On the axiomatic method and the algebraic method for dealing with propositional logics

By Tsutomu HOSOI

§ 1. Introduction.

We are mainly interested in the intermediate propositional logics which are between the classical and the intuitionistic. And in this paper, we always mean an intermediate logic simply by logic. We use logical symbols $\supset$, &, $\vee$ and $\neg$.

There are known two methods for dealing with logics. One is the axiomatic method which defines a logic by an axiomatic system, in which the notion of probability distinguishes some formulas as true from others in that system. Another is the algebraic method which defines a logic by giving a set of truth values and a set of designated values, which is a subset of the former set, and the truth tables for the logical symbols, which are functions for the set of truth values. We call as model the system consisting of these truth values, designated values and truth tables. In a model, a mapping f from a set of propositional variables into the set of truth values is called an assignment function or simply an assignment. The truth value $f(A)$ for a formula A can be calculated by using the truth tables if we are given the assignment f. A formula is said to be valid if it always gets a designated value by any assignments. This notion of validity distinguishes some formulas as true from others in that system. Usually a logic is defined by either method, but we often find it convenient to use both methods at the same time even when we are studying a logic defined by one of the methods since these two methods are deeply related to each other. And so, the aim of this paper is devoted to the study of these interconnecting two methods.

Together with other results, our main results of this paper are (1) that models with many designated values have equivalent models with only one designated value on certain natural conditions, and (2) that finite models are axiomatizable.

In this paper, we call the intuitionistic propositional logic as $\boldsymbol{LI}$ and the classical propositional logic as $\boldsymbol{LK}$. Let $\boldsymbol{M}$ and $\boldsymbol{N}$ be two logics defined by the axiomatic method or by the algebraic method. By $\boldsymbol{M}\supset\boldsymbol{N}$, we mean that the set of provable or valid formulas of $\boldsymbol{M}$ includes that of $\boldsymbol{N}$, and by $\boldsymbol{M}\supset\subset\boldsymbol{N}$ we mean that $\boldsymbol{M}\supset\boldsymbol{N}$ and $\boldsymbol{N}\supset\boldsymbol{M}$. By $\boldsymbol{L}+A_1+\cdots+A_k$, we mean an axiomatic system obtained by adding new axiom schemes $A_1, \cdots, A_k$ to an axiomatic system $\boldsymbol{L}$. By $\boldsymbol{L}\vdash A$ or by $\boldsymbol{L}\ni A$, we mean that a formula A is provable or valid in an axiomatic system or a model $\boldsymbol{L}$. By $\bigwedge_{1\leq i\leq n} A_i$ (or $\bigvee_{1\leq i\leq n} A_i$), we mean the formula $A_1\&\cdots\&A_k$ (or $A_1\vee\cdots\vee A_k$). If A and B are formulas, $A\equiv B$ expresses, as usual, the formula $(A\supset B)\&(B\supset A)$.

§ 2. Regular model.

Let $\boldsymbol{M}$ be a model. By $\mathbf{V}_M$, we mean the set of truth values of $\boldsymbol{M}$, by $\mathbf{D}_M$ the set of designated values of $\boldsymbol{M}$, and by $\mathbf{T}_M$ the truth tables of $\boldsymbol{M}$ (we omit the subscript $\boldsymbol{M}$ if there occurs no confusion). If $\mathbf{V}_M$ has n elements, we call $\boldsymbol{M}$ as n-valued. In this paper, we only deal with such k-regular models as defined below.

DEFINITION 2.1. *A model $\boldsymbol{M}$ is k-regular $(1\leq k\leq\infty)$ if it satisfies the following four conditions.*

(1) *$\mathbf{D}$ has k elements.*

(2) *$\boldsymbol{LK}\supset\boldsymbol{M}\supset\boldsymbol{LI}$.*

(3) *If $\mathbf{D}\ni v$ and $\mathbf{D}\ni v\supset w$, then $\mathbf{D}\ni w$.*

(4) *If $\mathbf{D}\ni v$, then $\mathbf{D}\ni w\supset v$ for any value w.*

In our arguments below, the condition (2) can be weakened, that is, $\boldsymbol{M}$ does not need satisfy all the intuitionistic theorems but only some of them. The condition (3) corresponds to modus ponens but it is rather stronger than the latter. The condition (4) might seem unnatural, but it can be deduced from the following quite natural condition (4′).

(4′) *If $\mathbf{D}\ni v$, then there exist an assignment f and a valid formula A of $\boldsymbol{M}$ such that $f(A)=v$.*

Suppose (4′) and let a be a propositional variable not contained in A. The way in which f assigns a value to the propositional variable a is not restricted in any way, so we can take w for the value $f(a)$. Since $\mathbf{D}\ni f(A)$, $\mathbf{D}\ni f(a\supset A)=f(a)\supset f(A)=w\supset v$. Hence the condition (4′) implies the condition (4). On the other hand, the condition (4) does not imply the condition (4′).

DEFINITION 2.2. *A model $\boldsymbol{M}$ is regular if it is 1-regular and it satisfies the following condition.*

(5) *If $\mathbf{D}\ni v\supset w$ and $\mathbf{D}\ni w\supset v$, then $v=w$.*

Examples of regular n-valued models are given in the appendix.

THEOREM 2.3. *For every k-regular model $\boldsymbol{M}$ $(2\leq k\leq\infty)$, there exists a regular model $\boldsymbol{N}$ such that $\boldsymbol{M}\supset\subset\boldsymbol{N}$.*

Before we prove this theorem, we must prepare some lemmas.

DEFINITION 2.4. *The relation* $v_s \sim v_t$ *between the elements of* $\mathbf{V}_M$ *is true if and only if* $\mathbf{D} \ni v_s \supset v_t$ *and* $\mathbf{D} \ni v_t \supset v_s$.

LEMMA 2.5. *The relation* $\sim$ *is an equivalence relation over* $\mathbf{V}_M$.

PROOF. (i) By the definition, $v_s \sim v_t$ implies $v_t \sim v_s$. (ii) Since $f(a \supset a) = f(a) \supset f(a) \in \mathbf{D}$ for any assignment f, $v_s \sim v_s$ holds for any v_s. (iii) Since $\boldsymbol{LI} \vdash (a \supset b) \supset ((b \supset c) \supset (a \supset c))$, $(v_s \supset v_t) \supset ((v_t \supset v_u) \supset (v_s \supset v_u)) \in \mathbf{D}$ and $(v_u \supset v_t) \supset ((v_t \supset v_s) \supset (v_u \supset v_s)) \in \mathbf{D}$. Using the condition (3), $v_s \sim v_u$ is deduced from $v_s \sim v_t$ and $v_t \sim v_u$.

LEMMA 2.6. *Let* $\mathbf{W}$ *be the set of the equivalence classes of* $\mathbf{V}_M$ *by the relation* $\sim$. *If* $\mathbf{W} \ni w_s, w_t$, *and* $w_s \ni v_s, v_{s'}$, *and* $w_t \ni v_t, v_{t'}$, *then there hold relations* $\neg v_s \sim \neg v_{s'}$ *and* $(v_s * v_t) \sim (v_{s'} * v_{t'})$, *where* $*$ *is* $\supset$, &, *or* $\vee$.

PROOF. In $\boldsymbol{LI}$, $\neg a \supset \neg b$ and $(a*c) \supset (b*d)$ are proved from $a \supset b$, $b \supset a$, $c \supset d$ and $d \supset c$. By substituting a, b, c, d, by $v_s, v_{s'}, v_t, v_{t'}$, respectively, and using the condition (3) and the assumptions of the lemma, the conclusion is obtained.

REMARK. By this lemma, we know that we can define operations $\supset$, &, $\vee$ and $\neg$ over $\mathbf{W}$ by the representative elements.

LEMMA 2.7. *There is an element* $w_d \in \mathbf{W}$, *which consists of all and just all elements of* $\mathbf{D}$.

PROOF. By (4), if $\mathbf{D} \ni v_s, v_t$, then $\mathbf{D} \ni v_s \supset v_t$ and $\mathbf{D} \ni v_t \supset v_s$, and so, all the elements of $\mathbf{D}$ are equivalent in the sense of N. We call the class which includes $\mathbf{D}$ as w_d. Let be that $\mathbf{D} \ni v_s$ and $w_d \ni v_t$, then it must be that $\mathbf{D} \ni v_s \supset v_t$. Then by (3), $\mathbf{D} \ni v_t$. So we have that $\mathbf{D} \supset w_d$.

PROOF OF THEOREM 2.3. We define a quotient model $N = \boldsymbol{M}/\boldsymbol{D}$ from $\boldsymbol{M}$ by $\sim$, taking $\mathbf{W}$ as $\mathbf{V}_N$, $\{w_d\}$ as $\mathbf{D}_N$ and logical operations defined over $\mathbf{W}$ as those of N. Let $w(v_s)$ be the element of $\mathbf{W}$ which contains v_s. If $v_s \supset v_t = v_u$ in $\boldsymbol{M}$, then $w(v_s) \supset w(v_t) = w(v_u)$ in N, and by the Lemma 2.6, similar relations hold concerning other logical operations. An assignment f_M of $\boldsymbol{M}$ can be regarded to be an assignment f_N of N by identifying equivalent elements. So $f_M(A) = v_s$ implies $f_N(A) = w(v_s)$. Hence a formula A is valid in $\boldsymbol{M}$ if and only if A is valid in N. Five conditions for regular models are easily seen to hold in N.

REMARK. The cardinal number of $\mathbf{V}_N$ is not greater than that of $\mathbf{V}_M$. If $\mathbf{V}_M$ has only finite elements, the cardinal number of $\mathbf{V}_N$ is necessarily smaller than that of $\mathbf{V}_M$.

For simplicity, we supposed in the above argument that $\boldsymbol{M}$ contains $\boldsymbol{LI}$. But as is seen in our proof, our theorem can be proved if only $\boldsymbol{M}$ contains the implicational fragment of $\boldsymbol{LI}$ even if it lacks in some other logical operations but $\supset$.

§ 3. Axiomatization of finite regular model.

In this § we prove that finitely many-valued regular models are axiomatizable. In the case of k-regular models, if they are reduced to finite regular models, they are, of course, axiomatizable. We reported previously, in [4] and [5], the axiomatization of Gödel's S_n (for its definition, cf. § 4), which are examples of linearly ordered regular models. Our result of this paper is an extension to those previous ones, but we do not suppose any knowledge of them.

Let M be a regular model with n truth values ($2 \leq n < \infty$). We will construct an axiomatic system LM by adding some new axiom schemes to the axiomatic intuitionistic system LI. The new axioms to be added are Z_M and X_n defined below. And we will prove that $M \supset\subset LM$.

We suppose that $\mathbf{V} = \{v_1, \cdots, v_n\}$ and that v_1 is the designated value, which we will write as 1 if there occurs no confusion. Let $\mathbf{P}_k$ be the set of propositional variables $\{a_1, \cdots, a_k\}$. Let f be an assignment function. In this paper, an assignment function f is always considered concerning certain formulas, in other words, concerning a certain set of propositional variables. By $\mathbf{V}(f)$, we mean the set of images by f of elements of the propositional variables with which the assignment f is concerned. Hereafter, in this §, an assignment f is always considered concerning the set of propositional variables $\mathbf{P}_n$ unless it is mentioned otherwise. In other words, f is a mapping from $\mathbf{P}_n$ into $\mathbf{V}$. It is important that the number of such mappings is finite. Let f^{-1} be the inverse mapping of f, which is usually one-to-many mapping. We define a mapping φ_f from $\mathbf{V}(f)$ into $\mathbf{P}_n$ by taking, for each $v_i \in \mathbf{V}(f)$, $\varphi_f(v_i)$ to be the element with the least index of the set $f^{-1}(v_i)$. This mapping φ_f is obviously one-to-one.

We suppose that we have at hand the truth tables $\mathbf{T}_M$ of M for the logical operations $\supset$, &, $\vee$ and $\neg$, each one of which we will call as $\mathbf{T}_i$, $\mathbf{T}_c$, $\mathbf{T}_d$ and $\mathbf{T}_n$, respectively.

Let $\mathbf{W}$ be a subset of $\mathbf{V}$. We call as completion of $\mathbf{V}$ and write as $\mathbf{W}^c$ the smallest subset of $\mathbf{V}$ generated by the four logical operations from the elements of $\mathbf{W}$. Let be that $\mathbf{W} \ni v_i, v_j$ but not that $\mathbf{W} \ni v_i \supset v_j = v_k$, for example. We must add v_k to $\mathbf{W}$ in order to obtain $\mathbf{W}^c$. But we add $v_i \supset v_j$ to $\mathbf{W}$ instead of v_k and regard it as an element of $\mathbf{V}$ added to $\mathbf{W}$ to get $\mathbf{W}^c$. In case that other combinations of $\mathbf{W}$ get the same value v_k, we refrain from adding them all but we add only one of them.

By $\mathbf{T}_i(f)$, we mean the part of $\mathbf{T}_i$ corresponding to $\mathbf{V}(f)^c$. $\mathbf{T}_i(f)$ is obtained by taking from $\mathbf{T}_i$ only those rows and columns for the elements of $\mathbf{V}(f)^c$.

Informally speaking, the truth table $\mathbf{T}_i(f)$ can be expressed by the following informal expression:

$$v_1 \supset v_1 = v_1 \text{ and } v_1 \supset v_2 = v_2 \text{ and } \cdots \text{ and } v_i \supset v_j = v_{ij} \text{ and } \cdots \text{ and } v_n \supset v_n = v_1 ,$$

where v_{ij} is the value determined for $v_i \supset v_j$ by the table, which might be $v_i \supset v_j$ itself in some cases. If $\mathbf{V}(f)^c \not\ni v_1$, we will find something of the form $v_k \supset v_k$ in the place of v_1. If $\mathbf{V}(f)^c \not\ni v_2$, there will not be the term $v_1 \supset v_2 = v_2$, etc.

In the above informal expression for $\mathbf{T}_i(f)$, let us substitute v_i's by the propositional variables expressed by $\varphi_f(v_i)$'s, $=$ by $\equiv$, 'and' by &. Then we obtain a formula, which we will write as $F_i(f)$.

Similarly we define formulas $F_c(f)$, $F_d(f)$ and $F_n(f)$ concerning $\mathbf{T}_c$, $\mathbf{T}_d$ and $\mathbf{T}_n$, respectively. It is easily seen that all of these formulas are satisfied by the assignment f.

DEFINITION 3.1. $E(f) = \bigwedge_{v_i \in \mathbf{V}(f)} \Big(\bigwedge_{a_j \in f^{-1}(v_i)} (\varphi_f(v_i) \equiv a_j) \Big)$.

$F(f) = F_i(f) \& F_c(f) \& F_d(f) \& F_n(f)$.

COROLLARY 3.2. $f(E(f)) = 1$.

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DEFINITION 3.3. $Z_M = \bigvee_f (E(f) \& F(f))$.

$X_n = \bigvee_{1 \le i < j \le n+1} (a_i \equiv a_j)$.

COROLLARY 3.4. *For any assignment* f, $f(Z_M) = 1$ *and* $f(X_n) = 1$.

DEFINITION 3.5. $\boldsymbol{LM} = \boldsymbol{LI} + Z_M + X_n$.

THEOREM 3.6. $\boldsymbol{M} \supset \boldsymbol{LM}$.

PROOF. Let g be any assignment of $\boldsymbol{M}$. Since $\boldsymbol{M}$ is regular, g satisfies the axioms of $\boldsymbol{LI}$. And by 3.4, g also satisfies Z_M and X_n. And again by the regularity of $\boldsymbol{M}$, the inferences of $\boldsymbol{LI}$ are met by g.

THEOREM 3.7. $\boldsymbol{LM} \supset \boldsymbol{M}$.

PROOF. Let A be a valid formula of $\boldsymbol{M}$. Then we will prove that A is provable in $\boldsymbol{LM}$. Without loss of generality, we can suppose that the propositional variables contained in A are $a_1, \cdots, a_m$.

