On the blowing up of solutions of the Cauchy problem for $u_t = \Delta u + u^{1+\alpha}$

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§ 1. Introduction and the result.

Solutions of initial value problems for quasi-linear parabolic partial differential equations may not exist for all time. In other words, these solutions may blow up in some sense or other. Recently in connection with problems for some class of quasi-linear parabolic equations, Kaplan [5], Itô [4] and Friedman [1] gave certain sufficient conditions under which the solutions blow up in a finite time. Although their results are not identical, we can say according to them that the solutions are apt to blow up when the initial values are sufficiently large.

On the other hand, it is commonly believed that the dimension of the x-space, x being the space variable, has a crucial influence on the conditions for the solutions of quasi-linear equations to exist for all time. As an example we can refer to the Navier-Stokes equation (for example, see [3, 6]), for which the situation concerning global existence is quite different according as the dimension of the x-space is 2 or 3.

The main objective of the present paper is to illustrate the following through the Cauchy problem in the title above: 1) Concerning quasi-linear initial value problems with an unbounded x-domain, the solution may blow up, however small the initial value may be. 2) The dimension of the x-space and the 'degree of non-linearity' of the equation have their combined effect on the global existence or the blowing up of solutions of these problems.

Now we pose our Cauchy problem with the unkonwn function $u = u(t, x)$ of $t \geq 0$ and $x \in R^m$, R^m being the m-dimensional Euclidean space:

$$u_t = \Delta u + u^{1+\alpha}, \qquad (t > 0, x \in R^m), \tag{1.1}$$

$$u|_{t=0} = a(x), \qquad (x \in R^m). \tag{1.2}$$

Here $a = a(x)$ is a given function called the initial value of u. α is a positive parameter. Throughout the present paper we shall deal only with non-negative solutions so that there is no ambiguity in the meaning of $u^{1+\alpha}$. The initial value problem (the Cauchy problem) consisting of (1.1) and (1.2) is denoted by IVP. Below in the second half of this section we shall specify the class of functions within which the solution is sought and shall state our main results in Theorems 1 and 2. However, let us describe the results here in a rough but intuitive way. If $0<m\alpha<2$, then every non-negative solution of IVP blows up eventually except the trivial solution $u\equiv 0$. If $2<m\alpha$, there are many non-negative initial values $a=a(x)$ which give global solutions. This appears somewhat remarkable inasmuch as the inevitable blowing up occurs rather in the case of smaller α. Also, the result for $\alpha<2/m$ forms a striking contrast to the fact that the initial value problem (1.1) and (1.2) has a global solution for a sufficiently small and sufficiently smooth $a(x)$ if R^m is replaced by a bounded domain with smooth boundary and if some appropriate boundary condition, say, the Dirichlet boundary condition is imposed.

We proceed to formulation of IVP.

DEFINITION 1.1. A non-negative function $u=u(t, x)$ is called a regular solution of IVP in $[0, T]$, T being a positive number, if u, $\nabla_x u$, $\nabla_x\nabla_x u$ and u_t all exist and are continuous in $Q_T=[0, T]\times R^m$ and if (1.1) and (1.2) are satisfied. A regular solution u of IVP in $[0, \infty)$ is a function whose restriction to $[0, T]\times R^m$ is a regular solution of IVP in $[0, T]$ for any $T>0$.

According to the definition above, a regular solution is smooth to some extent but nothing is assumed on its growth as $|x|\to+\infty$. We recall that even in the linear case ($\alpha=0$), we need some restriction concerning the growth of the solutions in order to have the uniqueness of the solution. In view of this, we shall restrict solutions of IVP into a class defined by Definition 1.2. At the same time, we note that the wider the class $\mathscr{E}$ in Theorem 1 is, the deeper the theorem is in its implication.

DEFINITION 1.2. T being a positive number, $\mathscr{E}[0, T]$ is the set of all continuous functions $u=u(t, x)$ defined in $[0,T]\times R^m$ satisfying

$$(1.3) \qquad |u(t, x)|\leq M\exp(|x|^\beta) \qquad (0\leq t\leq T, x\in R^m)$$

with some constants M and β subject to $M>0$ and $0<\beta<2$. M and β may depend on u. Furthermore, $\mathscr{E}[0, \infty)$ is the set of all u whose restriction to $[0, T]\times R^m$ belongs to $\mathscr{E}[0, T]$ for any $T>0$.

We note that $u^{1+\alpha}\in\mathscr{E}[0, T]$ if $u\in\mathscr{E}[0, T]$ and $u\geq 0$.

DEFINITION 1.3. If u is a regular solution of IVP in $[0, T]$ and at the same time $u\in\mathscr{E}[0, T]$, then u is called a regular soultion of IVP in $\mathscr{E}[0, T]$. Here we may replace $\mathscr{E}[0, T]$ by $\mathscr{E}[0, \infty)$. A regular solution u of IVP in $\mathscr{E}[0, \infty)$ is also called a global solution of IVP in $\mathscr{E}[0, \infty)$.

Next, we specify the class of initial values. As a standing assumption, we assume that the initial value $a=a(x)$ of IVP is taken from the class $\mathscr{A}$ defined by the following

DEFINITION 1.4. $\mathscr{A}$ is the set of all non-negative functions $a=a(x)$ on R^m such that $a, \nabla_x a$ and $\nabla_x \nabla_x a$ are all continuous and bounded there.

