Max Nöther's theorem on a regular projective algebraic variety

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Introduction. In the classical algebraic geometry, the following theorem due to Max Nöther is fundamental.

THEOREM. *Let C_1 and C_2 be plane algebraic curves defined by homogeneous polynomials φ and ψ of degree l and m respectively. We assume that C_1 and C_2 intersect at finite points $P_1, P_2, \cdots, P_s$ and that they have at most ordinary singularities of multiplicity r_j and q_j at P_j respectively $(j=1, 2, \cdots, s)$ with mutually distinct tangents. If a plane algebraic curve defined by a homogeneous polynomial f of degree n has the multiplicity at least r_j+q_j-1 at every P_j $(j=1, 2, \cdots, s)$, then f belongs to the graded ideal generated by φ and ψ, i.e. $f=A\varphi+B\psi$ where A and B are homogeneous polynomials of degree $n-l$ and $n-m$ respectively.*

In this paper we shall generalize the above theorem, motivated by [M] L. 13. A.[1]

THEOREM 1. *Let V be an n-dimensional regular algebraic variety[2] in a projective space P_k^N over an abstract field k, and V be defined by a graded prime ideal $\mathfrak{P}$ of the polynomial ring $k[X_0, X_1, \cdots, X_N]$. Let $H_{\varphi_1}, H_{\varphi_2}, \cdots, H_{\varphi_r}$ $(r\leqq n)$ be hypersurfaces in P_k^N, defined by homogeneous polynomials of degree $n_1, n_2, \cdots, n_r$ respectively, i.e. $H_{\varphi_i}=\mathrm{Proj}\, k[X_0, \cdots, X_N]/(\varphi_i)$ $(i=1, 2, \cdots, r)$. Let W be a scheme-theoretical intersection (the fibre product over P^N) of $V, H_{\varphi_1}, \cdots$ and H_{φ_r}, i.e.*

$$W=V\times_{P^N} H_{\varphi_1}\times_{P^N}\cdots\times_{P^N} H_{\varphi_r}\simeq\mathrm{Proj}\, k[X_0, X_1, \cdots, X_N]/(\mathfrak{P}, \varphi_1, \cdots, \varphi_r)\,.$$

We assume $\dim W=n-r$. *We decompose the above scheme W into the irreducible components $C_1, C_2, \cdots, C_s$, and assume further that each H_{φ_i} passes ordinarily through C_j with multiplicity $m_{i,j}$ $(j=1, \cdots, s)$. Finally we assume that $H_{\varphi_1}, H_{\varphi_2}, \cdots, H_{\varphi_r}$ intersect mutually ordinarily at every C_j. Then we can find an integer m_0 such that any homogeneous polynomial f of degree $m\geqq m_0$ belongs to the graded ideal $(\mathfrak{P}, \varphi_1, \cdots, \varphi_r)$ if the hypersurface H_f defined by f passes*

1) Here we discuss only the simple case and we do not come in the general case.

2) Regular prescheme means a prescheme whose stalk at every point is a regular local ring (cf. E. G. A. IV)

through every C_j with multiplicity at least $m_{1,j}+m_{2,j}+\cdots+m_{r,j}-(r-1)$.

"To pass ordinarily with multiplicity m" and "to intersect mutually ordinarily" will be defined precisely in the proof of Theorem 1. Estimations of m_0 in Theorem 1 are given in the following theorem.

THEOREM 2. (I) *If* $r=2$, *then* m_0 *can be chosen at most* $n_1+n_2+\nu(0,1)$.[3]

(II) *If* $r=n=3$, *then* m_0 *can be chosen at most* $n_1+n_2+n_3+\nu(0,1,2)$.[4]

(III) *If* V *is a complete intersection regular variety, then* m_0 *can be chosen* 0.

In Theorem 1, the scheme W is in general not a (reducible) variety, therefore we must use the notion of a scheme of A. Grothendieck. Next we investigate the infinitesimal structure of the scheme W.

THEOREM 3. *Let* p_j *be the generic point of* C_j *in Theorem* 1, *then* $m(\mathcal{O}_{W,p_j})$[5]$=m_{1,j}\cdot m_{2,j}\cdot\cdots\cdot m_{r,j}$ $(j=1,2,\cdots,s)$.

The following theorem is a ring-theoretic analogue of Theorem 1.

THEOREM 4. *Let* A *be a nötherian regular*[6] *integral domain over a field* k *of altitude* n, *and let* $\varphi_1,\cdots,\varphi_r$ *be elements of* A. *We assume that* $\mathrm{ht}(\varphi_1,\varphi_2,\cdots,\varphi_r)=r$.[7] *Let* $\mathfrak{p}_1,\mathfrak{p}_2,\cdots,\mathfrak{p}_s$ *be prime divisors of the ideal* $\mathfrak{a}=(\varphi_1,\cdots,\varphi_r)$. *We assume further that each* φ_i *is contained ordinarily in every* $\mathfrak{p}_j$ *with multiplicity* $m_{i,j}$. *Finally, we assume that* $\varphi_1,\varphi_2,\cdots,\varphi_r$ *are contained mutually ordinarily in every* $\mathfrak{p}_j$. *Then any element* f *of* A *belongs to the ideal* $\mathfrak{a}$ *if* f *is contained in each* $\mathfrak{p}_j$ *with multiplicity at least* $m_{1j}+m_{2j}+\cdots+m_{rj}-(r-1)$.