CASE 1. $(m \le n)$. Let g be an assignment function of $\boldsymbol{M}$. By the hypothesis, $g(A) = 1$. Hence $g(A) = g(a_1 \supset a_1)$. Since $m \le n$, we can regard g to be a mapping from $\mathbf{P}_n$ into $\mathbf{V}$. Let $C(B)$ be the formula obtained from a formula $C(A)$ by substituting some arbitrary occurrences of a subformula A in $C(A)$ by B. Then, as is well known, $C(A) \equiv C(B)$ can be proved in $\boldsymbol{LI}$ from $A \equiv B$.

Now we call as pc the process of calculation for $g(A) = g(a_1 \supset a_1)$. Usually pc is constructed by using the full tables $\mathbf{T}_i$, $\mathbf{T}_c$, $\mathbf{T}_d$ and $\mathbf{T}_n$. If a value v_i is not contained in $\mathbf{V}(g)$ but it is used in pc, let us substitute in pc all the occurrences of v_i by the expression added instead of v_i in the construction of $\mathbf{V}(g)^c$. Let us call as pd the process obtained from pc by substituting all the values not contained in $\mathbf{V}(g)$ by the expressions contained in $\mathbf{V}(g)^c$. It is obvious that this pd is also a process of calculation for $g(A)=g(a_1 \supset a_1)$.

Let us consider pd in more detail. The calculation is made by following the truth tables. If we take as the truth tables the informal expressions introduced above, the calculation consists of substitution of some expressions appearing on the left side of one of the equalities of the informal expressions by the right side of the same equality. So the calculation expresses a way of substitutions. Let us take the formula $E(g)\&F(g)$. The formula $E(g)$ consists of conjunctions of formulas of the form $a_i \equiv a_j$. In A, let us substitute a_i's by $\varphi_g(g(a_i))$'s and let us call the formula so obtained as A_g. Then it is obvious that $E(g) \supset (A \equiv A_g)$ is provable in $\boldsymbol{LI}$. In pd, let us substitute v_i's by $\varphi_g(v_i)$'s, and $=$ by $\equiv$. So changed pd expresses a substitution process for A_g, and finally A_g gets the form $\varphi_g(g(a_1)) \supset \varphi_g(g(a_1))$. If we have $F(g)$, the substitution process asserts that $A_g \equiv (\varphi_g(g(a_1)) \supset \varphi_g(g(a_1)))$. Hence we have that $E(g)\&F(g) \supset (A \equiv (a_g \supset a_g))$ is provable in $\boldsymbol{LI}$ for any assignment g of $\mathbf{P}_n$, where a_g is a propositional variable expressed by $\varphi_g(g(a_1))$. So we know that $E(g)\&F(g) \supset (A \equiv (a \supset a))$ is provable in $\boldsymbol{LI}$, since $\boldsymbol{LI} \vdash (a_g \supset a_g) \equiv (a \supset a)$. Hence $(\bigvee_f (E(f)\&F(f))) \supset (A \equiv (a \supset a))$ is provable in $\boldsymbol{LI}$. So $A \equiv (a \supset a)$ is provable in $\boldsymbol{LM}$. Hence A is provable in $\boldsymbol{LM}$.

Case 2. $(m>n)$. Let $\mathbf{G}$ be the set of one-to-one mappings from $\mathbf{P}_{n+1}$ into $\mathbf{P}_m$. It is obvious that $\mathbf{G}$ has only finite elements. By $X_n(g)$, where $\mathbf{G} \ni g$, we mean the formula obtained from X_n by substituting $a_1, \cdots, a_{n+1}$ by $g(a_1), \cdots, g(a_{n+1})$, respectively. We define $X_n{}^*$ to be $\bigwedge_{g \in G} X_n(g)$. This $X_n{}^*$ is of the form $\bigwedge\bigvee(a_i \equiv a_j)$. Using distributivity, $X_n{}^*$ can be expressed in the form $\bigwedge\bigvee(a_i \equiv a_j)$, which we write as $\bigvee_i Y_i$. Each Y_i determines an equivalence relation over the set $\mathbf{P}_m$ by the relation $\equiv$. By $\mathbf{Q}_i$, we mean the set of equivalence classes of $\mathbf{P}_m$ by Y_i. We define the representative elements for the elements of $\mathbf{Q}_i$ to be the elements with the least index. It can be proved that the number of elements of $\mathbf{Q}_i$ is n at the most, since, if there are mutually non-equivalent elements $a_{i_1}, \cdots, a_{i_{n+1}}$ of $\mathbf{P}_m$, it is contrary to $\bigvee_{1 \leq j<k \leq n+1} (a_{i_j} \equiv a_{i_k})$ in $X_n{}^*$.

We define A_i to be a formula obtained from A by substituting non-representative elements by their representatives concerning $\mathbf{Q}_i$. It is obvious that if A is satisfied by any assignments, A_i is also satisfied by any assignments. By this and by the fact that A_i has n distinct propositional variables at the most, A_i is provable in $\boldsymbol{LM}$. In $\boldsymbol{LI}$, we have $(Y_i \supset A) \equiv (Y_i \supset A_i)$. Hence for each i, $Y_i \supset A$ is provable in $\boldsymbol{LM}$. Hence $(\bigvee_i Y_i) \supset A$, and so A, is provable in $\boldsymbol{LM}$.

THEOREM 3.8. $\boldsymbol{M} \supset\subset \boldsymbol{LM}$.

Thus we proved the existence of axiomatic system $\boldsymbol{LM}$ for a regular finite model $\boldsymbol{M}$. But we do not claim that our axiomatization is best nor that it is simple. On the contrary, we will be able to simplify X_n and Z_M easily if we are given a model $\boldsymbol{M}$ in detail.

§ 4. Operations on regular models.

First, we introduce two operations $(\boldsymbol{M}_1, \cdots, \boldsymbol{M}_k)$ and $\Gamma(\boldsymbol{M})$ on regular models, which are those defined by Jaśkowski [6] and G. F. Rose [8] only with slight notational modifications. We have not, in this paper, a particular interest in those operations themselves. We introduce them only because we like to use them in § 5.

DEFINITION 4.1. *For $k \geqq 2$, and for regular models $\boldsymbol{M}_i$ $(1 \leq i \leq k)$, $(M_1, \cdots, M_k)$ is a model with truth values $(v_1, \cdots, v_k)$, where $v_i \in \mathbf{V}_{M_i}$ $(1 \leq i \leq k)$, and with a designated value $(d_1, \cdots, d_k)$, where $d_i \in \mathbf{D}_{M_i}$ $(1 \leq i \leq k)$, and with the operations defined by*

$$(v_1, \cdots, v_k)*(w_1, \cdots, w_k) = (v_1*w_1, \cdots, v_k*w_k)\ ,$$

$$\neg(v_1, \cdots, v_k) = (\neg v_1, \cdots, \neg v_k)\ ,$$

where $*$ is $\supset$, & or $\vee$.

If each M_i is equal to $\boldsymbol{M}$, we write $(\boldsymbol{M}_1, \cdots, \boldsymbol{M}_k)$ as $\boldsymbol{M}^k$.

DEFINITION 4.2. *Let $\boldsymbol{M}$ be a regular model whose designated value is d and whose set of non-designated values is $\mathbf{W}$. $\Gamma(\boldsymbol{M})$ is a model whose designated value is d and whose set of non-designated values is $\mathbf{W} \cup \{e\}$, where e is a letter different from d and not contained in $\mathbf{W}$. For any $v \in \mathbf{V}_M$, let $\alpha(v)$ be v or e according as $v \in \mathbf{W}$ or $v = d$. Logical operations of $\Gamma(\boldsymbol{M})$ are defined by those of $\boldsymbol{M}$ as follows:*

$x \supset y$	$y=d$	$y=\alpha(v)$
$x=d$	d	y
$x=\alpha(u)$	d	$u \supset v$

$x \,\&\, y$	$y=d$	$y=\alpha(u)$
$x=d$	d	y
$x=\alpha(u)$	x	$\alpha(u \,\&\, v)$

$x \vee y$	$y=d$	$y=\alpha(v)$
$x=d$	d	d
$x=\alpha(u)$	d	$\alpha(u \vee v)$

$\neg x$	
$x=d$	$\neg d$
$x=\alpha(u)$	$\neg u$.

The k-fold iteration of the operation Γ will be written as Γ^k.

LEMMA 4.3. *If $\boldsymbol{M}, \boldsymbol{M}_1, \cdots, \boldsymbol{M}_k$ are regular, then $\Gamma(\boldsymbol{M})$ and $(\boldsymbol{M}_1, \cdots, \boldsymbol{M}_k)$ are also regular.*

LEMMA 4.4. *If $M \vdash A$, then for any assignment f of $\Gamma(M)$, $f(A)$ gets the value d or e. And $M \supset \Gamma(M)$.*

PROOF. Let N be a model obtained from $\Gamma(M)$ by taking the set $\{d, e\}$ as $\mathbf{D}_N$. Then it is easily seen that $N \supset\subset M$. Hence, for any f, $f(A) \in \mathbf{D}_N$. And then $M \supset \Gamma(M)$ is immediate.

LEMMA 4.5. *The set of valid formulas of $(M_1, \cdots, M_k)$ is the intersection of those of $M_1, \cdots, M_k$.*

This is also obvious.

DEFINITION 4.6. *A model M is linear if it satisfies the following conditions.*

(1) *M is regular.*

(2) *$\mathbf{V}_M$ is linearly ordered by a relation $\geq$, and it has the least element 1 and the greatest element ω.*

(3) *For any $v_s, v_t \in \mathbf{V}_M$,*

$$v_s \supset v_t = \begin{cases} 1 & \text{if } v_s \geq v_t, \\ v_t & \text{otherwise,} \end{cases}$$

$$v_s \,\&\, v_t = \max(v_s, v_t),$$

$$v_s \vee v_t = \min(v_s, v_t),$$

$$\neg v_s = v_s \supset \omega.$$

DEFINITION 4.7. *S_n is an $n+1$-valued model which satisfies the following conditions.* ($n \geq 1$).

(1) *S_n is linear.*

(2) *The set $\mathbf{V}$ of truth values is $\{1, \cdots, n, \omega\}$, and the relation $\geq$ is the usual relation on integers except that ω is regarded to be greater than any positive integers.*

DEFINITION 4.8. *S_ω is an extension of S_n, where the set of truth values consists of all the positive integers and ω.*

REMARK. S_n and S_ω are the same as that of Gödel [3] and LC of Dummett [2], respectively.

LEMMA 4.9. (1) $LK \supset\subset S_1 \supset S_2 \supset \cdots \supset S_n \supset \cdots \supset S_\omega \supset LI.$

(2) $S_n \supset\subset \Gamma^{n-1}(S_1)$.

These are almost obvious.

DEFINITION 4.10. *Let M and N be regular models. Let d_M and d_N be the designated values of M and N, respectively. Let be that $\mathbf{V}_M \cap \mathbf{V}_N = \phi$. Let $\mathbf{V}'_N$ be the set of undesignated values of N. By $\supset_M (\supset_N)$, etc., we mean those logical operations of $M(N)$. For any $v \in \mathbf{V}_N$, let $\beta(v)$ be v or d_M according as $v \in \mathbf{V}'_N$ or $v = d_N$, and let $\gamma(v)$ be v or $\neg_M d_M$ according as $v \in \mathbf{V}'_N$, or $v = d_N$. Then $M \uparrow N$ is a model whose set of values is $\mathbf{V}_M \cup \mathbf{V}'_N$ and whose designated value is d_M*

and whose logical operations are defined as follows:

$x \supset y$	$y \in \mathbf{V}_M$	$y \in \mathbf{V}'_N$
$x \in \mathbf{V}_M$	$x \supset_M y$	y
$x \in \mathbf{V}'_N$	d_M	$\beta(x \supset_N y)$

$x \& y$	$y \in \mathbf{V}_M$	$y \in \mathbf{V}'_N$
$x \in \mathbf{V}_M$	$x \&_M y$	y
$x \in \mathbf{V}'_N$	x	$x \&_N y$

$x \vee y$	$y \in \mathbf{V}_M$	$y \in \mathbf{V}_{N'}$
$x \in \mathbf{V}_M$	$x \vee_M y$	x
$x \in \mathbf{V}'_N$	y	$\gamma(x \vee_N y)$

$\neg x$	
$x \in \mathbf{V}_M$	$\neg_M d_M$
$x \in \mathbf{V}'_N$	$\neg_N x$.

REMARK. $\boldsymbol{M} \uparrow \boldsymbol{N}$ is a model obtained by identifying the greatest value of $\boldsymbol{M}$ and the least value of $\boldsymbol{N}$ and by regarding any other values of $\boldsymbol{N}$ to be greater than any values of $\boldsymbol{M}$.

COROLLARY 4.11. *$\boldsymbol{M} \uparrow \boldsymbol{N}$ is regular.*

COROLLARY 4.12. *$\Gamma(\boldsymbol{M}) \supset\subset \boldsymbol{S}_1 \uparrow \boldsymbol{M}$, and $\boldsymbol{S}_m \uparrow \boldsymbol{S}_n \supset\subset \boldsymbol{S}_{m+n}$.*

COROLLARY 4.13. *$(\boldsymbol{M}_1 \uparrow \boldsymbol{M}_2) \uparrow \boldsymbol{M}_3 = \boldsymbol{M}_1 \uparrow (\boldsymbol{M}_2 \uparrow \boldsymbol{M}_3)$.*

Hence we can use an expression of the form $\boldsymbol{M}_1 \uparrow \boldsymbol{M}_2 \uparrow \boldsymbol{M}_3$, etc.

THEOREM 4.14. *Let $\boldsymbol{M}_1, \cdots, \boldsymbol{M}_k$ and $\boldsymbol{P}$ be regular models. Then,*

$$(\boldsymbol{M}_1, \cdots, \boldsymbol{M}_k) \uparrow \boldsymbol{P} \supset\subset (\boldsymbol{M}_1 \uparrow \boldsymbol{P}, \cdots, \boldsymbol{M}_k \uparrow \boldsymbol{P}) .$$

PROOF. It will be sufficient if we prove the theorem for the case of $k=2$. We put $\boldsymbol{Q}=(\boldsymbol{M}, \boldsymbol{N}) \uparrow \boldsymbol{P}$ and $\boldsymbol{R}=(\boldsymbol{M} \uparrow \boldsymbol{P}, \boldsymbol{N} \uparrow \boldsymbol{P})$, and we will prove that $\boldsymbol{Q} \supset\subset \boldsymbol{R}$.

We suppose that the sets of truth values of $\boldsymbol{M}$, $\boldsymbol{N}$ and $\boldsymbol{P}$ have been defined by using disjoint sets of letters. Let $\mathbf{V}'_P$ be the set of undesignated values of $\boldsymbol{P}$. We define four sets of values as follows:

$$\mathbf{V}_1=\{(x, y)\,;\ x \in \mathbf{V}_M, y \in \mathbf{V}_N\} ,$$

$$\mathbf{V}_2=\{(z, z)\,;\ z \in \mathbf{V}'_P\} ,$$

$$\mathbf{V}_3=\{(x, y)\,;\ x \in \mathbf{V}_M \cup \mathbf{V}'_P,\ y \in \mathbf{V}_N \cup \mathbf{V}'_P\} ,$$

$$\mathbf{V}_4=\{(x, y)\,;\ x \in \mathbf{V}_M \text{ and } y \in \mathbf{D}_N, \text{ or } x=y \in \mathbf{V}'_P\} .$$

We can regard $\mathbf{V}_3$ as the eet of truth values for $\boldsymbol{R}$. Since we have that $\mathbf{V}_3 \supset (\mathbf{V}_1 \cup \mathbf{V}_2) \supset \mathbf{V}_4$, and both $\mathbf{V}_1 \cup \mathbf{V}_2$ and $\mathbf{V}_4$ are complete in the sense of § 3 concerning the logical operations defined for $\mathbf{V}_3$ regarded as the set of truth values of $\boldsymbol{R}$ and since $\mathbf{V}_1 \cup \mathbf{V}_2$ and $\mathbf{V}_4$ can be regarded, with the logical operations for $\mathbf{V}_3$, as the sets of truth values for $\boldsymbol{Q}$ and $\boldsymbol{M} \uparrow \boldsymbol{P}$, respectively, so we can easily prove that $\boldsymbol{M} \uparrow \boldsymbol{P} \supset \boldsymbol{Q} \supset \boldsymbol{R}$. Similarly we can prove that $\boldsymbol{N} \uparrow \boldsymbol{P} \supset \boldsymbol{Q}$. Since the set of valid formulas of $\boldsymbol{R}$ is the intersection of those of $\boldsymbol{M} \uparrow \boldsymbol{P}$ and $\boldsymbol{N} \uparrow \boldsymbol{P}$, there holds the relation $\boldsymbol{R} \supset \boldsymbol{Q} \supset \boldsymbol{R}$. Hence $\boldsymbol{Q} \supset\subset \boldsymbol{R}$.

