscr{G}_a{}^0 \cap \mathscr{K}_a$ of $\mathscr{G}_a{}^0$ coincides with $\mathscr{K}_a{}^0$. Since $\mathscr{D} = \mathscr{G}_a{}^0/\mathscr{K}_a{}^0$ is homeomorphic to an open cell, $\mathscr{K}_a{}^0$ is a maximal compact subgroup of $\mathscr{G}_a{}^0$.

LEMMA 2.3. *Let E be a real algebraic set in a complex vector space L. Then the subgroup* $\mathscr{G}(E) = \{g \in \mathscr{A}(L) : gE = E\}$ *is real algebraic in* $\mathscr{A}(L)$.

PROOF Let us consider the complex vector space $\tilde{L}$ which contains L as a hyperplane not through the origin. As is easily seen, E is a real algebraic set in $\tilde{L}$. Hence the subgroup $\tilde{\mathscr{G}}(E) = \{g \in \mathscr{GL}(\tilde{L}_R) : gE = E\}$ is real algebraic, and $\mathscr{A}(L)$ can be regarded as an algebraic subgroup of $\mathscr{GL}(\tilde{L}_R)$. Then we have $\mathscr{G}(E) = \mathscr{A}(L) \cap \tilde{\mathscr{G}}(E)$, which implies that $\mathscr{G}(E)$ is real algebraic in $\mathscr{GL}(\tilde{L}_R)$.

LEMMA 2.4. *Let* $\mathscr{T}_1$ *be a maximal* ***R****-triangular subgroup of* $\mathscr{H}^0$. *Then we have*

$$\mathscr{H}^0 = \mathscr{K}_a{}^0 \cdot \mathscr{T}_1 \qquad \text{(semi-direct)}$$

and $\mathscr{T}_1$ *is simply transitive on V via the representation ρ.*

PROOF. We shall prove that $\mathscr{H}^0$ is the identity component of a real algebraic group. Since $E = \{(x+iy, u) \in R_c \times W : x = u = 0\}$ is a real algebraic set, it follows from Lemma 2.3 that $\tilde{\mathscr{H}} = \{g \in \mathscr{A}(R_c \times W) : gE = E\}$ is a real algebraic group. On the other hand we have $\mathscr{H} = \tilde{\mathscr{H}} \cap \mathscr{G}_a$, since $\mathscr{H}$ preserves the open set V in E. We denote by $\mathscr{N}$ the normalizer of $\mathscr{G}_a{}^0$ in $\mathscr{A}(R_c \times W)$. Then Lemma 2.1 shows that $\mathscr{H}^0 = (\mathscr{G}_a \cap \tilde{\mathscr{H}})^0 = (\mathscr{N} \cap \tilde{\mathscr{H}})^0$. By Lemma 2.2, $\mathscr{N}$ is real algebraic in $\mathscr{A}(R_c \times W)$, which implies that $\mathscr{N} \cap \tilde{\mathscr{H}}$ is real algebraic. It is well-known (cf. Vinberg [15]) that the identity component of a real algebraic group is the semi-direct product of a maximal compact subgroup and a maximal (connected) ***R***-triangular subgroup. Theorem 2.1 and Proposition 2.3 show that $\mathscr{K}_a{}^0$ is a maximal compact subgroup of $\mathscr{H}^0$, from which we have $\mathscr{H}^0 =$

$\mathscr{K}_a{}^0 \cdot \mathscr{T}_1$. Since V is represented as the coset space $\mathscr{H}^0/\mathscr{K}_a{}^0$, $\mathscr{T}_1$ is simply transitive on V. *q.e.d.*

We define the closed subgroup $\mathscr{T}$ of $\mathscr{G}_a{}^0$ by putting $\mathscr{T}=\mathscr{T}_1 \cdot RW$.

PROPOSITION 2.4. *Let $\mathscr{D}(V, F)$ be an affine homogeneous Siegel domain and $\mathscr{G}_a$ the affine automorphism group of $\mathscr{D}(V, F)$ and $\mathscr{K}_a$ be the isotropy subgroup of $\mathscr{G}_a$. Then*

$$\mathscr{G}_a{}^0=\mathscr{K}_a{}^0 \cdot \mathscr{T} \quad \text{(semi-direct)}$$

*Furthermore $\mathscr{T}$ is a maximal **R**-triangular subgroup of $\mathscr{G}_a{}^0$ and is simply transitive on $\mathscr{D}(V, F)$.*

PROOF. We choose the real base of $R_C \times W$ such that $h=(\rho(h), \sigma(h)) \in \mathscr{T}_1$ is represented by the matrix

$$\begin{pmatrix} \rho(h) & 0 & 0 \\ 0 & \rho(h) & 0 \\ 0 & 0 & \sigma(h)_R \end{pmatrix}$$

where $\sigma(h)_R$ is the real matrix corresponding to $\sigma(h)$. Then the linear part of the element $(h, a, c) \in \mathscr{T}=\mathscr{T}_1 \cdot RW$ is represented by the matrix

$$\begin{pmatrix} \rho(h) & 0 & * \\ 0 & \rho(h) & * \\ 0 & 0 & \sigma(h)_R \end{pmatrix}.$$

Since $\mathscr{T}_1$ is a $\boldsymbol{R}$-triangular subgroup of $\mathscr{GL}(R_C) \times \mathscr{GL}(W)$, all eigenvalues of matrices $\rho(h)$ and $\sigma(h)_R$ are real. Therefore by changing the real bases of R and W, $\rho(h)$ and $\sigma(h)_R$ are represented by $\boldsymbol{R}$-triangular matrices. It follows that the linear part of $\mathscr{T}$ is also $\boldsymbol{R}$-triangular, which shows that $\mathscr{T}$ is $\boldsymbol{R}$-triangular. On the other hand, we see $\mathscr{G}_a{}^0=\mathscr{K}_a{}^0 \cdot \mathscr{T}$ since $\mathscr{G}_a{}^0=\mathscr{H}^0 \cdot RW$ and $\mathscr{H}^0=\mathscr{K}_a{}^0 \cdot \mathscr{T}_1$. Therefore $\mathscr{T}$ is a maximal $\boldsymbol{R}$-triangular subgroup of $\mathscr{G}_a{}^0$. *q.e.d.*

REMARK. The choice of the maximal $\boldsymbol{R}$-triangular subgroup $\mathscr{T}_1$ of $\mathscr{H}^0$ is not unique. But $\mathscr{T}$ is uniquely determined by $\mathscr{D}(V, F)$ up to conjugateness in $\mathscr{G}_a{}^0$. In fact we consider the subgroup $\mathscr{T}'=\mathscr{T}_1' \cdot RW$ where $\mathscr{T}_1'$ is another maximal $\boldsymbol{R}$-triangular subgroup of $\mathscr{H}^0$. Then it is shown that $\mathscr{T}'$ is also maximal $\boldsymbol{R}$-triangular. Therefore $\mathscr{T}$ and $\mathscr{T}'$ are conjugate to each other in $\mathscr{G}_a{}^0$.

LEMMA 2.5. *Let $\mathscr{L}$ be a subgroup of $\mathscr{A}_{ut}{}^0 V$ (=the identity component of $\mathscr{A}_{ut} V$) which contains $\rho(\mathscr{T}_1)$. Then there exists no non-zero vector in R*

left fixed by $\mathscr{L}$.

PROOF. It is sufficient to remark that $\mathscr{L}$ contains the group $\boldsymbol{R}^+$ of positive real scalar matrices. As is proved in Vinberg [16], $\boldsymbol{R}^+$ is contained in each maximal $\boldsymbol{R}$-triangular subgroup of $\mathscr{A}_{ut}{}^0 V$, which proves Lemma 2.5.

PROPOSITION 2.5. *Let* $\mathscr{D}(V, F)$ *be an affine homogeneous Siegel domain of the second kind and* $\mathscr{G}_a$ *be the affine automorphism group of* $\mathscr{D}(V, F)$. *Let* $\mathscr{G}$ *be a Lie subgroup of* $\mathscr{G}_a$ *which contains* $\mathscr{T}$. *Then* $\mathscr{G}$ *is centreless. In particular* $\mathscr{G}_a$, $\mathscr{G}_a{}^0$ *and* $\mathscr{T}$ *are centreless.*

PROOF. Let $\mathscr{H}_1$ be the subgroup consisting of elements of $\mathscr{G}$ which leaves V stable. By the same method as in the proof of Theorem 2.1, we can prove that

$$\mathscr{G} = \mathscr{H}_1 \cdot RW \qquad \text{(semi-direct)}$$

and $\mathscr{H}_1$ contains $\mathscr{T}_1$. Let $g=(h, a, c) \in \mathscr{G}$ belong to the centre of $\mathscr{G}$. Then g commutes with each $g'=(h', a', c') \in \mathscr{G}$. We see by (2.1) that

$$A'a+a'-2\,\mathrm{Im}\, F(B'c, c') = Aa'+a-2\,\mathrm{Im}\, F(Bc', c) \tag{2.5}$$

$$B'c+c' = Bc'+c\,, \tag{2.6}$$

where

$$h=(\rho(h), \sigma(h))=(A, B),\ h'=(\rho(h'), \sigma(h'))=(A', B')\,.$$

If we consider the case $h'=$identity, then it follows that

$$a'-2\,\mathrm{Im}\, F(c, c') = Aa'-2\,\mathrm{Im}\, F(Bc', c) \tag{2.7}$$

$$Bc'=c'\,. \tag{2.8}$$

Since c' is an arbitrary element of W, B is reduced to the identity. If we put $c'=0$ in (2.7), then $Aa'=a'$ for all $a' \in R$, which shows that A is the identity. Therefore it follows from (2.5) and (2.6) that

$$A'a-2\,\mathrm{Im}\, F(B'c, c') = a-2\,\mathrm{Im}\, F(c', c) \tag{2.9}$$

$$B'c=c\,. \tag{2.10}$$

Putting $c'=0$, we see that $A'a=a$ for $A'=\rho(h')$, $h' \in \mathscr{H}_1$. Since $\rho(\mathscr{H}_1)$ contain $\rho(\mathscr{T}_1)$, it follows from Lemma 2.5 that $\rho(\mathscr{H}_1)$ leaves no non-zero vector fixed. This shows $a=0$. Hence we see from (2.9) that

$$\mathrm{Im}\, F(B'c, c') = \mathrm{Im}\, F(c', c) = \mathrm{Im}\, \overline{F(c, c')} = -\mathrm{Im}\, F(c, c')\,.$$

Using (2.10), we have for every $c' \in W$.

$$\mathrm{Im}\, F(B'c, c') + \mathrm{Im}\, F(c, c') = 2\,\mathrm{Im}\, F(c, c') = 0\,.$$

Hence we see

$$F(c, c)=\operatorname{Im} iF(c, c)=-\operatorname{Im} F(c, ic)=0 ,$$

which shows $c=0$. Thus we have proved that g is the identity. *q.e.d.*

From now on, we always suppose that Siegel domains are affine homogeneous. We shall study the structure of the Lie algebra G_a of $\mathscr{G}_a$. The representations of the Lie algebra H of $\mathscr{H}$ induced by ρ and σ are denoted by $\dot{\rho}$ and $\dot{\sigma}$ respectively and the subspaces of G_a corresponding to R and W are denoted by the same notations. It follows from Lemma 2.4 and Proposition 2.4 that

$$(2.11)\qquad G_a=K_a+T=K_a+T_1+R+W$$

$$(2.12)\qquad H=K_a+T_1 \qquad \text{(direct sum)} .$$

LEMMA 2.6. *The following relations are valid.* (cf. [17])

1) $[h, y]=\dot{\rho}(h)y$ *for* $h\in H,\ y\in R$.
2) $[h, c]=\dot{\sigma}(h)c$ *for* $h\in H,\ c\in W$.
3) $[c, c']=-4\operatorname{Im} F(c', c)$ *for* $c, c'\in W$.
4) $[a, c]=0$ *for* $a\in R, c\in W$.

In particular we have

$$\dot{\rho}(h)=\operatorname{ad}_R h, \qquad \dot{\sigma}(h)=\operatorname{ad}_W h \qquad \text{for } h\in H,$$

and

$$[H, R]\subset R \qquad [H, W]\subset W \qquad [W, W]\subset R \qquad [R, W]=(0) .$$

Furthermore R is an abelian ideal of G_a.

PROOF. 1) exp th and y can be represented by the following matrices with respect to the suitable C-base of $R_C\times W$.

$$\exp th=\begin{pmatrix} \rho(\exp th) & 0 & 0 \\ 0 & \sigma(\exp th) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$y=\begin{pmatrix} 0 & 0 & y \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Therefore

$$[h, y]=\lim_{t\to 0}\frac{(\exp th)\,y\,(\exp th)^{-1}-y}{t}=\dot{\rho}(h)y .$$

2) The one-parametre group generated by $c=(0, 0, c)$ is $(1, 0, tc)\in\mathscr{G}_a$. Using (2.2), the value of $(0, 0, c)$ at $(x+iy, u)$ is given by

$$(0,\,0,\,c)\,(x+iy,\,u)=\lim_{t\to 0}\frac{(1,\,0,\,tc)\,(x+iy,\,u)-(x+iy,\,u)}{t}=(2iF(u,\,c),\,c)\ ,$$

from which we see

$$(0,\,0,\,c)=\begin{pmatrix} 0 & f(c) & 0 \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix},$$

where $f(c)$ is the linear map $u\to 2iF(u,\,c)$. Therefore

$$(\exp th)\,c\,(\exp th)^{-1}=\begin{pmatrix} 0 & \# & 0 \\ 0 & 0 & \#' \\ 0 & 0 & 0 \end{pmatrix},$$

where $\#=\rho(\exp th)f(c)\sigma(\exp th)^{-1}=f(\sigma(\exp th)c)$ and $\#'=\sigma(\exp th)c$.
Therefore

$$[h,\,c]=\lim_{t\to 0}\frac{(\exp th)c(\exp th)^{-1}-c}{t}=\begin{pmatrix} 0 & f(\dot\sigma(h)c) & 0 \\ 0 & 0 & \dot\sigma(h)c \\ 0 & 0 & 0 \end{pmatrix}=(0,\,0,\,\dot\sigma(h)c)\,.$$

3)

$$\exp tc=(1,\,0,\,tc)=\begin{pmatrix} 1 & tf(c) & t^2 iF(c,\,c) \\ 0 & 1 & tc \\ 0 & 0 & 1 \end{pmatrix}$$

$$(\exp tc)c'\,(\exp tc)^{-1}=\begin{pmatrix} 0 & f(c') & 2it(F(c',c)-F(c,c')) \\ 0 & 0 & c' \\ 0 & 0 & 0 \end{pmatrix}=(0,\,2it(F(c',\,c)-F(c,\,c')),\,c').$$

Therefore

$$[c,\,c']=(0,\,-4\,\mathrm{Im}\,F(c',\,c),\,0)$$

4) is easily verified. *q.e.d.*

The vector field on $R_C\times W$ generated by $g\in G_a$ is denoted by D_g. Let z be an arbitrary point of a Siegel domain $\mathscr{D}$. Then the tangent space $T_z(\mathscr{D})$ at z is canonically identified with $R_C\times W$ under the parallel displacement with respect to the flat connection of $R_C\times W$.