When compared with $\mathscr{E}[0, T]$, $\mathscr{A}$ may seem to be composed of too restricted functions with respect to the growth order. However, we can avoid unessential difficulties by taking nice initial values. Moreover, our stress at the present paper is laid on the fact that under the assumption of Theorem 1, the blowing up takes place even when the initial value a is so nice.

We are ready to state our theorems.

THEOREM 1. *Let $0<m\alpha<2$. Suppose that $a\in\mathscr{A}$ does not vanish identically. Then there is no global solution of* IVP *in* $\mathscr{E}[0, \infty)$.

This theorem will be proved in § 2 by a method which was suggested by and is partly identical with Kaplan's method in [5]. The following theorem involves the Green function of the heat equation. We put here and hereafter

(1.4) $$H(t, x)=(4\pi t)^{-m/2}\exp\left(-\frac{|x|^2}{4t}\right), \qquad (t>0, x\in R^m).$$

THEOREM 2. *Let $2<m\alpha$. Take any positive number γ. Then there exists a positive number δ with the following property: if $a\in\mathscr{A}$ and $0\le a(x)\le\delta H(\gamma, x)$, then there exists a global solution $u=u(t, x)$ of* IVP *in* $\mathscr{E}[0, \infty)$, *which is subject to*

(1.5) $$0\le u(t, x)\le M\, H(t+\gamma, x)\ , \qquad (t\ge 0, x\in R^m),$$

for some positive constant M.

Theorem 2 will be proved in § 3. Several propositions used in §§ 2 and 3 will be established in Appendix.

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§ 2. Proof of Theorem 1.

As a preparation for the proof of Theorem 1, we state the following

LEMMA 2.1. *Let $u=u(t, x)$ be a regular solution of* IVP *in* $\mathscr{E}[0, T]$ *with a nontrivial initial value $a\in\mathscr{A}$. Then we have*

(2.1) $$J_0^{-\alpha}-u(t, 0)^{-\alpha}\ge\alpha t\ , \qquad (0\le t\le T),$$

where

(2.2) $$J_0=J_0(t)=\int_{R^m}H(t, x)a(x)dx.$$

PROOF. Let ε be a positive constant. Take t in $0 \leq t \leq T$ and fix it. Then we put

(2.3) $$v_\varepsilon = v_\varepsilon(s, x) = H(t-s+\varepsilon, x), \qquad (0 \leq s \leq t \leq T, x \in R^m),$$

and

(2.4) $$J_\varepsilon = J_\varepsilon(s) = \int_{R^m} v_\varepsilon(s, x) u(s, x) dx.$$

v_ε is regular in $[0, T] \times R^m$ and satisfies the adjoint heat equation

(2.5) $$\frac{\partial}{\partial s} v_\varepsilon = -\Delta v_\varepsilon.$$

We claim that $J_\varepsilon > 0$ for all $s \in [0, t]$. In fact, $v_\varepsilon(s, x)$ is positive everywhere in $[0, t] \times R^m$. $u(s, x)$ is also positive in $(0, t] \times R^m$ according to Proposition A2 in Appendix. By the assumption, $u(0, x) \equiv a(x) \geq 0$ and $a(x_0) > 0$ for some point $x_0 \in R^m$. Taking account of the continuity of v_ε and u, we see that $J_\varepsilon > 0$ if it exists. We now claim that J_ε exists and is continuous in s. Since $u \in \mathscr{E}[0, T]$, there are some positive constants M and $\beta < 2$ such that

(2.6) $$0 \leq u(s, x) \leq M \exp(|x|^\beta), \qquad (0 \leq s \leq T, x \in R^m).$$

Hence changing the variable of integration by

$$x = 2\sqrt{t-s+\varepsilon}\, \eta,$$

we have

(2.7) $$\begin{aligned} 0 \leq J_\varepsilon(s) &\leq M \int_{R^m} H(t-s+\varepsilon, x) \exp(|x|^\beta) dx \\ &= M\pi^{-\frac{m}{2}} \int_{R^m} \exp(-|\eta|^2) \exp(|2\sqrt{t-s+\varepsilon}|^\beta |\eta|^\beta) d\eta \\ &\leq M\pi^{-\frac{m}{2}} \int_{R^m} \exp(-|\eta|^2 + \gamma |\eta|^\beta) d\eta, \end{aligned}$$

where $\gamma = 2^\beta (t+\varepsilon)^{\beta/2}$. The last inequality shows that the integral defining J_ε exists uniformly and is a continuous function of s.

Furthermore, J_ε is continuously differentiable and satisfies the following equation.