REMARK. The similar relation that $\boldsymbol{P} \uparrow (\boldsymbol{M}, \boldsymbol{N}) \supset\subset (\boldsymbol{P} \uparrow \boldsymbol{M}, \boldsymbol{P} \uparrow \boldsymbol{N})$ does not hold in general. A counter example is the case when $P=M=N=S_1$.

LEMMA 4.15. *Let M be a linear model. Let A be a formula, which contains just n distinct propositional variables and let n be smaller than the cardinal number of V_M plus 2. Then $M \vdash A$ if and only if $S_{n+1} \vdash A$.*

This is almost obvious.

§ 5. Sequence of logics and its limit.

There are known some sequences of logics defined by the axiomatic method or by models. And for studying these sequences, we also find it convenient to deal with them in both ways. As an example, we will deal with an axiomatic sequence defined by Nagata [7].

DEFINITION 5.1. (Nagata's sequence.)

$$P_1 = ((a_1 \supset a_0) \supset a_1) \supset a_1 .$$

$$P_{i+1} = ((a_{i+1} \supset P_i) \supset a_{i+1}) \supset a_{i+1} , \quad (i \geqq 1) .$$

$$\mathbf{LP}_n = \mathbf{LI} + P_n .$$

COROLLARY 5.2. $\mathbf{LK} \supsetneqq \mathbf{LP}_1 \supsetneqq \mathbf{LP}_2 \supsetneqq \cdots \supsetneqq \mathbf{LP}_n \supsetneqq \cdots \supsetneqq \mathbf{LI}$.

To prove that $\mathbf{LP}_n \supsetneqq \mathbf{LP}_{n+1}$, Nagata used the following lemma concerning the sequence of models $\{S_n\}$.

LEMMA 5.3. $S_n \ni P_n$, *but* $S_{n+1} \not\ni P_n$.

In studying a sequence of logics, the notion of the limit of a sequence often comes to our mind. But it is difficult to define that notion since we have not the notion of distance between two logics. But in certain cases, it can be defined. For example, if a sequence $\{\mathbf{L}_n\}$ is monotoneously decreasing in the sence that for any n, $\mathbf{L}_n \supsetneqq \mathbf{L}_{n+1}$, it can be defined as follows.

DEFINITION 5.4. *Let $\{\mathbf{L}_n\}$ be a monotoneously decreasing sequence of logics and let $\mathbf{L}$ be a logic. $\mathbf{L} = \lim_{n\to\infty} \mathbf{L}_n$ if and only if the following two conditions hold.*

(1) *If $\mathbf{L} \ni A$, then for any i, $\mathbf{L}_i \ni A$.*

(2) *If $\mathbf{L} \not\ni A$, then there exists i such that $\mathbf{L}_i \not\ni A$.*

In this sence, it is obvious that $\lim_{n\to\infty} S_n = S_\omega$. But $\lim_{n\to\infty} \mathbf{LP}_n = S_\omega$ can be denied as follows.

LEMMA 5.5. (1) $S_n \supsetneqq \mathbf{LI} + P_n + (a \supset b) \vee (b \supset a)$.

(2) $S_\omega \supsetneqq \mathbf{LI} + (a \supset b) \vee (b \supset a)$, (cf. *Dummett* [2]).

(3) $\mathbf{LP}_n \not\ni (a \supset b) \vee (b \supset a)$, $(n \geq 2)$.

For the proof, cf. Hosoi [5].

COROLLARY 5.6. $\lim_{n\to\infty} (\boldsymbol{LP}_n + (a \supset b) \vee (b \supset a)) = \boldsymbol{S}_\omega$.

In reality, we have the following theorem.

THEOREM 5.7. $\lim_{n\to\infty} \boldsymbol{LP}_n = \boldsymbol{LI}$.

To prove this theorem, we borrow the following lemma 5.9 from Jaśkowski [6] and G. F. Rose [8].

DEFINITION 5.8. $\boldsymbol{J}_1 = \boldsymbol{S}_1$.

$\boldsymbol{J}_{i+1} = \Gamma(\boldsymbol{J}^i_i)$, $\qquad (i \geq 1)$.

LEMMA 5.9. $\lim_{n\to\infty} \boldsymbol{J}_n = \boldsymbol{LI}$.

LEMMA 5.10. *For any* n, $\boldsymbol{J}_n \ni P_n$.

PROOF. If $n=1$, $\boldsymbol{J}_n \ni P_n$ holds obviously. Next we suppose that $\boldsymbol{J}_m \ni P_m$. Since $\boldsymbol{J}_m \supset\subset \boldsymbol{J}^m_m$, $\boldsymbol{J}^m_m \ni P_m$. Let 1 be the designated value of $\boldsymbol{J}^m_m$ and let e be the value added in constructing $\Gamma(\boldsymbol{J}^m_m)$. Let f be any assignment of $\boldsymbol{J}_{m+1}$. By the hypothesis and by 4.4, $f(P_m)=1$ or e. If $f(a_{m+1})=1$ or $f(P_m)=1$, then $f(P_{m+1})=1$. If not so, $f(a_{m+1}) \supset e = 1$. So anyway $f(P_{m+1})=1$.

PROOF OF THEOREM 5.7. By the definition of $\boldsymbol{LP}_n$, $\boldsymbol{LI} \ni A$ implies $\boldsymbol{LP}_n \ni A$. If $\boldsymbol{LI} \not\ni A$, then there exists i such that $\boldsymbol{J}_i \not\ni A$. Since $\boldsymbol{J}_i \supset \boldsymbol{LP}_i$ by the lemma 5.10, $\boldsymbol{J}_i \not\ni A$ implies $\boldsymbol{LP}_i \not\ni A$. q. e. d.

Suppose that $\lim_{n\to\infty} \boldsymbol{L}_n = \boldsymbol{M}$ and that $\lim_{n\to\infty} \boldsymbol{L}_n = \boldsymbol{N}$. Then it follows that $\boldsymbol{M} \supset\subset \boldsymbol{N}$. But it does not necessarily follow that $\boldsymbol{M}$ and $\boldsymbol{N}$ are the same model. This can be easily seen.

DEFINITION 5.11. *Two models* $\boldsymbol{M}$ *and* $\boldsymbol{N}$ *are isomorphic if and only if there is a one-to-one correspondence between* V_M *and* V_N *which preserves the logical operations.*

COROLLARY 5.12. *If* $\boldsymbol{M}$ *and* $\boldsymbol{N}$ *are isomorphic then* $\boldsymbol{M} \supset\subset \boldsymbol{N}$.

The converse does not hold in general.

LEMMA 5.13. *Let* $\boldsymbol{M}$ *be a linear model. Let* $\mathfrak{m}$ *be the cardinal number of* V_M. *Then,*

(1) *if* $\mathfrak{m} < \aleph_0$, *then* $\boldsymbol{M} \supset\subset \boldsymbol{S}_{\mathfrak{m}-1}$,

(2) *if* $\mathfrak{m} \geqq \aleph_0$, *then* $\boldsymbol{M} \supset\subset \boldsymbol{S}_\omega$,

PROOF. The case (1) is obvious since $\boldsymbol{M}$ and $\boldsymbol{S}_{\mathfrak{m}-1}$ are isomorphic by the order-preserving correspondence. And by 4.15, $\lim_{n\to\infty} \boldsymbol{S}_n = \boldsymbol{M}$. Hence the case (2) is proved.

§ 6. Degree of complexity of logics,

We can define a notion which will describe, in a certain sence, the degree of complexity $c(\boldsymbol{L})$ of a logic $\boldsymbol{L}$, in terms of X_n, since X_n has the following property.

LEMMA 6.1. *If $m \leq n$, then X_n can be deduced from X_m in* $\boldsymbol{LI}$.

The proof is immediate, since $\boldsymbol{LI} \vdash X_m \supset X_n$.

DEFINITION 6.2. *If there exists an integer n such that $\boldsymbol{L} \ni X_n$, then $c(\boldsymbol{L})$ is the least n for which $\boldsymbol{L} \ni X_n$. Otherwise, $c(\boldsymbol{L}) = \infty$.*

COROLLARY 6.3. $c(\boldsymbol{LK}) = 2$,
$c(\boldsymbol{LI}) = \infty$,
$c(\boldsymbol{S}_n) = n+1$,
$c(\boldsymbol{S}_\omega) = \infty$.

COROLLARY 6.4. *If $\boldsymbol{M}$ is an n-valued model, then $c(\boldsymbol{M}) \leq n$.*

REMARK. Even if $c(\boldsymbol{M}) = n$, the model $\boldsymbol{M}$ is not necessarily n-valued. It can even be infinitely many-valued.

LEMMA 6.5. *If $\boldsymbol{M} \supset \boldsymbol{N}$, then $c(\boldsymbol{M}) \leq c(\boldsymbol{N})$.*

PROOF. *If* $c(\boldsymbol{N}) = n$, then $\boldsymbol{M} \supset \boldsymbol{N} \ni X_n$. Hence, $c(\boldsymbol{M}) \leq n = c(\boldsymbol{N})$. If $c(\boldsymbol{N}) = \infty$, then trivially $c(\boldsymbol{M}) \leq c(\boldsymbol{N})$.

LEMMA 6.6. *Let $\boldsymbol{M}$ be an n-valued regular model ($n \geq 2$). Then, $\boldsymbol{M} = \boldsymbol{S}_1$ or $\boldsymbol{M} = \boldsymbol{S}_1 \uparrow \boldsymbol{N}$ for certain $\boldsymbol{N}$ if and only if $c(\boldsymbol{M}) = n$.*

PROOF. Since $\boldsymbol{LK}$ is the one and only 2-valued logic, the case $n=2$ is trivial. $\boldsymbol{M} = \boldsymbol{S}_1 \uparrow \boldsymbol{N}$ means that for any undesignated values v and w, $v \vee w \neq 1$, where 1 is the designated value. By the regularity, if $v \neq w$, then $(v \equiv w) \neq 1$. Suppose $\boldsymbol{M} = \boldsymbol{S}_1 \uparrow \boldsymbol{N}$. And let f be an assignment of $\boldsymbol{M}$ such that $f(a_i) \neq f(a_j)$ if $i \neq j$ ($1 \leq i$, $j \leq n$). Then $f(a_i \equiv a_j) \neq 1$ if $i \neq j$. Hence $f(X_{n-1}) \neq 1$. Hence $n-1 < c(\boldsymbol{M})$. On the other hand, $c(\boldsymbol{M}) \leq n$, by 6.4. Hence $c(\boldsymbol{M}) = n$. If $\boldsymbol{M} \neq \boldsymbol{S}_1 \uparrow \boldsymbol{N}$, there exist undesignated values u and v such that $u \vee v = 1$. Let be that $w = u \& v$. Then $u \supset w = v$, $v \supset w = u$, $w \supset u = 1$ and $w \supset v = 1$ (cf. 7.2.). So $(u \equiv w) \vee (v \equiv w) = v \vee u = 1$. Hence by any assignment f, $f(X_{n-1}) = 1$. Hence $c(M) \leq n-1$.

§7. Some comments on Appendix.

In the appendix we show examples of regular n-valued models for $2 \leq n \leq 9$. Those tables were hunted up exhaustively by using an old, slow and small-sized electronic computer TOSBAC 3300 installed in the Department of Mathematics, University of Tokyo. This work was scheduled as a back ground job for such periods as when the computer did not have particular jobs to do, and it took about four months, from September of 1966 to the end of that year, to obtain the tables shown below.

The search had its basis on the following lemma.

LEMMA 7.1. *A regular model coincides with a relatively pseudo-complemented lattice with a smallest element if we define $v \geq w$ by the relation $\mathbf{D} \ni v \supset w$.*

Since this lemma is well known and the proof of it is not important in this paper, we will not give the proof (cf. Birkhoff [1]).

In our program, a partially ordered set of n elements was generated first. And then, it was tried to define the truth tables for them by the equalities as follows:

DEFINITION 7.2. $x \& y = \min\{z ; z \geq x, z \geq y\}$,
$x \vee y = \max\{z ; x \geq z, y \geq z\}$,
$x \supset y = \min\{z ; z \& x \geq y\}$.

(If $\supset$ has been defined, $\neg x$ can be defined as $x \supset \omega$, where ω is the greatest element.)

During and after the process to define the truth tables, the conditions for regular models were tested. After one partially ordered set was tested if it was a regular model, another such set was generated, and so on.

Lemmas as follows were useful in making the truth tables and in testing the regularity.

LEMMA 7.3. (1) *For any* v, $\neg 1 \geq v \geq 1$.
(2) $v \supset v = 1$.
(3) $1 \supset v = v$.
(4) $v \supset (v \supset w) = v \supset w$.

LEMMA 7.4. *If* $v \neq w$, *then* $v \supset w \neq w \supset v$.

PROOF. Suppose that $v \supset w = w \supset v = x$. Then $v \supset x = v \supset (w \supset v) = 1$, on the other hand, $v \supset x = v \supset (v \supset w) = v \supset w = x$. Hence $x = 1$. This is contrary to the regularity.

LEMMA 7.5. *If* $v \neq \neg 1$, *then, for any* w, $w \supset v \neq \neg 1$.

PROOF. If $w \supset v = \neg 1$, then $1 = v \supset (w \supset v) = v \supset \neg 1$. Hence $v = \neg 1$. This is contrary to the hypothesis.

The results obtained are shown in the appendix, with names of the form $L_{i,j}$, where i shows the number of truth values and j is the index for the i-valued models though it is not so systematic.

Simple cases, such as S_n, are shown only by names and their tables are omitted. The designated element is expressed by 1 and ω expresses the greatest element. Tables for the logical operator $\neg$ are omitted since the value $\neg x$ can been obtained as $x \supset \omega$. The figures show the structure of models as lattice. Isomorphic models are omitted. By $c=3$, etc., we show the degree of complexity of models.

OBSERVATION 7.6. *Three operations for models defined in* § *4 are not enough to describe the models by* S_1 *and those operations, since* $L_{8,14}$, *for example, cannot be described.* If $L_{8,14}$ should be expressed as $(M_1, \cdots, M_k)$, it must be either S_1^3,

(S_1, S_3) or (S_3, S_1) since $L_{8,14}$ has 2^3 elements. But they are all distinct from $L_{8,14}$. And $L_{8,14}$ is not obviously of the form $M \uparrow N$.

DEFINITION 7.7. *A logical operation $a*b$ (* is $\supset$, & or $\vee$) or $\neg a$ is independent from others in a logic L, if there is not a formula A such that $L \vdash (a*b) \equiv A$ or $L \vdash \neg a \equiv A$ and in which that operation is not contained.*

OBSERVATION 7.8. *There is a finitely many valued model in which all the logical operations are independent from others.* $L_{6,3}$ is one of those models. For example, $3 \supset 5$ cannot be expressed by 3, 5, &, $\vee$ and $\neg$. Also $3 \& 4$, $3 \vee 4$ and $\neg 1$ cannot be expressed by others.