"Contained ordinarily" and "contained mutually ordinarily" will be also defined precisely in the proof of Theorem 4.

§1. Proof of Theorem 4 and Theorem 1.

First we shall prove Theorem 4, and then reduce Theorem 1 to Theorem 4. Since the unmixedness theorem holds for a nötherian regular integral domain (theorem of Cohen, [N] 25.14, [N] 25.6), every prime divisor $\mathfrak{p}_j$ of $\mathfrak{a}$ is minimal and $\mathrm{ht}\,\mathfrak{p}_j=r$. Put $\mathfrak{q}_j=A\cap\mathfrak{a}A_{\mathfrak{p}_j}$, and we get the shortest primary decomposition $\mathfrak{a}=\mathfrak{q}_1\cap\mathfrak{q}_2\cap\cdots\cap\mathfrak{q}_s$ ([N] p.20, 7.3, p.22, 8.7). From this decomposition, we have $\mathfrak{a}A_{\mathfrak{p}_j}=\mathfrak{q}_jA_{\mathfrak{p}_j}$. Since $\mathrm{alt}(A_{\mathfrak{p}_j}/\mathfrak{a}A_{\mathfrak{p}_j})=\mathrm{alt}(A_{\mathfrak{p}_j}/\mathfrak{q}_jA_{\mathfrak{p}_j})=0$, $A_{\mathfrak{p}_j}/\mathfrak{a}A_{\mathfrak{p}_j}$ is discrete with respect

3), 4) $\nu(0,1)$ and $\nu(0,1,2)$ are integers defined as follows. Let S be the graded ring $k[X_0,X_1,\cdots,X_N]/\mathfrak{P}$ which defines V, and S_n be the homogeneous part of degree n. Then $\nu(0,1)$ and $\nu(0,1,2)$, $(\nu(0,1)\leqq\nu(0,1,2))$ are integers such that $S_n\cong H^0(V,\mathcal{O}_V(n))$ and $H^1(V,\mathcal{O}_V(n))=0$ hold for any integer $n\geqq\nu(0,1)$, and that $H^2(V,\mathcal{O}_V(n))=0$ holds for any integer $n\geqq\nu(0,1,2)$.

5) $m(R)$ means the multiplicity of a local ring R (cf. [N]).

6) A commutative ring A with 1 is called regular if Spec A is a regular scheme.

7) $\mathrm{ht}\,\mathfrak{a}$ means the height of an ideal $\mathfrak{a}$. As to the terminology of local rings we follow the usage in [N].

to the canonical topology in the theory of local rings. Therefore $A\mathfrak{p}_j/\mathfrak{a}A\mathfrak{p}_j \simeq (A\mathfrak{p}_j/\mathfrak{a}A\mathfrak{p}_j)^{\wedge}$[8] $\simeq \hat{A}\mathfrak{p}_j/\mathfrak{a}\hat{A}\mathfrak{p}_j$. For an element f of A (1) $f \in \mathfrak{a}$ if and only if $f \in \mathfrak{q}_j$, $j = 1, 2, \cdots, s$. (2) $f \in \mathfrak{q}_j$ if and only if $f \in \mathfrak{a}A\mathfrak{p}_j$ and (3) $f \in \mathfrak{a}A\mathfrak{p}_j$ if and only if $f \in \mathfrak{a}\hat{A}\mathfrak{p}_j$.

Since $A\mathfrak{p}_j$ is a regular local ring, $\hat{A}\mathfrak{p}_j$ is a complete regular local ring, containing a field k. Therefore $\hat{A}\mathfrak{p}_j$ is isomorphic to the formal power series ring $K[[X_1, \cdots, X_r]]$, where K is the residue field of $A\mathfrak{p}_j$ ([N] p.106, 31.1). We say that *φ_i is contained in $\mathfrak{p}_j$ with multiplicity $m_{i,j}$ in the case*; $m(A\mathfrak{p}_j/\varphi_i A\mathfrak{p}_j) = m_{i,j}$. We remark here that $m(A\mathfrak{p}_j/\varphi_i A\mathfrak{p}_j) = m(\hat{A}\mathfrak{p}_j/\varphi_i \hat{A}\mathfrak{p}_j) = m([[X_1, \cdots, X_r]]/(\varphi_i))$.

LEMMA 1. *Let f be a non-unit element of the formal power series ring $K[[X_1, X_2, \cdots, X_r]]$. Then $m(K[[X_1, \cdots, X_r]]/(f)) = \deg f$.*[9]

PROOF. If one wants to show the above directly, one can use the preparation theorem of Weierstrass in a formal power series ring. But one can find a direct generalization of Lemma 1 in [N] p.154, 40.2.