LEMMA 2.7. *Under the above identification, the following relations are valid.*

$$D_a(x+iy,\,u)=(a,\,0) \qquad \text{for } a\in R.$$

$$D_h(x+iy,\,u)=(\dot\rho(h)x+i\dot\rho(h)y,\,\dot\sigma(h)u) \qquad \text{for } h\in H.$$

$$D_c(x+iy,\,u)=(2iF(u,\,c),\,c) \qquad \text{for } c\in W.$$

Using (2.2) these are verified by direct calculations.

Let J be the complex structure of the Siegel domain $\mathscr{D}=\mathscr{G}_a/\mathscr{K}_a$. This is the restriction to $\mathscr{D}$ of the natural complex structure of $R_C\times W$. Let π be the canonical projection of $\mathscr{G}_a$ onto $\mathscr{G}_a/\mathscr{K}_a$ and $z_0=(ir, 0)$ be the point left fixed by $\mathscr{K}_a$. We define the mapping π' of G_a onto the tangent space $T_{z_0}(\mathscr{D})$ by putting $\pi'(g)=\pi(g(1))$, $g\in G_a$ where $g(1)$ is the value at $1\in\mathscr{G}_a$ of the left invariant vector field g. Note that $\operatorname{Ker}\pi'=K_a$. There exists the linear endomorphisms j of G_a satisfying the conditions:

$$\pi' j=J_{z_0}\pi' \tag{2.13}$$

$$jK_a\subset K_a \tag{2.14}$$

where J_{z_0} is the linear endomorphism on $T_{z_0}(\mathscr{D})$ induced by J. Let (j) be the collection of the endomorphisms j satisfying the additional condition

$$jW\subset W\,. \tag{2.15}$$

Since J is a homogeneous complex structure, each j of (j) satisfies the axioms of j-algebra. The linear form ω of j-algebra is defined on G_a (cf. §1). It follows that $\{G_a, K_a, (j)\}$ is an effective j-algebra. We prove the following proposition for completeness.

PROPOSITION 2.6. ([17]) *Let $\{G_a, K_a, (j)\}$ be the j-algebra of the affine automorphism group of an affine homogeneous Siegel domain $\mathscr{D}(V, F)$ in $R_C\times W$. Then*

$$G_a=K_a+jR+R+W \qquad \text{(direct sum)}.$$

PROOF. We should note that

$$\pi'(g)=D_g(z_0) \qquad g\in G_a\,,$$

where $z_0=(ir, 0)$, $r\in V$. According to (2.11) and (2.12), it is sufficient to prove $H=K_a+jR$. By Lemma 2.7 and (2.13),

$$D_{ja}(z_0)=J_{z_0}\cdot D_a(z_0)=i(a, 0)=(ia, 0)$$

for any $a\in R$. $\mathscr{H}$ being transitive on V, there exists an $h\in H$ such that $\rho(h)r=a$. Therefore

$$D_{h-ja}(z_0)=D_h(z_0)-D_{ja}(z_0)=(i\rho(h)r, 0)-(ia, 0)=0\,,$$

from which we see $h-ja\in\operatorname{Ker}\pi'=K_a$. This shows $jR\subset H$. If we choose an arbitrary $h\in H$ and put $\rho(h)r=a\in R$, then it follows that $D_{h-ja}(z_0)=0$. This shows $H=K_a+jR$. *q.e.d.*

REMARK 2.1. If $r\in R$ is the point left fixed by K_a, then for any $a\in R$

$$[ja, r]=a .$$

In fact,

$$(ia, 0)=J_{z_0}D_a(z_0)=D_{ja}(z_0)=D_{ja}(ir, 0)=(i\dot\rho(ja)r, 0) .$$

Therefore $\dot\rho(ja)r=a$, which proves the above assertion. (cf. Lemma 2.6.)

REMARK 2.2. Since $j^2=-1$ is valid on the subspace W of G_a, the complex vector space (W, j) is canonically identified with the subspace W of $R_C\times W$. Under this identification, F is a V-hermitian form of $(W, j)\times(W, j)$ into $R+iR$. Then using Lemma 2.6 we have

$$F(c, c')=\frac{1}{4}([jc, c']+i[c, c']) . \tag{2.16}$$

LEMMA 2.8. *If we consider R and W as subspaces of G_a, then*

$$D_a(x+iy, u)=(a, 0) \qquad \textit{for } a\in R.$$

$$D_h(x+iy, u)=([h, x]+i[h, y], [h, u]) \qquad \textit{for } h\in H.$$

$$D_c(x+iy, u)=\left(\frac{1}{2}(-[u, c]+i[ju, c]), c\right) \quad \textit{for } c\in W.$$

These are direct consequences of Lemma 2.6, Lemma 2.7 and (2.16).

Suppose that an affine homogeneous Siegel domain $\mathscr{D}(V, F)$ is given. One should remark that $\mathscr{D}(V, F)$ is reconstructed by giving the decomposition $G_a=K_a+jR+R+W$. Let us define D_a, D_h and D_c by the equalities in Lemma 2.8. For each $g=h+a+c\in G_a$ we define the infinitesimal affine transformation D_g on the complex vector space $R+iR+W$ by putting $D_g=D_h+D_a+D_c$. Then we can deduce by direct calculations that the mapping $G_a\ni g\to D_g\in A(R+iR+W)$ is a faithful representation of G_a. $\mathscr{G}_a^0$ coincides with the analytic subgroup of $\mathscr{A}(R+iR+W)$ generated by the image of G_a. Let us consider the subalgebra $H'=\mathrm{ad}_R H$ of $GL(R)$ and $\mathscr{H}'$ be the analytic subgroup of $\mathscr{GL}(R)$ corresponding to H'. Then V is the orbit $\mathscr{H}'\cdot r$.

$$V=\mathscr{H}'\cdot r . \tag{2.17}$$

$$\mathscr{D}(V, F)=\left\{x+iy+u\in R+iR+W: y-\frac{1}{4}[ju, u]\in V\right\} . \tag{2.18}$$

§ 3. The holomorphic automorphism groups of Siegel domains

Let $\mathscr{D}$ be an affine homogeneous Siegel domain. In this section we denote by $\mathscr{G}_h$ (resp. $\mathscr{G}_a$) the identity component of the holomorphic (resp. affine) automorphism group of $\mathscr{D}$. Since $\mathscr{D}$ is homeomorphic to an open cell, the isotropy subgroup $\mathscr{K}_h$ of $\mathscr{G}_h$ at $z_0=(ir, 0)\in\mathscr{D}$ is a maximal compact subgroup of $\mathscr{G}_h$. Since the group $\mathscr{T}$, constructed in § 2, is closed in $\mathscr{G}_a$, $\mathscr{T}$ is a closed subgroup of $\mathscr{G}_h$. Let $\mathscr{G}$ be a closed connected subgroup of $\mathscr{G}_h$ containing $\mathscr{T}$ and $\mathscr{K}$ be the isotropy subgroup of $\mathscr{G}$ at z_0. $\mathscr{K}$ is a maximal compact subgroup of $\mathscr{G}$. We see

$$\mathscr{G}=\mathscr{K}\cdot\mathscr{T} \qquad \text{(semi-direct)}.$$

In particular,

$$\mathscr{G}_h=\mathscr{K}_h\cdot\mathscr{T} \qquad \text{(semi-direct)}.$$

The following theorem is a generalization of the well-known fact in the case of symmetric domains.

THEOREM 3.1. *Let $\mathscr{G}$ be a connected closed subgroup of the holomorphic automorphism group $\mathscr{G}_h$ of an affine homogeneous Siegel domain of the second kind. Suppose that $\mathscr{G}$ contains the group $\mathscr{T}$. Then $\mathscr{G}$ is centreless. In particular $\mathscr{G}_h$ is centreless.*

PROOF. We denote by $\mathscr{G}^*$ the adjoint group of $\mathscr{G}$ and denote by $\mathscr{K}^*$ and $\mathscr{T}^*$ the images of $\mathscr{K}$ and $\mathscr{T}$ in $\mathscr{G}^*$ respectively. We shall prove that $\mathscr{T}^*$ is closed in $\mathscr{G}^*$. $\mathscr{T}$ being simply connected, $\mathscr{A}ut\,\mathscr{T}\cong\mathscr{A}ut\,T$, where $\mathscr{A}ut\,\mathscr{T}$ and $\mathscr{A}ut\,T$ are automorphism groups of $\mathscr{T}$ and of its Lie algebra T respectively. $\mathscr{A}ut\,T$ is an algebraic subgroup of $\mathscr{G}\mathscr{L}(T)$ and so it is closed in $\mathscr{G}\mathscr{L}(T)$. On the other hand since T is $\boldsymbol{R}$-triangular, all eigen-values of each element of T are real. It is true for each element of $\mathrm{ad}\,T$. Therefore $\mathrm{ad}\,T$ is $\boldsymbol{R}$-triangular and so $\mathrm{Ad}\,\mathscr{T}$ is a $\boldsymbol{R}$-triangular subgroup of $\mathscr{G}\mathscr{L}(T)$. Hence $\mathrm{Ad}\,\mathscr{T}$ is closed in $\mathscr{G}\mathscr{L}(T)$, which shows that $\mathrm{Ad}\,\mathscr{T}$ is closed in $\mathscr{A}ut\,T$. If we denote by $\mathscr{I}nn\,\mathscr{T}$ the inner automorphism group of $\mathscr{T}$, then $\mathscr{I}nn\,\mathscr{T}\cong\mathrm{Ad}\,\mathscr{T}$, which implies that $\mathscr{I}nn\,\mathscr{T}$ is a closed subgroup of $\mathscr{A}ut\,\mathscr{T}$. Let ϖ be the natural homomorphism of $\mathscr{G}$ onto $\mathscr{G}^*$. $\mathrm{Ker}\,\varpi=\mathscr{Z}$ is the centre of $\mathscr{G}$ and $\mathscr{Z}\cap\mathscr{T}$ is contained in the centre of $\mathscr{T}$. Since $\mathscr{T}$ is centreless by Proposition 2.5, $\mathscr{Z}\cap\mathscr{T}=(1)$, which shows that $\mathscr{T}\cong\mathscr{T}^*$. Therefore by a theorem of Goto [4] $\mathscr{T}^*$ is a closed subgroup of $\mathscr{G}^*$. $\mathscr{T}$ and of course $\mathscr{T}^*$ contain no compact subgroups. Since $\mathscr{K}^*\cap\mathscr{T}^*$ is compact, $\mathscr{K}^*\cap\mathscr{T}^*=(1)$. Take an element $z\in\mathscr{Z}$ and let $z=kt$ where $k\in\mathscr{K}$, $t\in\mathscr{T}$. Then $\varpi(z)=\varpi(k)\varpi(t)=1$. Hence $\varpi(k)=\varpi(t^{-1})\in\mathscr{K}^*\cap\mathscr{T}^*$, which implies $\varpi(t)=1$. Since ϖ restricted to $\mathscr{T}$ gives an isomorphism, we see $\mathscr{Z}\subset\mathscr{K}$. $\mathscr{G}$ being effective on the Siegel domain, $\mathscr{Z}$ is reduced to the identity. *q.e.d.*

Let G be a subalgebra of End L (L is a real vector space.). By the "*algebraic hull*" of G we shall mean the smallest algebraic Lie subalgebra of End L containing G. A j-algebra $\{G, K, (j)\}$ is called "*linear*" if G is a linear Lie algebra. A j-algebra $\{G, K, (j)\}$ is called "*proper*" if any compact semi-simple j-subalgebra G' of G is contained in K (cf. [17]). The following theorem plays an essential role for the later considerations.

THEOREM A (*Vinberg, Gindikin, Piatetskij-Šapiro* [17]) *Let* $\{G, K, (j), \omega\}$ *be an effective real linear j-algebra. Then there exists a linear j-algebra* $\{\hat{G}, \hat{k}, (\hat{j}), \hat{\omega}\}$ *satisfying the following conditions:*

1) $\hat{G}$ *is the algebraic hull of* G *and* $\hat{K}$ *is an algebraic subalgebra of* $\hat{G}$.

2) $\hat{G}=G+\hat{K}$ *and* $G\cap\hat{K}=K$.

3) $\{G, K, (j)\}$ *is a j-subalgebra of* $\{\hat{G}, \hat{K}, (\hat{j})\}$.

And if G *is proper, then* $\hat{G}$ *is proper.*

LEMMA 3.1. *Let* $\{G, K, (j)\}$ *and* $\{G', K', (j')\}$ *be j-algebras and* φ *be a j-homomorphism of* G *to* G'. *Let* $\{G_1, K_1, (j_1)\}$ *be a j-subalgebra of* G. *If* $\mathrm{Ker}\,\varphi\subset K$, *then the pair* $\{\varphi(G_1), \varphi(K_1)\}$ *has the structure of a j-subalgebra of* G'.

PROOF. In order to prove the lemma, it is sufficient to show that φ is a j-homomorphism of G_1 into G' (cf. §1). The property $\varphi j_1\equiv j'\varphi \pmod{K'}$ is easily verified. To show $\varphi(K_1)=\varphi(G_1)\cap K'$, take an element $\varphi(x_1)\in\varphi(G_1)\cap K'$, $x\in G_1$. φ being a j-homomorphism of G to G', we have $\varphi(x_1)\in\varphi(G)\cap K'=\varphi(K)$. Hence there exists a $k\in K$ such that $\varphi(x_1)=\varphi(k)$, from which we have $x_1-k\in\mathrm{Ker}\,\varphi\subset K$. Therefore $x_1\in G_1\cap K=K_1$, from which we obtain $\varphi(K_1)=\varphi(G_1)\cap K'$. *q.e.d.*

LEMMA 3.2. *Let* $\{G, K, (j), \omega\}$ *and* $\{\hat{G}, \hat{K}, (\hat{j}), \hat{\omega}\}$ *be the j-algebras in Theorem A. Then there exists an algebraic ideal* N *of* $\hat{G}$ *contained in* $\hat{K}$ *and the rational representation* φ *of* $\hat{G}$ *with the kernel* N, *satisfying the following conditions:*

1) $\varphi(\hat{G})=\varphi(G)+\varphi(\hat{K})$, $\varphi(G)\cap\varphi(\hat{K})=\varphi(K)$.

2) *The pair* $\{\varphi(\hat{G}), \varphi(\hat{K})\}$ *has the structure of an effective j-algebra such that* φ *is a j-homomorphism of* $\hat{G}$ *onto* $\varphi(\hat{G})$.

PROOF. Let N be the maximal ideal of $\hat{G}$ contained in $\hat{K}$. Then N is an algebraic Lie algebra. In fact, let $\hat{N}$ be the algebraic hull of N. Then $\hat{K}$ being algebraic, $\hat{N}$ is an ideal of $\hat{G}$ contained in $\hat{K}$. From the maximality of N, it follows $\hat{N}=N$. Let $\hat{\mathscr{G}}$ (resp. $\mathscr{N}$) be the irreducible algebraic group whose Lie algebra is $\hat{G}$ (resp. N). Then by Chevalley [3], $\mathscr{N}$ is a normal subgroup of $\hat{\mathscr{G}}$, and there exists the rational represetantion φ of $\hat{\mathscr{G}}$ with the kernel $\mathscr{N}$ ([3]). The rational representation of $\hat{G}$ induced by φ with the kernel N is denoted by the same notation. 1) is an immediate consequence of Theorem A. To prove

2) we consider the subcollection $(\hat{j})'$ of $(\hat{j})$ each $\hat{j}$ of which leaves N stable. Each $\hat{j}\in(\hat{j})'$ induced the linear endomorphism $\hat{j}'$ on G/N such that $\hat{j}'\varphi=\varphi\hat{j}$. Each $\hat{j}'$ satisfies axioms of j-algebra. $\{G, K, (j), \omega\}$ being effective, $G\cap N=(0)$. As is mentioned in [17], $\hat{\omega}$ is defined by extending ω arbitrarily to $\hat{G}$. Therefore we may define $\hat{\omega}$ so that it may satisfy the additional condition $\hat{\omega}(N)=0$. If we define the linear form $\tilde{\omega}$ on $\varphi(\hat{G})$ by putting

$$\tilde{\omega}(\varphi(x))=\hat{\omega}(x) \qquad \text{for } x\in G,$$

then $\tilde{\omega}$ satisfies axioms of j-algebra, which proves that $\{\varphi(\hat{G}), \varphi(\hat{K})\}$ is a j-algebra.