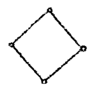
Appendix

$L_{2,1} = S_1$. $(c=2)$

$L_{3,1} = S_2$. $(c=3)$

$L_{4,1} = S_3$. $(c=4)$

$L_{4,2} = S_1^2$. $(c=2)$



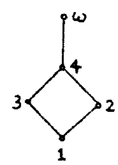
$L_{5,1} = S_4$. $(c=5)$

$L_{5,2} = S_1^2 \uparrow S_1 \supset \subset S_2$. $(c=3)$

$\supset$	ω	1	2	3	4
ω	1	1	1	1	1
1	ω	1	2	3	4
2	ω	1	1	3	3
3	ω	1	2	1	2
4	ω	1	1	1	1

&	ω	1	2	3	4
ω	ω	ω	ω	ω	ω
1	ω	1	2	3	4
2	ω	2	2	4	4
3	ω	3	4	3	4
4	ω	4	4	4	4

$\vee$	ω	1	2	3	4
ω	ω	1	2	3	4
1	1	1	1	1	1
2	2	1	2	1	2
3	3	1	1	3	3
4	4	1	2	3	4

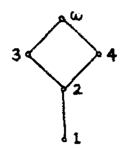


$L_{5,3} = S_1 \uparrow S_1^2$. $(c=5)$

$\supset$	ω	1	2	3	4
ω	1	1	1	1	1
1	ω	1	2	3	4
2	ω	1	1	3	4
3	4	1	1	1	4
4	3	1	1	3	1

&	ω	1	2	3	4
ω	ω	ω	ω	ω	ω
1	ω	1	2	3	4
2	ω	2	2	3	4
3	ω	3	3	3	ω
4	ω	4	4	ω	4

$\vee$	ω	1	2	3	4
ω	ω	1	2	3	4
1	1	1	1	1	1
2	2	1	2	2	2
3	3	1	2	3	2
4	4	1	2	2	4



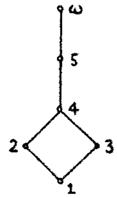
$L_{6,1} = S_5$. $(c=6)$

$L_{6,2} = S_1^2 \uparrow S_2 \supset \subset S_3$. $(c=4)$

$\supset$	ω	1	2	3	4	5
ω	1	1	1	1	1	1
1	ω	1	2	3	4	5
2	ω	1	1	3	3	5
3	ω	1	2	1	2	5
4	ω	1	1	1	1	5
5	ω	1	1	1	1	1

&	ω	1	2	3	4	5
ω	ω	ω	ω	ω	ω	ω
1	ω	1	2	3	4	5
2	ω	2	2	4	4	5
3	ω	3	4	3	4	5
4	ω	4	4	4	4	5
5	ω	5	5	5	5	5

$\vee$	ω	1	2	3	4	5
ω	ω	1	2	3	4	5
1	1	1	1	1	1	1
2	2	1	2	1	2	2
3	3	1	1	3	3	3
4	4	1	2	3	4	4
5	5	1	2	3	4	5

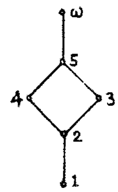


$L_{6,3} = S_1 \uparrow S_1^2 \uparrow S_1$. $(c=6)$

$\supset$	ω	1	2	3	4	5
ω	1	1	1	1	1	1
1	ω	1	2	3	4	5
2	ω	1	1	3	4	5
3	ω	1	1	1	4	4
4	ω	1	1	3	1	3
5	ω	1	1	1	1	1

&	ω	1	2	3	4	5
ω	ω	ω	ω	ω	ω	ω
1	ω	1	2	3	4	5
2	ω	2	2	3	4	5
3	ω	3	3	3	5	5
4	ω	4	4	5	4	5
5	ω	5	5	5	5	5

$\vee$	ω	1	2	3	4	5
ω	ω	1	2	3	4	5
1	1	1	1	1	1	1
2	2	1	2	2	2	2
3	3	1	2	3	2	3
4	4	1	2	2	4	4
5	5	1	2	3	4	5

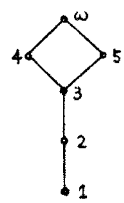


$L_{6,4} = S_2 \uparrow S_1^2$. $(c=6)$

$\supset$	ω	1	2	3	4	5
ω	1	1	1	1	1	1
1	ω	1	2	3	4	5
2	ω	1	1	3	4	5
3	ω	1	1	1	4	5
4	5	1	1	1	1	5
5	4	1	1	1	4	1

&	ω	1	2	3	4	5
ω	ω	ω	ω	ω	ω	ω
1	ω	1	2	3	4	5
2	ω	2	2	3	4	5
3	ω	3	3	3	4	5
4	ω	4	4	4	4	ω
5	ω	5	5	5	ω	5

$\vee$	ω	1	2	3	4	5
ω	ω	1	2	3	4	5
1	1	1	1	1	1	1
2	2	1	2	2	2	2
3	3	1	2	3	3	3
4	4	1	2	3	4	3
5	5	1	2	3	3	5

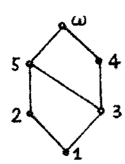


$L_{6,5} = (S_1,\ S_2) \supset \subset S_2$. $(c=3)$

$\supset$	ω	1	2	3	4	5
ω	1	1	1	1	1	1
1	ω	1	2	3	4	5
2	4	1	1	3	4	3
3	ω	1	2	1	4	2
4	2	1	2	1	1	2
5	4	1	1	1	4	1

&	ω	1	2	3	4	5
ω	ω	ω	ω	ω	ω	ω
1	ω	1	2	3	4	5
2	ω	2	2	5	ω	5
3	ω	3	5	3	4	5
4	ω	4	ω	4	4	ω
5	ω	5	5	5	ω	5

$\vee$	ω	1	2	3	4	5
ω	ω	1	2	3	4	5
1	1	1	1	1	1	1
2	2	1	2	1	1	2
3	3	1	1	3	3	3
4	4	1	1	3	4	3
5	5	1	2	3	3	5



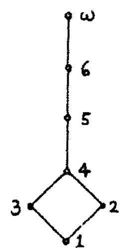
$L_{7.1} = S_6$. $(c=7)$

$L_{7.2} = S_1^2 \uparrow S_2 \supset \subset S_4$. $(c=5)$

$\supset$	ω	1	2	3	4	5	6
ω	1	1	1	1	1	1	1
1	ω	1	2	3	4	5	6
2	ω	1	1	3	3	5	6
3	ω	1	2	1	2	5	6
4	ω	1	1	1	1	5	6
5	ω	1	1	1	1	1	6
6	ω	1	1	1	1	1	1

&	ω	1	2	3	4	5	6
ω	ω	ω	ω	ω	ω	ω	ω
1	ω	1	2	3	4	5	6
2	ω	2	2	4	4	5	6
3	ω	3	4	3	4	5	6
4	ω	4	4	4	4	5	6
5	ω	5	5	5	5	5	6
6	ω	6	6	6	6	6	6

$\vee$	ω	1	2	3	4	5	6
ω	ω	1	2	3	4	5	6
1	1	1	1	1	1	1	1
2	2	1	2	1	2	2	2
3	3	1	1	3	3	3	3
4	4	1	2	3	4	4	4
5	5	1	2	3	4	5	5
6	6	1	2	3	4	5	6

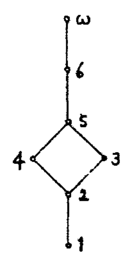


$L_{7.3} = S_1 \uparrow S_1^2 \uparrow S_2$. $(c=7)$

$\supset$	ω	1	2	3	4	5	6
ω	1	1	1	1	1	1	1
1	ω	1	2	3	4	5	6
2	ω	1	1	3	4	5	6
3	ω	1	1	1	4	4	6
4	ω	1	1	3	1	3	6
5	ω	1	1	1	1	1	6
6	ω	1	1	1	1	1	1

&	ω	1	2	3	4	5	6
ω	ω	ω	ω	ω	ω	ω	ω
1	ω	1	2	3	4	5	6
2	ω	2	2	3	4	5	6
3	ω	3	3	3	5	5	6
4	ω	4	4	5	4	5	6
5	ω	5	5	5	5	5	6
6	ω	6	6	6	6	6	6

$\vee$	ω	1	2	3	4	5	6
ω	ω	1	2	3	4	5	6
1	1	1	1	1	1	1	1
2	2	1	2	2	2	2	2
3	3	1	2	3	2	3	3
4	4	1	2	2	4	4	4
5	5	1	2	3	4	5	5
6	6	1	2	3	4	5	6

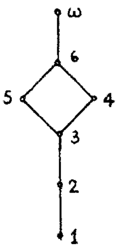


$L_{7,4} = S_2 \uparrow S_1^2 \uparrow S_1$. (c=7)

⊃	ω	1	2	3	4	5	6
ω	1	1	1	1	1	1	1
1	ω	1	2	3	4	5	6
2	ω	1	1	3	4	5	6
3	ω	1	1	1	4	5	6
4	ω	1	1	1	1	5	5
5	ω	1	1	1	4	1	4
6	ω	1	1	1	1	1	1

&	ω	1	2	3	4	5	6
ω	ω	ω	ω	ω	ω	ω	ω
1	ω	1	2	3	4	5	6
2	ω	2	2	3	4	5	6
3	ω	3	3	3	4	5	6
4	ω	4	4	4	4	6	6
5	ω	5	5	5	6	5	6
6	ω	6	6	6	6	6	6

∨	ω	1	2	3	4	5	6
ω	ω	1	2	3	4	5	6
1	1	1	1	1	1	1	1
2	2	1	2	2	2	2	2
3	3	1	2	3	3	3	3
4	4	1	2	3	4	3	4
5	5	1	2	3	3	5	5
6	6	1	2	3	4	5	6

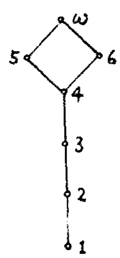


$L_{7,5} = S_3 \uparrow S_1^2$. (c=7)

⊃	ω	1	2	3	4	5	6
ω	1	1	1	1	1	1	1
1	ω	1	2	3	4	5	6
2	ω	1	1	3	4	5	6
3	ω	1	1	1	4	5	6
4	ω	1	1	1	1	5	6
5	6	1	1	1	1	1	6
6	5	1	1	1	1	5	1

&	ω	1	2	3	4	5	6
ω	ω	ω	ω	ω	ω	ω	ω
1	ω	1	2	3	4	5	6
2	ω	2	2	3	4	5	6
3	ω	3	3	3	4	5	6
4	ω	4	4	4	4	5	6
5	ω	5	5	5	5	5	ω
6	ω	6	6	6	6	ω	6

∨	ω	1	2	3	4	5	6
ω	ω	1	2	3	4	5	6
1	1	1	1	1	1	1	1
2	2	1	2	2	2	2	2
3	3	1	2	3	3	3	3
4	4	1	2	3	4	4	4
5	5	1	2	3	4	5	4
6	6	1	2	3	4	4	6

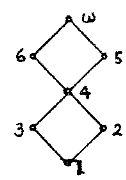


$L_{7,6}=S_1^2 \uparrow S_1^2\supset \subset (S_1 \uparrow S_1^2)^2\supset \subset L_{5,3}$. $(c=5)$

$\supset$	ω	1	2	3	4	5	6
ω	1	1	1	1	1	1	1
1	ω	1	2	3	4	5	6
2	ω	1	1	3	3	5	6
3	ω	1	2	1	2	5	6
4	ω	1	1	1	1	5	6
5	6	1	1	1	1	1	6
6	5	1	1	1	1	5	1

&	ω	1	2	3	4	5	6
ω	ω	ω	ω	ω	ω	ω	ω
1	ω	1	2	3	4	5	6
2	ω	2	2	4	4	5	6
3	ω	3	4	3	4	5	6
4	ω	4	4	4	4	5	6
5	ω	5	5	5	5	5	ω
6	ω	6	6	6	6	ω	6

$\vee$	ω	1	2	3	4	5	6
ω	ω	1	2	3	4	5	6
1	1	1	1	1	1	1	1
2	2	1	2	1	2	2	2
3	3	1	1	3	3	3	3
4	4	1	2	3	4	4	4
5	5	1	2	3	4	5	4
6	6	1	2	3	4	4	6

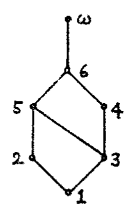


$L_{7,7}=(S_1,\ S_2)\uparrow S_1\supset \subset S_3$. $(c=4)$

$\supset$	ω	1	2	3	4	5	6
ω	1	1	1	1	1	1	1
1	ω	1	2	3	4	5	6
2	ω	1	1	3	4	3	4
3	ω	1	2	1	4	2	6
4	ω	1	2	1	1	2	2
5	ω	1	1	1	4	1	4
6	ω	1	1	1	1	1	1

&	ω	1	2	3	4	5	6
ω	ω	ω	ω	ω	ω	ω	ω
1	ω	1	2	3	4	5	6
2	ω	2	2	5	6	5	6
3	ω	3	5	3	4	5	6
4	ω	4	6	4	4	6	6
5	ω	5	5	5	6	5	6
6	ω	6	6	6	6	6	6

$\vee$	ω	1	2	3	4	5	6
ω	ω	1	2	3	4	5	6
1	1	1	1	1	1	1	1
2	2	1	2	1	1	2	2
3	3	1	1	3	3	3	3
4	4	1	1	3	4	3	4
5	5	1	2	3	3	5	5
6	6	1	2	3	4	5	6

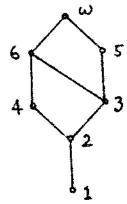


$L_{7,8} = S_1 \uparrow (S_1,\ S_2)$. $(c=7)$

$\supset$	ω	1	2	3	4	5	6
ω	1	1	1	1	1	1	1
1	ω	1	2	3	4	5	6
2	ω	1	1	3	4	5	6
3	ω	1	1	1	4	5	4
4	5	1	1	3	1	5	3
5	4	1	1	1	4	1	4
6	5	1	1	1	1	5	1

&	ω	1	2	3	4	5	6
ω	ω	ω	ω	ω	ω	ω	ω
1	ω	1	2	3	4	5	6
2	ω	2	2	3	4	5	6
3	ω	3	3	3	6	5	6
4	ω	4	4	6	4	ω	6
5	ω	5	5	5	ω	5	ω
6	ω	6	6	6	6	ω	6

$\vee$	ω	1	2	3	4	5	6
ω	ω	1	2	3	4	5	6
1	1	1	1	1	1	1	1
2	2	1	2	2	2	2	2
3	3	1	2	3	2	3	3
4	4	1	2	2	4	2	4
5	5	1	2	3	2	5	3
6	6	1	2	3	4	3	6



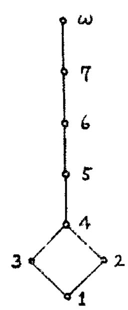
$L_{8,1} = S_7$. $(c=8)$

$L_{8,2} = S_1{}^2 \uparrow S_4 \supset \subset S_5$. $(c=6)$

$\supset$	ω	1	2	3	4	5	6	7
ω	1	1	1	1	1	1	1	1
1	ω	1	2	3	4	5	6	7
2	ω	1	1	3	3	5	6	7
3	ω	1	2	1	2	5	6	7
4	ω	1	1	1	1	5	6	7
5	ω	1	1	1	1	1	6	7
6	ω	1	1	1	1	1	1	7
7	ω	1	1	1	1	1	1	1

&	ω	1	2	3	4	5	6	7
ω	ω	ω	ω	ω	ω	ω	ω	ω
1	ω	1	2	3	4	5	6	7
2	ω	2	2	4	4	5	6	7
3	ω	3	4	3	4	5	6	7
4	ω	4	4	4	4	5	6	7
5	ω	5	5	5	5	5	6	7
6	ω	6	6	6	6	6	6	7
7	ω	7	7	7	7	7	7	7

$\vee$	ω	1	2	3	4	5	6	7
ω	ω	1	2	3	4	5	6	7
1	1	1	1	1	1	1	1	1
2	2	1	2	1	2	2	2	2
3	3	1	1	3	3	3	3	3
4	4	1	2	3	4	4	4	4
5	5	1	2	3	4	5	5	5
6	6	1	2	3	4	5	6	6
7	7	1	2	3	4	5	6	7

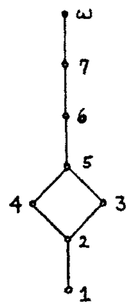


$L_{8,3} = S_1 \uparrow S_1^2 \uparrow S_3$. $(c=8)$

⊃	ω	1	2	3	4	5	6	7
ω	1	1	1	1	1	1	1	1
1	ω	1	2	3	4	5	6	7
2	ω	1	1	3	4	5	6	7
3	ω	1	1	1	4	4	6	7
4	ω	1	1	3	1	3	6	7
5	ω	1	1	1	1	1	6	7
6	ω	1	1	1	1	1	1	7
7	ω	1	1	1	1	1	1	1