(2.8) $$\frac{d}{ds} J_\varepsilon(s) = \int_{R^m} v_\varepsilon(s, x) u(s, x)^{1+\alpha} dx.$$

This can be seen as follows. Take a function $\rho \in C_0^\infty(R^m)$ such that $1 \geq \rho \geq 0$ and

$$\rho(x)=\begin{cases} 1, & (|x|\leq 1), \\ 0, & (|x|\geq 2). \end{cases}$$

Then we define ρ_N $(N=1, 2, \cdots)$ by

$$\rho_N(x)=\rho\left(\frac{x}{N}\right) \tag{2.9}$$

and call them the truncating functions. We put

$$j^{(N)}(s)=\int_{R^m} v_\varepsilon\cdot u\cdot\rho_N dx \qquad (N=1, 2, \cdots). \tag{2.10}$$

By what we have seen above, $j^{(N)}(s)$ tends to $J_\varepsilon(s)$ as $N\to\infty$ uniformly in s. Since its integrand is compactly supported, $j^{(N)}(s)$ can be differentiated under the integral sign. Namely we have

$$\begin{aligned}\frac{d}{ds}j^{(N)}(s)&=\int_{R^m}\left(\frac{\partial}{\partial s}v_\varepsilon\cdot u+v_\varepsilon\cdot\frac{\partial}{\partial s}u\right)\rho_N dx\\ &=\int_{R^m}\{-\Delta v_\varepsilon\cdot u+v_\varepsilon\cdot\Delta u\}\rho_N dx\\ &\quad+\int_{R^m}v_\varepsilon\cdot u^{1+\alpha}\cdot\rho_N dx\\ &\equiv I_1+I_2.\end{aligned}$$

Here use has been made of (1.1) and (2.5). I_2 tends to the integral on the right side of (2.8) as $N\to\infty$ uniformly, since $u^{1+\alpha}\in\mathscr{E}[0, T]$. We have to show $I_1\to 0$. By integration by parts we get

$$I_1=-2\int_{R^m}v_\varepsilon\cdot\nabla\rho_N\cdot\nabla u dx-\int_{R^m}v_\varepsilon\cdot u\cdot\Delta\rho_N dx. \tag{2.11}$$

The last integral on the right side of (2.11) can be estimated with the aid of (2.6) as

$$p_N\equiv\left|\int_{R^m}v_\varepsilon\cdot u\cdot\Delta\rho_N dx\right|\leq CN^{-2}\int_{R^m}v_\varepsilon\cdot u dx\leq CN^{-2},$$

since $|\Delta\rho_N|\leq CN^{-2}$. Here and hereafter we shall denote positive constants indifferently by one and the same symbol C. Thus $p_N\to 0$ uniformly as $N\to\infty$. In order to deal with the first integral on the right side of (2.11) we need Proposition A3 in Appendix, which asserts that $\frac{\partial u}{\partial x_k}$ is in $\mathscr{E}[0, T]$ also, $(k=1, 2, \cdots, m)$. Therefore, as in (2.7), we have

$$\int_{R^m}v_\varepsilon\cdot|\nabla u|dx\leq C,$$

whence follows

$$q_N=\left|\int_{R^m} v_\varepsilon \cdot \nabla\rho_N \cdot \nabla u dx\right| \leq CN^{-1}$$

in virtue of $|\nabla\rho_N|\leq CN^{-1}$. Thus $q_N \to 0$ uniformly as $N\to\infty$. In this way, we obtain the uniform convergence of $\frac{dj^{(N)}}{ds}$ to the right side of (2.8) and have (2.8) consequently. At the same time, the continuity of $\frac{dJ_\varepsilon}{ds}$ follows from that of $\frac{dj^{(N)}}{ds}$.

We proceed to derivation of the differential inequality

$$\frac{dJ_\varepsilon}{ds} \geq J_\varepsilon^{1+\alpha}, \qquad (0\leq s\leq t). \tag{2.12}$$

Noting that v_ε is positive and satisfies

$$\int_{R^m} v_\varepsilon(s, x)dx=1,$$

we apply Jensen's inequality to the right side of (2.8) and obtain (2.12). Here essential use has been made of the convexity of $u^{1+\alpha}$ as a function of the real variable u. Solving (2.12), we have

$$J_\varepsilon(0)^{-\alpha}-J_\varepsilon(t)^{-\alpha}\geq \alpha t. \tag{2.13}$$

Finally we make $\varepsilon\to 0$. Then we have

$$J_\varepsilon(t)\to u(t, 0) \quad \text{and} \quad J_\varepsilon(0)\to J_0 \qquad (\varepsilon\to 0), \tag{2.14}$$

owing to the well-known properties of the Green function H. Combining (2.13) and (2.14), we end up with (2.1).

PROOF OF THEOREM 1. We are going to prove Theorem 1 by reduction to absurdity. On the contrary to the theorem, suppose that there exists a global solution u of IVP in $\mathscr{E}[0, \infty)$ and that the initial value a of u is not trivial. Then (2.1) is applicable with any positive t and gives

$$J_0^{-\alpha}\geq u(t, 0)^{-\alpha}+\alpha t\geq \alpha t. \tag{2.15}$$

We need an estimate of J_0 from below. Without loss of generality we can assume that $a(x)$ is positive in a neighborhood of the origin. Then we can choose positive constants γ and δ such that $|x|<2\delta$ implies $a(x)\geq\gamma$. From now on, we restrict ourselves to $t\geq\delta^2$. Then we can estimate as

$$J_0=\int_{R^m} H(t, x)a(x)dx \geq \int_{|x|<2\delta} H(t, x)a(x)dx$$
$$\geq \gamma\int_{|x|<2\delta}(4\pi t)^{-m/2}e^{-1}dx.$$

Consequently,

(2.16) $$J_0 \geq c_1 t^{-m/2}, \qquad (t\geq\delta^2),$$

with a certain positive constant c_1. Substitution of (2.16) into (2.15) yields

(2.17) $$t^{(m\alpha)/2} \geq \alpha c_1^{\alpha} t, \qquad (t\geq\delta^2).$$

However, (2.17) is impossible for sufficiently large t, because $\frac{m\alpha}{2}<1$ by the assumption. This completes the proof of Theorem 1.