By Lemma 1, the integer $m_{i,j}$ is equal to $\deg \varphi_i$ in $\hat{A}\mathfrak{p}_j \simeq K[[X_1, X_2, \cdots, X_r]]$. Let Ω be an algebraically closed field extending a field K, then the completed tensor product of $\hat{A}\mathfrak{p}_j$ and Ω over K: $\hat{A}\mathfrak{p}_j \hat{\otimes}_K \Omega$ is isomorphic to $\Omega[[X_1, X_2, \cdots, X_r]]$. Then we say that φ_i is contained ordinarily in $\mathfrak{p}_j$ if φ_i can be expressed as a product of elements of degree 1: $\varphi_i = \varphi_{i,1} \cdots \varphi_{i,m_{i\ j}}$ in $\Omega[[X_1, \cdots, X_r]]$. We say also that $\varphi_1, \cdots, \varphi_r$ are contained mutually ordinarily if $J(\varphi_{1,p_1}, \varphi_{2,p_2}, \cdots, \varphi_{r,p_r}) \neq 0$[10] for integers $1 \leqq p_j \leqq m_{i,j}$.

LEMMA 2. *Let Σ be an extension of a field K, and let $f, f_1, \cdots, f_r$ be elements of the formal power series ring $K[[X_1, \cdots, X_n]]$. If $f \in (f_1, f_2, \cdots, f_r)$ in $\Sigma[[X_1, \cdots, X_n]]$, then $f \in (f_1, f_2, \cdots, f_r)$ in $K[[X_1, \cdots, X_n]]$.*

PROOF. Let R and R_1 be $K[[X_1, \cdots, X_n]]$ and $\Sigma[[X_1, \cdots, X_n]]$ respectively. If one knows that R_1 is R-faithfully flat, one gets Lemma 2 immediately. In fact, put $\mathfrak{a} = (f_1, \cdots, f_r)$ in R, and $\mathfrak{a} \otimes_R R_1$ can be considered as $\mathfrak{a}R_1$. Therefore $(\mathfrak{a}, f)/\mathfrak{a} \otimes_R R_1 \simeq (\mathfrak{a}R_1, f)/\mathfrak{a}R_1 = 0$, hence $(\mathfrak{a}, f) = \mathfrak{a}$, i.e. $f \in \mathfrak{a}$. On the other hand flatness follows from Corollary 5.7 in S. G. A. IV "Morphisms plats".

COROLLARY 5.7. *Let $B \to A$ be a local homomorphism between nötherian local rings $(B, \mathfrak{n})$*[11] *and $(A, \mathfrak{m})$. We assume that $\mathfrak{m}B$ is $\mathfrak{n}$-primary. Then B is A-flat if and only if $\hat{B}$ is $\hat{A}$-flat.*

In our case it is obvious that the polynomial ring $\Sigma[X_1, \cdots, X_n]$ is $K[X_1, \cdots, X_n]$-

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8) $\hat{R}$ means the completion of the local ring R with respect to its maximal ideal.

9) By $\deg f$ we mean the least index of f with non-zero coefficients. If necessary, we put $\deg 0 = \infty$.

10) $J(\psi_1, \psi_2, \cdots, \psi_r)$ means the value at zero of the Jacobian of $\psi_1, \psi_2, \cdots, \psi_r \in K[[X_1, \cdots, X_r]]$. Namely, if $\psi_i = C_{i1}X_1 + \cdots + C_{ir}X_r + (\text{high order})$, then $J(\psi_1, \cdots, \psi_r) = \det(C_{i,j})$.

11) $(B, \mathfrak{n})$ means that the local ring B has the unique maximal ideal $\mathfrak{n}$.

flat. Localizing by the maximal ideal $(X_1, X_2, \cdots, X_n)$, $\Sigma[X_1, \cdots, X_n]_{(X)}$ is also $K[X_1, \cdots, X_n]_{(X)}$-flat. From the above corollary and $K[X_1, \cdots, X_n]\hat{}_{(X)} \simeq K[[X_1, \cdots, X_n]]$, we get the assertion.

LEMMA 2 shows us that an element f of A belongs to $\mathfrak{a}A_{\mathfrak{p}_j}$ if and only if f belongs to $\mathfrak{a}\Omega[[X_1, \cdots, X_r]]$. Therefore Theorem 4 follows from the following proposition:

PROPOSITION $M(m_1, m_2, \cdots, m_r)$. *Let R be a formal power series ring $K[[X_1, \cdots, X_r]]$ and let $\varphi_1, \cdots, \varphi_r$ be elements of R such that φ_i is the product of elements $\varphi_{i,1}, \cdots, \varphi_{i,m_i}$ of degree 1, i.e. $\varphi_i = \varphi_{i,1}\varphi_{i,2} \cdots \varphi_{i,m_i}$, $\deg \varphi_{i,j} = 1$ and that $J(\varphi_{1,p_1}, \cdots, \varphi_{r,p_r}) \neq 0$ for any integer $1 \leqq p_j \leqq m_j$. If the degree s of $f \geqq m_1 + m_2 + \cdots + m_r - (r-1)$, then we have an expression $f = A_1\varphi_1 + \cdots + A_r\varphi_r$ where $A_i \in R$ and $\deg A_i \geqq s - m_i$.*