&	ω	1	2	3	4	5	6	7
ω	ω	ω	ω	ω	ω	ω	ω	ω
1	ω	1	2	3	4	5	6	7
2	ω	2	2	3	4	5	6	7
3	ω	3	3	3	5	5	6	7
4	ω	4	4	5	4	5	6	7
5	ω	5	5	5	5	5	6	7
6	ω	6	6	6	6	6	6	7
7	ω	7	7	7	7	7	7	7

∨	ω	1	2	3	4	5	6	7
ω	ω	1	2	3	4	5	6	7
1	1	1	1	1	1	1	1	1
2	2	1	2	2	2	2	2	2
3	3	1	2	3	2	3	3	3
4	4	1	2	2	4	4	4	4
5	5	1	2	3	4	5	5	5
6	6	1	2	3	4	5	6	6
7	7	1	2	3	4	5	6	7

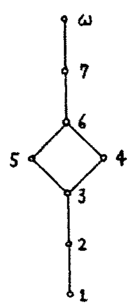


$L_{8,4} = S_2 \uparrow S_1^2 \uparrow S_2$. $(c=8)$

⊃	ω	1	2	3	4	5	6	7
ω	1	1	1	1	1	1	1	1
1	ω	1	2	3	4	5	6	7
2	ω	1	1	3	4	5	6	7
3	ω	1	1	1	4	5	6	7
4	ω	1	1	1	1	5	5	7
5	ω	1	1	1	4	1	4	7
6	ω	1	1	1	1	1	1	7
7	ω	1	1	1	1	1	1	1

&	ω	1	2	3	4	5	6	7
ω	ω	ω	ω	ω	ω	ω	ω	ω
1	ω	1	2	3	4	5	6	7
2	ω	2	2	3	4	5	6	7
3	ω	3	3	3	4	5	6	7
4	ω	4	4	4	4	6	6	7
5	ω	5	5	5	6	5	6	7
6	ω	6	6	6	6	6	6	7
7	ω	7	7	7	7	7	7	7

∨	ω	1	2	3	4	5	6	7
ω	ω	1	2	3	4	5	6	7
1	1	1	1	1	1	1	1	1
2	2	1	2	2	2	2	2	2
3	3	1	2	3	3	3	3	3
4	4	1	2	3	4	3	4	4
5	5	1	2	3	3	5	5	5
6	6	1	2	3	4	5	6	6
7	7	1	2	3	4	5	6	7

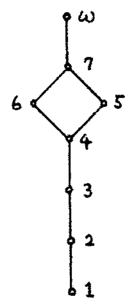


$L_{8.5}=S_3 \uparrow S_1^2 \uparrow S_1$. $(c=8)$

⊃	ω	1	2	3	4	5	6	7
ω	1	1	1	1	1	1	1	1
1	ω	1	2	3	4	5	6	7
2	ω	1	1	3	4	5	6	7
3	ω	1	1	1	4	5	6	7
4	ω	1	1	1	1	5	6	7
5	ω	1	1	1	1	1	6	6
6	ω	1	1	1	1	5	1	5
7	ω	1	1	1	1	1	1	1

&	ω	1	2	3	4	5	6	7
ω	ω	ω	ω	ω	ω	ω	ω	ω
1	ω	1	2	3	4	5	6	7
2	ω	2	2	3	4	5	6	7
3	ω	3	3	3	4	5	6	7
4	ω	4	4	4	4	5	6	7
5	ω	5	5	5	5	5	7	7
6	ω	6	6	6	6	7	6	7
7	ω	7	7	7	7	7	7	7

∨	ω	1	2	3	4	5	6	7
ω	ω	1	2	3	4	5	6	7
1	1	1	1	1	1	1	1	1
2	2	1	2	2	2	2	2	2
3	3	1	2	3	3	3	3	3
4	4	1	2	3	4	4	4	4
5	5	1	2	3	4	5	4	5
6	6	1	2	3	4	4	6	6
7	7	1	2	3	4	5	6	7

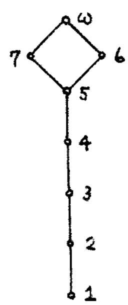


$L_{8.6}=S_4 \uparrow S_1^2$. $(c=8)$

⊃	ω	1	2	3	4	5	6	7
ω	1	1	1	1	1	1	1	1
1	ω	1	2	3	4	5	6	7
2	ω	1	1	3	4	5	6	7
3	ω	1	1	1	4	5	6	7
4	ω	1	1	1	1	5	6	7
5	ω	1	1	1	1	1	6	7
6	7	1	1	1	1	1	1	7
7	6	1	1	1	1	1	6	1

&	ω	1	2	3	4	5	6	7
ω	ω	ω	ω	ω	ω	ω	ω	ω
1	ω	1	2	3	4	5	6	7
2	ω	2	2	3	4	5	6	7
3	ω	3	3	3	4	5	6	7
4	ω	4	4	4	4	5	6	7
5	ω	5	5	5	5	5	6	7
6	ω	6	6	6	6	6	6	ω
7	ω	7	7	7	7	7	ω	7

∨	ω	1	2	3	4	5	6	7
ω	ω	1	2	3	4	5	6	7
1	1	1	1	1	1	1	1	1
2	2	1	2	2	2	2	2	2
3	3	1	2	3	3	3	3	3
4	4	1	2	3	4	4	4	4
5	5	1	2	3	4	5	5	5
6	6	1	2	3	4	5	6	5
7	7	1	2	3	4	5	5	7

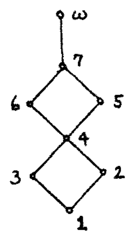


$L_{8,7}=S_1^2 \uparrow S_1^2 \uparrow S_1 \supset \subset S_1 \uparrow S_1^2 \uparrow S_1 = L_{6,3}$. $(c=6)$

⊃	ω	1	2	3	4	5	6	7
ω	1	1	1	1	1	1	1	1
1	ω	1	2	3	4	5	6	7
2	ω	1	1	3	3	5	6	7
3	ω	1	2	1	2	5	6	7
4	ω	1	1	1	1	5	6	7
5	ω	1	1	1	1	1	6	6
6	ω	1	1	1	1	5	1	5
7	ω	1	1	1	1	1	1	1

&	ω	1	2	3	4	5	6	7
ω	ω	ω	ω	ω	ω	ω	ω	ω
1	ω	1	2	3	4	5	6	7
2	ω	2	2	4	4	5	6	7
3	ω	3	4	3	4	5	6	7
4	ω	4	4	4	4	5	6	7
5	ω	5	5	5	5	5	7	7
6	ω	6	6	6	6	7	6	7
7	ω	7	7	7	7	7	7	7

∨	ω	1	2	3	4	5	6	7
ω	ω	1	2	3	4	5	6	7
1	1	1	1	1	1	1	1	1
2	2	1	2	1	2	2	2	2
3	3	1	1	3	3	3	3	3
4	4	1	2	3	4	4	4	4
5	5	1	2	3	4	5	4	5
6	6	1	2	3	4	4	6	6
7	7	1	2	3	4	5	6	7

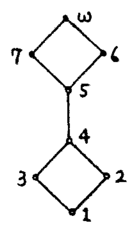


$L_{8,8}=S_1^2 \uparrow S_1 \uparrow S_1^2 \supset \subset S_2 \uparrow S_1^2 = L_{6,4}$. $(c=6)$

⊃	ω	1	2	3	4	5	6	7
ω	1	1	1	1	1	1	1	1
1	ω	1	2	3	4	5	6	7
2	ω	1	1	3	3	5	6	7
3	ω	1	2	1	2	5	6	7
4	ω	1	1	1	1	5	6	7
5	ω	1	1	1	1	1	6	7
6	7	1	1	1	1	1	1	7
7	6	1	1	1	1	1	6	1

&	ω	1	2	3	4	5	6	7
ω	ω	ω	ω	ω	ω	ω	ω	ω
1	ω	1	2	3	4	5	6	7
2	ω	2	2	4	4	5	6	7
3	ω	3	4	3	4	5	6	7
4	ω	4	4	4	4	5	6	7
5	ω	5	5	5	5	5	6	7
6	ω	6	6	6	6	6	6	ω
7	ω	7	7	7	7	7	ω	7

∨	ω	1	2	3	4	5	6	7
ω	ω	1	2	3	4	5	6	7
1	1	1	1	1	1	1	1	1
2	2	1	2	1	2	2	2	2
3	3	1	1	3	3	3	3	3
4	4	1	2	3	4	4	4	4
5	5	1	2	3	4	5	5	5
6	6	1	2	3	4	5	6	5
7	7	1	2	3	4	5	5	7

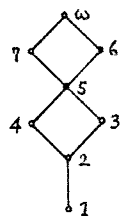


$L_{8,9}=S_1\uparrow S_1^2\uparrow S_1^2$. $(c=8)$

⊃	ω	1	2	3	4	5	6	7
ω	1	1	1	1	1	1	1	1
1	ω	1	2	3	4	5	6	7
2	ω	1	1	3	4	5	6	7
3	ω	1	1	1	4	4	6	7
4	ω	1	1	3	1	3	6	7
5	ω	1	1	1	1	1	6	7
6	7	1	1	1	1	1	1	7
7	6	1	1	1	1	1	6	1

&	ω	1	2	3	4	5	6	7
ω	ω	ω	ω	ω	ω	ω	ω	ω
1	ω	1	2	3	4	5	6	7
2	ω	2	2	3	4	5	6	7
3	ω	3	3	3	5	5	6	7
4	ω	4	4	5	4	5	6	7
5	ω	5	5	5	5	5	6	7
6	ω	6	6	6	6	6	6	ω
7	ω	7	7	7	7	7	ω	7

∨	ω	1	2	3	4	5	6	7
ω	ω	1	2	3	4	5	6	7
1	1	1	1	1	1	1	1	1
2	2	1	2	2	2	2	2	2
3	3	1	2	3	2	3	3	3
4	4	1	2	2	4	4	4	4
5	5	1	2	3	4	5	5	5
6	6	1	2	3	4	5	6	5
7	7	1	2	3	4	5	5	7

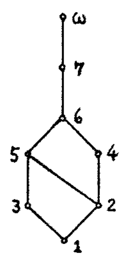


$L_{8,10}=(S_1,\ S_2)\uparrow S_2\supset\subset S_4$. $(c=5)$

⊃	ω	1	2	3	4	5	6	7
ω	1	1	1	1	1	1	1	1
1	ω	1	2	3	4	5	6	7
2	ω	1	1	3	4	3	6	7
3	ω	1	2	1	4	2	4	7
4	ω	1	1	3	1	3	3	7
5	ω	1	1	1	4	1	4	7
6	ω	1	1	1	1	1	1	7
7	ω	1	1	1	1	1	1	1

&	ω	1	2	3	4	5	6	7
ω	ω	ω	ω	ω	ω	ω	ω	ω
1	ω	1	2	3	4	5	6	7
2	ω	2	2	5	4	5	6	7
3	ω	3	5	3	6	5	6	7
4	ω	4	4	6	4	6	6	7
5	ω	5	5	5	6	5	6	7
6	ω	6	6	6	6	6	6	7
7	ω	7	7	7	7	7	7	7

∨	ω	1	2	3	4	5	6	7
ω	ω	1	2	3	4	5	6	7
1	1	1	1	1	1	1	1	1
2	2	1	2	1	2	2	2	2
3	3	1	1	3	1	3	3	3
4	4	1	2	1	4	2	4	4
5	5	1	2	3	2	5	5	5
6	6	1	2	3	4	5	6	6
7	7	1	2	3	4	5	6	7

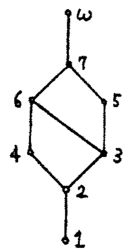


$L_{8,11} = S_1 \uparrow (S_1,\ S_2) \uparrow S_1$. $(c=8)$

⊃	ω	1	2	3	4	5	6	7
ω	1	1	1	1	1	1	1	1
1	ω	1	2	3	4	5	6	7
2	ω	1	1	3	4	5	6	7
3	ω	1	1	1	4	5	4	7
4	ω	1	1	3	1	5	3	5
5	ω	1	1	1	4	1	4	4
6	ω	1	1	1	1	5	1	5
7	ω	1	1	1	1	1	1	1

&	ω	1	2	3	4	5	6	7
ω	ω	ω	ω	ω	ω	ω	ω	ω
1	ω	1	2	3	4	5	6	7
2	ω	2	2	3	4	5	6	7
3	ω	3	3	3	6	5	6	7
4	ω	4	4	6	4	7	6	7
5	ω	5	5	5	7	5	7	7
6	ω	6	6	6	6	7	6	7
7	ω	7	7	7	7	7	7	7

∨	ω	1	2	3	4	5	6	7
ω	ω	1	2	3	4	5	6	7
1	1	1	1	1	1	1	1	1
2	2	1	2	2	2	2	2	2
3	3	1	2	3	2	3	3	3
4	4	1	2	2	4	2	4	4
5	5	1	2	3	2	5	3	5
6	6	1	2	3	4	3	6	6
7	7	1	2	3	4	5	6	7

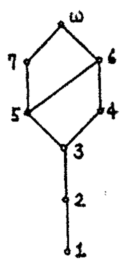


$L_{8,12} = S_2 \uparrow (S_1,\ S_2)$. $(c=8)$

⊃	ω	1	2	3	4	5	6	7
ω	1	1	1	1	1	1	1	1
1	ω	1	2	3	4	5	6	7
2	ω	1	1	3	4	5	6	7
3	ω	1	1	1	4	5	6	7
4	7	1	1	1	1	5	5	7
5	ω	1	1	1	4	1	4	7
6	7	1	1	1	1	1	1	7
7	4	1	1	1	4	1	4	1

&	ω	1	2	3	4	5	6	7
ω	ω	ω	ω	ω	ω	ω	ω	ω
1	ω	1	2	3	4	5	6	7
2	ω	2	2	3	4	5	6	7
3	ω	3	3	3	4	5	6	7
4	ω	4	4	4	4	6	6	ω
5	ω	5	5	5	6	5	6	7
6	ω	6	6	6	6	6	6	ω
7	ω	7	7	7	ω	7	ω	7

∨	ω	1	2	3	4	5	6	7
ω	ω	1	2	3	4	5	6	7
1	1	1	1	1	1	1	1	1
2	2	1	2	2	2	2	2	2
3	3	1	2	3	3	3	3	3
4	4	1	2	3	4	3	4	3
5	5	1	2	3	3	5	5	5
6	6	1	2	3	4	5	6	5
7	7	1	2	3	3	5	5	7

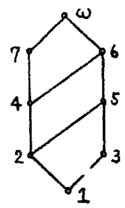


$L_{8.13}=(S_1,\ S_3)\supset\subset S_3$. $(c=4)$

$\supset$	ω	1	2	3	4	5	6	7
ω	1	1	1	1	1	1	1	1
1	ω	1	2	3	4	5	6	7
2	ω	1	1	3	4	3	6	7
3	7	1	2	1	4	2	4	7
4	ω	1	1	3	1	3	3	7
5	7	1	1	1	4	1	4	7
6	7	1	1	1	1	1	1	7
7	3	1	1	3	1	3	3	1

&	ω	1	2	3	4	5	6	7
ω	ω	ω	ω	ω	ω	ω	ω	ω
1	ω	1	2	3	4	5	6	7
2	ω	2	2	5	4	5	6	7
3	ω	3	5	3	6	5	6	ω
4	ω	4	4	6	4	6	6	7
5	ω	5	5	5	6	5	6	ω
6	ω	6	6	6	6	6	6	ω
7	ω	7	7	ω	7	ω	ω	7

$\vee$	ω	1	2	3	4	5	6	7
ω	ω	1	2	3	4	5	6	7
1	1	1	1	1	1	1	1	1
2	2	1	2	1	2	2	2	2
3	3	1	1	3	1	3	3	1
4	4	1	2	1	4	2	4	4
5	5	1	2	3	2	5	5	2
6	6	1	2	3	4	5	6	4
7	7	1	2	1	4	2	4	7