§ 3. Proof of Theorem 2.

In this section we shall prove Theorem 2 by constructing the global solution mentioned there. First of all, suppose that we are given a function $a \in \mathscr{S}$ subject to

(3.1) $$0\leq a(x)\leq \delta H(\gamma, x)$$

for some positive constants γ and δ. We fix γ here and will determine δ later. The theorem will follow with the aid of Proposition A4, if we construct a global solution of the following integral equation associated with IVP.

(3.2) $$u(t, x)=u_0(t, x)+\int_0^t ds\int_{R^m} H(t-s, x-y)u(s, y)^{1+\alpha}dy,$$

where

(3.2)′ $$u_0(t, x)=\int_{R^m} H(t, x-y)a(y)dy.$$

This integral equation is denoted by IE. We are going to solve IE by iteration in the class $\mathscr{S}[0, \infty)$ specified below.

DEFINITION 3.1. $\mathscr{S}[0, \infty)$ is the set of all non-negative continuous functions $u=u(t, x)$ defined in $[0, \infty)\times R^m$ such that the inequality

(3.3) $$0\leq u(t, x)\leq MH(t+\gamma, x), \qquad (t\geq 0, x\in R^m),$$

is satisfied for some constant M which may depend on u.

We put

(3.4) $$\|v\|=\sup_{x\in R^m, 0\leq t}\frac{|v(t, x)|}{\rho(t, x)}$$

for any function v with $|v| \in \mathscr{S}[0, \infty)$, $\rho(t, x)$ being $H(t+\gamma, x)$. Then obviously we have

$$|v(t, x)| \leq ||v|| \rho(t, x) .$$

For instance, (3.1) implies

$$0 \leq u_0(t, x) \leq \delta \int_{R^m} H(t, x-y) H(\gamma, y) dy = \delta H(t+\gamma, x)$$
$$= \delta \rho(t, x)$$

and thus $||u_0|| \leq \delta$.

We study the non-linear integral transformation Φ on the right side of (3.2):

$$(\Phi u)(t, x) = \int_0^t ds \int_{R^m} H(t-s, x-y) u^{1+\alpha}(s, y) dy. \tag{3.5}$$

LEMMA 3.1. *Let* $m\alpha > 2$. *Then,* $\Phi\rho \in \mathscr{S}[0, \infty)$ *and*

$$||\Phi\rho|| \leq c_0, \tag{3.6}$$

where $\rho = H(t+\gamma, x)$ *and* c_0 *is a constant given by*

$$c_0 = (4\pi)^{-m\alpha/2} \int_0^\infty (s+\gamma)^{-m\alpha/2} ds \tag{3.7}$$
$$= \frac{2\gamma}{2-m\alpha} (4\pi\gamma)^{-m\alpha/2}.$$

PROOF. The continuity of $\Phi\rho$ is clear. $\Phi\rho \geq 0$ is obvious. We note

$$\rho^\alpha = H^\alpha(s+\gamma, y) \leq (4\pi(s+\gamma))^{-m\alpha/2}, \tag{3.8}$$

for $\exp\left(-\frac{\alpha |y|^2}{4(s+\gamma)}\right) \leq 1$. (3.8) yields

$$0 \leq \Phi\rho \leq \int_0^t ds \int_{R^m} H(t-s, x-y)(4\pi(s+\gamma))^{-m\alpha/2} H(s+\gamma, y) dy$$
$$= (4\pi)^{-m\alpha/2} \int_0^t (s+\gamma)^{-m\alpha/2} H(t+\gamma, x) ds$$
$$\leq \left((4\pi)^{-m\alpha/2} \int_0^\infty (s+\gamma)^{-m\alpha/2} ds\right) H(t+\gamma, x)$$
$$= c_0 \rho(t, x).$$

Here use has been made of the evolution property of the Green function H. We note that the existence of the integral in (3.7) is based on the assumption $m\alpha > 2$. Now (3.6) is clear and we have established the lemma.

As corollaries of Lemma 3.1, we have the following Lemmas 3.2 and 3.3, where c_0 means the same as above.

LEMMA 3.2. *Let $m\alpha>2$ and let $u\in\mathscr{S}[0,\infty)$. Then $\Phi u\in\mathscr{S}[0,\infty)$ and we have*

$$\|\Phi u\|\le c_0\|u\|^{1+\alpha}. \tag{3.9}$$

PROOF. (3.9) is clear in view of

$$0\le(\Phi u)(t,x)\le\|u\|^{1+\alpha}(\Phi\rho)(t,x)\le c_0\|u\|^{1+\alpha}\rho(t,x)$$

due to the preceding lemma.

