The axiomatization of the intermediate propositional systems S_n of Gödel

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Introduction.

In [2] Gödel showed that there is a series of propositional systems S_n between the intuitionistic and the classical. His systems were defined by using finite models as follows. (We give them here with a slight notational change.)

Values of S_n are integers $1, 2, \cdots, n$ and ω, where only 1 is the designated value and $\omega > i$ for any positive integer i. And logical operations $\supset$, $\wedge$, $\vee$ and $\neg$ are defined as follows.

$$v_1 \supset v_2 = \begin{cases} 1 & \text{if } v_1 \geqq v_2 , \\ v_2 & \text{otherwise} . \end{cases}$$

$$v_1 \wedge v_2 = \max(v_1, v_2) .$$

$$v_1 \vee v_2 = \min(v_1, v_2) .$$

$$\neg v = \begin{cases} 1 & \text{if } v = \omega , \\ \omega & \text{otherwise} . \end{cases}$$

An extension of S_n is $\boldsymbol{LC}$ of Dummett [1], in which the set of values is defined to be all the positive integers and ω. And concerning the sets of valid formulas in these systems, there holds, as is well known, the following relation,

$$S_1 \supset S_2 \supset \cdots \supset S_n \supset \cdots \supset \boldsymbol{LC} \supset \boldsymbol{LI} ,$$

where $\boldsymbol{LI}$ is the intuitionistic system.

The axiomatizations of S_1, which is the classical system, and of $\boldsymbol{LI}$ are well known, and that of $\boldsymbol{LC}$ has been given by Dummett by adding a new axiom scheme $(a_1 \supset a_2) \vee (a_2 \supset a_1)$ to $\boldsymbol{LI}$. In this paper, we give an axiomatization for each S_n by adding new axiom schemes to the axiomatic system of $\boldsymbol{LC}$ of Dummett.

§ 1. Preliminaries.

By $S \vdash A$, we mean that a formula A is provable (*or* valid) in the axiomatic system (*or* model) S. By $S + A_1 + \cdots + A_k$, we mean an axiomatic system obtained by adding the axiom schemes $A_1, \cdots, A_k$ to an axiomatic system S. If $\boldsymbol{T}_1$ and $\boldsymbol{T}_2$ are two systems axiomatic or defined by a model, we mean by $\boldsymbol{T}_1 \supset \boldsymbol{T}_2$ that the set of all the provable or valid formulas of $\boldsymbol{T}_2$ is included in that of $\boldsymbol{T}_1$.

And $T_1 \supset\subset T_2$ means that $T_1 \supset T_2$ and $T_2 \supset T_1$. If f is an assignment function of a model, we mean by $f(A)$ the value calculated for the formula A by the assignment f. We define formulas X_n, T_n, F_n and Y_n as follows.

DEFINITION 1.1.

$$X_n = \bigvee_{1 \leqq i < j \leqq n+1} (a_i \supset a_j) \wedge (a_j \supset a_i) .$$

$$T_n = \bigvee_{1 \leqq i \leqq n+1} a_i .$$

$$F_n = \bigvee_{1 \leqq i \leqq n+1} \neg a_i .$$

$$Y_n = X_n \vee (T_n \wedge F_n) .$$

LEMMA 1.2. $S_m \vdash X_{n+1}$, *if* $m \leqq n$.

PROOF. By any assignment function f of S_m, there are always some distinct propositional variables a_i and a_j to which is assigned the same value, since the number of values is not greater than $n+1$. And so X_{n+1} always gets the designated value.

REMARK. We only use the case of $m=n$ afterwards. And the meaning of $f(X_n)=1$ is that the number of distinct values is n at the most.

LEMMA 1.3. $S_m \vdash Y_n$, *if* $m \leqq n$.

PROOF. Let f be an assignment function of S_m, and let be that $f(X_n) \neq 1$. Then the values $v_1, \cdots, v_{n+1}$ assigned to $a_1, \cdots, a_{n+1}$ must be all different. But S_m has only $m+1$ ($m \leqq n$) different values, so there must be 1 and ω among $v_1, \cdots, v_{n+1}$. Hence $S_m \vdash Y_n$.

REMARK. Again, only the case of $m=n$ is needed afterwards. The meaning of $f(Y_n)=1$ is that if the number of values assigned by f is more than n, there always exist 1 and ω among them.

DEFINITION 1.4. $L_n = LC + X_{n+1} + Y_n$.

COROLLARY 1.5. $S_n \supset L_n$.

PROOF. Since we have $S_n \supset LC$ and 1.2 and 1.3, the corollary holds obviously.

The purpose of this paper is to prove the converse, that is, to prove that L_n is an axiomatization for S_n. But, before we prove it, we provide some more lemmas.