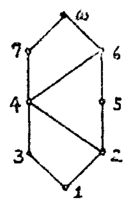


$L_{8.14}$. $(c=5)$

$\supset$	ω	1	2	3	4	5	6	7
ω	1	1	1	1	1	1	1	1
1	ω	1	2	3	4	5	6	7
2	ω	1	1	3	3	5	6	7
3	ω	1	2	1	2	5	5	7
4	ω	1	1	1	1	5	5	7
5	7	1	1	3	3	1	3	7
6	7	1	1	1	1	1	1	7
7	5	1	1	1	1	5	5	1

&	ω	1	2	3	4	5	6	7
ω	ω	ω	ω	ω	ω	ω	ω	ω
1	ω	1	2	3	4	5	6	7
2	ω	2	2	4	4	5	6	7
3	ω	3	4	3	4	6	6	7
4	ω	4	4	4	4	6	6	7
5	ω	5	5	6	6	5	6	ω
6	ω	6	6	6	6	6	6	ω
7	ω	7	7	7	7	ω	ω	7

$\vee$	ω	1	2	3	4	5	6	7
ω	ω	1	2	3	4	5	6	7
1	1	1	1	1	1	1	1	1
2	2	1	2	1	2	2	2	2
3	3	1	1	3	3	1	3	3
4	4	1	2	3	4	2	4	4
5	5	1	2	1	2	5	5	2
6	6	1	2	3	4	5	6	4
7	7	1	2	3	4	2	4	7

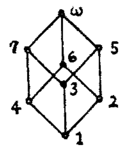


$L_{8,15}=S_1{}^3\supset\subset S_1$. $(c=2)$

$\supset$	ω	1	2	3	4	5	6	7
ω	1	1	1	1	1	1	1	1
1	ω	1	2	3	4	5	6	7
2	7	1	1	3	4	3	4	7
3	6	1	2	1	4	2	6	4
4	5	1	2	3	1	5	2	3
5	4	1	1	1	4	1	4	4
6	3	1	1	3	1	3	1	3
7	2	1	2	1	1	2	2	1

&	ω	1	2	3	4	5	6	7
ω	ω	ω	ω	ω	ω	ω	ω	ω
1	ω	1	2	3	4	5	6	7
2	ω	2	2	5	6	5	6	ω
3	ω	3	5	3	7	5	ω	7
4	ω	4	6	7	4	ω	6	7
5	ω	5	5	5	ω	5	ω	ω
6	ω	6	6	ω	6	ω	6	ω
7	ω	7	ω	7	7	ω	ω	7

$\vee$	ω	1	2	3	4	5	6	7
ω	ω	1	2	3	4	5	6	7
1	1	1	1	1	1	1	1	1
2	2	1	2	1	1	2	2	1
3	3	1	1	3	1	3	1	3
4	4	1	1	1	4	1	4	4
5	5	1	2	3	1	5	2	3
6	6	1	2	1	4	2	6	4
7	7	1	1	3	4	3	4	7



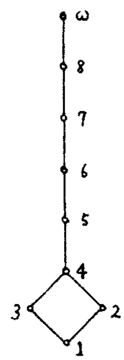
$L_{9,1}=S_8$. $(c=9)$

$L_{9,2}=S_1{}^2\uparrow S_5\supset\subset S_6$. $(c=7)$

$\supset$	ω	1	2	3	4	5	6	7	8
ω	1	1	1	1	1	1	1	1	1
1	ω	1	2	3	4	5	6	7	8
2	ω	1	1	3	3	5	6	7	8
3	ω	1	2	1	2	5	6	7	8
4	ω	1	1	1	1	5	6	7	8
5	ω	1	1	1	1	1	6	7	8
6	ω	1	1	1	1	1	1	7	8
7	ω	1	1	1	1	1	1	1	8
8	ω	1	1	1	1	1	1	1	1

&	ω	1	2	3	4	5	6	7	8
ω	ω	ω	ω	ω	ω	ω	ω	ω	ω
1	ω	1	2	3	4	5	6	7	8
2	ω	2	2	4	4	5	6	7	8
3	ω	3	4	3	4	5	6	7	8
4	ω	4	4	4	4	5	6	7	8
5	ω	5	5	5	5	5	6	7	8
6	ω	6	6	6	6	6	6	7	8
7	ω	7	7	7	7	7	7	7	8
8	ω	8	8	8	8	8	8	8	8

$\vee$	ω	1	2	3	4	5	6	7	8
ω	ω	1	2	3	4	5	6	7	8
1	1	1	1	1	1	1	1	1	1
2	2	1	2	1	2	2	2	2	2
3	3	1	1	3	3	3	3	3	3
4	4	1	2	3	4	4	4	4	4
5	5	1	2	3	4	5	5	5	5
6	6	1	2	3	4	5	6	6	6
7	7	1	2	3	4	5	6	7	7
8	8	1	2	3	4	5	6	7	8

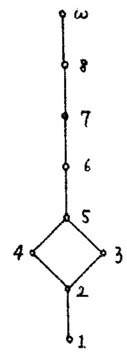


$L_{9,3} = S_1 \uparrow S_1^2 \uparrow S_4$. $(c=9)$

⊃	ω	1	2	3	4	5	6	7	8
ω	1	1	1	1	1	1	1	1	1
1	ω	1	2	3	4	5	6	7	8
2	ω	1	1	3	4	5	6	7	8
3	ω	1	1	1	4	4	6	7	8
4	ω	1	1	3	1	3	6	7	8
5	ω	1	1	1	1	1	6	7	8
6	ω	1	1	1	1	1	1	7	8
7	ω	1	1	1	1	1	1	1	8
8	ω	1	1	1	1	1	1	1	1

&	ω	1	2	3	4	5	6	7	8
ω	ω	ω	ω	ω	ω	ω	ω	ω	ω
1	ω	1	2	3	4	5	6	7	8
2	ω	2	2	3	4	5	6	7	8
3	ω	3	3	3	5	5	6	7	8
4	ω	4	4	5	4	5	6	7	8
5	ω	5	5	5	5	5	6	7	8
6	ω	6	6	6	6	6	6	7	8
7	ω	7	7	7	7	7	7	7	8
8	ω	8	8	8	8	8	8	8	8

∨	ω	1	2	3	4	5	6	7	8
ω	ω	1	2	3	4	5	6	7	8
1	1	1	1	1	1	1	1	1	1
2	2	1	2	2	2	2	2	2	2
3	3	1	2	3	2	3	3	3	3
4	4	1	2	2	4	4	4	4	4
5	5	1	2	3	4	5	5	5	5
6	6	1	2	3	4	5	6	6	6
7	7	1	2	3	4	5	6	7	7
8	8	1	2	3	4	5	6	7	8

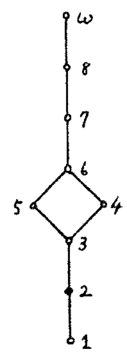


$L_{9,4} = S_2 \uparrow S_1^2 \uparrow S_3$. $(c=9)$

⊃	ω	1	2	3	4	5	6	7	8
ω	1	1	1	1	1	1	1	1	1
1	ω	1	2	3	4	5	6	7	8
2	ω	1	1	3	4	5	6	7	8
3	ω	1	1	1	4	5	6	7	8
4	ω	1	1	1	1	5	5	7	8
5	ω	1	1	1	4	1	4	7	8
6	ω	1	1	1	1	1	1	7	8
7	ω	1	1	1	1	1	1	1	8
8	ω	1	1	1	1	1	1	1	1

&	ω	1	2	3	4	5	6	7	8
ω	ω	ω	ω	ω	ω	ω	ω	ω	ω
1	ω	1	2	3	4	5	6	7	8
2	ω	2	2	3	4	5	6	7	8
3	ω	3	3	3	4	5	6	7	8
4	ω	4	4	4	4	6	6	7	8
5	ω	5	5	5	6	5	6	7	8
6	ω	6	6	6	6	6	6	7	8
7	ω	7	7	7	7	7	7	7	8
8	ω	8	8	8	8	8	8	8	8

∨	ω	1	2	3	4	5	6	7	8
ω	ω	1	2	3	4	5	6	7	8
1	1	1	1	1	1	1	1	1	1
2	2	1	2	2	2	2	2	2	2
3	3	1	2	3	3	3	3	3	3
4	4	1	2	3	4	3	4	4	4
5	5	1	2	3	3	5	5	5	5
6	6	1	2	3	4	5	6	6	6
7	7	1	2	3	4	5	6	7	7
8	8	1	2	3	4	5	6	7	8

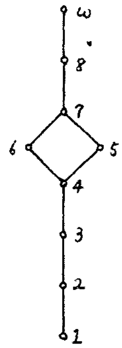


$L_{9,5} = S_3 \uparrow S_1^2 \uparrow S_2$. $(c=9)$

⊃	ω	1	2	3	4	5	6	7	8
ω	1	1	1	1	1	1	1	1	1
1	ω	1	2	3	4	5	6	7	8
2	ω	1	1	3	4	5	6	7	8
3	ω	1	1	1	4	5	6	7	8
4	ω	1	1	1	1	5	6	7	8
5	ω	1	1	1	1	1	6	6	8
6	ω	1	1	1	1	5	1	5	8
7	ω	1	1	1	1	1	1	1	8
8	ω	1	1	1	1	1	1	1	1

&	ω	1	2	3	4	5	6	7	8
ω	ω	ω	ω	ω	ω	ω	ω	ω	ω
1	ω	1	2	3	4	5	6	7	8
2	ω	2	2	3	4	5	6	7	8
3	ω	3	3	3	4	5	6	7	8
4	ω	4	4	4	4	5	6	7	8
5	ω	5	5	5	5	5	7	7	8
6	ω	6	6	6	6	7	6	7	8
7	ω	7	7	7	7	7	7	7	8
8	ω	8	8	8	8	8	8	8	8

∨	ω	1	2	3	4	5	6	7	8
ω	ω	1	2	3	4	5	6	7	8
1	1	1	1	1	1	1	1	1	1
2	2	1	2	2	2	2	2	2	2
3	3	1	2	3	3	3	3	3	3
4	4	1	2	3	4	4	4	4	4
5	5	1	2	3	4	5	4	5	5
6	6	1	2	3	4	4	6	6	6
7	7	1	2	3	4	5	6	7	7
8	8	1	2	3	4	5	6	7	8

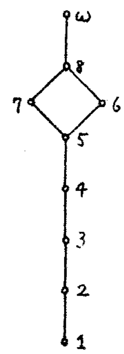


$L_{9,6} = S_4 \uparrow S_1^2 \uparrow S_1$. $(c=9)$

⊃	ω	1	2	3	4	5	6	7	8
ω	1	1	1	1	1	1	1	1	1
1	ω	1	2	3	4	5	6	7	8
2	ω	1	1	3	4	5	6	7	8
3	ω	1	1	1	4	5	6	7	8
4	ω	1	1	1	1	5	6	7	8
5	ω	1	1	1	1	1	6	7	8
6	ω	1	1	1	1	1	1	7	7
7	ω	1	1	1	1	1	6	1	6
8	ω	1	1	1	1	1	1	1	1

&	ω	1	2	3	4	5	6	7	8
ω	ω	ω	ω	ω	ω	ω	ω	ω	ω
1	ω	1	2	3	4	5	6	7	8
2	ω	2	2	3	4	5	6	7	8
3	ω	3	3	3	4	5	6	7	8
4	ω	4	4	4	4	5	6	7	8
5	ω	5	5	5	5	5	6	7	8
6	ω	6	6	6	6	6	6	8	8
7	ω	7	7	7	7	7	8	7	8
8	ω	8	8	8	8	8	8	8	8

∨	ω	1	2	3	4	5	6	7	8
ω	ω	1	2	3	4	5	6	7	8
1	1	1	1	1	1	1	1	1	1
2	2	1	2	2	2	2	2	2	2
3	3	1	2	3	3	3	3	3	3
4	4	1	2	3	4	4	4	4	4
5	5	1	2	3	4	5	5	5	5
6	6	1	2	3	4	5	6	5	6
7	7	1	2	3	4	5	5	7	7
8	8	1	2	3	4	5	6	7	8

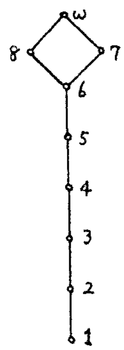


$L_{9,7} = S_5 \uparrow S_1^2$. $(c=9)$

⊃	ω	1	2	3	4	5	6	7	8
ω	1	1	1	1	1	1	1	1	1
1	ω	1	2	3	4	5	6	7	8
2	ω	1	1	3	4	5	6	7	8
3	ω	1	1	1	4	5	6	7	8
4	ω	1	1	1	1	5	6	7	8
5	ω	1	1	1	1	1	6	7	8
6	ω	1	1	1	1	1	1	7	8
7	8	1	1	1	1	1	1	1	8
8	7	1	1	1	1	1	1	7	1

&	ω	1	2	3	4	5	6	7	8
ω	ω	ω	ω	ω	ω	ω	ω	ω	ω
1	ω	1	2	3	4	5	6	7	8
2	ω	2	2	3	4	5	6	7	8
3	ω	3	3	3	4	5	6	7	8
4	ω	4	4	4	4	5	6	7	8
5	ω	5	5	5	5	5	6	7	8
6	ω	6	6	6	6	6	6	7	8
7	ω	7	7	7	7	7	7	7	ω
8	ω	8	8	8	8	8	8	ω	8

∨	ω	1	2	3	4	5	6	7	8
ω	ω	1	2	3	4	5	6	7	8
1	1	1	1	1	1	1	1	1	1
2	2	1	2	2	2	2	2	2	2
3	3	1	2	3	3	3	3	3	3
4	4	1	2	3	4	4	4	4	4
5	5	1	2	3	4	5	5	5	5
6	6	1	2	3	4	5	6	6	6
7	7	1	2	3	4	5	6	7	6
8	8	1	2	3	4	5	6	6	8

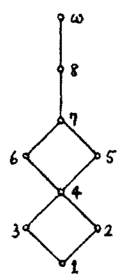


$L_{9,8} = S_1^2 \uparrow S_1^2 \uparrow S_2 \supset \subset S_1 \uparrow S_1^2 \uparrow S_2 = L_{7,8}$. $(c=7)$

⊃	ω	1	2	3	4	5	6	7	8
ω	1	1	1	1	1	1	1	1	1
1	ω	1	2	3	4	5	6	7	8
2	ω	1	1	3	3	5	6	7	8
3	ω	1	2	1	2	5	6	7	8
4	ω	1	1	1	1	5	6	7	8
5	ω	1	1	1	1	1	6	6	8
6	ω	1	1	1	1	5	1	5	8
7	ω	1	1	1	1	1	1	1	8
8	ω	1	1	1	1	1	1	1	1

&	ω	1	2	3	4	5	6	7	8
ω	ω	ω	ω	ω	ω	ω	ω	ω	ω
1	ω	1	2	3	4	5	6	7	8
2	ω	2	2	4	4	5	6	7	8
3	ω	3	4	3	4	5	6	7	8
4	ω	4	4	4	4	5	6	7	8
5	ω	5	5	5	5	5	7	7	8
6	ω	6	6	6	6	7	6	7	8
7	ω	7	7	7	7	7	7	7	8
8	ω	8	8	8	8	8	8	8	8

∨	ω	1	2	3	4	5	6	7	8
ω	ω	1	2	3	4	5	6	7	8
1	1	1	1	1	1	1	1	1	1
2	2	1	2	1	2	2	2	2	2
3	3	1	1	3	3	3	3	3	3
4	4	1	2	3	4	4	4	4	4
5	5	1	2	3	4	5	4	5	5
6	6	1	2	3	4	4	6	6	6
7	7	1	2	3	4	5	6	7	7
8	8	1	2	3	4	5	6	7	8

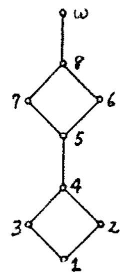


$L_{9,9}=S_1^2 \uparrow S_1 \uparrow S_1^2 \uparrow S_1 \supset \subset S_1 \uparrow S_1 \uparrow S_1^2 \uparrow S_1 = L_{7,4}$. $(c=7)$