PROOF. We notice the following lemma:

LEMMA 3. *Let $\phi_1, \phi_2, \cdots, \phi_r$ be non-zero and non-unit elements of the formal power series ring $K[[X_1, \cdots, X_r]]$ such that $J(\phi_1, \phi_2, \cdots, \phi_r) \neq 0$. Then the map $T: K[[X_1, \cdots, X_r]] \to K[[X_1, \cdots, X_r]]$, defined by $T(F[X_1, \cdots, X_r]) = F[\phi_1(X), \cdots, \phi_r(X)]$, is a ring-automorphism and homeomorphism. In other words, we can consider $\phi_1, \phi_2, \cdots, \phi_r$ as a set of new independent variables instead of $X_1, X_2, \cdots, X_r$.*

From Lemma 3, we shall show Proposition M inductively. First we shall prove $M(1, 1, \cdots, 1)$. Assume $s = \deg f \geqq 1$. Then we can develop f with respect to the new variables $\varphi_1, \varphi_2, \cdots, \varphi_r$ and we can express f as follows. $f = F_1(\varphi_1)\varphi_1 + F_2(\varphi_1, \varphi_2)\varphi_2 + \cdots + F_r(\varphi_1, \cdots, \varphi_r)\varphi_r$. Since these terms $F_1(\varphi_1)\varphi_1$, $F_2(\varphi_1, \varphi_2)\varphi_2, \cdots$ never cancel each other, we have $\deg F_i \geqq s-1$. Second, we obtain $M(m+1, 1, \cdots, 1)$ from $M(1, 1, \cdots, 1)$ and $M(m, 1, 1, \cdots, 1)$. In fact, we can express φ_1 as a product of $\varphi_{1,1}$ and φ_1' of degree 1 and m respectively, i.e. $\varphi_1 = \varphi_{1,1}\varphi_1'$. Assume $s = \deg f \geqq m+1 \geqq m$. We can apply $M(m, 1, \cdots, 1)$ to f with respect to $\varphi_1', \varphi_2, \cdots, \varphi_r$. Then we have $f = B_1\varphi_1' + B_2\varphi_2 + \cdots + B_r\varphi_r$, $\deg B_1 \geqq s-m$, $\deg B_2 \geqq s-1, \cdots, \deg B_r \geqq s-1$. Since $\deg B_1 \geqq s-m \geqq 1$ we can apply $M(1, 1, \cdots, 1)$ to B_1 with respect to $\varphi_{1,1}, \varphi_2, \cdots, \varphi_r$. Therefore we have $B_1 = C_1\varphi_{1,1} + C_2\varphi_2 + \cdots + C_r\varphi_r$, $\deg C_i \geqq \deg B_1 - 1 \geqq s-m-1$. Then we have $f = (C_1\varphi_{1,1} + C_2\varphi_2 + \cdots + C_r\varphi_r)\varphi_1' + B_2\varphi_2 + \cdots + B_r\varphi_r = C_1\varphi_1 + (C_2\varphi_1' + B_2)\varphi_2 + \cdots + (C_r\varphi_1' + B_r)\varphi_r$ where $\deg C_1 \geqq s-(m+1)$, $\deg(C_2\varphi_1' + B_2) \geqq s-1, \cdots, \deg(C_r\varphi_1' + B_r) \geqq s-1$. Thus this case is accomplished. Finally, let $N(i)$ be the new assertion $M(m_1, \cdots, m_i, 1, \cdots, 1)$ and we can show $N(i)$ inductively. We have proved $N(1)$ already. The proof by induction is parallel to the case of $N(1)$ and we omit the detail.

REDUCTION OF THEOREM 1 TO THEOREM 4. Let D_i be $D_+(X_i) \cap V$ in P^N and I

be a subset of $\{0, 1, 2, \cdots, N\}$ such that $i \in I$ if and only if $W \cap D_i \neq \phi$. From the assumption follows $\dim_{i\in I}(W \cap D_i) \leqq n-r$(♯). In the universal domain we can take a generic point in the classical sense $(\xi_1, \xi_2, \cdots, \xi_N)$ of D_i. Then $D_i \simeq$ Spec $k[\xi_1, \cdots, \xi_N]/(\varphi_1(\xi_1, \cdots, \xi_i, 1, \xi_{i+1}, \cdots, \xi_N), \cdots, \varphi_r(\xi_1, \cdots, \xi_i, 1, \xi_{i+1}, \cdots, \xi_N))$.

LEMMA 4. *Let A be an affine ring (namely, A is a finitely generated integral domain over a field k) of altitude n, and $\mathfrak{a}$ be the ideal generated by r elements $\varphi_1, \varphi_2, \cdots, \varphi_r$ in A. Then* 1) $\operatorname{alt}(A/\mathfrak{a}) \geqq n-r$, 2) *If* $\operatorname{alt}(A/\mathfrak{a}) = n-r$, *then* $\operatorname{ht}\mathfrak{a} = r$.