In this paper, an assignment function f is always considered with relation to some formulas, in other words, to a set of some propositional variables $\{a_1, \cdots, a_m\}$. Let v_i be the value assigned to a_i by f ($1 \leqq i \leqq m$). By $V(f)$, we mean the set $\{v_1, \cdots, v_m\}$, and by $M_f(i)$ the i-th maximum value of $V(f)$ (we omit the subscript f if there occurs no confusion), and by $H(f)$ the number of different values of $V(f)$, and by f_k an assignment function which assigns to a_i the value 1 if

$f(a_i)\leqq k$ and the value $f(a_i)$ otherwise.

LEMMA 1.6. *If an assignment function f of S_m assigns v_i to a_i $(1\leqq i\leqq n+1)$, then*

$$f(X_n)=\begin{cases}1 & \text{if there exist } v_i \text{ and } v_j \text{ such that } v_i=v_j\ ,\\ M(n) & \text{otherwise}\ .\end{cases}$$

PROOF. The case of $f(X_n)=1$ is obvious. In the latter case, $(v_i\supset v_j)\wedge(v_j\supset v_i)=\max(v_i, v_j)$ for any distinct v_i and v_j. And since $v_1, \cdots, v_{n+1}$ are all distinct, $f(X_n)=\min\limits_{1\leqq i<j\leqq n+1}(\max(v_i, v_j))=M(n)$.

DEFINITION 1.7. *Let A be a formula with propositional variables $b_1, \cdots, b_k$ and without any other propositional variables. (We write this as $A(b_1, \cdots, b_k)$.) Let ρ be a mapping function of $\{b_1, \cdots, b_k\}$ into $\{a_1, \cdots, a_m\}$. Then we mean by A^* (concerning $\{a_1, \cdots, a_m\}$) the conjunction*

$$\bigwedge_{all\,\rho} A(\rho(b_1), \cdots, \rho(b_k))\ .$$

LEMMA 1.8. *$L_n\vdash X^*_{n+1}$ and $L_n\vdash Y^*_n$.*

PROOF. By 1.2-5, this lemma holds obviously.

LEMMA 1.9. *In LC, if A is a formula only with propositional variables $a_1, \cdots, a_m$ and f is an assignment function which assigns v_i to a_i $(1\leqq i\leqq m)$ and k is a value distinct from ω, then*

$$f(k\supset A)=f_k(A)\ .$$

PROOF. We prove the lemma by the induction on the number of logical symbols of A. If A has no logical symbols, the lemma holds obviously.

If $A=A_1\wedge A_2$, $\vdash(B\supset A)\equiv((B\supset A_1)\wedge(B\supset A_2))$ for any B. Hence $f(k\supset A_1\wedge A_2)=f(k\supset A_1)\wedge f(k\supset A_2)=f_k(A_1)\wedge f_k(A_2)=f_k(A_1\wedge A_2)$.

If $A=A_1\vee A_2$ or $A=A_1\supset A_2$, we can treat them similarly, since we have

$$\vdash(B\supset A_1\vee A_2)\equiv((B\supset A_1)\vee(B\supset A_2))$$

and

$$\vdash(B\supset(A_1\supset A_2))\equiv((B\supset A_1)\supset(B\supset A_2))$$

for any B.

If $A=\neg A_1$, let a be a propositional variable contained in A_1. Then $f(k\supset\neg A_1)=f(k\supset(A_1\supset a\wedge\neg a))=f((k\supset A_1)\supset(k\supset a\wedge\neg a))=f(k\supset A_1)\supset f(k\supset a\wedge\neg a)=f_k(A_1)\supset f_k(a\wedge\neg a)=f_k(\neg A_1)$ since $f(k\supset a\wedge\neg a)=f(a\wedge\neg a)=f_k(a\wedge\neg a)$ if $k\neq\omega$.

LEMMA 1.10. *If $S_n\vdash A$ and f is an assignment function of LC satisfying one of the following three conditions,*

(i) $H(f)\leqq n-1$,

(ii) $H(f)=n$, *and* $V(f)\ni 1$ *or* $V(f)\ni\omega$,

(iii) $H(f)=n+1$, *and* $V(f)\ni 1$ *and* $V(f)\ni\omega$,

then $f(A)=1$.

PROOF. The calculation of $f(A)$ just goes as in S_n under one of the above conditions.

§ 2. Main result.

Now we prove the converse of 1.5.

LEMMA 2.1. $L_n \supset S_n$.

PROOF. Let A be a valid formula of S_n in which are contained only the propositional variables $a_1, \cdots, a_m$. Then we prove that

$$LC \vdash X^*_{n+1} \supset (Y^*_n \supset A)\,.$$

Without loss of generality, we can assume that $m \geqq n+2$, since, if not, we can take as A the formula $A \wedge (a_{m+1} \supset a_{m+1}) \wedge \cdots \wedge (a_{n+2} \supset a_{n+2})$ which is equivalent to A. Let f be an assignment function of LC which assigns v_i to a_i $(1 \leqq i \leqq m)$.