⊃	ω	1	2	3	4	5	6	7	8
ω	1	1	1	1	1	1	1	1	1
1	ω	1	2	3	4	5	6	7	8
2	ω	1	1	3	3	5	6	7	8
3	ω	1	2	1	2	5	6	7	8
4	ω	1	1	1	1	5	6	7	8
5	ω	1	1	1	1	1	6	7	8
6	ω	1	1	1	1	1	1	7	7
7	ω	1	1	1	1	1	6	1	6
8	ω	1	1	1	1	1	1	1	1

&	ω	1	2	3	4	5	6	7	8
ω	ω	ω	ω	ω	ω	ω	ω	ω	ω
1	ω	1	2	3	4	5	6	7	8
2	ω	2	2	4	4	5	6	7	8
3	ω	3	4	3	4	5	6	7	8
4	ω	4	4	4	4	5	6	7	8
5	ω	5	5	5	5	5	6	7	8
6	ω	6	6	6	6	6	6	8	8
7	ω	7	7	7	7	7	8	7	8
8	ω	8	8	8	8	8	8	8	8

∨	ω	1	2	3	4	5	6	7	8
ω	ω	1	2	3	4	5	6	7	8
1	1	1	1	1	1	1	1	1	1
2	2	1	2	1	2	2	2	2	2
3	3	1	1	3	3	3	3	3	3
4	4	1	2	3	4	4	4	4	4
5	5	1	2	3	4	5	5	5	5
6	6	1	2	3	4	5	6	5	6
7	7	1	2	3	4	5	5	7	7
8	8	1	2	3	4	5	6	7	8

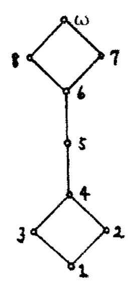


$L_{9,10}=S_1^2 \uparrow S_2 \uparrow S_{12} \supset \subset S_1 \uparrow S_2 \uparrow S_1^2 = L_{7,5}$. $(c=7)$

⊃	ω	1	2	3	4	5	6	7	8
ω	1	1	1	1	1	1	1	1	1
1	ω	1	2	3	4	5	6	7	8
2	ω	1	1	3	3	5	6	7	8
3	ω	1	2	1	2	5	6	7	8
4	ω	1	1	1	1	5	6	7	8
5	ω	1	1	1	1	1	6	7	8
6	ω	1	1	1	1	1	1	7	8
7	8	1	1	1	1	1	1	1	8
8	7	1	1	1	1	1	1	7	1

&	ω	1	2	3	4	5	6	7	8
ω	ω	ω	ω	ω	ω	ω	ω	ω	ω
1	ω	1	2	3	4	5	6	7	8
2	ω	2	2	4	4	5	6	7	8
3	ω	3	4	3	4	5	6	7	8
4	ω	4	4	4	4	5	6	7	8
5	ω	5	5	5	5	5	6	7	8
6	ω	6	6	6	6	6	6	7	8
7	ω	7	7	7	7	7	7	7	ω
8	ω	8	8	8	8	8	8	ω	8

∨	ω	1	2	3	4	5	6	7	8
ω	ω	1	2	3	4	5	6	7	8
1	1	1	1	1	1	1	1	1	1
2	2	1	2	1	2	2	2	2	2
3	3	1	1	3	3	3	3	3	3
4	4	1	2	3	4	4	4	4	4
5	5	1	2	3	4	5	5	5	5
6	6	1	2	3	4	5	6	6	6
7	7	1	2	3	4	5	6	7	6
8	8	1	2	3	4	5	6	6	8

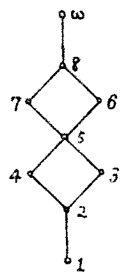


$L_{9,11} = S_1 \uparrow S_1^2 \uparrow S_1^2 \uparrow S_1$. $(c=9)$

⊃	ω	1	2	3	4	5	6	7	8
ω	1	1	1	1	1	1	1	1	1
1	ω	1	2	3	4	5	6	7	8
2	ω	1	1	3	4	5	6	7	8
3	ω	1	1	1	4	4	6	7	8
4	ω	1	1	3	1	3	6	7	8
5	ω	1	1	1	1	1	6	7	8
6	ω	1	1	1	1	1	1	7	7
7	ω	1	1	1	1	1	6	1	6
8	ω	1	1	1	1	1	1	1	1

&	ω	1	2	3	4	5	6	7	8
ω	ω	ω	ω	ω	ω	ω	ω	ω	ω
1	ω	1	2	3	4	5	6	7	8
2	ω	2	2	3	4	5	6	7	8
3	ω	3	3	3	5	5	6	7	8
4	ω	4	4	5	4	5	6	7	8
5	ω	5	5	5	5	5	6	7	8
6	ω	6	6	6	6	6	6	8	8
7	ω	7	7	7	7	7	8	7	8
8	ω	8	8	8	8	8	8	8	8

∨	ω	1	2	3	4	5	6	7	8
ω	ω	1	2	3	4	5	6	7	8
1	1	1	1	1	1	1	1	1	1
2	2	1	2	2	2	2	2	2	2
3	3	1	2	3	2	3	3	3	3
4	4	1	2	2	4	4	4	4	4
5	5	1	2	3	4	5	5	5	5
6	6	1	2	3	4	5	6	5	6
7	7	1	2	3	4	5	5	7	7
8	8	1	2	3	4	5	6	7	8

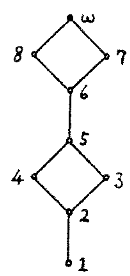


$L_{9,12} = S_1 \uparrow S_1^2 \uparrow S_1 \uparrow S_1^2$. $(c=9)$

⊃	ω	1	2	3	4	5	6	7	8
ω	1	1	1	1	1	1	1	1	1
1	ω	1	2	3	4	5	6	7	8
2	ω	1	1	3	4	5	6	7	8
3	ω	1	1	1	4	4	6	7	8
4	ω	1	1	3	1	3	6	7	8
5	ω	1	1	1	1	1	6	7	8
6	ω	1	1	1	1	1	1	7	8
7	8	1	1	1	1	1	1	1	8
8	7	1	1	1	1	1	1	7	1

&	ω	1	2	3	4	5	6	7	8
ω	ω	ω	ω	ω	ω	ω	ω	ω	ω
1	ω	1	2	3	4	5	6	7	8
2	ω	2	2	3	4	5	6	7	8
3	ω	3	3	3	5	5	6	7	8
4	ω	4	4	5	4	5	6	7	8
5	ω	5	5	5	5	5	6	7	8
6	ω	6	6	6	6	6	6	7	8
7	ω	7	7	7	7	7	7	7	ω
8	ω	8	8	8	8	8	8	ω	8

∨	ω	1	2	3	4	5	6	7	8
ω	ω	1	2	3	4	5	6	7	8
1	1	1	1	1	1	1	1	1	1
2	2	1	2	2	2	2	2	2	2
3	3	1	2	3	2	3	3	3	3
4	4	1	2	2	4	4	4	4	4
5	5	1	2	3	4	5	5	5	5
6	6	1	2	3	4	5	6	6	6
7	7	1	2	3	4	5	6	7	6
8	8	1	2	3	4	5	6	6	8

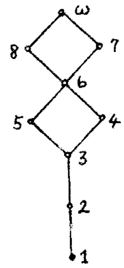


$L_{9.13} = S_2 \uparrow S_1^2 \uparrow S_1^2$. $(c=9)$

⊃	ω	1	2	3	4	5	6	7	8
ω	1	1	1	1	1	1	1	1	1
1	ω	1	2	3	4	5	6	7	8
2	ω	1	1	3	4	5	6	7	8
3	ω	1	1	1	4	5	6	7	8
4	ω	1	1	1	1	5	5	7	8
5	ω	1	1	1	4	1	4	7	8
6	ω	1	1	1	1	1	1	7	8
7	8	1	1	1	1	1	1	1	8
8	7	1	1	1	1	1	1	7	1

&	ω	1	2	3	4	5	6	7	8
ω	ω	ω	ω	ω	ω	ω	ω	ω	ω
1	ω	1	2	3	4	5	6	7	8
2	ω	2	2	3	4	5	6	7	8
3	ω	3	3	3	4	5	6	7	8
4	ω	4	4	4	4	6	6	7	8
5	ω	5	5	5	6	5	6	7	8
6	ω	6	6	6	6	6	6	7	8
7	ω	7	7	7	7	7	7	7	ω
8	ω	8	8	8	8	8	8	ω	8

∨	ω	1	2	3	4	5	6	7	8
ω	ω	1	2	3	4	5	6	7	8
1	1	1	1	1	1	1	1	1	1
2	2	1	2	2	2	2	2	2	2
3	3	1	2	3	3	3	3	3	3
4	4	1	2	3	4	3	4	4	4
5	5	1	2	3	3	5	5	5	5
6	6	1	2	3	4	5	6	6	6
7	7	1	2	3	4	5	6	7	6
8	8	1	2	3	4	5	6	6	8

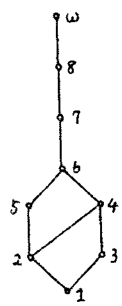


$L_{9.14} = (S_1, S_2) \uparrow S_3 \supset \subset S_5$. $(c=6)$

⊃	ω	1	2	3	4	5	6	7	8
ω	1	1	1	1	1	1	1	1	1
1	ω	1	2	3	4	5	6	7	8
2	ω	1	1	3	3	5	6	7	8
3	ω	1	2	1	2	5	5	7	8
4	ω	1	1	1	1	5	5	7	8
5	ω	1	1	3	3	1	3	7	8
6	ω	1	1	1	1	1	1	7	8
7	ω	1	1	1	1	1	1	1	8
8	ω	1	1	1	1	1	1	1	1

&	ω	1	2	3	4	5	6	7	8
ω	ω	ω	ω	ω	ω	ω	ω	ω	ω
1	ω	1	2	3	4	5	6	7	8
2	ω	2	2	4	4	5	6	7	8
3	ω	3	4	3	4	6	6	7	8
4	ω	4	4	4	4	6	6	7	8
5	ω	5	5	6	6	5	6	7	8
6	ω	6	6	6	6	6	6	7	8
7	ω	7	7	7	7	7	7	7	8
8	ω	8	8	8	8	8	8	8	8

∨	ω	1	2	3	4	5	6	7	8
ω	ω	1	2	3	4	5	6	7	8
1	1	1	1	1	1	1	1	1	1
2	2	1	2	1	2	2	2	2	2
3	3	1	1	3	3	1	3	3	3
4	4	1	2	3	4	2	4	4	4
5	5	1	2	1	2	5	5	5	5
6	6	1	2	3	4	5	6	6	6
7	7	1	2	3	4	5	6	7	7
8	8	1	2	3	4	5	6	7	8

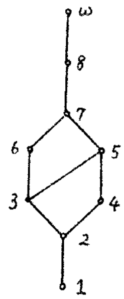


$L_{9,15} = S_1 \uparrow (S_1,\ S_2) \uparrow S_2$. $(c=9)$

⊃	ω	1	2	3	4	5	6	7	8
ω	1	1	1	1	1	1	1	1	1
1	ω	1	2	3	4	5	6	7	8
2	ω	1	1	3	4	5	6	7	8
3	ω	1	1	1	4	4	6	7	8
4	ω	1	1	3	1	3	6	6	8
5	ω	1	1	1	1	1	6	6	8
6	ω	1	1	1	4	4	1	4	8
7	ω	1	1	1	1	1	1	1	8
8	ω	1	1	1	1	1	1	1	1

&	ω	1	2	3	4	5	6	7	8
ω	ω	ω	ω	ω	ω	ω	ω	ω	ω
1	ω	1	2	3	4	5	6	7	8
2	ω	2	2	3	4	5	6	7	8
3	ω	3	3	3	5	5	6	7	8
4	ω	4	4	5	4	5	7	7	8
5	ω	5	5	5	5	5	7	7	8
6	ω	6	6	6	7	7	6	7	8
7	ω	7	7	7	7	7	7	7	8
8	ω	8	8	8	8	8	8	8	8

∨	ω	1	2	3	4	5	6	7	8
ω	ω	1	2	3	4	5	6	7	8
1	1	1	1	1	1	1	1	1	1
2	2	1	2	2	2	2	2	2	2
3	3	1	2	3	2	3	3	3	3
4	4	1	2	2	4	4	2	4	4
5	5	1	2	3	4	5	3	5	5
6	6	1	2	3	2	3	6	6	6
7	7	1	2	3	4	5	6	7	7
8	8	1	2	3	4	5	6	7	8

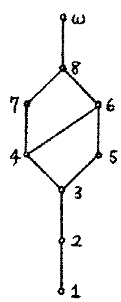


$L_{9,16} = S_2 \uparrow (S_1,\ S_2) \uparrow S_1$. $(c=9)$

⊃	ω	1	2	3	4	5	6	7	8
ω	1	1	1	1	1	1	1	1	1
1	ω	1	2	3	4	5	6	7	8
2	ω	1	1	3	4	5	6	7	8
3	ω	1	1	1	4	5	6	7	8
4	ω	1	1	1	1	5	5	7	8
5	ω	1	1	1	4	1	4	7	7
6	ω	1	1	1	1	1	1	7	7
7	ω	1	1	1	1	5	5	1	5
8	ω	1	1	1	1	1	1	1	1

&	ω	1	2	3	4	5	6	7	8
ω	ω	ω	ω	ω	ω	ω	ω	ω	ω
1	ω	1	2	3	4	5	6	7	8
2	ω	2	2	3	4	5	6	7	8
3	ω	3	3	3	4	5	6	7	8
4	ω	4	4	4	4	6	6	7	8
5	ω	5	5	5	6	5	6	8	8
6	ω	6	6	6	6	6	6	8	8
7	ω	7	7	7	7	8	8	7	8
8	ω	8	8	8	8	8	8	8	8

∨	ω	1	2	3	4	5	6	7	8
ω	ω	1	2	3	4	5	6	7	8
1	1	1	1	1	1	1	1	1	1
2	2	1	2	2	2	2	2	2	2
3	3	1	2	3	3	3	3	3	3
4	4	1	2	3	4	3	4	4	4
5	5	1	2	3	3	5	5	3	5
6	6	1	2	3	4	5	6	4	6
7	7	1	2	3	4	3	4	7	7
8	8	1	2	3	4	5	6	7	8

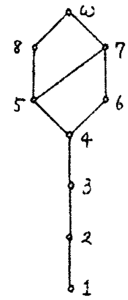


$L_{9,17} = S_3 \uparrow (S_1,\ S_2)$. $(c=9)$

⊃	ω	1	2	3	4	5	6	7	8
ω	1	1	1	1	1	1	1	1	1
1	ω	1	2	3	4	5	6	7	8
2	ω	1	1	3	4	5	6	7	8
3	ω	1	1	1	4	5	6	7	8
4	ω	1	1	1	1	5	6	7	8
5	ω	1	1	1	1	1	6	6	8
6	8	1	1	1	1	5	1	5	8
7	8	1	1	1	1	1	1	1	8
8	6	1	1	1	1	1	6	6	1

&	ω	1	2	3	4	5	6	7	8
ω	ω	ω	ω	ω	ω	ω	ω	ω	ω
1	ω	1	2	3	4	5	6	7	8
2	ω	2	2	3	4	5	6	7	8
3	ω	3	3	3	4	5	6	7	8
4	ω	4	4	4	4	5	6	7	8
5	ω	5	5	5	5	5	7	7	8
6	ω	6	6	6	6	7	6	7	ω
7	ω	7	7	7	7	7	7	7	ω
8	ω	8	8	8	8	8	ω	ω	8

∨	ω	1	2	3	4	5	6	7	8
ω	ω	1	2	3	4	5	6	7	8
1	1	1	1	1	1	1	1	1	1
2	2	1	2	2	2	2	2	2	2
3	3	1	2	3	3	3	3	3	3
4	4	1	2	3	4	4	4	4	4
5	5	1	2	3	4	5	4	5	5
6	6	1	2	3	4	4	6	6	4
7	7	1	2	3	4	5	6	7	5
8	8	1	2	3	4	5	4	5	8