REMARK. $\varphi(\hat{G})$ is an algebraic Lie algebra, since $\hat{G}$ is algebraic and φ is rational (cf. Chevalley [3]).

Let G_h and K_h be the Lie algebras of $\mathscr{G}_h$ and $\mathscr{K}_h$ respectively. From now on we consider exclusively the case that $G=\mathrm{ad}\,G_h$ and $K=\mathrm{ad}_{G_h}K_h$. Since $N\cap G=(0)$, φ restricted to G gives an isomorphism. We consider the linear representation $\tau=\varphi\circ\mathrm{ad}$ of G_h, which is faithful by Theorem 3.1. From Lemma 3.1 it follows that $\{\tau(G_h), \tau(K_h)\}$ is a j-subalgebra of $\{\varphi(\hat{G}), \varphi(\hat{K})\}$ and is j-isomorphic to $\{G_h, K_h, (j)\}$. For simplicity we write $\hat{G}'$ and $\hat{K}'$ for $\varphi(\hat{G})$ and $\varphi(\hat{K})$ respectively, and G_h' and K_h' for $\tau(G_h)$ and $\tau(K_h)$ respectively. Let $\tilde{\hat{\mathscr{G}}}'$ be the simply connected Lie group generated by $\hat{G}'$ and $\tilde{\hat{\mathscr{K}}}'$ be the analytic subgroup of $\tilde{\hat{\mathscr{G}}}'$ corresponding to $\hat{K}'$. Since $\hat{K}$ is an algebraic subalgebra of $\hat{G}$ and φ is rational, $\hat{K}'$ is an algebraic subalgebra of G'. Therefore $\tilde{\hat{\mathscr{K}}}'$ is closed in $\tilde{\hat{\mathscr{G}}}'$. We consider the coset space $\hat{\mathscr{D}}=\tilde{\hat{\mathscr{G}}}'/\tilde{\hat{\mathscr{K}}}'$, on which $\tilde{\hat{\mathscr{G}}}'$ acts almost effectively by Lemma 3.2. Let Γ be a maximal (discrete) normal subgroup of $\tilde{\hat{\mathscr{G}}}'$ contained in $\tilde{\hat{\mathscr{K}}}'$. We put $\hat{\mathscr{G}}'=\tilde{\hat{\mathscr{G}}}'/\Gamma$ and $\hat{\mathscr{K}}'=\tilde{\hat{\mathscr{K}}}'/\Gamma$. The simply connected manifold $\hat{\mathscr{D}}$ is represented by the coset space $\hat{\mathscr{G}}'/\hat{\mathscr{K}}'$ on which $\hat{\mathscr{G}}'$ acts effectively.

LEMMA 3.3. *$\hat{\mathscr{D}}=\hat{\mathscr{G}}'/\hat{\mathscr{K}}'$ is a homogeneous Kähler manifold.*

PROOF. Since $\{\hat{G}', \hat{K}', (\hat{j}'), \tilde{\omega}\}$ is a j-algebra and $\hat{\mathscr{K}}'$ is connected, $\hat{\mathscr{D}}$ has the homogeneous complex structure J induced by $\hat{j}'$. The linear form $\tilde{\omega}$ on $\hat{G}'$ is extended to the left invariant 1-form on $\hat{\mathscr{G}}'$ which is denoted by the same notation. $\tilde{\omega}$ is right invariant by $\hat{\mathscr{K}}'$, since $\tilde{\omega}([k, x])=0$ for $k\in\hat{K}'$, $x\in\hat{G}'$. The left invariant 2-form $\tilde{\rho}=d\tilde{\omega}$ is right invariant by $\hat{\mathscr{K}}'$. For left invariant vector fields x, y on $\hat{\mathscr{G}}'$ we have

$$\tilde{\rho}(x, y)=-\tilde{\omega}([x, y]) .$$

Therefore $\tilde{\rho}\equiv0$ mod $\hat{K}'$. If we denote by π the canonical projection of $\hat{\mathscr{G}}'$ onto $\hat{\mathscr{D}}$, then there exists the 2-form ρ on $\hat{\mathscr{D}}$ such that $\pi^*\rho=\tilde{\rho}$. ρ is a $\hat{\mathscr{G}}'$-

invariant and closed 2-form. We define the hermitian form h by putting $h(x', y')=\rho(x', Jy')$. Then h is $\hat{\mathscr{G}}'$-invariant Kähler metric on $\hat{\mathscr{D}}$. q.e.d.

LEMMA 3.4. *Let $\mathscr{G}_h'$ be the analytic subgroup of $\hat{\mathscr{G}}'$ corresponding to G_h'. Then $\mathscr{G}_h'$ is transitive and effective on $\hat{\mathscr{D}}$. $\hat{\mathscr{D}}$ is represented by the coset space $\mathscr{G}_h'/\mathscr{K}_h'$, where $\mathscr{K}_h'$ is the analytic subgroup of $\mathscr{G}_h'$ corresponding to K_h'.*

PROOF. We denote by o the point of $\hat{\mathscr{D}}$ corresponding to $\hat{\mathscr{K}}'$. The orbit $\mathscr{D}'=\mathscr{G}_h'\cdot o$ is an open subset of $\hat{\mathscr{D}}$, since we obtain $\hat{G}'=G_h'+\hat{K}'$ from Lemma 3.2. $\hat{\mathscr{D}}$ being a homogeneous Kähler manifolds, it is a complete metric space. $\mathscr{D}'$ is $\mathscr{G}_h'$-homogeneous Kähler manifold with respect to the induced metric and the injection $\mathscr{D}'\to\mathscr{D}$ is isometric. Therefore $\mathscr{D}'$ is a complete subspace of $\hat{\mathscr{D}}$, which shows that $\mathscr{D}'$ is closed in $\hat{\mathscr{D}}$. Therefore $\mathscr{G}_h'$ is transitive on $\hat{\mathscr{D}}$. Since $\hat{\mathscr{D}}$ is simply connected and $G_h'\cap\hat{K}'=K_h'$ by Lemma 3.2, $\mathscr{G}_h'\cap\hat{\mathscr{K}}'=\mathscr{K}_h'$. q.e.d.

THEOREM 3.2. *Let $\mathscr{D}$ be an affine homogeneous Siegel domain of the second kind and $\mathscr{G}_h$ be the identity component of the holomorphic automorphism group of $\mathscr{D}$. Then there exists a faithful linear representation τ of $\mathscr{G}_h$ such that $\tau(\mathscr{G}_h)$ is the identity component of a real algebraic group.*

PROOF. Since $\mathscr{G}_h$ is isomorphic to the adjoint group of G_h by Theorem 3.1, there exists the covering homomorphism π of $\mathscr{G}_h'$ onto $\mathscr{G}_h$. Then we have $\pi(\mathscr{K}_h')=\mathscr{K}_h$. $\mathscr{G}_h'/\mathscr{K}_h'$ is a covering manifold of $\mathscr{G}_h/\mathscr{K}_h\cong\mathscr{G}_h'/\pi^{-1}(\mathscr{K}_h)$. But $\mathscr{G}_h/\mathscr{K}_h$ being simply connected, $\mathscr{G}_h'/\mathscr{K}_h'$ is diffeomorphic to $\mathscr{G}_h/\mathscr{K}_h$. Furthermore $\mathscr{G}_h'/\mathscr{K}_h'$ is holomorphically equivalent to $\mathscr{G}_h/\mathscr{K}_h$, since $\{G_h', K_h'\}$ is j-isomorphic to $\{G_h, K_h\}$. From the fact that $\mathscr{G}_h'$ operates effectively on $\hat{\mathscr{D}}$, it follows that $\mathscr{G}_h'$ is isomorphic to $\mathscr{G}_h$. If $\hat{\mathscr{G}}'\supsetneqq\mathscr{G}_h'$, then the strictly larger connected group $\hat{\mathscr{G}}'$ than $\mathscr{G}_h$ acts holomorphically on $\mathscr{D}$ under the identification $\mathscr{G}_h'=\mathscr{G}_h$. This contradicts the fact that $\mathscr{G}_h$ is the identity component of the group of all holomorphic transformations of $\mathscr{D}$. Therefore $\hat{\mathscr{G}}'=\mathscr{G}_h'$, which shows $\varphi(\hat{G})=\tau(G_h)$. In the following we use the notation in the proof of Lemma 3.2. $\tau=\varphi\circ\mathrm{ad}$ being an isomorphism, G_h is isomorphic to $\varphi(\hat{G})$ by τ. Since $\varphi(\hat{G})$ is an algebraic Lie algebra, $\varphi(\hat{\mathscr{G}})^0$ is the identity component of algebraic group. We denote by the same notation τ the representation $\varphi\circ\mathrm{Ad}$ of $\hat{\mathscr{G}}$ corresponding to $\tau=\varphi\circ\mathrm{ad}$. Then it follows $\tau(\mathscr{G}_h)=\varphi(\hat{\mathscr{G}})^0$. On the other hand τ is a covering homomorphism. But by Theorem 3.1 $\mathscr{G}_h$ is centreless. Therefore τ is an isomorphism, which proves the theorem.

§ 4. The uniqueness of the simply transitive group $\mathscr{T}$

In this section we denote by $\mathscr{G}_h$ (resp. $\mathscr{G}_a$) the identity component of the holomorphic (resp. affine) automorphism group of an affine homogeneous Siegel domain. We shall begin with some results of Vinberg, Gindikin, Piatetskij-Šapiro [17]. Let $\{G, K, (j), \omega\}$ be an effective j-algebra and R be an abelian ideal of G. Then

$$(4.1) \qquad R\cap K=jR\cap R=K\cap jR=(0)$$

is valid. A symmetric bilinear form on R is defined by putting

$$(4.2) \qquad (a, b)=\omega([ja, b]) \qquad a, b\in R.$$

This is independent of the choice of $j\in(j)$ and is positive definite. The element $r^*\in R$ is defined by the following equality:

$$(4.3) \qquad (a, r^*)=\omega(a) \qquad \text{for all } a\in R.$$

Then it follows that

$$(4.4) \qquad [K, r^*]=0$$

$$(4.5) \qquad [jr^*, r^*]=r^* \qquad \text{for each } j\in(j).$$

Furtheremore there exists a $j\in(j)$ for which the relation

$$(4.6) \qquad [jr^*, K]=0$$

is satisfied. We consider a linear transformation $\mathrm{ad}\, jr^*$ on G, which is not a semi-simple operator in general. The complexification G^c of G is decomposed into the direct sum of eigenspaces in the wide sense by the operator $\mathrm{ad}\, jr^*$ (so-called Jordan decomposition). We denote by $\tilde{G}^\lambda$ the direct sum of the eigen-spaces in the wide sense such that the real parts of the corresponding eigenvalues are equal to λ. Then we have

$$G^c=\sum\tilde{G}^\lambda \qquad \text{(direct sum)}.$$

Each $\tilde{G}^\lambda$, as well as G, is stable under the conjugation of G^c with respect to G. Therefore it follows that

$$(4.7) \qquad G=\sum G^\lambda \qquad \text{(direct sum)}$$

where $G^\lambda=\tilde{G}^\lambda\cap G$. It is known [17] that there exists an $\mathrm{ad}_G K$-invariant positive definite inner product on G. Hence each element of $\mathrm{ad}_G K$ is a skew-symmetric operator. From this and (4.6) it follows that the decomposition (4.7) is inde-pendent of the choice of $j\in(j)$ and is uniquely determind by R. One should note that

$$[G^\lambda, G^\mu] \subset G^{\lambda+\mu} . \tag{4.8}$$

It is known [17] that if $G^\lambda \neq (0)$, then λ has to be equal to 0, 1, or 1/2. From (4.7) we see

$$G = G^0 + G^1 + G^{1/2} \quad \text{(direct sum)}. \tag{4.9}$$

Furthermore the following relations are valid.

$$R = R^1 + R^{1/2} \quad \text{where } R^\lambda = G^\lambda \cap R. \tag{4.10}$$

$$[ja, r^*] = a \quad \text{for all } a \in R^1. \tag{4.11}$$

$$[ja, r^*] = 0 \quad \text{for all } a \in R^{1/2}. \tag{4.12}$$

DEFINITION 4.1. An abelian ideal R of G is called "*of the first kind*" if there exists an element $r \in R$ satisfying the condition:

$$[ja, r] = a \quad \text{for all } a \in R. \tag{4.13}$$

r is called the *unit* of the abelian ideal R.

If such an $r \in R$ exists, then it is unique. In fact if another $r' \in R$ satisfies (4.13), then $(a, r) = (a, r')$ for all $a \in R$, which implies $r = r'$. We should remark that R is of the first kind if and only if $R = R^1$. In fact for the unit r of R it follows from (4.13) that $(a, r) = \omega(a)$ for all $a \in R$, which shows $r = r^*$. From (4.11) and (4.12) we have $R = R^1$. Conversely if $R = R^1$, then r^* is the unit of R. From now on, we suppose that the abelian ideal R is of the first kind and that r is the unit of R. The linear operator ad jr gives the decomposition (4.9). It is known that the following relations are valid.

$$G^0 \supset K + jR , \quad G^1 = R . \tag{4.14}$$

We define the subspace Q by putting

$$Q = \{q \in G ;\ [q, r] = [jq, r] = 0\} .$$

Then it is verified that

$$Q \supset K , \quad jQ \subset Q , \quad [jr, Q] \subset Q .$$

Furthermore

$$Q = Q^0 + Q^{1/2} \quad \text{where } Q^\lambda = G^\lambda \cap Q \tag{4.15}$$

$$Q^{1/2} = G^{1/2} \tag{4.16}$$

$$jQ^\lambda \subset Q^\lambda + K \quad \text{for each } j \in (j) \tag{4.17}$$

$$Q^0 \supset K .$$

THEOREM B. (*Vinberg, Gindikin, Piatetskij-Šapiro* [17]) *Let* $\{G, K, (j), \omega\}$ *be an effective proper j-algebra, R a maximal abelian ideal of the first kind and r be the unit of R. Then there exists a j-ideal* G_1 *and a j-subalgebra* S_1, *which is a semi-simple subalgebra of non-compact type, such that G is a semi-direct sum of* G_1 *and* S_1. *Furthermore if* $G = G^0 + G^1 + G^{1/2}$ *is the decomposition with respect to the operator* $\mathrm{ad}\, jr$, *then*

1) $$G_1 = G_1 \cap G^0 + G^1 + G^{1/2} \qquad (\textit{semi-direct})$$

2) $$G^0 = S_1 + G_1 \cap G^0 \qquad (\textit{semi-direct})$$

3) $$G_1 \cap G^0 = G_1 \cap K + jR \qquad \textit{for } j \textit{ satisfying } jG_1 \subset G_1 \ (\textit{semi-direct})$$

4) $$K = S_1 \cap K + G_1 \cap K \qquad (\textit{semi-direct})$$

5) $$[S_1, G^1] = 0 , \qquad G^1 = R .$$

From now on we preserve the notations in §2. Let $\mathscr{D}(V, F)$ be an affine homogeneous Siegel domain. Then the j-algebra $\{G_a, K_a\}$ of the affine automorphism group $\mathscr{G}_a$ is effective and by Lemma 2.6, R is an abelian ideal of G_a. Furthermore if $r \in R$ is the point left fixed by the isotropy subgroup $\mathscr{K}_a$ of $\mathscr{G}_a$, then r is the unit of R (see Remark 2.1). In particular the abelian ideal R is of the first kind. Therefore the above considerations are applicable to the j-algebra $\{G_a, K_a\}$.