PROOF. Let $\mathfrak{p}_1, \mathfrak{p}_2, \cdots, \mathfrak{p}_t$ be minimal prime divisors of $\mathfrak{a}$. Then theorem of Krull ([N] p.26, 9.3) shows $\operatorname{ht}\mathfrak{p}_i \leqq r$ $(i=1, 2, \cdots, t)$, and [N] p.46, 14.6 shows $\operatorname{alt}(A/\mathfrak{p}_i) = n - \operatorname{ht}\mathfrak{p}_i \geqq n-r$. On the one hand we have $\operatorname{alt}(A/\mathfrak{a}) = \max_{1\leqq i\leqq t} \operatorname{alt}(A/\mathfrak{p}_i) \geqq n-r$. This proves 1). If $\operatorname{alt}(A/\mathfrak{a}) = n-r$, then $\operatorname{alt} A/\mathfrak{p}_i = n-r$. Therefore, for every minimal prime divisor $\mathfrak{p}_i$ of $\mathfrak{a}$, $\operatorname{ht}\mathfrak{p}_i = r$. This proves (2).

From the inequality (♯) and (1) in Lemma 4, we get $\dim(D_i \cap W) = n-r$ for $i \in I$ and further from (2) in Lemma 4 we get $\operatorname{ht}(\varphi_1(\xi), \cdots, \varphi_r(\xi)) = r$. Fix a suffix $i \in I$, and let $\mathfrak{p}_j$ be the prime ideal in $k[\xi_1, \cdots, \xi_N]$ corresponding to $D_i \cap C_j$, then $\mathfrak{p}_j$ is a (minimal) prime divisor of $(\varphi_1(\xi), \cdots, \varphi_r(\xi))$. Thus we have arrived at the situation in Theorem 4. "To pass ordinarily with multiplicity m" and "to intersect mutually ordinarily" can be defined in the case of Theorem 1 as in Theorem 4. We remark that these do not depend on a fixed suffix $i \in I$, because $k[\xi_1, \cdots, \xi_N]_{\mathfrak{p}_j} \simeq \mathcal{O}_{V, \mathfrak{p}_j}$. Now, let $\mathfrak{A}$ be a graded ideal $(\mathfrak{P}, \varphi_1, \cdots, \varphi_r)$ modulo $\mathfrak{P}$ in $S = k[X_0, \cdots, X_N]/\mathfrak{P}$, then we get the following exact sequence for any integer m:

$$0 \longrightarrow \mathfrak{A}(m) \longrightarrow S(m) \longrightarrow S/\mathfrak{A}(m) \longrightarrow 0 . \tag{1}$$

From the sequence (1), follows the exact sequence of $\mathcal{O}_V$-modules (2):

$$0 \longrightarrow \widetilde{\mathfrak{A}}(m) \longrightarrow \mathcal{O}_V(m) \longrightarrow \mathcal{O}_W(m) \longrightarrow 0 . \tag{2}$$

Combining the sequences (1) and (2), we get the following commutative diagram (3):

$$\begin{array}{ccccccccc} 0 \longrightarrow & \mathfrak{A}_m & \longrightarrow & S_m & \longrightarrow & S_m/\mathfrak{A}_m & \longrightarrow 0 & & \\ & \downarrow & & \downarrow & & \downarrow & & & \\ 0 \longrightarrow & H^0(V, \mathfrak{A}(m)) & \longrightarrow & H^0(V, \mathcal{O}_V(m)) & \longrightarrow & H^0(W, \mathcal{O}_W(m)) & \longrightarrow & H^1(V, \widetilde{\mathfrak{A}}(m)) \end{array} \tag{3}$$

where the vertical arrows mean the homomorphisms α_m, defined in E.G.A. II, §2. If m_0 is sufficiently large, then for any integer $m \geqq m_0$, we have $S_m \simeq H^0(V, \mathcal{O}_V(m))$ and $\mathfrak{A}_m \simeq H^0(V, \widetilde{\mathfrak{A}}(m))$. Then we get the following exact sequence (4):

$$0 \longrightarrow \mathfrak{A}_m \longrightarrow S_m \longrightarrow H^0(W, \mathcal{O}_W(m)) . \tag{4}$$

For any homogeneous element f of degree m of S, f belongs to $\mathfrak{A}_m$ if and only if $f \mid W=0$. But $f \mid W=0$ if and only if $f \mid W \cap D_i=0$ for any integer in I. We know also $\mathcal{O}_W(m)\mid_{W\cap D_i} \simeq \mathcal{O}_W\mid_{W\cap D_i}$, $\mathcal{O}_W\mid_{W\cap D_i} \simeq (k[\xi_1, \cdots, \xi_N]/(\varphi_1(\xi), \cdots, \varphi_r(\xi)))^{\sim}$[12] and $H^0(W\cap D_i, \mathcal{O}_W\mid_{W\cap D_i}) \simeq k[\xi_1, \cdots, \xi_N]/(\varphi_1(\xi), \cdots, \varphi_r(\xi))$. Therefore, $f \mid W\cap D_i=0$ if and only if $f(\xi) \in (\varphi_1(\xi), \cdots, \varphi_r(\xi))$. We have thus reduced Theorem 1 to Theorem 4 and we have completed the proof of Theorem 1.