LEMMA 3.3. *Let $m\alpha>2$. Suppose that u and v are in $\mathscr{S}[0,\infty)$ and they satisfy*

$$\|u\|\le M \quad\text{and}\quad \|v\|\le M \tag{3.10}$$

for a positive number M. Then we have

$$\|\Phi u-\Phi v\|\le c_0(1+\alpha)M^\alpha\|u-v\|. \tag{3.11}$$

PROOF. Making use of the elementary inequality

$$|p^{1+\alpha}-q^{1+\alpha}|\le(1+\alpha)|p-q|\max\{p^\alpha,q^\alpha\}, \qquad (p\ge0,\ q\ge0),$$

we have

$$\begin{aligned}|u^{1+\alpha}(s,y)-v^{1+\alpha}(s,y)|&\le(1+\alpha)M^\alpha\rho^\alpha(s,y)\,|u(s,y)-v(s,y)|\\&\le(1+\alpha)M^\alpha\|u-v\|\rho^{1+\alpha}(s,y).\end{aligned}$$

Hence it follows that

$$\begin{aligned}\|(\Phi u)(t,x)-(\Phi v)(t,x)\|&\le(1+\alpha)M^\alpha\|u-v\|(\Phi\rho)(t,x)\\&\le c_0(1+\alpha)M^\alpha\|u-v\|\rho(t,x),\end{aligned}$$

in virtue of Lemma 3.1. Thus we have (3.11).

We proceed to the iteration, setting

$$u_{n+1}=u_0+\Phi u_n \qquad (n=0,1,\cdots) \tag{3.12}$$

with u_0 given by (3.2)′. We assume $m\alpha>2$. According to Lemma 3.2, we can continue the iteration indefinitely within $\mathscr{S}[0,\infty)$ and get the inequalities

$$\|u_{n+1}\|\le\delta+c_0\|u_n\|^{1+\alpha}, \qquad (n=0,1,2,\cdots), \tag{3.13}$$

recalling $\|u_0\|\le\delta$. By an elementary theory of recurrent inequalities, (3.13) implies that the sequence $\|u_n\|$ is bounded for sufficiently small $\delta>0$. Namely, we have

$$\|u_n\|\le M, \qquad (n=1,2,\cdots) \tag{3.14}$$

with a constant $M=M(\delta)$ for sufficiently small δ, where $M(\delta)\to0$ as $\delta\to0$. We choose a δ such that

$$r \equiv c_0(1+\alpha)M^\alpha < 1. \tag{3.15}$$

Since $u_{n+2}-u_{n+1}=\Phi u_{n+1}-\Phi u_n$ by (3.12), we have

$$\| u_{n+2}-u_{n+1} \| \leq r \| u_{n+1}-u_n \| , \qquad (n=0, 1, \cdots) \tag{3.16}$$

by means of Lemma 3.3. (3.16) implies the convergence of $\sum_n^\infty \| u_{n+1}-u_n \|$ in virtue of $r<1$. Thus u_n converges with respect to the norm $\| \ \|$, that is, u_n/ρ converges uniformly in $[0, \infty)\times R^m$. Hence there exists a function $u \in \mathscr{S}[0, \infty)$ such that

$$\| u_n-u \| \to 0 \quad \text{as} \quad n\to\infty. \tag{3.17}$$

Making use of (3.12) and (3.17), it is quite easy to verify that u is a solution of IE. In this way we have established the following

LEMMA 3.4. *The statement of Theorem* 2 *with a global solution of* IVP *in* $\mathscr{E}[0, \infty)$ *replaced by a solution of* IE *in* $\mathscr{S}[0, \infty)$ *is true.*

PROOF OF THEOREM 2. The solution u of IE constructed above is the required global solution of IVP, since $\mathscr{S}[0, \infty)\subset\mathscr{E}[0, \infty)$ and since a solution of IE in $\mathscr{S}[0, \infty)$ is a regular solution of IVP according to Proposition A4 in Appendix.

Appendix

Here we shall prove several propositions used in the preceding sections.

PROPOSITION A1. *Let u be a regular solution of* IVP *in* $\mathscr{E}[0, T]$ *for* $T>0$. *Then u satisfies the integral equation* (3.2) $u=u_0+\Phi u$ *in* $0\leq t\leq T$.

PROOF. Let ρ_N $(N=1, 2, \cdots)$ be the truncating functions introduced by (2.9). We put $v_N=\rho_N u$ $(N=1, 2, \cdots)$. Then we have

$$\frac{\partial v_N}{\partial t}=\Delta v_N+\rho_N u^{1+\alpha}-2\nabla\rho_N\cdot\nabla u-(\Delta\rho_N)u \tag{A.1}$$

and $v_N(0, x)=\rho_N(x)a(x)$. Since every term in (A.1) has a compact support with respect to x, we have the following integral representation

$$v_N(t, x)=V_1+V_2-2V_3-V_4, \tag{A.2}$$

where

$$V_1=\int_{R^m} H(t, x-y)\rho_N\cdot a(y)dy,$$

$$V_2=\int_0^t ds\int_{R^m} H(t-s, x-y)\rho_N\cdot u(s, y)^{1+\alpha}dy,$$

$$V_3=\int_0^t ds\int_{R^m} H(t-s,\, x-y)\nabla_y\rho_N\cdot\nabla_y u(s,\, y)dy,$$

and

$$V_4=\int_0^t ds\int_{R^m} H(t-s,\, x-y)(\Delta_y\rho_N)u(s,\, y)dy,$$

where the argument of ρ_N is y. Since $a\in\mathscr{S}$ by our standing assumption, we have immediately

$$V_1\to\int_{R^m} H(t,\, x-y)a(y)dy \quad \text{as} \quad N\to\infty. \tag{A.3}$$