If $H(f) \leqq n-1$, $f(A)=1$ by 1.10.

If $H(f)=n$, and $V(f)\ni 1$ or $V(f)\ni\omega$, $f(A)=1$ by 1.10.

If $H(f) \geqq n$ and $V(f) \not\ni \omega$, then $f(F_n)=\omega$. Hence $f(Y^*_n \supset A) = f(X^*_n \supset A) = f(M(n) \supset A) = f_{M(n)}(A)=1$, since $H(f_{M(n)})=n$ and $V(f_{M(n)})\ni 1$.

If $H(f)=n+1$ and $V(f)\ni 1, \omega$, $f(A)=1$ by 1.10.

If $H(f)=n+1$ and $V(f)\ni\omega$ but $V(f)\not\ni 1$, then $f(F_n)=1$ and $f(T_n)=M(n+1)$ and $f(X_n)=M(n)$. Hence $f(Y^*_n \supset A) = f(M(n+1) \supset A) = f_{M(n+1)}(A)=1$, since $V(f_{M(n+1)})\ni 1, \omega$ and $H(f_{M(n+1)})=n+1$.

If $H(f)>n+1$, $f(X^*_{n+1})=M(n+1)$. Hence $f(X^*_{n+1} \supset (Y^*_n \supset A)) = f(M(n+1) \supset (Y^*_n \supset A)) = f_{M(n+1)}(Y^*_n \supset A)$. And this case is reduced into the cases of $H(f)=n+1$.

Hence we have the

THEOREM 2.2. $L_n \supset\subset S_n$.

§ 3. Another axiomatization.

Obviously holds the following

LEMMA 3.1. *If a formula A is interdeducible with $X_{n+1} \wedge Y_n$ in* **LC**, *then* $LC + A \supset\subset S_n$.

An example of such interdeducible formulas of the above lemma is the following

$$R_n\colon a_1 \vee (a_1 \supset a_2) \vee \cdots \vee (a_{n-1} \supset a_n) \vee \neg a_n\,.$$

The system $LI + R_n$ was studied by Umezawa and he obtained the following two lemmas. (For the proof, cf. [3].)

LEMMA 3.2. (*By Umezawa.*) $LI+R_n \supset LC$.

LEMMA 3.3. (*By Umezawa.*) $S_n \vdash R_n$, *that is*, $S_n \supset LI+R_n$.

He did not prove the converse of 3.3. But we have the

LEMMA 3.4. $S_n \supset\subset LI+R_n$.

PROOF. We only need to prove that $LC+R_n \vdash X_{n+1}$ and that $LC+R_n \vdash Y_n$.

Let R_n^* be constructed as in 1.7 concerning $\{a_1, \cdots, a_{n+2}\}$. And let f be an assignment function of LC which assigns v_i to a_i $(1 \leqq i \leqq n+2)$. If $H(f) \leqq n+1$, $f(X_{n+1})=1$. If $H(f)=n+2$, $f(X_{n+1})=M(n+1)$. On the other hand, $f(R_n) \neq 1$ if and only if $1<v_1<v_2< \cdots <v_n<\omega$, and then $f(R_n)=M(n)$. So if $V(f) \ni \omega$, $f(R_n^*)=M(n+1)$, and if $V(f) \not\ni \omega$, $f(R_n^*)=M(n)$. Anyway $f(R_n^*) \geqq f(X_{n+1})$, hence $f(R_n^* \supset X_{n+1})=1$.

Next we construct R_n^* concerning $\{a_1, \cdots, a_{n+1}\}$. Let f be defined as above. If $H(f) \leqq n$, $f(X_n)=1$. If $H(f)=n+1$ and if $V(f) \ni \omega$, $f(R_n^*)=M(n+1)$ and $f(F_n)=1$ and $f(T_n)=M(n+1)$. If $H(f)=n+1$ and $V(f) \not\ni \omega$, $f(R_n^*)=f(X_n)=M(n)$. Hence $f(R_n^* \supset Y_{n+1})=1$.

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References

[1] M. Dummett, A propositional calculus with denumerable matrix, J. Symbolic Logic, **24** (1959), 97-106.

[2] K. Gödel, Zum intuitionistischen Aussagenkalkül, Akademie der Wissenschaften in Wien, Mathematisch-naturwissenschaftliche Klasse, Anzeiger, **69** (1932), 65-66. Reprinted in Ergebnisse eines mathematischen Kolloquiums, Heft 4 (for 1931-2, pub. 1933), 40.

[3] T. Umezawa, On intermediate many-valued logics, J. Math. Soc. Japan, **11** (1959), 116-128.

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