On Finite solvable groups with $t(G)=4$ *or* 5

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Introduction:

In [3] Iwahori defined the number $t(G)$ for finite group G, determined all groups with $t(G)=2$ and proposed an interesting problem of determining the structure of all finite groups with a given $t(G)$. This problem was solved for $t(G)=3$ by Iwahori and Kondo [4]. In this paper, we shall solve this problem for solvable groups for the case $t(G)=4$ and 5. First we shall recall the definitions given by Iwahori [3].

Let G be a finite group acting on a set M, we call such a set M a G-space. For any element σ in G, we denote by M_σ the subset of M consisting of the fixed points by σ. For any non empty subset S of G, we denote by M_S the intersection of the sets M_σ for all σ in S. The cardinality of a set A will be denoted by $|A|$. Now let k be a positive integer. A G-space M is called of type k if the following two conditions are satisfied.

(1) $|M_\sigma|=k$ for all σ in $G^\sharp$, $G^\sharp$ being the set consisting of all elements in G different from the identity.

(2) $|M_G|=0$.

If G admits a G-space M of type k, then we say that G is of type k on M. The number $t(G)$ is the minimum of the types of G-spaces. (If there is no G-space of a positive type, then we put $t(G)=0$. It is easily proved that there is no group with $t(G)=1$.)

In dealing with the problem of determining G with a given $t(G)$, we may and shall assume that the G-spaces M in consideration are "pure G-spaces" i.e. such that any element in M is fixed by at least one element σ in $G^\sharp$.

In §1 we shall determine the structures of all finite solvable groups G with $t(G)=4$ and in §2 those of all finite solvable groups G with $t(G)=5$.

We shall give here several notations used throughout this paper. p will always denote a prime. We denote by Sp a Sylow p-group of G. If N is a subgroup of G, we denote by $(Sp)_N$ a Sylow p-subgroup of N. For any subset S of G the normalizer of S in G is denoted by $N_G(S)$, the centralizer of S in G by $C_G(S)$. For any σ in G and any subgroup N of G, $\sigma^{-1}N\sigma$ is denoted by N^σ, we denote by $\mathfrak{S}_n$ the symmetric group on n letters. and by $\mathfrak{A}_n$ the alternating group on n letters.

We shall have to use the following lemmas several times in the course of this paper.

LEMMA 1. *Let G be a finite group admitting a G-space M of type $k>0$. If there is a normal subgroup N of index p, and M_N consists of k points, then any element σ of $G-N$ is of order p, and the order of $C_G(\sigma)$ is less than kp.*

PROOF. This is clear from [4].

LEMMA 2. *Let G be a finite group, M a G-space of type $k>0$, and A a G-orbit in M. Let φ be the homomorphism from G to the symmetric group $\mathfrak{S}_{|A|}$, and N the kernel of φ, i.e. normal subgroup of G, consisting of all elements which fix all points of A. If an element $\bar{\sigma}=\sigma N$ ($\sigma \notin N$) in the quotient group $\bar{G}=G/N$ is of order m in $\bar{G}$, then any element $\sigma\tau$ of σN is also of order m.*

PROOF. If σ is of order m, then for any $\sigma\tau$ in σN, $(\sigma\tau)^m$ is in N. If $(\sigma\tau)^m \neq 1$, $M_{\sigma\tau}=M_{(\sigma\tau)^m}=A$, therefore $\sigma\tau$ is contained in N, which is impossible. Hence $\sigma\tau$ is of order m.