LEMMA 4.1. *The decomposition of* G_a *in Proposition 2.6 coincides with that of (4.9) with respect to* $\mathrm{ad}\, jr$, *more precisely if the decomposition (4.9) for* G_a *is given by*

$$G_a = G_a{}^0 + G_a{}^1 + G_a{}^{1/2} ,$$

then $G_a{}^0 = K_a + jR$, $G_a{}^1 = R$ *and* $G_a{}^{1/2} = W$.

PROOF. Since W is stable under j and $[W, R] = 0$, we have $W \subset Q$. Furthermore W being stable under $\mathrm{ad}\, jr$, it follows that

$$W = W \cap Q^0 + W \cap Q^{1/2} .$$

To show $W \cap Q^0 = (0)$, take an element $w \in W \cap Q^0$. On the other hand (4.14) and (4.17) show $R = G_a{}^1$ and $jQ^0 \subset Q^0$ respectively. Hence it follows that

$$[jw, w] \in [W, W] \cap Q^0 \subset R \cap Q^0 = G_a{}^1 \cap Q^0 = (0) .$$

Consequently $\omega([jw, w]) = 0$, which implies $w = 0$. Therefore $W \subset Q^{1/2} = G_a{}^{1/2}$.

On the other hand we know from (4.14) that $G_a^0 \supset K_a + jR$. These arguments show that $G_a^0 = K_a + jR$ and $G_a^{1/2} = W$. *q.e.d.*

LEMMA 4.2. *R is a maximal abelian ideal of the first kind of G_a.*

PROOF. Let R' be an abelian ideal of the first kind containing R and r' be the unit of R'. Then R' being stable by $\operatorname{ad} jr$, it follows from Lemma 4.1 that

$$R' = (K_a + jR) \cap R' + R + R' \cap W .$$

To show $(K_a + jR) \cap R' = (0)$, take an element $k + ja \in (K_a + jR) \cap R'$, where $k \in K_a$ and $a \in R$. Then from (4.4) it follows that $[k + ja, r'] = a$. The left-hand side is equal to zero, which shows $a = 0$ and $k \in R'$. By (4.1) we see $k = 0$, which implies $(K_a + jR) \cap R' = (0)$. Therefore $R' = R + R' \cap W$. Let $r' = b + w$, where $b \in R$, $w \in R' \cap W$. For every $a \in R$

$$(b, a) = (r', a) - (w, a) = \omega([ja, r']) - \omega([jw, a]) = \omega(a) = \omega([ja, r]) = (r, a) ,$$

which shows $b = r$. Using (4.5) we see

$$r' = r + w = [j(r+w), \ r+w] = r + [jr, w] + [jw, w] .$$

Since $[jw, w] \in R$ and $[jr, w] \in W$, we have $[jw, w] = 0$. Therefore $\omega([jw, w]) = 0$, which shows $w = 0$ and so $r' = r$. For each $u \in R' \cap W$, we see $[ju, r] = [ju, r'] = u$. But the left-hand side is equal to zero, which shows $R' = R$. *q.e.d.*

Let τ be the faithful representation of $\mathscr{G}_h$ constructed in Theorem 3.2. We denote by the same notation the representation of the Lie algebra G_h induced by τ and denote by $\hat{G}_a$ the algebraic hull of $\tau(G_a)$. Since by Theorem 3.2 $\tau(G_h)$ is an algebraic Lie algebra, $\hat{G}_a$ is a subalgebra of $\tau(G_h)$. Let $\hat{\mathscr{G}}_a$ be the analytic subgroup of $\tau(\mathscr{G}_h)$ generated by $\hat{G}_a$. Then $\hat{\mathscr{G}}_a$ acts on the Siegel domain $\mathscr{D}(V, F)$ effectively and transitively. We denote by $\hat{\mathscr{K}}_a$ the isotropy subgroup of $\hat{\mathscr{G}}_a$ at the point left fixed by $\mathscr{K}_a$. Then the pair $\{\hat{G}_a, \hat{K}_a\}$ is an effective j-algebra, which is denoted by $\{\hat{G}_a, \hat{K}_a, (\hat{j})\}$, where $\hat{K}_a$ is a Lie algebra of $\hat{\mathscr{K}}_a$. $\hat{\mathscr{K}}_a$ is a maximal compact subgroup of $\hat{\mathscr{G}}_a$ and we see

$$\hat{\mathscr{G}}_a = \hat{\mathscr{K}}_a \cdot \tau(\mathscr{T}) \qquad \text{(semi-direct)}, \tag{4.18}$$

where $\mathscr{T}$ is the simply transitive subgroup constructed in § 2. Since $\mathscr{D}(V, F)$ contains no compact analytic set with positive dimension, the j-algebra $\{\hat{G}_a, \hat{K}_a\}$ is proper.

PROPOSITION 4.1. *Under the identification of $\tau(G_a)$ with G_a, R is an abelian ideal of $\hat{G}_a$ and $\hat{G}_a$ is decomposed into the direct sum of subspaces:*

$$\hat{G}_a = \hat{K}_a + \hat{j}R + R + W .$$

Moreover

$$[\hat{K}_a+\hat{j}R,\ W]\subset W\ .$$

PROOF. Since $\hat{G}_a$ is the algebraic hull of G_a, the ideal R of G_a is also an ideal of $\hat{G}_a$ (cf. Chevalley [2]). R is a maximal abelian ideal of the first kind of $\hat{G}_a$. In fact, let R' be an abelian ideal of the first kind containing R and r' be the unit of R'. We consider the decomposition of G_a with respect to ad jr':

$$\hat{G}_a=\hat{G}'^0+\hat{G}'^1+\hat{G}'^{1/2}\ .$$

Then adjr' is non-singular endomorphism on $G'^1=R'$. Therefore by the property of the algebraic hull,

$$R'=[\hat{j}r',\ R']\subset[\hat{G}_a,\ \hat{G}_a]\subset G_a\ .$$

But by Lemma 4.2, R is a maximal abelian ideal of the first kind in G_a, which implies $R'=R$. Applying Theorem B to the effective proper j-algebra $\{\hat{G}_a,\ \hat{K}_a,\ (\hat{j})\}$ and the ideal R, we obtain the following decomposition with respect to ad $\hat{j}r$:

$$\begin{aligned}(4.19)\qquad \hat{G}_a&=\hat{G}^0+\hat{G}^1+\hat{G}^{1/2}\\ &=\hat{S}_1+\hat{G}_1=\hat{S}_1+\hat{K}_a\cap\hat{G}_1+\hat{j}R+R+\hat{G}^{1/2}\ ,\end{aligned}$$

where $\hat{S}_1$ is a semi-simple j-subalgebra of non-compact type and $\hat{G}_1$ is a j-ideal. We shall prove that $G_a\cap\hat{G}^\lambda=G_a^\lambda$ $(\lambda=0,\ 1,\ 1/2)$. First we have $[\hat{j}r,\ G_a]\subset[\hat{G}_a,\ \hat{G}_a]\subset G_a$, that is to say, G_a is stable under ad$\hat{j}r$. Hence G_a is decomposed by ad$\hat{j}r$ as follows,

$$G_a=\hat{G}^0\cap G_a+\hat{G}^1\cap G_a+\hat{G}^{1/2}\cap G_a\ .$$

$\hat{\mathscr{K}}_a$ being compact, each element of $\mathrm{ad}_{G_a}\hat{K}_a$ is a skew-symmetric operator. By (4.6) there exists a $\hat{j}\in(\hat{j})$ satisfying $[\hat{j}r,\ \hat{K}_a]=0$. Since $jr\equiv\hat{j}r$ (mod. $\hat{K}_a$), we see that ad$\hat{j}r$ and adjr give the same decomposition of G_a, which shows $G_a\cap\hat{G}^\lambda=G_a^\lambda$. In particular by Lemma 4.1 we have $G_a\cap\hat{G}^{1/2}=W$. Since ad$\hat{j}r$ is non-singular on $\hat{G}^{1/2}$, we have $\hat{G}^{1/2}=[\hat{j}r,\ \hat{G}^{1/2}]\subset[\hat{G}_a,\ \hat{G}_a]\subset G_a$, which implies $\hat{G}^{1/2}=W$. Next we shall prove $\hat{S}_1=(0)$. We obtain

$$\hat{S}_1=[\hat{S}_1,\ \hat{S}_1]\subset G_a\cap\hat{G}^0=G_a^0=K_a+jR\ .$$

Let $s\in\hat{S}_1$ and $s=k+ja$, where $k\in K_a$, $a\in R$. Theorem B (4′) shows that $ja=s-k\in\hat{S}_1+\hat{G}_1\cap\hat{K}_a$. From Theorem B (2) (3) it follows $s=k$. Hence $\hat{S}_1\subset K_a$. If $\hat{S}_1\neq(0)$, then $\hat{S}_1$ generates a semi-simple group without compact simple factors, while $\mathscr{K}_a$ is compact. From this we see $\hat{S}_1=(0)$. By (4.19) we have $\hat{G}_a=\hat{K}_a+\hat{j}R+R+W$. The bracket relation in the proposition is easily verified. *q.e.d.*

Let us define an affine representation of $\hat{G}_a$ on the complex vector space $R+iR+W$. For each $g=h+a+c\in\hat{G}_a$, where $h\in\hat{H}=\hat{K}_a+\hat{j}R$, $a\in R$, $c\in W$, we define an infinitesimal affine transformation $D_g\in A(R+iR+W)$ by putting

$$D_g(x+iy, u)=([h, x]+a+i([h, y]+\frac{1}{2}[\hat{j}u, c]+\frac{i}{2}[u, c]), [h, u]+c),$$

where $x, y\in R$, $u\in W$ (cf. Lemma 2.8.).

LEMMA 4.3. *The mapping $g\to D_g$ is a faithful representation of $\hat{G}_a$ into $A(R+iR+W)$.*

PROOF. Let $g=h+a+c$ and $g'=h'+a'+c'$, where $h, h'\in\hat{H}$, $a, a'\in R$ and $c, c'\in W$. Then D_g and $D_{g'}$ are represented by the following matrices:

$$D_g=\begin{pmatrix}\mathrm{ad}_R h & f(c) & a\\ 0 & \mathrm{ad}_W h & c\\ 0 & 0 & 0\end{pmatrix} \qquad D_{g'}=\begin{pmatrix}\mathrm{ad}_R h' & f(c') & a'\\ 0 & \mathrm{ad}_W h' & c'\\ 0 & 0 & 0\end{pmatrix}$$

where for a fixed $w\in W$, $f(w)$ is a linear mapping of W into $R+iR$ defined by

$$f(w)=\frac{1}{2}(\mathrm{ad}\, w-i(\mathrm{ad}\, w)\hat{j}).$$

The property

$$(\mathrm{ad}_R h)f(c')=f(c')(\mathrm{ad}_W h)+f([h, c'])$$

is easily verified. Using this we have

$$[D_g, D_{g'}]=\begin{pmatrix}\mathrm{ad}_R[h, h'] & f([h, c'])-f([h', c]) & [h, a']-[h', a]+[c, c']\\ 0 & \mathrm{ad}_W[h, h'] & [h, c']-[h' c]\\ 0 & 0 & 0\end{pmatrix}$$

$$=D_{[h,h']}+D_{[h,a']}+D_{[a,h']}+D_{[c,c']}+D_{[h,c']}+D_{[c,h']}=D_{[g,g']},$$

which shows that D is a homomorphism. $\mathrm{Ker}D\subset\hat{H}$ is easily seen. The element $h\in\mathrm{Ker}\, D$ and $h=k+\hat{j}a$ where $k\in\hat{K}_a$, $a\in R$. Then $[h, r]=a=0$, which implies $h\in\hat{H}$ is contained in $\mathrm{Ker}D$ if and only if $[h, R]=[h, W]=0$. Let $\mathrm{Ker}\, D\subset\hat{K}_a$. But the j-algebra $\{\hat{G}_a, \hat{K}_a\}$ being effective, $\mathrm{Ker}\, D=(0)$. This proves that D is faithful. *q.e.d.*

We denote by $\hat{G}_a{}^*$ and $\hat{K}_a{}^*$ the D-images of $\hat{G}_a$ and $\hat{K}_a$ respectively. It is obvious from the definition of D_g and Lemma 2.8 that $\hat{G}_a{}^*$ contains G_a.

PROPOSITION 4.2. *Let $\mathscr{D}=\mathscr{D}(V, F)$ be an affine homogeneous Siegel domain. Let $\hat{\mathscr{G}}_a{}^*$ and $\hat{\mathscr{K}}_a{}^*$ be the analytic subgroup of $\mathscr{A}(R+iR+W)$ corresponding to $\hat{G}_a{}^*$ and $\hat{K}_a{}^*$ respectively. Then $\hat{\mathscr{G}}_a{}^*$ acts on $\mathscr{D}$ effectively and transitively as affine automorphisms. Moreover $\hat{\mathscr{G}}_a{}^*=\hat{\mathscr{K}}_a{}^*\cdot\mathscr{T}$ (semi-direct) and $\hat{\mathscr{G}}_a{}^*$ is isomorphic to $\hat{\mathscr{G}}_a$, where $\mathscr{T}$ is the simply transitive group con-*

structured in § 2.

PROOF. Since $\hat{\mathscr{G}}_a$ is a closed subgroup of $\tau(\mathscr{G}_h)$ containing $\tau(\mathscr{T})$, $\hat{\mathscr{G}}_a$ is the adjoint group of $\hat{G}_a$ by Theorem 3.1. Therefore there exists the covering homomorphism π of $\hat{\mathscr{G}}_a^*$ onto $\hat{\mathscr{G}}_a$. Then $\mathscr{D}=\hat{\mathscr{G}}_a/\hat{\mathscr{K}}_a=\hat{\mathscr{G}}_a^*/\pi^{-1}(\hat{\mathscr{K}}_a)$ and $\hat{\mathscr{G}}_a^*/\hat{\mathscr{K}}_a^*$ is the covering manifold of $\mathscr{D}$. $\mathscr{D}$ being simply connected, $\hat{\mathscr{K}}_a^*=\pi^{-1}(\hat{\mathscr{K}}_a)$, which shows $\mathrm{Ker}\,\pi\subset\hat{\mathscr{K}}_a^*$. It is easily verified that $\hat{\mathscr{K}}_a^*$ is the isotropy subgroup of $\hat{\mathscr{G}}_a^*$ at $z_0=(ir,0)\in R+iR+W$. Moreover $\hat{\mathscr{G}}_a^*$ contains $\mathscr{G}_a$ and of course $\mathscr{T}$. Therefore the orbit $\hat{\mathscr{G}}_a^*\cdot z_0$ is open in $R+iR+W$. It follows that $\hat{\mathscr{G}}_a^*$ acts on $\hat{\mathscr{G}}_a^*\cdot z_0=\hat{\mathscr{G}}_a^*/\hat{\mathscr{K}}_a^*$ effectively, which shows $\mathrm{Ker}\,\pi=(0)$ and so $\hat{\mathscr{G}}_a^*\cong\hat{\mathscr{G}}_a$. On the other hand $\hat{\mathscr{G}}_a^*/\hat{\mathscr{K}}_a^*=\hat{\mathscr{G}}_a/\hat{\mathscr{K}}_a=\tau(\mathscr{G}_a)/\tau(\mathscr{K}_a)=\mathscr{G}_a/\mathscr{K}_a=\mathscr{D}$. Hence we have $\hat{\mathscr{G}}_a^*=\hat{\mathscr{K}}_a^*\mathscr{T}$ (semi-direct), which shows $\hat{\mathscr{G}}_a^*\cdot z_0=\mathscr{T}\cdot z_0=\mathscr{D}$. *q.e.d.*

Using the above considerations we establish main theorems in this section.