§ 2. Proof of Theorem 2 and Theorem 3.

PROOF OF THEOREM 2 (I).[13] Consider the following sequence:

$$0 \longrightarrow S(-l-m) \xrightarrow{\ d\ } S(-l) \oplus S(-m) \xrightarrow{\ d'\ } \mathfrak{A} \longrightarrow 0 \tag{5}$$

where d means the homomorphism of degree 0 which maps an element ε of $S(-l-m)$ to an element $(\varepsilon\psi, -\varepsilon\varphi)$ and similarly d' maps an element (α, β) to an element $\alpha\varphi+\beta\psi$ of $\mathfrak{A}$. It is trivial that $d' \circ d$ is a null homomorphism, d is injective and that d' is surjective. Now we get the following sequence of $\mathcal{O}_V$-modules (6) from the sequence (5):

$$0 \longrightarrow \mathcal{O}_V(-l-m) \longrightarrow \mathcal{O}_V(-l) \oplus \mathcal{O}_V(-m) \longrightarrow \tilde{\mathfrak{A}} \longrightarrow 0 . \tag{6}$$

The exactness of the sequence (6) comes from the exactness of $(6)_v$ at every point v of V:

$$0 \longrightarrow \mathcal{O}_v \longrightarrow \mathcal{O}_v^2 \longrightarrow \mathfrak{A}_v \longrightarrow 0 \qquad (6)_v$$

The exactness of $(6)_v$ can be proved as follows. Let (α, β) be an element of $\mathcal{O}_v^2$ such that $\alpha\varphi+\beta\psi=0$ in $\mathcal{O}_v$. From the unique factorization theorem of the regular local ring ([N] p.99, 28.7) and from that φ and ψ cannot have common factors[14] follows $\alpha=\varepsilon\psi$, $\beta=-\varepsilon\varphi$ where ε is an element of $\mathcal{O}_v$. Then we have the following exact sequence (7) for any integer k from the sequence (6):

$$0 \longrightarrow \mathcal{O}_V(k-l-m) \longrightarrow \mathcal{O}_V(k-l) \oplus \mathcal{O}_V(k-m) \longrightarrow \tilde{\mathfrak{A}}(k) \longrightarrow 0 . \tag{7}$$

From the above sequence (7) follows the exact sequence (8):

$$0 \longrightarrow H^0(V, \mathcal{O}_V(k-l-m)) \longrightarrow H^0(V, \mathcal{O}_V(k-l)) \oplus H^0(V, \mathcal{O}_V(k-m)) \longrightarrow H^0(V, \tilde{\mathfrak{A}}(k)) \longrightarrow H^1(V, \mathcal{O}_V(k-l-m)) . \tag{8}$$

If $n \geqq \nu(0, 1)+l+m$, then the following commutative diagram (9) holds:

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12) $A^{\sim}$ means a structure sheaf of an affine scheme Spec A corresponding to a commutative ring A, cf. E. G. A. I.

13) For brevity we write l, m, n and φ, ψ, χ instead of n_1, n_2, n_3 and φ_1, φ_2, φ_3, respectively.

14) If φ or ψ is a unit, the exactness is obvious.

$$\begin{array}{ccccccccc}
0 & \to & S_{k-l-m} & \to & S_{k-l}\oplus S_{k-m} & \to & \mathfrak{A}_k & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & H^0(V, \mathcal{O}_V(k-l-m)) & \to & H^0(V, \mathcal{O}_V(k-l))\oplus H^0(V, \mathcal{O}_V(k-m)) & \to & H^0(V, \tilde{\mathfrak{A}}(k)) & \to & 0
\end{array} \quad (9)$$

where the vertical arrows mean the natural homomorphisms. Hence, from the following Lemma 5 we get $\mathfrak{A}_n \simeq H^0(V, \tilde{\mathfrak{A}}(n))$, which accomplishes the proof of Theorem 2, (I).

LEMMA 5. *Let A be an arbitrary commutative ring, L, L', L'', M, M', M'' be A-modules and $d, d', \delta, \delta', \alpha, \beta, \gamma$ be A-homomorphism such that the following commutative diagram holds*

$$\begin{array}{ccccccc}
L & \xrightarrow{d} & L' & \xrightarrow{d'} & L'' & \longrightarrow & 0 \\
\downarrow \alpha & & \downarrow \beta & & \downarrow \gamma & & \\
M & \xrightarrow{\delta} & M' & \xrightarrow{\delta} & M'' & \longrightarrow & 0
\end{array} \quad (10)$$

where one knows only the bottom sequence is exact, $d'\circ d$ is null, d' is surjective and that α and β are A-isomorphisms. Then, γ is an A-isomorphism.

PROOF OF THEOREM 2 (II).