§1. Determination of finite solvable groups with $t(G)=4$

We shall first prove:

LEMMA 3. *Let G be a finite group acting on M and M a G-space of type 4.*

a) *If there is a G-orbit A consisting of 4 points in M, then G is one of the following groups.*

1) *An elementary abelian group of order 8 or of order 16.*

2) *A Frobenius group with the kernel N which is abelian and of order $m\equiv 1$ (mod. 4), such that $G=N\cdot H$ (semi-direct product), where $H=\langle 1, \sigma, \sigma^2, \sigma^3\rangle$ is cyclic of order 4 and the order of $\sigma\tau$ and $\sigma^{-1}\tau$ for any τ in N is 4, while the order of $\sigma^2\tau$ is 2. (In this case M has no other G-orbit consisting of 4 points than A.)*

b) *If there are three G-orbits consisting of 2 points in M respectively, G is an elementary abelian group of order 8.*

PROOF. a) The action of G on A defines a homomorphism φ from G to $\mathfrak{S}_4$. Let K be the kernel of φ. Then $(G: K)$ will be 4, 8, 12 or 24 by the transitivity of G on A. In any case the quotient group $\bar{G}=G/K$ has an element of order 2, so by lemma 2, K is an abelian group. In particular if there is an elementary abelian subgroup of order 4 in G, then K is an elementary abelian 2-group. Therefore if $(G: K)=8$ or 24, then there is an element of order 4 in $G-K$, which contradicts to the definition of K. (Cf. [3], Lemma 1) Also if $\bar{G}$ is of order 12, namely $\bar{G}$ is isomorphic to the alternating group of 4 letters, then we can easily verify that $t(\bar{G})=3$. Hence $\bar{G}$ has to be a cyclic group or an elementary abelian group of order 4.

Case 1. $\bar{G}$ is a cyclic group of order 4.

Let $\bar{\sigma}$ be a generator of $\bar{G}$, so by Lemma 1 the order of $\sigma\tau$ and $\sigma^{-1}\tau$ is 4, the order of $\sigma^2\tau$ is 2 for all τ in K. Hence σ^2 transforms any element τ of K into τ^{-1}. As K is the kernel of φ, σ induces a fix-point-free automorphism of K, hence G is a Frobenius group and the order m of K is congruent to 1 modulo 4.

Case 2. $\bar{G}$ is an elementary abelian group of order 4.

K is an elementary abelian 2-group, so is G by Lemma 1. Then G is an elementary abelian group of order 8 or of order 16. In particular when there is a G-orbit B different from A in M. We denote by K' the kernel of a homomorphism G to $\mathfrak{S}_4$ defined by the action of G on B. Then K and K' have the trivial intersection, and G is not an elementary abelian group of order 4. Hence Case 1 cannot occur and G is an elementary abelian group of order 8 or of order 16.

b) By the assumption G has three different abelian normal subgroups N_i $(i=1,2,3)$ of index 2. Any element of $G-N_i$ $(i=1,2,3)$ is of order 2 by Lemma 2, and the intersection of N_1, N_2 and N_3 is the identity. So G is an elementary abelian 2-group. Therefore we can conclude that G is an elementary abelian group of order 8. Hence this lemma is proved.

Remark. Conversely these groups in Lemma 3 admit G-spaces of type 4 clearly.

As for the case G is a finite solvable group of type 4 on a G-space M, we shall prove later, by Lemmas 5 and 6, there is a 4-points-G-orbit in M or otherwise G has some special properties. In the next Lemma 4 we shall show that groups with these "special properties" are in fact of type 4.

Lemma 4. *The following groups G are of type* 4.

(1) *G has a normal subgroup N of index* 2 *such that any element of $G-N$ is of order* 2 *(hence N is abelian), and $(S_2)_N$ is a direct product of two cyclic groups. (We call this group an A_1-group if $4<|N|$)*

(2) *Two Frobenius groups of order* 56 *or of order* 80, *with an elementary abelian normal Sylow* 2*-subgroups as Frobenius kernel. (We call these groups an A_2-group, an A_3-group respectively.)*

Proof. It is sufficient to construct G-spaces of type 4.

(1) N is abelian, so all elements of order 2 in N form a subgroup of order 4. So if we put $N^2=\{\tau^2\colon \tau\in N\}$, then $(N\colon N^2)=4$. Fix any element σ of $G-N$, and from four left cosets of N by N^2, we choose τ_1, τ_2, τ_3 and 1 respectively.