THEOREM 4.1. *Let $\mathscr{D}$ be an affine homogeneous Siegel domain of the second kind and $\mathscr{G}_h$ (resp. $\mathscr{G}_a$) be the identity component of the holomorphic (resp. affine) automorphism group of $\mathscr{D}$. Let τ be the faithful linear representation of $\mathscr{G}_h$ in Theorem 3.2. Then $\tau(\mathscr{G}_a)$ is the identity component of an algebraic group. Furthermore let $\mathscr{T}$ be the subgroup of $\mathscr{G}_a$ constructed in* § 2 *which is simply transitive on $\mathscr{D}$. Then $\tau(\mathscr{T})$ is a maximal **R**-triangular subgroup of $\tau(\mathscr{G}_h)$.*

PROOF. Since $\hat{G}_a$ is an algebraic subalgebra of $\tau(G_h)$, $\hat{\mathscr{G}}_a$ is the identity component of an algebraic group. If $\tau(\mathscr{G}_a)\subsetneqq\hat{\mathscr{G}}_a$, then by Proposition 4.2 the strictly larger group $\hat{\mathscr{G}}_a^*$ than $\mathscr{G}_a$ acts effectively on $\mathscr{D}$ as affine automorphisms. This contradicts the fact that $\mathscr{G}_a$ is the identity component of the group of all affine automorphisms of $\mathscr{D}$. Hence $\tau(\mathscr{G}_a)=\hat{\mathscr{G}}_a$. We shall prove the latter half of the theorem. One should note that

$$\dot{\tau}\cdot\mathrm{Ad}\,\mathscr{G}_a\cdot\dot{\tau}^{-1}=\mathrm{Ad}\,\tau(\mathscr{G}_a)\,, \tag{4.20}$$

where $\dot{\tau}$ is the representation of G_a induced by τ. By Proposition 2.1, $\mathscr{G}_a$ is the identity component of a real algebraic group and the adjoint representation of $\mathscr{G}_a$ is rational. Therefore it follows from a theorem of Rosenlicht [12] that for any element $g\in\mathscr{G}_a$ the eigenvalues of the matrix $\mathrm{Ad}\,g$ are power products of the eigenvalues of the matrix g. $\mathscr{T}$ being $\boldsymbol{R}$-triangular, all eigenvalues of each $t\in\mathscr{T}$ are real. Therefore all eigenvalues of $\mathrm{Ad}_{G_a}t$ are real, which implies that $\mathrm{Ad}_{G_a}\mathscr{T}$ is $\boldsymbol{R}$-triangular. It is obvious that $\mathrm{Ad}_{G_a}\mathscr{T}$ is a maximal $\boldsymbol{R}$-triangular subgroup of $\mathrm{Ad}\,\mathscr{G}_a$. From (4.20) we have $\mathrm{Ad}_{\tau(G_a)}\tau(\mathscr{T})=\dot{\tau}\cdot\mathrm{Ad}_{G_a}\mathscr{T}\cdot\dot{\tau}^{-1}$, which shows that $\mathrm{Ad}_{\tau(G_a)}\tau(\mathscr{T})$ is a maximal $\boldsymbol{R}$-triangular subgroup of $\mathrm{Ad}\,\tau(\mathscr{G}_a)$. Let $\tilde{\mathscr{T}}$ be an arbitrary maximal $\boldsymbol{R}$-triangular subgroup of $\tau(\mathscr{G}_a)$. Then $\tilde{\mathscr{T}}$ is also a maximal $\boldsymbol{R}$-triangular subgroup of $\tau(\mathscr{G}_h)$. Since $\tau(\mathscr{G}_a)$ is the idendity component of an algebraic group, the same argument as above shows that $\mathrm{Ad}_{\tau(G_a)}\tilde{\mathscr{T}}$ is a maximal $\boldsymbol{R}$-triangular subgroup of $\mathrm{Ad}\,\tau(\mathscr{G}_a)$. By the conjugateness theorem of maximal $\boldsymbol{R}$-triangular subgroups (cf. [15]), there exists an element $g\in\tau(\mathscr{G}_a)$ such that

$$\mathrm{Ad}_{\tau(G_a)}\tau(\mathscr{T})=\mathrm{Ad}\,g\cdot\mathrm{Ad}_{\tau(G_a)}\tilde{\mathscr{T}}(\mathrm{Ad}\,g)^{-1}=\mathrm{Ad}_{\tau(G_a)}(g\tilde{\mathscr{T}}g^{-1})\,.$$

By Proposition 2.5 the adjoint representation is faithful. Hence it follows that

$$\tau(\mathscr{T})=g\tilde{\mathscr{T}}g^{-1}\,,$$

which implies that $\tau(\mathscr{T})$ is a maximal $\boldsymbol{R}$-triangular subgroup of $\tau(\mathscr{G}_h)$. *q.e.d.*

REMARK 4.1. If a Siegel domain of the second kind $\mathscr{D}$ is a symmetric domain, then $\mathscr{G}_h$ is semi-simple. And $\mathrm{Ad}\,\mathscr{G}_h$ is the identity component of $\mathscr{A}_{ut}G_h$ which is an algebraic group. Therefore from the proof of Theorem 3.2 it follows that the representation τ of $\mathscr{G}_h$ coincides with the adjoint representation. We consider a Cartan decomposition of G_h and the Iwasawa subgroup $\mathscr{T}^*$ of $\mathrm{Ad}\,\mathscr{G}_h$ associated to this Cartan decomposition. Then $\mathscr{T}^*$ is a maximal $\boldsymbol{R}$-triangular subgroup. Hence $\tau(\mathscr{T})$ and $\mathscr{T}^*$ are conjugate to each other in $\mathrm{Ad}\,\mathscr{G}_h$. By using the conjugateness of Cartan decomposition of G_h, it can be proved that, $\tau(\mathscr{T})$ is the Iwasawa subgroup of $\mathrm{Ad}\,\mathscr{G}_h$ associated to some Cartan decomposition.

THEOREM 4.2. (*The uniqueness of the group* $\mathscr{T}$) *Let* $\mathscr{D}$ *and* $\mathscr{D}'$ *be the affine homogeneous Siegel domains of the second kind. Let* $\mathscr{T}$ *and* $\mathscr{T}'$ *be the simply transitive affine automorphism of* $\mathscr{D}$ *and* $\mathscr{D}'$, *constructed in* § 2 *respectively. Suppose that* $\mathscr{D}'$ *is holomorphically equivalent to* $\mathscr{D}$ *under a map* ϕ. *Then* $\mathscr{T}$ *is conjugate to* $\phi\mathscr{T}'\phi^{-1}$ *in* $\mathscr{G}_h$, *where* $\mathscr{G}_h$ *is the identity component of the holomorphic automorphism group of* $\mathscr{D}$.

PROOF. We denote by $\mathscr{G}_h'$ the identity component of the holomorphic automorphism group of $\mathscr{D}'$. Then $\phi\mathscr{G}_h'\phi^{-1}=\mathscr{G}_h$. We define the isomorphism $\tilde{\phi}$ of $\mathscr{G}_h'$ onto $\mathscr{G}_h$ by putting

$$\tilde{\phi}(g')=\phi\cdot g'\cdot\phi^{-1}\qquad g'\in\mathscr{G}_h'\,.$$

Let τ and τ' be the faithful representations of $\mathscr{G}_h$ and $\mathscr{G}_h'$ constructed in Theorem 3.2 respectively. Then $\hat{\phi}=\tau\cdot\tilde{\phi}\cdot\tau'^{-1}$ is an isomorphism of $\tau'(\mathscr{G}_h')$ onto $\tau(\mathscr{G}_h)$. By Theorem 4.1 $\tau(\mathscr{T})$ and $\tau'(\mathscr{T}')$ are maximal $\boldsymbol{R}$-triangular subgroups of $\tau(\mathscr{G}_h)$ and $\tau'(\mathscr{G}_h')$ respectively. Since $\tau(\mathscr{G}_h)$ and $\tau'(\mathscr{G}_h')$ are the identity components of algebraic groups, the same arguments as in the proof of Theorem

4.1 show that $\mathrm{Ad}_{\tau(G_h)}\tau(\mathscr{T})$ and $\mathrm{Ad}_{\tau'(G_h')}\tau'(\mathscr{T}')$ are maximal $\boldsymbol{R}$-triangular subgroups of $\mathrm{Ad}\,\tau(\mathscr{G}_h)$ and $\mathrm{Ad}\,\tau'(\mathscr{G}_h')$ respectively. It follows from (4.20) that

$$\hat{\phi}\,\mathrm{Ad}\,\tau'(\mathscr{G}_h')\hat{\phi}^{-1}=\mathrm{Ad}\,\tau(\mathscr{G}_h)\,.$$

$\hat{\phi}\cdot\mathrm{Ad}_{\tau'(G_h')}\tau'(\mathscr{T}')\cdot\hat{\phi}^{-1}$ is a maximal $\boldsymbol{R}$-triangular subgroup of $\mathrm{Ad}\,\tau(\mathscr{G}_h)$. Therefore by the conjugateness theorem, there exists an element $g\in\mathscr{G}_h$ such that

$$\hat{\phi}\cdot\mathrm{Ad}_{\tau'(G_h')}\tau'(\mathscr{T}')\cdot\hat{\phi}^{-1}=\mathrm{Ad}\,\tau(g)\,\mathrm{Ad}_{\tau(G_h)}\tau(\mathscr{T})\cdot\mathrm{Ad}\,\tau(g)^{-1}\,,$$

or equivalently

$$\mathrm{Ad}_{\tau(G_h)}\hat{\phi}\tau'(\mathscr{T}')=\mathrm{Ad}_{\tau(G_h)}\tau(g\mathscr{T}g^{-1})\,.$$

Since the adjoint representation is faithful by Theorem 3.1, we see

$$\hat{\phi}\tau'(\mathscr{T}')=\tau(g\mathscr{T}g^{-1})$$

from which we have $\phi\mathscr{T}'\phi^{-1}=g\mathscr{T}g^{-1}$. *q.e.d.*

§ 5. The uniqueness of realizations

Let $\mathscr{D}(V,F)$ be an affine homogeneous Siegel domain in $R_C\times W$ and $\mathscr{T}$ be the simply transitive affine automorphism group of $\mathscr{D}(V,F)$ constructed in § 2. We shall begin with some studies of the Lie algebra T of $\mathscr{T}$. As is known in § 2,

$$T=T_1+R+W \qquad \text{(direct sum)}$$

and T has a structure of the j-algebra which is sometimes denoted by $\{T,j\}$. By the analogous way as in the proof of Proposition 2.6 we can show $T_1=jR$. Consequently

$$T=jR+R+W \qquad \text{(direct sum)}. \tag{5.1}$$

The following relations are valid.

$$[jR,W]\subset W \qquad [R,W]=(0) \qquad [W,W]\subset R \qquad jW=W\,. \tag{5.2}$$

It is proved that R is a maximal abelian ideal of the first kind of T (cf. Lemma 4.2). Let r be the unit of R. On the other hand the Lie algebra G_a is algebraic and T is a maximal $\boldsymbol{R}$-triangular subalgebra of G_a (§ 2). Therefore T is an algebraic Lie algebra. Applying the results of § 4 in [17] we see that the operator $\mathrm{ad}\,jr$ is semi-simple and that its eigenvalues are all real. According to (4.9) T is decomposed by $\mathrm{ad}\,jr$ as follows,

$$T=T^0+T^1+T^{1/2} \qquad \text{(direct sum)}. \tag{5.3}$$

Each T^λ ($\lambda=0,1,1/2$) coincides with the eigenspace of $\mathrm{ad}\,jr$ corresponding to the eigenvalue λ. As in Lemma 4.1

$$T^0=jR, \qquad T^1=R, \qquad T^{1/2}=W \tag{5.4}$$

are valid. The Siegel domain $\mathscr{D}(V, F)$ can be reconstructed in the j-algebra T (cf. (2.16)—(2.18)). In fact the cone V is the orbit of the unit $r\in R$ under the operation of $\mathrm{Ad}_R\mathscr{T}$. The V-hermitian form F is given by

$$F(u, v)=\frac{1}{4}([ju, v]+j[u, v]) \qquad u, v\in W. \tag{5.5}$$

Moreover

$$\mathscr{D}(V, F)=\left\{x+jy+u\in R+jR+W=T: y-\frac{1}{4}[ju, u]\in V\right\}. \tag{5.6}$$

LEMMA 5.1. *Let $\tilde{R}$ and R' be abelian ideals of the first kind of a j-algebra T_1. Suppose that T_1 admits the following two decompositions:*

$$T_1=j\tilde{R}+\tilde{R}+\tilde{W}=jR'+R' \qquad \text{(direct sum)},$$

where $[j\tilde{R}, \tilde{W}]\subset\tilde{W}$, $[\tilde{R}, \tilde{W}]=(0)$, $[\tilde{W}, \tilde{W}]\subset\tilde{R}$, $j\tilde{W}=\tilde{W}$. Then $\tilde{R}=R'$ and $\tilde{W}=(0)$.

PROOF. Let $\tilde{r}$ and r' be the units of $\tilde{R}$ and R' respectively. Let $T=T'^0+T'^1+T'^{1/2}$ be the decomposition (4.9) of T with respect to ad jr'. Then we see from (4.14) that $T'^0=jR'$, $T'^1=R'$ and $T'^{1/2}=(0)$. Since $\tilde{R}$ is stable by ad jr', we have

$$\tilde{R}=\tilde{R}\cap jR'+\tilde{R}\cap R'.$$

To show $jR'\cap\tilde{R}=(0)$, take $ja'\in jR'\cap\tilde{R}$, $a'\in R'$. Then $a'=[ja', r']\in[\tilde{R}, R']\subset\tilde{R}$, from which $[ja', \tilde{r}]=a'$. The left-hand side is equal to zero. Hence $a'=0$, which implies $\tilde{R}\subset R'$. We can show that $\tilde{R}$ is a maximal abelian ideal of the first kind. Therefore $\tilde{R}=R'$ and $\tilde{W}=(0)$. *q.e.d.*

PROPOSITION 5.1. *Let R' be an abelian ideal of the first kind of the j-algebra T. Suppose that T is decomposed into the following direct sum*

$$T=jR'+R'+W'$$

where $[jR', W']\subset W'$, $[R', W']=(0)$, $[W', W']\subset R'$, $jW'=W'$. Then $R=R'$ and $W=W'$.