In this case we have the following sequence (11):

$$\begin{aligned}
0 \longrightarrow S(-l-m-n) &\xrightarrow{d} S(-m-n)\oplus S(-n-l)\oplus S(-l-m) \\
&\xrightarrow{d'} S(-l)\oplus S(-m)\oplus S(-n) \xrightarrow{d''} \mathfrak{A} \longrightarrow 0\,.
\end{aligned} \quad (11)$$

where d means also the homomorphism of degree 0 which maps an element ε of S to an element $(\varepsilon\varphi, \varepsilon\psi, \varepsilon\chi)$, d' maps (λ, μ, ν) to $(\mu\chi-\nu\psi, \nu\varphi-\lambda\chi, \lambda\psi-\mu\varphi)$ and d'' maps (α, β, γ) to $\alpha\varphi+\beta\psi+\gamma\chi$. We can get also the following sequence of $\mathcal{O}_V$-modules from the sequence (11):

$$\begin{aligned}
0 \longrightarrow \mathcal{O}_V(-l-m-n) &\longrightarrow \mathcal{O}_V(-m-n)\oplus \mathcal{O}_V(-n-l)\oplus \mathcal{O}_V(-l-m) \\
&\longrightarrow \mathcal{O}_V(-l)\oplus \mathcal{O}_V(-m)\oplus \mathcal{O}_V(-n) \longrightarrow \tilde{\mathfrak{A}} \longrightarrow 0\,.
\end{aligned} \quad (12)$$

The exactness of the sequence (12) comes from the unique fractorization theorem and the following lemma:

LEMMA $K(s, t, u)$. *Let $\mathcal{O}_v$ be the local ring of V at v and φ, ψ, χ be the elements of $\mathcal{O}_v$ such that* $\deg\varphi=s$, $\deg\psi=t$, *and* $\deg\chi=u$ *(in the case $r=n=3$ of Theorem* 1). *Then $\mathcal{O}_v^3 \xrightarrow{d'} \mathcal{O}_v^3 \xrightarrow{d''} \mathfrak{A}_v$ is exact.*

Proof of $K(s, t, u)$ is parallel to that of $M(m_1, m_2, \cdots, m_r)$, which we shall not repeat here.

Thus we have the following short exact sequences from the sequence (12)

$$0\longrightarrow\mathcal{O}_V(-l-m-n)\longrightarrow\mathcal{O}_V(-m-n)\oplus\mathcal{O}_V(-n-l)\oplus\mathcal{O}_V(-l-m)\longrightarrow\mathcal{M}\longrightarrow 0 \tag{13}$$

$$0\longrightarrow\mathcal{M}\longrightarrow\mathcal{O}_V(-l)\oplus\mathcal{O}_V(-m)\oplus\mathcal{O}_V(-n)\longrightarrow\tilde{\mathfrak{A}}\longrightarrow 0 \tag{14}$$

where $\mathcal{M}$ is an $\mathcal{O}_V$-Module. By tensoring each terms of the sequence (13) with $\mathcal{O}_V(k)$, we get $H^1(V, \mathcal{M}(k))=0$ for any integer $k\geqq l+m+n+\nu(0, 1, 2))$. Hence we get the following long exact sequence for an integer k above:

$$0\longrightarrow M^0\longrightarrow M^1\longrightarrow M^2\longrightarrow M^3\longrightarrow 0 \tag{15}$$

where $M^0=H^0(V, \mathcal{O}_V(k-l-m-n)), M^1=H^0(V, \mathcal{O}_V(k-m-n))\oplus H^0(V, \mathcal{O}_V(k-n-l))$ $\oplus H^0(V_V, \mathcal{O}_V(k-l-m)), M^2=H^0(V, \mathcal{O}_V(k-l))\oplus H^0(V, \mathcal{O}_V(k-m))\oplus H^0(V, \mathcal{O}_V(k-n))$ and $M^3=H^0(V, \tilde{\mathfrak{A}}(k))$. As in the proof of Theorem 2(I), we have the following commutative diagram (16):

$$\begin{array}{ccccccc} L^1 & \longrightarrow & L^2 & \longrightarrow & L^3 & \longrightarrow & 0 \\ \alpha^1\downarrow & & \alpha^2\downarrow & & \alpha^3\downarrow & & \\ M^1 & \longrightarrow & M^2 & \longrightarrow & M^3 & \longrightarrow & 0 \end{array} \tag{16}$$

where $L^1=S_{k-m-n}\oplus S_{k-n-l}\oplus S_{k-l-m}$, $L^2=S_{k-l}\oplus S_{k-m}\oplus S_{k-n}$, $L^3=\mathfrak{A}_k$, M^i is one and the same with the above M^i $(1\leqq i\leqq 3)$ and α^i is a natural homomorphism $(1\leqq i\leqq 3)$. Hence Theorem 2 (II) is proved by the above diagram (16) and Lemma 5.

PROOF OF THEOREM 2 (III).

Theorem 2 (III) is an immediate consequence of the following Proposition $P(s)$ and Corollary $L(s)$.

PROPOSITION $P(s)$. *Let X_s be a complete intersection scheme*[15] *defined by homogeneous polynomials $f_1, \cdots, f_s$, then $H^i(X_s, \mathcal{O}_{X_s}(m))=0$ for any integer m and any integer $i\neq 0$, $N-s$.*

We prove Proposition $P(s)$ inductively. $P(0)$ can be found in E.G.A. III, §2 and $P(s)$ can be easily proved by $P(s-1)$, so we omit the detail.