$u^{1+\alpha}$ satisfies, for some $M>0$ and some β in $0<\beta<2$, the inequality

$$|u^{1+\alpha}(s,\, y)|\le M\exp(|y|^\beta), \qquad (0\le s\le T,\, y\in R^m), \tag{A.4}$$

because $u^{1+\alpha}\in\mathscr{E}[0,\, T]$ is implied by $u\in\mathscr{E}[0,\, T]$. By the same estimation as in (2.7), we see $(\Phi u)(t,\, x)<+\infty$. Futhermore, we have

$$\begin{aligned}|V_2(t,\, x)-(\Phi u)(t,\, x)| &\le M\int_0^t ds\int_{|y|\ge N} H(t-s,\, x-y)\exp(|y|^\beta)dy\\ &=M\int_0^t \varphi_N(s)ds,\end{aligned}$$

where

$$\varphi_N(s)=\int_{|y|\ge N} H(t-s,\, x-y)\exp(|y|^\beta)dy.$$

For each x and t, $\varphi_N(s)$ is bounded uniformly by a constant, since

$$\begin{aligned}0\le\varphi_N(s) &\le\int_{R^m} H(t-s,\, x-y)\exp(|y|^\beta)dy\\ &=C\int_{R^m}\exp(-|\eta|^2)\exp(|x+2\sqrt{t-s}\,\eta|^\beta)d\eta\\ &\le C\int_{R^m}\exp(-|\eta|^2)\exp\{(|x|+2\sqrt{t}\,|\eta|)^\beta\}d\eta.\end{aligned}$$

Therefore we can apply the convergence theorem of Lebesgue, in order to have

$$|V_2(t,\, x)-(\Phi u)(t,\, x)|\le M\int_0^t\varphi_N(s)ds\to 0, \qquad (N\to\infty).$$

We now claim $V_3\to 0$ and $V_4\to 0$. V_4 is dealt with easily by means of $|\Delta\rho_N(y)|\le CN^{-2}$, since $u\in\mathscr{E}[0,\, T]$. Concerning V_3 we rewrite it by integration by parts as

$$V_3=-\tilde{V}_3-V_4, \tag{A.5}$$

where

$$\tilde{V}_3 = \int_0^t ds \int_{R^m} \nabla_x H(t-s, x-y)\cdot\nabla\rho_N\cdot u(s, y)dy.$$

In order to estimate $|\tilde{V}_3|$, we note $|\nabla\rho_N(y)| \leq CN^{-1}$ and also

$$|\nabla_x H(t, x)| \leq Ct^{-(m+1)/2}\exp\left(-\frac{|x|^2}{9t}\right). \tag{A.6}$$

We may suppose that

$$|u(s, y)| \leq M\exp(|y|^\beta), \qquad (0 \leq s \leq T, y \in R^m),$$

with some constants $M>0$ and $\beta\in(0, 2)$, since $u\in\mathscr{E}[0, T]$. Then changing the variable of integration by $y-x=3\sqrt{t-s}\,\eta$, we have

$$|\tilde{V}_3| \leq CN^{-1}\int_0^t (t-s)^{-1/2}ds\int_{R^m}\exp(-|\eta|^2)\exp\{(|x|+3\sqrt{t-s}\,|\eta|)^\beta\}d\eta$$
$$\leq C'N^{-1}$$

with C' depending only on x and t. Thus we have $\tilde{V}_3\to 0$ and consequently $V_3\to 0$. In this way, we obtain $u=u_0+\Phi u$ from (A.2).

PROPOSITION A2. *Let u be as in the preceding proposition. In addition, assume that the initial value a of u is not trivial. Then $u=u(t, x)>0$ if $t>0$.*

PROOF. According to Proposition A1, $u=u_0+\Phi u$. On the other hand, $(\Phi u)(t, x)$ is non-negative and $u_0(t, x)$ is positive for $t>0$ as is well-known. Thus the proposition is true.

PROPOSITION A3. *Let u be a regular solution of* IVP *in $\mathscr{E}[0, T]$. Then we have*

$$\frac{\partial u}{\partial x_j}\in\mathscr{E}[0, T], \qquad (j=1, 2, \cdots, m),$$

PROOF. Suppose that v is a function in $\mathscr{E}[0, T]$ satisfying

$$|v(s, y)| \leq M\exp(|y|^\beta), \qquad (0 \leq s \leq T, y \in R^m), \tag{A.7}$$

for some constants $M>0$ and $\beta\in(0, 2)$. Put

$$w(t, x) = \int_0^t ds\int_{R^m} H(t-s, x-y)v(s, y)dy. \tag{A.8}$$

Then by means of (A.6) we can easily verify

$$\frac{\partial w}{\partial x_j} = \int_0^t ds\int_{R^m}\frac{\partial}{\partial x_j}H(t-s, x-y)v(s, y)dy. \tag{A.9}$$