PROOF. R' being stable under ad jr, it follows from (5.4) that

$$R'=jR\cap R'+R\cap R'+W\cap R'.$$

By the same method as in the proof of Lemma 5.1 we have $jR\cap R'=(0)$. Therefore

$$R'=R\cap R'+W\cap R'. \tag{5.7}$$

We consider the j-subalgebra $T_1=jR'+R'$. (5.7) implies

$$T_1=j\tilde{R}+\tilde{R}+\tilde{W},$$

where $\tilde{R}=R\cap R'$, $\tilde{W}=W\cap R'+j(W\cap R')$. We shall prove that the abelian ideal $\tilde{R}$ is of the first kind. Let ω be the linear form of the j-algebra T. We define the element $\tilde{r}\in\tilde{R}$ as follows,

$$(a, \tilde{r})=\omega(a) \qquad \text{for all } a\in\tilde{R}.$$

Let r' be the unit of R' and let $r'=b+w$, where $b\in\tilde{R}$, $w\in W\cap R'$. Then for every $a\in R'$

$$\omega(a)=(a, r')=(a, b)+(a, w).$$

On the other hand $(a, w)=\omega([jw, a])=0$ for $a\in R$, which implies that $(a, b)=\omega(a)$ for $a\in\tilde{R}$. Therefore $b=\tilde{r}$ and we have $r'=\tilde{r}+w$. As is mentioned in § 4, in order to prove that $\tilde{R}$ is of the first kind it is sufficient to show that $\tilde{R}$ is the eigenspace of ad$j\tilde{r}$ corresponding to the eigenvalue 1. Since R' is the eigenspace of ad jr' corresponding to the eigenvalue 1,

$$[j\tilde{r}, a]=[jr', a]-[jw, a]=a \qquad \text{for each } a\in\tilde{R},$$

which shows that $\tilde{R}$ is of the first kind. Moreover by direct verifications we see

$$[j\tilde{R}, \tilde{W}]\subset\tilde{W}, \qquad [\tilde{R}, \tilde{W}]=(0), \qquad [\tilde{W}, \tilde{W}]\subset\tilde{R}.$$

Therefore from Lemma 5.1 we obtain $\tilde{R}=R'$, which shows $R'\subset R$. By the maximality of the abelian ideal R' we see $R'=R$. Hence $r'=r$ which implies $W'=W$. *q.e.d.*

PROPOSITION 5.2. *Let $\mathscr{D}$ and $\mathscr{D}'$ be affine homogeneous Siegel domains, $\mathscr{T}$ and $\mathscr{T}'$ the simply transitive affine automorphism groups of $\mathscr{D}$ and $\mathscr{D}'$ respectively, and T and T' be the Lie algebras of $\mathscr{T}$ and $\mathscr{T}'$ respectively. Then $\mathscr{D}'$ is holomorphically equivalent to $\mathscr{D}$ if and only if the j-algebra $\{T', j'\}$ is j-isomorphic to the j-algebra $\{T, j\}$.*

PROOF. First we assume that $\mathscr{D}'$ is holomorphically equivalent to $\mathscr{D}$. Then by Theorem 4.2 we may choose a holomorphic homeomorphism ψ of $\mathscr{D}'$ onto $\mathscr{D}$ such that

$$\psi\mathscr{T}'\psi^{-1}=\mathscr{T}. \tag{5.8}$$

We denote by $\tilde{\psi}$ the isomorphism of $\mathscr{T}'$ onto $\mathscr{T}$ induced by ψ. Then we obtain the following commutative diagram:

$$\begin{array}{ccc} T' & \xrightarrow{\tilde{\varphi}} & T \\ \downarrow{\scriptstyle \pi'} & & \downarrow{\scriptstyle \varpi'} \\ T_{z_0'}(\mathscr{D}') & \xrightarrow{\varphi} & T_{z_0}(\mathscr{D}) \end{array},$$

where $z_0=\varphi(z_0')$ and $T_{z_0}(\mathscr{D})$ (resp. $T_{z'_0}(\mathscr{D}')$) is the tangent space of $\mathscr{D}$ at z_0 (resp. $\mathscr{D}'$ at z_0'), and π' and ϖ' are projections defined by the same manner as in (2.13). From this it follows $j\tilde{\varphi}=\tilde{\varphi}j'$, which shows that $\tilde{\varphi}$ is a j-isomorphism of T' onto T. As for the converse, j and j' are extended to the left invariant complex structures on $\mathscr{T}$ and $\mathscr{T}'$ respectively. The j-isomorphism of T' onto T is also extended to the holomorphic homeomorphism of $\mathscr{T}'$ onto $\mathscr{T}$. Furthermore the complex manifolds $\mathscr{T}$ and $\mathscr{T}'$ are holomorphically equivalent to $\mathscr{D}$ and $\mathscr{D}'$ respectively. *q.e.d.*

DEFINITION 5.1. Let L and L' be real (resp. complex) vector spaces. Let U and U' be domains in L and L' respectively. U is called "*linearly equivalent*" to U' if there exists a real (resp. complex) linear isomorphism of L onto L' by which U is carried to U'.

The next theorem has been stated in Piatetskij-Šapiro [10] without proof.

THEOREM 5.1. (The uniqueness of realizations) *Let $\mathscr{D}(V, F)$ and $\mathscr{D}(V', F')$ be affine homogeneous Siegel domains. Then they are holomorphically equivalent if and only if they are linearly equivalent. In particular let $\mathscr{D}$ be a homogeneous bounded domain in C^n. Then affine homogeneous Siegel domains, which are holomorphically equivalent to $\mathscr{D}$, are unique up to their linear equivalence.*

PROOF. Let $\mathscr{T}$ and $\mathscr{T}'$ be the simply transitive affine automorphism groups of $\mathscr{D}(V, F)$ and $\mathscr{D}(V', F')$ respectively. Then by (5.1) we have the following decomposition of Lie algebras

$$T=jR+R+W, \qquad T'=j'R'+R'+W'.$$

We choose a holomorphic homeomorphism φ of $\mathscr{D}(V', F')$ onto $\mathscr{D}(V, F)$ so that it may satisfy (5.8). Then it follows from Proposition 5.2 that T' is j-isomorphic to T under $\tilde{\varphi}$. Therefore by Proposition 5.1 we obtain $\tilde{\varphi}(R')=R$ and $\tilde{\varphi}(W')=W$. Let r and r' be the units of abelian ideals R and R' respectively. Then $\tilde{\varphi}(r')=r$. In fact for any $\tilde{\varphi}(a')\in R$, $a'\in R'$,

$$[j\tilde{\varphi}(a'), \tilde{\varphi}(r')]=\tilde{\varphi}([j'a', r'])=\tilde{\varphi}(a').$$

This implies that $\tilde{\varphi}(r')$ is the unit of R, which shows $\tilde{\varphi}(r')=r$. Moreover we have $\tilde{\varphi}(V')=V$. In fact

$$\tilde{\phi}(V') = \tilde{\phi}((\mathrm{Ad}_{R'}\mathscr{T}')\cdot r') = (\mathrm{Ad}_R \mathscr{T})\tilde{\phi}(r') = (\mathrm{Ad}_R \mathscr{T})\cdot r = V\,. \tag{5.9}$$

Using (5.5) and (5.6) we see $\tilde{\phi}(\mathscr{D}(V', F')) = \mathscr{D}(V, F)$. The "if" part of the theorem is trivial. *q.e.d.*

COROLLARY 5.1. *Let $\mathscr{D}(V, F)$ and $\mathscr{D}(V', F')$ be affine homogeneous Siegel domains. If they are holomorphically equivalent, then cones V and V' are linearly equivalent.*

PROPOSITION 5.3. *Let $\mathscr{D}(V, F)$ be an affine homogeneous Siegel domain of the second kind. If $\mathscr{D}(V, F)$ is holomorphically equivalent to an affine homogeneous Siegel domain of the first kind $\mathscr{D}(V')$, then $\mathscr{D}(V, F)$ is reduced to the Siegel domain of the first kind $\mathscr{D}(V)$.*

PROOF. Let $\mathscr{T}$ and $\mathscr{T}'$ be the simply transitive affine automorphism group of $\mathscr{D}(V, F)$ and $\mathscr{D}(V')$ respectively. Then the Lie algebra T and T' admit the following decompositions:

$$T = jR + R + W\,, \qquad T' = j'R' + R'\,,$$

where R and R' are abelian ideals of the first kind of T and T' respectively. By Proposition 5.2 there exists a j-isomorphism $\tilde{\phi}$ of T' onto T. Therefore from Lemma 5.1 we see $W = (0)$, which shows that $\mathscr{D}(V, F)$ is reduced to $\mathscr{D}(V)$. *q.e.d.*

A convex cone V in a vector space R is called "*self-adjoint*" if there exists an inner product $\langle\ ,\ \rangle$ on R such that

$$V = \{a \in R : \ \langle a, b\rangle > 0 \ \text{ for all non-zero } \ b \in \bar{V}\}\,,$$

where $\bar{V}$ is the closure of V. As a corollary of Theorem 5.1 we obtain a theorem of Rothaus [14].

COROLLARY 5.2. *Let $\mathscr{D}(V)$ be an affine homogeneous Siegel domain of the first kind associated to the convex cone V. If $\mathscr{D}(V)$ is holomorphically equivalent to a symmetric domain, then V is a homogeneous self-adjoint cone.*

PROOF. Since $\mathscr{D}(V)$ is affine homogeneous, Theorem 2.1 shows that V is homogeneous cone. According to a theorem of Koranyi-Wolf [8] (or Hano [5]) $\mathscr{D}(V)$ is holomorphically equivalent to an affine homogeneous Siegel domain of the second kind $\mathscr{D}(V', F')$, where V' is a homogeneous self-adjoint cone. Proposition 5.3 and Corollary 5.1 show that $\mathscr{D}(V', F')$ is reduced to the Siegel domain of the first kind $\mathscr{D}(V')$ and that V is linearly equivalent to V'. Since V' is self-adjoint, V is also self-adjoint. *q.e.d.*

§ 6. The irreducible decomposition of homogeneous bounded domains

Let $\mathscr{D}$ be a homogeneous bounded domain in C^n. In this section we denote by $\mathscr{G}$ the identity component of the holomorphic automorphism group of $\mathscr{D}$. We consider the affine homogeneous Siegel domain $\mathscr{D}(V, F)$ which is holomorphically equivalent to $\mathscr{D}$. According to Theorem 4.2, the solvable group $\mathscr{T}$, constructed in § 2, which is simply transitive on $\mathscr{D}(V, F)$, is uniquely determined by $\mathscr{D}$ up to the conjugateness in $\mathscr{G}$. From now on $\mathscr{T}$ will be called the *Iwasawa group* of $\mathscr{D}$ and the Lie algebra T of $\mathscr{T}$ is called the *Iwasawa algebra* of $\mathscr{D}$. (cf. Remark 4.1.)

The homogeneous bounded domain $\mathscr{D}$ is a simply connected homogeneous Kähler manifold with respect to the Bergman metric (See Appendix). Therefore by a theorem of Hano-Matsushima [6], $\mathscr{D}$ is uniquely decomposed into the direct product of simply connected homogeneous Kähler manifolds $\mathscr{D}_i$'s.

$$\mathscr{D}=\mathscr{D}_1\times\cdots\times\mathscr{D}_s \tag{6.1}$$

where each $\mathscr{D}_i$ is irreducible in the sense of Kähler geometry. This is the de Rham decomposition of $\mathscr{D}$. We should remark that a complex Euclidean space does not appear among $\mathscr{D}_i$'s, since the Ricci curvature of $\mathscr{D}$ is non-degenerate. We denote by $\mathscr{A}$ and $\mathscr{A}_i$ the identity components of the automorphism groups of the homogeneous Kähler manifolds $\mathscr{D}$ and $\mathscr{D}_i$ respectively. Then it is known [6] that

$$\mathscr{A}\cong\mathscr{A}_1\times\cdots\times\mathscr{A}_s \tag{6.2}$$

LEMMA 6.1. *Let $\mathscr{G}$ and $\mathscr{G}_i$ be the identity components of the holomorphic automorphism groups of $\mathscr{D}$ and $\mathscr{D}_i$ respectively. Then*

$$\mathscr{G}\cong\mathscr{G}_1\times\cdots\times\mathscr{G}_s\,. \tag{6.3}$$

And each $\mathscr{G}_i$ is centreless.

PROOF. Since $\mathscr{D}$ is a homogeneous bounded domain, we see $\mathscr{G}=\mathscr{A}$. Each element $g_i\in\mathscr{G}_i$ is written as follows

$$g_i=(a_1,\cdots,a_i,\cdots a_s)\,,$$

where $a_i\in\mathscr{A}_i$. But every element of $\mathscr{G}_i$ acts on $\mathscr{D}_j$ $(j\neq i)$ as the identity map. Therefore we have $a_j=1$ $(j\neq i)$, which implies $\mathscr{G}_i\subset\mathscr{A}_i$. The inverse inclusion is trivial. Hence Lemma 6.1 follows from (6.2). $\mathscr{G}$ is centreless by Theorem 3.1, and so $\mathscr{G}_i$ is centreless. *q.e.d.*

LEMMA 6.2. *Let τ be the faithful representation of $\mathscr{G}$, constructed in Theorem 3.2. Then each $\tau(\mathscr{G}_i)$ is the identity component of an algebraic group.*

PROOF. It is is sufficient to prove Lemma 6.2 in the case of $i=1$. By Lemma 6.1, we can put

$$\mathscr{G}\cong\mathscr{G}_1\times\mathscr{G}'$$

where $\mathscr{G}'=\mathscr{G}_2\times\cdots\times\mathscr{G}_s$. Then we prove that $\mathscr{G}_1$ is the centralizer $\mathscr{Z}_{\mathscr{G}}(\mathscr{G}')$ of $\mathscr{G}'$ in $\mathscr{G}$. Each element $g\in\mathscr{Z}_{\mathscr{G}}(\mathscr{G}')$ is written as follows

$$g=(g_1, g'), \qquad g_1\in\mathscr{G}_1, \qquad g'\in\mathscr{G}'.$$

For every element $g_2\in\mathscr{G}'$, we have $(g_1, g')(1, g_2)=(1, g_2)(g_1, g')$. Therefore we see that $g'g_2=g_2g'$ for each $g_2\in\mathscr{G}'$, which means that g' belongs to the center of $\mathscr{G}'$. Since $\mathscr{G}'$ is centerless by Lemma 6.1, we have $g'=1$ and so $g\in\mathscr{G}_1$. Hence $\mathscr{G}_1=\mathscr{Z}_{\mathscr{G}}(\mathscr{G}')$. We denote by $\hat{\mathscr{G}}$ the algebraic group whose identity component coincides with $\tau(\mathscr{G})$. And we denote by $\mathscr{Z}(\tau(\mathscr{G}'))$ the centralizer of $\tau(\mathscr{G}')$ in the general linear group of the vector space on which $\tau(\mathscr{G}')$ operates. Then we have

$$\begin{aligned}\tau(\mathscr{G}_1)&=\mathscr{Z}_{\tau(\mathscr{G})}(\tau(\mathscr{G}'))=(\mathscr{Z}(\tau(\mathscr{G}'))\cap\tau(\mathscr{G}))^0\\&=(\mathscr{Z}(\tau(\mathscr{G}'))\cap\hat{\mathscr{G}}^0)^0=(\mathscr{Z}(\tau(\mathscr{G}'))\cap\hat{\mathscr{G}})^0.\end{aligned}$$

Since $\mathscr{Z}(\tau(\mathscr{G}'))$ and $\hat{\mathscr{G}}$ are algebraic groups, $\mathscr{Z}(\tau(\mathscr{G}'))\cap\hat{\mathscr{G}}$ is also algebraic. *q.e.d.*

PROPOSITION 6.1. *Let $\mathscr{D}$ be a homogeneous bounded domain, $\mathscr{D}=\mathscr{D}_1\times\cdots\times\mathscr{D}_s$ be the de Rham decomposition of $\mathscr{D}$, and $\mathscr{T}$ be the Iwasawa group of $\mathscr{D}$. Then*

$$\mathscr{T}\cong\mathscr{T}_1\times\cdots\times\mathscr{T}_s$$

where $\mathscr{T}_i=\mathscr{G}_i\cap\mathscr{T}$ $(1\leqq i\leqq s)$. And each $\mathscr{T}_i$ is simply transitive on $\mathscr{D}_i$.