COROLLARY $L(s)$. *$R_m^{(s)}\simeq H^0(X_s, \mathcal{O}_{X_s}(m))$ for any integer m,* where $R^{(s)}=k[X_0, \cdots, X_N]/(f_1, \cdots, f_s)$.

$L(0)$ can be found also in E.G.A. III §2 and $L(s)$ can be easily proved by $L(s-1)$ and $P(s-1)$ so we omit the detail.

PROOF OF THEOREM 3. Theorem 3 is an immediate consequence of Associativity formula of Chevalley ([N] p.81, 24.7).

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15) We call a k-scheme which is k-isomorphic to $\mathrm{Proj}\, k[X_0, X_1, \cdots, X_N]/(f_1, \cdots, f_s)$ of dimension $N-s$ a complete intersection scheme in P_k^N defined by homogeneous polynomials $f_1, f_2, \cdots, f_s$. We remark that $\dim \mathrm{Proj}\, k[X_0, X_1, \cdots, X_N]/(f_1, \cdots, f_i)=N-i$ $(0<i<s)$ follows from $\dim \mathrm{Proj}\, k[X_0, X_1, \cdots, X_N]/(f_1, \cdots, f_s)=N-s$.

ASSOCIATIVITY FORMULA. *Let R be a local ring (of course, nötherian) of altitude n, $x_1, x_2, \cdots, x_n$ be a parameter system of R, $\mathfrak{q}$ be a primary ideal generated by $x_1, x_2, \cdots, x_n$, $\mathfrak{a}$ be an ideal generated by a subset $x_1, x_2, \cdots, x_s$ of $x_1, x_2, \cdots, x_n$ and $\mathfrak{p}_1, \cdots, \mathfrak{p}_r$ be all minimal prime divisors of $\mathfrak{a}$. Then the following relation among multiplicities holds*

$$\mu(\mathfrak{q}) = \sum_i \mu(\mathfrak{a}R\mathfrak{p}_i) \cdot \mu((\mathfrak{q}, \mathfrak{p}_i)/\mathfrak{p}_i) .$$

In our case,

$$m(\mathcal{O}_{W,p_j}) = m(\hat{\mathcal{O}}_{W,p_j}) = m(\hat{\mathcal{O}}_{V,p_j}/(\varphi_1, \cdots, \varphi_r)) = m(\Omega[[X_1, \cdots, X_r]]/(\varphi_1, \cdots, \varphi_r))$$
$$= \mu((\varphi_1, \cdots, \varphi_r); \Omega[[X_1, \cdots, X_r]]) .$$

Let R be $\Omega[[X_1, \cdots, X_r]]$, $\mathfrak{q}$ be $(\varphi_1, \cdots, \varphi_r)$ and $\mathfrak{a}$ be (φ_1). Then $\varphi_1 = \varphi_{1,1}^{e_1} \cdot \cdots \cdot \varphi_{1,t}^{e_t} \cdot \varepsilon$[16] where $(\varphi_{1,1}), \cdots, (\varphi_{1,t})$ are all distinct prime divisors of (φ_1), and ε is a unit. We may remark $\mu(\varphi_1 R\mathfrak{p}_i) = e_i$.[17] Hence from the associativity formula follows $\mu((\varphi_1, \cdots, \varphi_r)) = \sum_{i=1}^{t} e_i \cdot \mu((\varphi_{1,i}, \cdots, \varphi_r)/(\varphi_{1,i}))$. If (φ_1) is a prime ideal, we get $\mu((\varphi_1, \varphi_2, \cdots, \varphi_r)) = \mu((\varphi_1, \cdots, \varphi_r)/(\varphi_1))$. Finally we have

$$\mu((\varphi_1, \cdots, \varphi_r)) = \sum_{i=1}^{m_{1,j}} \mu((\varphi_{1,i}, \varphi_2, \cdots, \varphi_r)) = \cdots = \sum_{1 \leq p_i \leq m_{i,j}} \mu((\varphi_{1,p_1}, \varphi_{2,p_2}, \cdots, \varphi_{r,p_r}))$$
$$= m_{1,j} m_{2,j} \cdot \cdots \cdot m_{r,j} .$$

The last equality follows from $J(\varphi_{1,p_1}, \cdots, \varphi_{r,p_r}) \neq 0$, Lemma 3, regularity of the local ring $\Omega[[X_1, X_2, \cdots, X_r]]$ and that the multiplicity of a regular local ring $=1$.

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References

E. G. A.: A. Grothendieck et J. Dieudonné; Eléments de géométrie algébrique, I, II, III-1, Publ. Math. I. H. E. S. No. 4, 8, 11.

[M]: D. Mumford; Lectures on Curves on an Algebraic Surface, Harvard Univ. (mimeographed), 1964.

[N]: M. Nagata; Local rings, Interscience Publ., 1962.

S. G. A.: A. Grothendieck; Séminaire de géométrie algébrique, I. H. E. S., 1960/61.

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16) $\sum_{i=1}^{t} e_i = m_{1,j}$

17) $\mathfrak{p}_i = (\varphi_{1,i})$