Actually in virtue of (A.7)

(A.10) $$w^{(\varepsilon)}(t,x)=\int_0^{t-\varepsilon}ds\int_{R^m}H(t-s,x-y)v(s,y)dy$$

with $\varepsilon>0$ is seen to converge to w uniformly in t and locally uniformly in x as $\varepsilon\to 0$. This kind of convergence occurs also to

(A.11) $$\frac{\partial}{\partial x_j}w^{(\varepsilon)}(t,x)=\int_0^{t-\varepsilon}ds\int_{R^m}\frac{\partial}{\partial x_j}H(t-s,x-y)v(s,y)dy$$

with the aid of (A.6) and (A.7). Furthermore, noting that for any positive constant λ we can choose $N>0$ and γ in $\beta<\gamma<2$ satisfying

(A.12) $$\exp(\lambda|x|^\beta)\le N\exp(|x|^\gamma), \qquad (x\in R^m),$$

we see that

$$\left|\frac{\partial w}{\partial x_j}(t,x)\right|\le C\int_0^t\frac{ds}{\sqrt{t-s}}\int_{R^m}\exp\{-|\eta|^2+(|x|+3\sqrt{t}|\eta|)^\beta\}dy$$
$$\le C\exp(|x|^\gamma)\int_0^t\frac{ds}{\sqrt{t-s}}\int_{R^m}\exp\{-|\eta|^2+C|\eta|^\beta\}d\eta.$$

Hence we have the continuity of $\dfrac{\partial w}{\partial x_j}$ and

$$\left|\frac{\partial w}{\partial x_j}(t,x)\right|\le C\exp(|x|^\gamma), \qquad (0\le t\le T, x\in R^m),$$

with some constants $C>0$ and γ in $\beta<\gamma<2$. This shows

(A.13) $$\frac{\partial w}{\partial x_j}\in\mathscr{E}[0,T], \qquad (j=1,2,\cdots,m).$$

On the other hand, it is quite easy to show that $u_0(t,x)$ in (3.2) is continuously differentiable in x and $\nabla_x u_0(t,x)$ is bounded in $[0,T]\times R^m$. (Recall the definition of $\mathscr{A}$ in §1.) We are now ready to establish the proposition. In fact, according to Proposition A1 we have $u=u_0+\Phi u$. Since $u^{1+\alpha}\in\mathscr{E}[0,T]$ along with u, we can apply (A.13) to $w=\Phi u$. Hence $\dfrac{\partial}{\partial x_j}\Phi u\in\mathscr{E}[0,T]$. This implies $\dfrac{\partial}{\partial x_j}u\in\mathscr{E}[0,T]$ by the remark stated just above and the proposition has been proved.

Proposition A4. *Let u be a non-negative continuous solution of the integral equation* (3.2) *in $Q_T=[0,T]\times R^m$. Suppose that u is bounded in Q_T. Then we have the following.*

i) *$\nabla_x u, \nabla_x\nabla_x u$ and u_t are continuous and bounded in Q_T.*

ii) *u is a regular solution of* IVP *in* $[0,T]$.

Proof. This proposition might be regarded as a special case of an essentially well-known differentiability theorem for solutions of quasi-linear parabolic equations. However, the simple character of our problem enables us to give a simple proof, which we sketch here.

In the same way as in the preceding proposition or more easily, we can show that $\frac{\partial u}{\partial x_j}$ is continuous and bounded in Q_T and that it is expressible as

$$\frac{\partial u}{\partial x_j}=\frac{\partial u_0}{\partial x_j}+\int_0^t ds\int_{R^m}\frac{\partial}{\partial x_j}H(t-s,x-y)u(s,y)^{1+\alpha}dy. \tag{A.14}$$

Noting $\frac{\partial}{\partial x_j}H(t-s,x-y)=-\frac{\partial}{\partial y_j}H(t-s,x-y)$, we can transform the integral in (A.14) by integration by parts, which is justified by the same procedure as that in the proof of Lemma 2.1. The result is

$$\frac{\partial u}{\partial x_j}=\frac{\partial u_0}{\partial x_j}+(1+\alpha)\int_0^t ds\int_{R^m}H(t-s,x-y)u^\alpha\frac{\partial u}{\partial y_j}dy. \tag{A.15}$$

On account of the boundedness of $u^\alpha\frac{\partial u}{\partial x_j}$, (A.15) yields

$$\frac{\partial^2 u}{\partial x_j\partial x_k}=\frac{\partial^2 u_0}{\partial x_j\partial x_k}+(1+\alpha)\int_0^t ds\int_{R^m}\frac{\partial}{\partial x_k}H(t-s,x-y)u^\alpha\frac{\partial u}{\partial y_j}dy \tag{A.16}$$

$$(j,k=1,2,\cdots,m).$$

From this it follows that $\frac{\partial^2 u}{\partial x_j\partial x_k}$ is continuous and bounded in Q_T, since $u^\alpha\frac{\partial u}{\partial x_j}$ is so.