PROOF. We choose a closed subgroup $\tilde{\mathscr{T}}_i$ of $\mathscr{G}_i$ such that $\tau(\tilde{\mathscr{T}}_i)$ is a maximal $\boldsymbol{R}$-triangular subgroup of $\tau(\mathscr{G}_i)$. We consider the direct product

$$\tilde{\mathscr{T}}=\tilde{\mathscr{T}}_1\times\cdots\times\tilde{\mathscr{T}}_s.$$

Each element A of $\tau(\tilde{\mathscr{T}})$ is expressed as the product of elements $A_1, \cdots, A_s$ where $A_i\in\tau(\tilde{\mathscr{T}}_i)$ and each A_i commutes with each other. Since $\tau(\tilde{\mathscr{T}}_i)$ is $\boldsymbol{R}$-triangular, all the eigenvalues of the matrices A_i are real. Each eigenvalue of A is expressed as the product of the eigenvalues of $A_1, \cdots, A_s$. Therefore every eigenvalue of A is real, which shows that $\tau(\tilde{\mathscr{T}})$ is $\boldsymbol{R}$-triangular group of $\tau(\mathscr{G})$. We denote by $\mathscr{K}$ the isotropy subgroup of $\mathscr{G}$ at a point $z=(z_1, \cdots, z_s)\in\mathscr{D}$. Then the isotropy subgroup $\mathscr{K}_i$ of $\mathscr{G}_i$ at the point $z_i\in\mathscr{D}_i$ is $\mathscr{K}\cap\mathscr{G}_i$. Therefore $\mathscr{K}_i$ is compact. On the other hand $\mathscr{D}$ is homeomorphic to an open cell, and so it is true for each $\mathscr{D}_i$, which implies that $\mathscr{K}_i$ is a maximal compact subgroup of $\mathscr{G}_i$. Since $\mathscr{G}_i$ is isomorphic to the identity component of a real algebraic group and $\tau(\tilde{\mathscr{T}}_i)$ is a maximal $\boldsymbol{R}$-triangular subgroup of $\tau(\mathscr{G}_i)$, we see

$$\mathscr{G}_i=\mathscr{K}_i\cdot\tilde{\mathscr{T}}_i \quad \text{(semi-direct)} \quad 1\leq i\leq s\ .$$

Therefore $\tilde{\mathscr{T}}_i$ is simply transitive on $\mathscr{D}_i$ and so $\tilde{\mathscr{T}}$ is simply transitive on $\mathscr{D}$. Hence we have $\mathscr{G}=\mathscr{K}\cdot\tilde{\mathscr{T}}$ (semi-direct), which shows that a $\boldsymbol{R}$-triangular subgroup $\tau(\tilde{\mathscr{T}})$ is a maximal $\boldsymbol{R}$-triangular subgroup of $\tau(\mathscr{G})$. According to Theorem 4.1, $\tau(\mathscr{T})$ is a maximal $\boldsymbol{R}$-triangular subgroup of $\tau(\mathscr{G})$. Therefore by the conjugateness of maximal $\boldsymbol{R}$-triangular subgroups [15], there exists an element $g\in\mathscr{G}$ such that $\mathscr{T}=g\tilde{\mathscr{T}}g^{-1}$. If we write g in the form $g=(g_1\cdots,g_s)$, $g_i\in\mathscr{G}_i$ and put $\mathscr{T}_i=g_i\tilde{\mathscr{T}}_ig_i^{-1}$ $(1\leq i\leq s)$, then we obtain

$$\mathscr{T}\cong\mathscr{T}_1\times\cdots\times\mathscr{T}_s$$

and

$$\mathscr{T}_i=\mathscr{T}\cap\mathscr{G}_i\ .$$

Since $\tilde{\mathscr{T}}_i$ is simply transitive on $\mathscr{D}_i$, $\mathscr{T}_i$ is also simply transitive on $\mathscr{D}_i$.

PROPOSITION 6.2. *Let $\mathscr{D}$ be a homogeneous bounded domain and the decomposition $\mathscr{D}=\mathscr{D}_1\times\cdots\times\mathscr{D}_s$ be the de Rham decomposition of $\mathscr{D}$. Then each irreducible factor $\mathscr{D}_i$ is holomorphically equivalent to a homogeneous bounded domain of lower dimension.*

PROOF. Let T and T_i $(1\leqq i\leqq s)$ be the Lie algebra of $\mathscr{T}$ and $\mathscr{T}_i$ $(1\leqq i\leqq s)$ respectively. Each T_i is a j-ideal of T. There exists a non-trivial abelian ideal in T_i, since T_i is solvable. As is mentioned in [17], we can suppose that the abelian ideal is of the first kind. We choose a maximal abelian ideal of the first kind of T_i, which is denoted by R_i. Let r_i be the unit of R_i. As is easily seen, the j-algebra T_i is effective and proper. Therefore, using Theorem B, T_i is decomposed by $\mathrm{ad}\,jr_i$ as follows:

$$\left.\begin{array}{l} T_i=T_i^0+T_i^1+T_i^{1/2} \\ T_i^0=jR_i\ , \qquad T_i^1=R_i \end{array}\right\} \quad (1\leqq i\leqq s)\ .$$

Putting $W_i=T_i^{1/2}$, then

$$T_i=jR_i+R_i+W_i$$
$$jW_i=W_i\ ,\quad [jR_i,W_i]\subset W_i\ ,\quad [R_i,W_i]=(0)\ ,\quad [W_i,W_i]\subset R_i$$

are valid. Therefore we see from a theorem of Vinberg, Gindikin, Piatetskij-Šapiro (Theorem 2 in [17]) that there exists a Siegel domain of the second kind $\mathscr{D}(V_i,F_i)$ in the complex vector space T_i (with complex structure j) such that $\mathscr{T}_i$ acts simply transitively on $\mathscr{D}(V_i,F_i)$ as affine automorphisms (See §5).

On the other hand, by Proposition 6.1 $\mathcal{T}_i$ acts simply transitively on $\mathcal{D}_i$ as holomorphic transformations. Therefore $\mathcal{D}_i$ is holomorphically equivalent to $\mathcal{D}(V_i, F_i)$, which proves Proposition 6.2.

DEFINITION 6.1. A homogeneous bounded domain $\mathcal{D}$ is called "*reducible*" if $\mathcal{D}$ is holomorphically equivalent to the direct product of two homogeneous bounded domains. $\mathcal{D}$ is called "*irreducible*" unless it is reducible.

EXAMPLES. The polydisc $\{|z_1|<1, \cdots, |z_n|<1\}$ is reducible, while the open ball $|z_1|^2+\cdots+|z_n|^2<1$ is irreducible.

LEMMA 6.3. *A homogeneous bounded domain $\mathcal{D}$ is irreducible if and only if it is irreducible in the sense of Kähler geometry (with respect to the Bergman metric).*

PROOF. If $\mathcal{D}$ is reducible in the sense of Kähler geometry, then $\mathcal{D}$ is the direct product of irreducible homogeneous Kähler manifolds. From Proposition 6.2 they are holomorphically equivalent to bounded domains, which shows that $\mathcal{D}$ is reducible. Conversely let $\mathcal{D}$ be a direct product of two homogeneous bounded domains $\mathcal{D}_1$ and $\mathcal{D}_2$. Then by a result of Kobayashi [7], the Bergman metric of $\mathcal{D}$ is the direct sum of those of $\mathcal{D}_1$ and $\mathcal{D}_2$. This shows that $\mathcal{D}$ is reducible in the sense of Kähler geometry. *q.e.d.*

From Proposition 6.1 and the above lemma, we obtain the following theorem, which is the generalization of the well-known theorem in the case of symmetric domains [1].

THEOREM 6.1. (The irreducible decomposition of the homogeneous bounded domain) *Every homogeneous bounded domain* in C^n *is holomorphically decomposed into the direct product of irreducible homogeneous bounded domains. This decomposition is unique up to the order.*

Next we shall give a criterion of irreducibility of homogeneous bounded domains.

DEFINITION 6.2. ([16]) Let V_1 and V_2 be convex cones in vector spaces R_1 and R_2 respectively. The subset $\{(x_1, x_2): x_1 \in V_1, x_2 \in V_2\}$ of R_1+R_2 is a convex cone in R_1+R_2, which is called the "*direct sum*" of V_1 and V_2.

DEFINITION 6.3. ([16]) A homogeneous convex cone V in a vector space R is called "*reducible*" if V is linearly equivalent to the direct sum of two homogeneous convex cones. V is called "*irreducible*" unless it is reducible.

Next theorem is a generalization of Theorem 5.1.

THEOREM 6.2. *Let $\mathcal{D}(V, F)$ and $\mathcal{D}(V_i, F_i)$, $1 \leq i \leq s$, be affine homogeneous Siegel domains. Suppose that $\mathcal{D}(V, F)$ is holomorphically equivalent to the direct product $\mathcal{D}(V_1, F_1)\times\cdots\times\mathcal{D}(V_s, F_s)$. Then $\mathcal{D}(V, F)$ is linearly equiva-*

lent to $\mathscr{D}(V_1, F_1)\times\cdots\times\mathscr{D}(V_s, F_s)$, *and the cone* V *is linearly equivalent to the direct sum* $V_1+\cdots+V_s$. *In particular if* V *is irreducible, then* $\mathscr{D}(V, F)$ *is irreducible as a homogeneous bounded domain.*

PROOF. Let each $\mathscr{D}(V_i, F_i)$ be a Siegel domain in a complex vector space $R_{ic}\times W_i$ $(1\leq i\leq s)$. We consider the convex cone $V'=V_1+\cdots+V_s$ and define the V'-hermitian form F' by putting

$$F'(u_1+\cdots+u_s, v_1+\cdots+v_s)=\sum_{i=1}^{s} F_i(u_i, v_i) ,$$

where $u_i, v_i\in W_i$. Then the direct product $\mathscr{D}(V_1, F_1)\times\cdots\times\mathscr{D}(V_s, F_s)$ is canonically identified with the affine homogeneous Siegel domain $\mathscr{D}(V', F')$. Since $\mathscr{D}(V, F)$ and $\mathscr{D}(V', F')$ are holomorphically equivalent, they are linearly equivalent by Theorem 5.1, which implies that $\mathscr{D}(V, F)$ is linearly equivalent to $\mathscr{D}(V_1, F_1)\times\cdots\times\mathscr{D}(V_s, F_s)$. By Corollary 5.1 V is linearly equivalent to $V_1+\cdots+V_s$. *q.e.d.*

COROLLARY 6.1. *Let* $\mathscr{D}(V)$ *be an affine homogeneous Siegel domain of the first kind. Then each irreducible factor of* $\mathscr{D}(V)$ *is also a Siegel domain of the first kind.*

PROOF. Let $\mathscr{D}(V)=\mathscr{D}(V_1, F_1)\times\cdots\times\mathscr{D}(V_s, F_s)$ be the irreducible decomposition of $\mathscr{D}(V)$, where each $\mathscr{D}(V_i, F_i)$ is affine homogeneous. Let T and T_i $(1\leq i\leq s)$ be Iwasawa algebras of $\mathscr{D}(V)$ and $\mathscr{D}(V_i, F_i)$ respectively. By Theorem 6.2 $\mathscr{D}(V)$ is linearly equivalent to $\mathscr{D}(V_1, F_1)\times\cdots\times\mathscr{D}(V_s, F_s)$. Hence T is j-isomorphic to the direct sum $T_1+\cdots+T_s$. As is seen in § 5,

$$T=jR+R , \qquad T_i=jR_i+R_i+W_i ,$$

where R and R_i are abelian ideals of the first kind. By Lemma 5.1 we have

$$R=R_1+\cdots+R_s \qquad W_1=W_2=\cdots=W_s=(0) ,$$

which shows that each $\mathscr{D}(V_i, F_i)$ is reduced to the Siegel domain of the first kind $\mathscr{D}(V_i)$. *q.e.d.*

From now on we shall consider the converse of Theorem 6.2. For this we need the theory of the root system of the j-algebra, due to Vinberg, Gindikin, Piatetskij-Šapiro [17]. Let T be the Iwasawa algebra of an affine homogeneous Siegel domain $\mathscr{D}(V, F)$. Then T is an algebraic Lie algebra and

$$T=jR+R+W .$$

It is known that R is decomposed into the direct sum of subspaces $R_{\alpha\beta}$:

(6.7)
$$R=\sum_{\alpha\leq\beta} R_{\alpha\beta}$$

Each $R_{\alpha\alpha}$ is of one dimension, and there exists an element r_α in $R_{\alpha\alpha}$ such that for every $x\in R_{\beta\gamma}$

(6.8)
$$[jr_\alpha, x]=\frac{1}{2}(\delta_{\alpha\beta}+\delta_{\alpha\gamma})x \qquad [jr_\alpha, jx]=\frac{1}{2}(\delta_{\alpha\beta}-\delta_{\alpha\gamma})jx$$

(6.9)
$$r=\sum_\alpha r_\alpha ,$$

where r is the unit of R. Since T is $\boldsymbol{R}$-triangular, all eigenvalues of ad jr_α are real. Furthermore each ad jr_α is a semi-simple operator on T, and ad jr_α's commute with each other. Therefore T is decomposed into the direct sum of common eigenspaces T^Λ of operators ad jr_α:

$$T=\sum_\Lambda T^\Lambda$$

where Λ is the system of m real numbers $\{\Lambda_1, \cdots, \Lambda_m\}$ such that for every $x\in T^\Lambda$

$$[jr_\alpha, x]=\Lambda_\alpha x .$$

Λ is called a "*root*" if $T^\Lambda\neq(0)$. The relation

$$[T^\Lambda, T^M]\subset T^{\Lambda+M}$$

is valid. We put $\Delta^{(\beta)}=\{\delta_{1\beta}, \delta_{2\beta}, \cdots, \delta_{m\beta}\}$. The subspaces jR, R and W are stable under ad jr_α's, and they are spanned by the root spaces T^Λ. The explicit values of roots are as follows:

(6.10)
$$\begin{cases} \Lambda=\frac{1}{2}(\Delta^{(\alpha)}+\Delta^{(\beta)}) & \text{on } R \quad 1\leqq\alpha\leqq\beta\leqq m \\ \Lambda=\frac{1}{2}(\Delta^{(\alpha)}-\Delta^{(\beta)}) & \text{on } jR \quad 1\leqq\alpha\leqq\beta\leqq m \\ \Lambda=\frac{1}{2}\Delta^{(\alpha)} & \text{on } W \quad 1\leqq\alpha\leqq m . \end{cases}$$