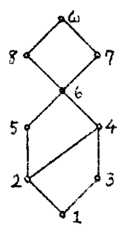


$L_{9,18} = (S_1,\ S_2) \uparrow S_1{}^2 \supset \subset (S_1{}^3,\ S_2 \uparrow S_1{}^2) \supset \subset L_{6,4}$. $(c=6)$

⊃	ω	1	2	3	4	5	6	7	8
ω	1	1	1	1	1	1	1	1	1
1	ω	1	2	3	4	5	6	7	8
2	ω	1	1	3	3	5	6	7	8
3	ω	1	2	1	2	5	5	7	8
4	ω	1	1	1	1	5	5	7	8
5	ω	1	1	3	3	1	3	7	8
6	ω	1	1	1	1	1	1	7	8
7	8	1	1	1	1	1	1	1	8
8	7	1	1	1	1	1	1	7	1

&	ω	1	2	3	4	5	6	7	8
ω	ω	ω	ω	ω	ω	ω	ω	ω	ω
1	ω	1	2	3	4	5	6	7	8
2	ω	2	2	4	4	5	6	7	8
3	ω	3	4	3	4	6	6	7	8
4	ω	4	4	4	4	6	6	7	8
5	ω	5	5	6	6	5	6	7	8
6	ω	6	6	6	6	6	6	7	8
7	ω	7	7	7	7	7	7	7	ω
8	ω	8	8	8	8	8	8	ω	8

∨	ω	1	2	3	4	5	6	7	8
ω	ω	1	2	3	4	5	6	7	8
1	1	1	1	1	1	1	1	1	1
2	2	1	2	1	2	2	2	2	2
3	3	1	1	3	3	1	3	3	3
4	4	1	2	3	4	2	4	4	4
5	5	1	2	1	2	5	5	5	5
6	6	1	2	3	4	5	6	6	6
7	7	1	2	3	4	5	6	7	6
8	8	1	2	3	4	5	6	6	8

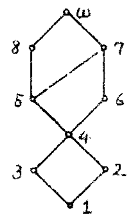


$L_{9,19} = S_1^2 \uparrow (S_1, S_2) \supset \subset S_1 \uparrow (S_1, S_2) = L_{7,8}$. $(c=7)$

⊃	ω	1	2	3	4	5	6	7	8
ω	1	1	1	1	1	1	1	1	1
1	ω	1	2	3	4	5	6	7	8
2	ω	1	1	3	3	5	6	7	8
3	ω	1	2	1	2	5	6	7	8
4	ω	1	1	1	1	5	6	7	8
5	ω	1	1	1	1	1	6	6	8
6	8	1	1	1	1	5	1	5	8
7	8	1	1	1	1	1	1	1	8
8	6	1	1	1	1	1	6	6	1

&	ω	1	2	3	4	5	6	7	8
ω	ω	ω	ω	ω	ω	ω	ω	ω	ω
1	ω	1	2	3	4	5	6	7	8
2	ω	2	2	4	4	5	6	7	8
3	ω	3	4	3	4	5	6	7	8
4	ω	4	4	4	4	5	6	7	8
5	ω	5	5	5	5	5	7	7	8
6	ω	6	6	6	6	7	6	7	ω
7	ω	7	7	7	7	7	7	7	ω
8	ω	8	8	8	8	8	ω	ω	8

∨	ω	1	2	3	4	5	6	7	8
ω	ω	1	2	3	4	5	6	7	8
1	1	1	1	1	1	1	1	1	1
2	2	1	2	1	2	2	2	2	2
3	3	1	1	3	3	3	3	3	3
4	4	1	2	3	4	4	4	4	4
5	5	1	2	3	4	5	4	5	5
6	6	1	2	3	4	4	6	6	4
7	7	1	2	3	4	5	6	7	5
8	8	1	2	3	4	5	4	5	8

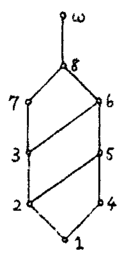


$L_{9,20} = (S_1, S_3) \uparrow S_1 \supset \subset S_4$. $(c=5)$

⊃	ω	1	2	3	4	5	6	7	8
ω	1	1	1	1	1	1	1	1	1
1	ω	1	2	3	4	5	6	7	8
2	ω	1	1	3	4	4	6	7	8
3	ω	1	1	1	4	4	4	7	8
4	ω	1	2	3	1	2	3	7	7
5	ω	1	1	3	1	1	3	7	7
6	ω	1	1	1	1	1	1	7	7
7	ω	1	1	1	4	4	4	1	4
8	ω	1	1	1	1	1	1	1	1

&	ω	1	2	3	4	5	6	7	8
ω	ω	ω	ω	ω	ω	ω	ω	ω	ω
1	ω	1	2	3	4	5	6	7	8
2	ω	2	2	3	5	5	6	7	8
3	ω	3	3	3	6	6	6	7	8
4	ω	4	5	6	4	5	6	8	8
5	ω	5	5	6	5	5	6	8	8
6	ω	6	6	6	6	6	6	8	8
7	ω	7	7	7	8	8	8	7	8
8	ω	8	8	8	8	8	8	8	8

∨	ω	1	2	3	4	5	6	7	8
ω	ω	1	2	3	4	5	6	7	8
1	1	1	1	1	1	1	1	1	1
2	2	1	2	2	1	2	2	2	2
3	3	1	2	3	1	2	3	3	3
4	4	1	1	1	4	4	4	1	4
5	5	1	2	2	4	5	5	2	5
6	6	1	2	3	4	5	6	3	6
7	7	1	2	3	1	2	3	7	7
8	8	1	2	3	4	5	6	7	8

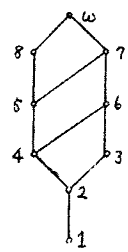


$L_{9,21} = S_1 \uparrow (S_1, S_3)$ $(c=9)$

⊃	ω	1	2	3	4	5	6	7	8
ω	1	1	1	1	1	1	1	1	1
1	ω	1	2	3	4	5	6	7	8
2	ω	1	1	3	4	5	6	7	8
3	8	1	1	1	4	5	4	5	8
4	ω	1	1	3	1	5	3	7	8
5	ω	1	1	3	1	1	3	3	8
6	8	1	1	1	1	5	1	5	8
7	8	1	1	1	1	1	1	1	8
8	3	1	1	3	1	1	3	3	1

&	ω	1	2	3	4	5	6	7	8
ω	ω	ω	ω	ω	ω	ω	ω	ω	ω
1	ω	1	2	3	4	5	6	7	8
2	ω	2	2	3	4	5	6	7	8
3	ω	3	3	3	6	7	6	7	ω
4	ω	4	4	6	4	5	6	7	8
5	ω	5	5	7	5	5	7	7	8
6	ω	6	6	6	6	7	6	7	ω
7	ω	7	7	7	7	7	7	7	ω
8	ω	8	8	ω	8	8	ω	ω	8

∨	ω	1	2	3	4	5	6	7	8
ω	ω	1	2	3	4	5	6	7	8
1	1	1	1	1	1	1	1	1	1
2	2	1	2	2	2	2	2	2	2
3	3	1	2	3	2	2	3	3	2
4	4	1	2	2	4	4	4	4	4
5	5	1	2	2	4	5	4	5	5
6	6	1	2	3	4	4	6	6	4
7	7	1	2	3	4	5	6	7	5
8	8	1	2	2	4	5	4	5	8

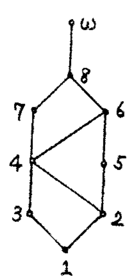


$L_{9,22} = L_{8,14} \uparrow S_1$. $(c=6)$

⊃	ω	1	2	3	4	5	6	7	8
ω	1	1	1	1	1	1	1	1	1
1	ω	1	2	3	4	5	6	7	8
2	ω	1	1	3	3	5	6	7	8
3	ω	1	2	1	2	5	5	7	8
4	ω	1	1	1	1	5	5	7	8
5	ω	1	1	3	3	1	3	7	7
6	ω	1	1	1	1	1	1	7	7
7	ω	1	1	1	1	5	5	1	5
8	ω	1	1	1	1	1	1	1	1

&	ω	1	2	3	4	5	6	7	8
ω	ω	ω	ω	ω	ω	ω	ω	ω	ω
1	ω	1	2	3	4	5	6	7	8
2	ω	2	2	4	4	5	6	7	8
3	ω	3	4	3	4	6	6	7	8
4	ω	4	4	4	4	6	6	7	8
5	ω	5	5	6	6	5	6	8	8
6	ω	6	6	6	6	6	6	8	8
7	ω	7	7	7	7	8	8	7	8
8	ω	8	8	8	8	8	8	8	8

∨	ω	1	2	3	4	5	6	7	8
ω	ω	1	2	3	4	5	6	7	8
1	1	1	1	1	1	1	1	1	1
2	2	1	2	1	2	2	2	2	2
3	3	1	1	3	3	1	3	3	3
4	4	1	2	3	4	2	4	4	4
5	5	1	2	1	2	5	5	2	5
6	6	1	2	3	4	5	6	4	6
7	7	1	2	3	4	2	4	7	7
8	8	1	2	3	4	5	6	7	8

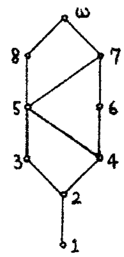


$L_{9,23} = S_1 \uparrow L_{8,14}$. $(c=9)$

$\supset$	ω	1	2	3	4	5	6	7	8
ω	1	1	1	1	1	1	1	1	1
1	ω	1	2	3	4	5	6	7	8
2	ω	1	1	3	4	5	6	7	8
3	ω	1	1	1	4	4	6	6	8
4	ω	1	1	3	1	3	6	7	8
5	ω	1	1	1	1	1	6	6	8
6	8	1	1	3	1	3	1	3	8
7	8	1	1	1	1	1	1	1	8
8	6	1	1	1	1	1	6	6	1

&	ω	1	2	3	4	5	6	7	8
ω	ω	ω	ω	ω	ω	ω	ω	ω	ω
1	ω	1	2	3	4	5	6	7	8
2	ω	2	2	3	4	5	6	7	8
3	ω	3	3	3	5	5	7	7	8
4	ω	4	4	5	4	5	6	7	8
5	ω	5	5	5	5	5	7	7	8
6	ω	6	6	7	6	7	6	7	ω
7	ω	7	7	7	7	7	7	7	ω
8	ω	8	8	8	8	8	ω	ω	8

$\vee$	ω	1	2	3	4	5	6	7	8
ω	ω	1	2	3	4	5	6	7	8
1	1	1	1	1	1	1	1	1	1
2	2	1	2	2	2	2	2	2	2
3	3	1	2	3	2	3	2	3	3
4	4	1	2	2	4	4	4	4	4
5	5	1	2	3	4	5	4	5	5
6	6	1	2	2	4	4	6	6	4
7	7	1	2	3	4	5	6	7	5
8	8	1	2	3	4	5	4	5	8

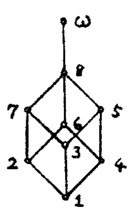


$L_{9,24} = S_1^3 \uparrow S_1 \supset \subset S_2$. $(c=3)$

$\supset$	ω	1	2	3	4	5	6	7	8
ω	1	1	1	1	1	1	1	1	1
1	ω	1	2	3	4	5	6	7	8
2	ω	1	1	3	4	5	4	3	5
3	ω	1	2	1	4	4	6	2	6
4	ω	1	2	3	1	3	2	7	7
5	ω	1	2	1	1	1	2	2	2
6	ω	1	1	3	1	3	1	3	3
7	ω	1	1	1	4	4	4	1	4
8	ω	1	1	1	1	1	1	1	1

&	ω	1	2	3	4	5	6	7	8
ω	ω	ω	ω	ω	ω	ω	ω	ω	ω
1	ω	1	2	3	4	5	6	7	8
2	ω	2	2	7	6	8	6	7	8
3	ω	3	7	3	5	5	8	7	8
4	ω	4	6	5	4	5	6	8	8
5	ω	5	8	5	5	5	8	8	8
6	ω	6	6	8	6	8	6	8	8
7	ω	7	7	7	8	8	8	7	8
8	ω	8	8	8	8	8	8	8	8

$\vee$	ω	1	2	3	4	5	6	7	8
ω	ω	1	2	3	4	5	6	7	8
1	1	1	1	1	1	1	1	1	1
2	2	1	2	1	1	1	2	2	2
3	3	1	1	3	1	3	1	3	3
4	4	1	1	1	4	4	4	1	4
5	5	1	1	3	4	5	4	3	5
6	6	1	2	1	4	4	6	2	6
7	7	1	2	3	1	3	2	7	7
8	8	1	2	3	4	5	6	7	8

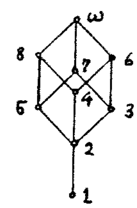


$L_{9.25} = S_1 \uparrow S_1^3$. $(c=9)$

⊃	ω	1	2	3	4	5	6	7	8
ω	1	1	1	1	1	1	1	1	1
1	ω	1	2	3	4	5	6	7	8
2	ω	1	1	3	4	5	6	7	8
3	8	1	1	1	4	5	4	5	8
4	7	1	1	3	1	5	3	7	5
5	6	1	1	3	4	1	6	3	4
6	5	1	1	1	1	5	1	5	5
7	4	1	1	1	4	1	4	1	4
8	3	1	1	3	1	1	3	3	1

&	ω	1	2	3	4	5	6	7	8
ω	ω	ω	ω	ω	ω	ω	ω	ω	ω
1	ω	1	2	3	4	5	6	7	8
2	ω	2	2	3	4	5	6	7	8
3	ω	3	3	3	6	7	6	7	ω
4	ω	4	4	6	4	8	6	ω	8
5	ω	5	5	7	8	5	ω	7	8
6	ω	6	6	6	6	ω	6	ω	ω
7	ω	7	7	7	ω	7	ω	7	ω
8	ω	8	8	ω	8	8	ω	ω	8

∨	ω	1	2	3	4	5	6	7	8
ω	ω	1	2	3	4	5	6	7	8
1	1	1	1	1	1	1	1	1	1
2	2	1	2	2	2	2	2	2	2
3	3	1	2	3	2	2	3	3	2
4	4	1	2	2	4	2	4	2	4
5	5	1	2	2	2	5	2	5	5
6	6	1	2	3	4	2	6	3	4
7	7	1	2	3	2	5	3	7	5
8	8	1	2	2	4	5	4	5	8

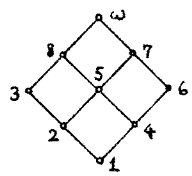


$L_{9.26} = S_2^2 \supset \subset S_2$. $(c=3)$

⊃	ω	1	2	3	4	5	6	7	8
ω	1	1	1	1	1	1	1	1	1
1	ω	1	2	3	4	5	6	7	8
2	ω	1	1	3	4	4	6	6	8
3	6	1	1	1	4	4	6	6	4
4	ω	1	2	3	1	2	6	7	3
5	ω	1	1	3	1	1	6	6	3
6	3	1	2	3	1	2	1	2	3
7	3	1	1	3	1	1	1	1	3
8	6	1	1	1	1	1	6	6	1

&	ω	1	2	3	4	5	6	7	8
ω	ω	ω	ω	ω	ω	ω	ω	ω	ω
1	ω	1	2	3	4	5	6	7	8
2	ω	2	2	3	5	5	7	7	8
3	ω	3	3	3	8	8	ω	ω	8
4	ω	4	5	8	4	5	6	7	8
5	ω	5	5	8	5	5	7	7	8
6	ω	6	7	ω	6	7	6	7	ω
7	ω	7	7	ω	7	7	7	7	ω
8	ω	8	8	8	8	8	ω	ω	8

∨	ω	1	2	3	4	5	6	7	8
ω	ω	1	2	3	4	5	6	7	8
1	1	1	1	1	1	1	1	1	1
2	2	1	2	2	1	2	1	2	2
3	3	1	2	3	1	2	1	2	3
4	4	1	1	1	4	4	4	4	4
5	5	1	2	2	4	5	4	5	5
6	6	1	1	1	4	4	6	6	4
7	7	1	2	2	4	5	6	7	5
8	8	1	2	3	4	5	4	5	8



University of Tokyo

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