Next, we claim that $u=u(t,x)$ is Hölder continuous in t in the following sense: there exists a constant C independent of t and x such that

$$|u(t+h,x)-u(t,x)|\le C\sqrt{h},\qquad (0\le t<t+h\le T,\ x\in R^m).$$

In order to show this, it is enough to deal with $w=\Phi u$ since the corresponding inequality for u_0 is obvious. We have

$$w(t+h,x)-w(t,x)=I_1+I_2,$$

where

$$I_1=\int_t^{t+h}ds\int_{R^m}H(t+h-s,x-y)u(s,y)^{1+\alpha}dy$$

and

$$I_2=\int_0^t ds\int_{R^m}H(t-s,x-y)v(s,y)dy$$

with

$$v(s,y)=\int_{R^m}H(h,y-z)u(s,z)^{1+\alpha}dz-u(s,y)^{1+\alpha}.$$

In view of the boundedness of u, it is obvious that

$$|I_1| \leq Ch.$$

On the other hand, we have

$$|v(s, y)| \leq C\int_{R^m} \exp(-|\eta|^2)\,|u(s, y+2\sqrt{h}\eta)^{1+\alpha} - u(s, y)^{1+\alpha}|\,d\eta$$
$$\leq C\sqrt{h},$$

because $|\nabla_x(u(s, x)^{1+\alpha})| = (1+\alpha)u^\alpha |\nabla_x u|$ is bounded in in Q_T. Hence we have $|I_2| \leq C\sqrt{h}$ and thus we have established the uniform Hölder continuity of u in t.

We turn to u_t. Taking a small positive number ε, we put

$$(\Phi_\varepsilon u)(t, x) = \int_0^{t-\varepsilon} ds \int_{R^m} H(t-s, x-y)u^{1+\alpha}dy, \qquad (\varepsilon \leq t \leq T, x \in R^m). \tag{A.17}$$

Since $u^{1+\alpha}$ is bounded, $\Phi_\varepsilon u$ tends to Φu as $\varepsilon \to 0$ uniformly in $[\delta, T] \times R^m$, δ being an arbitrary positive number. Recalling

$$\frac{\partial}{\partial t} H(t-s, x-y) = \Delta_x H(t-s, x-y) = \Delta_y H(t-s, x-y),$$

we have

$$\frac{\partial}{\partial t}(\Phi_\varepsilon u)(t, x) = \int_{R^m} H(\varepsilon, x-y)u(t-\varepsilon, y)^{1+\alpha}dy \tag{A.18}$$
$$+ \int_0^{t-\varepsilon} ds \int_{R^m} H(t-s, x-y)(\Delta u^{1+\alpha})dy$$
$$\equiv \tilde{I}_1 + \tilde{I}_2.$$

$\tilde{I}_1$ tends to $u(t, x)^{1+\alpha}$ uniformly in t and x as $\varepsilon \to 0$, since $u^{1+\alpha}$ is bounded and uniformly continuous in Q_T. On the other hand, $\tilde{I}_2$ converges uniformly in Q_T to

$$\varphi(t, x) = \int_0^t ds \int_{R^m} H(t-s, x-y)(\Delta u^{1+\alpha})dy, \tag{A.19}$$

because $(\Delta u^{1+\alpha})(s, x)$ is bounded in Q_T. From (A.19) it is obvious that $\varphi(t, x)$ is continuous and bounded in Q_T. At this stage, we have

$$\frac{\partial}{\partial t}\Phi u(t, x) = u(t, x)^{1+\alpha} + \varphi(t, x), \tag{A.20}$$

while $\varphi(t, x)$ is nothing but $\Delta_x(\Phi u)$ as is verified by integration by parts. Namely, we have

(A.21) $$\frac{\partial}{\partial t}\Phi u = u^{1+\alpha} + \Delta\Phi u,$$

and furthermore,

(A.22) $$\frac{\partial}{\partial t}u = \Delta u + u^{1+\alpha},$$

because $u = u_0 + \Phi u$ and $\frac{\partial}{\partial t}u_0 = \Delta u_0$. At the same time, we get the continuity and boundedness of u_t in Q_T. Finally, we note that $u \to a$ as $t \to 0$ uniformly in R^m, since $u_0 \to a$ and $\Phi u \to 0$ as $t \to 0$ uniformly in R^m. In this way, we have established i) and ii) in the proposition.

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References

[1] A. Friedman, Remarks on non-linear parabolic equations, Proc. of Symposia in Appl. Math., vol. 17 (Applications of non-linear partial differential equations in mathematical physics), 3-23, AMS, 1965.
[2] A. Friedman, Partial differential equations of parabolic type, Prentice-Hall, 1964.
[3] H. Fujita and T. Kato, On the Navier-Stokes initial value problem I, Arch. Rat. Mech. Anal., **16** (1964), 269-315.
[4] S. Itô, On the blowing up of solutions of semi-linear parabolic equations, Bulletin of the Mathematical Society of Japan (Sûgaku), **18** (1966), No. 1, 44-47 (in Japanese).
[5] S. Kaplan, On the growth of solutions quasi-linear parabolic equations, Comm. Pure Appl. Math., **16** (1963), 305-330.
[6] T. Kato and H. Fujita, On the non-stationary Navier-Stokes system, Rend. Sem. Mat. Univ. Padova, **32** (1962), 243-260.
[7] J. L. Lions, Équations différentielles opérationelles et problème aux limites, Springer, 1961.

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