THEOREM 6.3. (A criterion of irreducibility) *Let $\mathscr{D}$ be the homogeneous bounded domain in C^n. Suppose that $\mathscr{D}$ is holomorphically equivalent to an affine homogeneous Siegel domain $\mathscr{D}(V, F)$. Then $\mathscr{D}$ is irreducible if and only if the cone V is irreducible.*

PROOF. The "if" part of the theorem has been proved in Theorem 6.2. We shall prove the converse of this. Let $T=jR+R+W$ be the Iwasawa algebra of $\mathscr{D}(V, F)$. Suppose that the cone V in R is the direct sum of homogeneous convex cones V_1 in R_1 and V_2 in R_2. To the affine homogeneous Siegel domains of the first kind $\mathscr{D}(V)$ and $\mathscr{D}(V_i)$ $(i=1, 2)$ there correspond the Iwasawa algebras

$$T'=jR+R\,,\qquad T_i'=jR_i+R_i\qquad(i=1,2)$$

respectively, where R and R_i are abelian ideals of the first kind. The Iwasawa algebra of $\mathscr{D}(V_1)\times\mathscr{D}(V_2)$ is the direct sum $T_1'+T_2'$. Since $V=V_1+V_2$, $\mathscr{D}(V)$ is linearly equivalent to $\mathscr{D}(V_1)\times\mathscr{D}(V_2)$. Therefore T' is the direct sum of two j-ideals T_1' and T_2' (via the identification under the j-isomorphism). From (6.7) we see that

$$R_1=\sum_{\alpha\leq\beta}R_{\alpha\beta}\qquad 1\leqslant\alpha\leqslant\beta\leqslant m_1$$

$$R_2=\sum_{\mu\leq\nu}R_{\mu\nu}\qquad m_1+1\leqslant\mu\leqslant\nu\leqslant m_2\,.$$

We can suppose that α, β run through $1, 2, \cdots, m_1$ and μ, ν run through $m_1+1, \cdots, m_2$. The explicit value of roots are

$$(6.11)\qquad \begin{cases} \Lambda=\frac{1}{2}(\Delta^{(\alpha)}+\Delta^{(\beta)}) & \text{on } R_1 \\ \Lambda=\frac{1}{2}(\Delta^{(\mu)}+\Delta^{(\nu)}) & \text{on } R_2 \\ \Lambda=\frac{1}{2}(\Delta^{(\alpha)}-\Delta^{(\beta)}) & \text{on } jR_1 \\ \Lambda=\frac{1}{2}(\Delta^{(\mu)}-\Delta^{(\nu)}) & \text{on } jR_2 \\ \Lambda=\frac{1}{2}\Delta^{(\alpha)} \text{ or } \frac{1}{2}\Delta^{(\mu)} & \text{on } W\,. \end{cases}$$

We define the subspaces W_1 and W_2 of W by putting

$$W_1=\sum_{\Lambda=\Delta^{(\alpha)}/2}T^{\Lambda}\qquad W_2=\sum_{\Lambda=\Delta^{(\mu)}/2}T^{\Lambda}\,.$$

And we put

$$T_i=jR_i+R_i+W_i\qquad(i=1,2)\,.$$

From (6.11) we see that $[jR_i, W_i]\subset W_i$, $[W_i, W_i]\subset R_i$, which implies that T_1, and T_2 are subalgebras of T. Furthermore $[jR_1, W_2]=0$ and $[jR_2, W_1]=0$, since $(\Delta^{(\alpha)}-\Delta^{(\beta)}+\Delta^{(\mu)})/2$ and $(\Delta^{(\mu)}-\Delta^{(\nu)}+\Delta^{(\alpha)})/2$ are not roots. And $[W_1, W_2]=0$, since $(\Delta^{(\alpha)}+\Delta^{(\mu)})/2$ is not a root. Therefore $[T_1, T_2]=0$. Using a axiom 3) of the j-algebra we obtain for every $u_1\in T^{\Lambda}\subset W_1$,

$$[jr_\alpha, ju_1]=j[jr_\alpha, u_1]=j(\Lambda_\alpha u_1)=\Lambda_\alpha(ju_1)\,,$$

which shows that W_1 is stable by j. Analogously W_2 is stable by j. Therefore T_1 and T_2 are j-ideals of T. If we put

$$F_i(u, v)=\frac{1}{4}([ju, v]+j[u, v]) \qquad u, v\in W_i \quad (i=1, 2)\ ,$$

then F_i is V_i-hermitian form. In fact $F_i(u, u)=[ju, u]/4\in\bar{V}\cap R_i=\bar{V}_i$ since $[W_i, W_i]\subset R_i$. From the above argument, we can construct a Siegel domain $D(V_i, F_i)$ in T_i as follows

$$D(V_i, F_i)=\{x+jy+u\in R_i+jR_i+W_i : y-F_i(u, u)\in V_i\}$$

It follows that the simply connected group $\mathscr{T}_i$ corresponding to T_i acts simply transitively on $\mathscr{D}(V_i, F_i)$ as affine automorphism (cf. § 5). Thus $\mathscr{D}(V, F)$ is the direct product of affine homogeneous Siegel domains $\mathscr{D}(V_1, F_1)$ and $\mathscr{D}(V_2, F_2)$.

Appendix

In § 6 we used the fact that the homogeneous bounded domain is an open cell. This fact is based on a theorem of Vinberg, Gindikin and Piatetskij-Sapiro (Theorem 6 in [17]); every homogeneous bounded domain is holomorphically equivalent to some Siegel domain of the second kind. Since the proof given in [17] is very difficult to follow, we should like to give a sketch of the proof, supplying some facts which are omitted.

Let $\mathscr{D}$ be a homogeneous bounded domain in C^n and $\tilde{\mathscr{D}}$ be the universal covering manifold of $\mathscr{D}$, and let $\mathscr{G}$ be the identity component of the holomorphic automorphism group of $\mathscr{D}$ and $\mathscr{K}$ be the isotropy subgroup of $\mathscr{G}$. Then $\mathscr{D}=\mathscr{G}/\mathscr{K}$ and the pair $\{G, K\}$ of the Lie algebra of $\mathscr{G}$, $\mathscr{K}$ is an effective j-algebra. Let $\tilde{\mathscr{G}}'$ be the universal covering group of $\mathscr{G}$ and $\tilde{\mathscr{K}}'$ be the analytic subgroup of $\tilde{\mathscr{G}}'$ corresponding to K. Since $\{G, K\}$ is a j-algebra, $\tilde{\mathscr{D}}$ is represented as a complex homogeneous space $\tilde{\mathscr{G}}'/\tilde{\mathscr{K}}'$. Let π be the covering homomorphism of $\tilde{\mathscr{G}}'$ onto $\mathscr{G}$ and $\varDelta$ (resp. $\varDelta_0$) be the maximal normal subgroup of $\tilde{\mathscr{G}}'$ contained in $\pi^{-1}(\mathscr{K})$ (resp. $\pi^{-1}(\mathscr{K})^0=\tilde{\mathscr{K}}'$). We see $\varDelta_0\subset\varDelta$ and $\mathscr{G}=\tilde{\mathscr{G}}'/\varDelta$. If we put $\tilde{\mathscr{G}}=\tilde{\mathscr{G}}'/\varDelta_0$ and $\tilde{\mathscr{K}}=\tilde{\mathscr{K}}'/\varDelta_0$, then $\tilde{\mathscr{D}}$ is represented as the coset space $\tilde{\mathscr{G}}/\tilde{\mathscr{K}}$. $\tilde{\mathscr{G}}$ acts on $\tilde{\mathscr{D}}$ effectively and holomorphically. Since $G\cong\mathrm{ad}\, G$, G is considered as a linear effective j-algebra. Hence if we preserve the notations in Lemma 3.2, then there exists an effective algebraic j-algebra $\{\varphi(\hat{G}), \varphi(\hat{K})\}$ in which $\{G, K\}$ is contained as a j-subalgebra, as is mentioned in § 3. For simplicity we write $\hat{G}'$ and $\hat{K}'$ for $\varphi(\hat{G})$ and $\varphi(\hat{K})$ respectively.

PROPOSITION A. *Let $\mathscr{D}$ be a homogeneous bounded domain in C^n. Then the universal covering manifold $\tilde{\mathscr{D}}$ of $\mathscr{D}$ is holomorphically equivalent to an affine homogeneous Siegel domain of the second kind.*

The sketch of the proof. By the analogous method as in § 3, we can prove that the group $\hat{\mathscr{G}}'$ whose Lie algebra is $\hat{G}'$ acts on $\tilde{\mathscr{D}}$ effectively, holomorphically and transitively. We choose the subgroup $\mathscr{G}^*$ of $\hat{\mathscr{G}}'$ with the least dimension which is transitive on $\tilde{\mathscr{D}}$ and whose Lie algebra G^* is algebraic. Since $\{\hat{G}', \hat{K}'\}$ is proper, the j-algebra G^* is proper. Therefore the proof given in [17] is applicable to G^*, from which we obtain the proposition.

Next we shall show that every homogeneous bounded domain $\mathscr{D}$ is simply connected. Let $\mathscr{G}_h$ be the holomorphic automorphism group of $\tilde{\mathscr{D}}$ and $\mathscr{K}_h$ be the isotropy subgroup of $\mathscr{G}_h$. Since the j-algebra $\{G_h, K_h\}$ is effective, $G_h \cong \operatorname{ad} G_h$. By Theorem B in § 3 there exists the j-algebra $\{\hat{G}, \hat{K}\}$ such that $\hat{G}$ is the algebraic hull of $\operatorname{ad} G_h$ and $\operatorname{ad} G_h$ is a j-subalgebra of $\hat{G}$.

LEMMA A. *Let N be the maximal ideal of $\hat{G}$ contained in $\hat{K}$. Then N is abelian and $\hat{G} = \operatorname{ad} G_h + N$ (direct sum of ideals). Furthermore* $\operatorname{ad}(\operatorname{ad} G_h)$ *is an algebraic Lie algebra.*

PROOF. We put $\operatorname{ad} G_h = G_h^*$ and $\operatorname{ad}_{G_h} K_h = K_h^*$. $\{G_h^*, K_h^*\}$ being effective, $G_h^* \cap N = (0)$. As is shown in the proof of Theorem 3.2, G_h^* is isomorphic to $\hat{G}/N$. By the property $[\hat{G}, \hat{G}] \subset G_h^*$, we obtain $[G_h^*, N] = (0)$ and $[N, N] = (0)$. Since $\hat{G}$ is algebraic, the Lie algebra $\operatorname{ad}_{\hat{G}} G_h^* = \operatorname{ad}_{\hat{G}} \hat{G}$ is algebraic. Hence we see that $\operatorname{ad} G_h^*$ is algebraic. *q.e.d.*

We denote by ρ the representations $\operatorname{Ad} \circ \operatorname{Ad}$ of $\mathscr{G}_h$. Since $\tilde{\mathscr{D}}$ is holomorphically equivalent to a bounded domain, we see from Theorem 3.1 that the representation ρ is faithful on $\mathscr{G}_h^0$. Let Γ be the fundamental group of $\mathscr{D}$. Then Γ is considered as a subgroup of $\mathscr{G}_h$.

LEMMA B. *Let $\mathscr{Z}_{\mathscr{G}_h}(\Gamma)$ be the centralizer of Γ in $\mathscr{G}_h$. Then the identity component $\mathscr{Z}_{\mathscr{G}_h}(\Gamma)^0$ is isomorphic to the identity component of some real algebraic group.*

PROOF. We shall prove that $\rho(\mathscr{Z}_{\mathscr{G}_h}(\Gamma)^0) = \mathscr{Z}_{\rho(\mathscr{G}_h)}(\rho(\Gamma))^0$. Since $\rho(\mathscr{G}_h)^0 = \rho(\mathscr{G}_h^0)$, $\mathscr{Z}_{\rho(\mathscr{G}_h)}(\rho(\Gamma))^0 = \mathscr{Z}_{\rho(\mathscr{G}_h{}^0)}(\rho(\Gamma))^0$. Choose an element $\rho(g) \in \mathscr{Z}_{\rho(\mathscr{G}_h{}^0)}(\rho(\Gamma))^0$ where $g \in \mathscr{G}_h^0$. Then for every $\gamma \in \Gamma$

$$\rho(\gamma g \gamma^{-1}) = \rho(g) .$$

Taking account that $\gamma g \gamma^{-1} \in \mathscr{G}_h^0$, we obtain $\gamma g \gamma^{-1} = g$ for every $\gamma \in \Gamma$, which shows that $\rho(g) \in \rho(\mathscr{Z}_{\mathscr{G}_h}(\Gamma))$. This implies that

$$\mathscr{Z}_{\rho(\mathscr{G}_h)}(\rho(\Gamma))^0 \subset \rho(\mathscr{Z}_{\mathscr{G}_h}(\Gamma))^0 = \rho(\mathscr{Z}_{\mathscr{G}_h}(\Gamma)^0) .$$

The inverse inclusion is trivial. Let $\hat{\mathscr{G}}$ be the real algebraic group corresponding to the algebraic Lie algebra $\rho(G_h)$ and $\mathscr{Z}(\rho(\Gamma))$ be the centralizer of $\rho(\Gamma)$ in the general linear group. Then we see $\mathscr{X}_{\rho(\mathscr{G}_h)}(\rho(\Gamma))^0=(\mathscr{X}(\rho(\Gamma))\cap\hat{\mathscr{G}})^0$ using the fact $\hat{\mathscr{G}}^0=\rho(\mathscr{G}_h{}^0)$ (cf. Theorem 3.2). Therefore $\mathscr{X}_{\mathscr{G}_h}(\Gamma)^0$ is isomorphic to the identity component of the algebraic group $\mathscr{X}(\rho(\Gamma))\cap\hat{\mathscr{G}}$ by the representation ρ.

PROPOSITION B. *Let $\mathscr{D}$ be a homogeneous bounded domain in C^n. Then $\mathscr{D}$ is simply connected.*

PROOF. We can easily show that $\tilde{\mathscr{G}}=\mathscr{X}_{\mathscr{G}_h}(\Gamma)^0$. Since $\tilde{\mathscr{D}}=\tilde{\mathscr{G}}/\tilde{\mathscr{K}}$ is an open cell and $\tilde{\mathscr{G}}$ is a closed subgroup of $\mathscr{G}_h$, $\tilde{\mathscr{K}}$ is a maximal compact subgroup of $\tilde{\mathscr{G}}$. Then we obtain

$$\tilde{\mathscr{G}}=\tilde{\mathscr{K}}\cdot\mathscr{T}\qquad\text{(semi-direct)}$$

where the group $\mathscr{T}$ is isomorphic to a maximal $\boldsymbol{R}$-triangular subgroup of $\rho(\tilde{\mathscr{G}})$ (=the identity component of a real algebraic group). By the analogous way as in §4, we see that $\mathscr{T}$ is the Iwasawa group of $\tilde{\mathscr{D}}$. Therefore by Theorem 3.1, $\tilde{\mathscr{G}}$ is centreless, which shows $\tilde{\mathscr{G}}\cong\mathscr{G}$. We obtain $\tilde{\mathscr{D}}=\mathscr{G}/\mathscr{K}^0$. And $\mathscr{K}^0$ is a maximal compact subgroup, which implies that $\mathscr{K}=\mathscr{K}^0$ and $\mathscr{D}=\mathscr{G}/\mathscr{K}$ is simply connected.

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ADDED IN PROOF

The analogous results as in our Theorem 3.2 and Theorem 4.2 are found in the introduction of Piatetskij-Šapiro's paper which appeared in Uspehi Math. Nauk 20 (1965).