And we put

$$M_1=\{\tau\langle\sigma\rangle : N\ni\tau\}$$
$$M_2=\{\tau\langle\sigma\tau_1\rangle : N\ni\tau\}$$
$$M_3=\{\tau\langle\sigma\tau_2\rangle : N\ni\tau\}$$
$$M_4=\{\tau\langle\sigma\tau_3\rangle : N\ni\tau\}$$
$$M_5=M_6=\{N, \sigma N\}$$

Then $M=\bigcup_{i=1}^{6} M_i$ is a G-space of type 4.

(2) If G is an A_2-group, then

$$G=\{\sigma^7=1,\ F(G)=\langle\tau\rangle\times\langle\rho\rangle\times\langle\varepsilon\rangle : \sigma \text{ transforms cyclically } \tau, \rho, \varepsilon, \tau\rho, \rho\varepsilon, \tau\rho\varepsilon, \tau\varepsilon\} .$$

So if we put

M_1= the set of the left cosets of G by $\langle\tau\rangle$
$M_2=M_3=M_4=M_5$= the set of the left cosets of G by $\langle\sigma\rangle$.

Then $M=\bigcup_{i=1}^{5} M_i$ is a G-space of type 4. Also if G is an A_3-group, we can easily construct a G-space M of type 4 as above.

LEMMA 5. *Let G be a finite solvable group and M a G-space of type 4. Then the following holds:*

(1) *If G has an A_1-group as a maximal normal subgroup, then G itself is an A_1-group or there is a G-orbit consisting of 4 points in M.*

(2) *If G has an A_2-group or an an A_3-group as a maximal normal subgroup, then G has another maximal normal subgroup which is neither an A_2-group nor an A_3-group.*

PROOF. Let N be a maximal normal subgrup of G, then $(G:N)$ is a prime p. In our case $|M_N|=0, 2$ or 4. If $|M_N|=4$, then by Lemma 1 $p=2$, and any element σ of $G-N$ is of order 2 and $|C_G(\sigma)|\leqq 8$. If we prove $|C_G(\sigma)|=8$, then $(S_2)_N$ is a direct product of two cyclic group, hence G is an A_1-group. If $|C_G(\sigma)|\leqq 4$, $(S_2)_N$ is a cyclic group, so G is generalized dihedral group, which is impossible since generalized dihedral groups are of type 2 (cf. [3]). Hence $|C_G(\sigma)|=8$. If $|M_N|=2$, then also $p=2$ and N is of type 2 on $M-M_N$, so G is an elementary abelian group of order 8. Therefore we may assume $|M_N|=0$, namely N is of type 4 on M.

(1) Since N is an A_1-group, N has a normal abelian subgroup H of index

2 and of order more than 4. Then $|M_H|=0, 2$ or 4. If $|M_H|=0$, then H is an elementary abelian group of order 8 or of order 16, which contradicts to the definition of N. Also if $|M_H|=2$, N is an elementary abelian group of order 8, which contradicts to the fact that N is an A_1-group. Hence $|M_H|=4$. As $(N: H)=2$ and N is of type 4 on M, M_H is decomposed into two N-orbits M_1, M_2 consisting of 2 points respectively. By Lemma 3 b), M_1 and M_2 are all of N-orbits consisting of two points, and there is not a G-orbit consisting of two points in M (because if there is a 2-points-G-orbit in M, then G has a normal subgroup N_1 such that $M_{N_1}\neq\phi$), therefore $p=2$ and $M_1\cup M_2=M_H$ is a G-orbit consisting of four points.

(2) When N is an A_2-group or an A_3-group, $(S_2)_N$ is normal in G. In our case $M_{(S_2)_N}$ is empty.

Case 1. N is an A_2-group.

Since the order of the automorphism group of $(S_2)_N$ is $2^3\cdot 3\cdot 7$, if $p\neq 2, 3$ or 7, an element of order p in G induces an identical automorphism of $(S_2)_N$. Hence $M_{(S_2)_N}\neq\phi$, which is impossible. Therefore we examine only $p=2, 3$ and 7.

$p=2$: $N_G(S_7)$ is of order 14, so there are σ of order 7 and τ of order 2 in $N_G(S_7)$. τ and $(S_2)_N$ generate S_2. Put Z the center of S_2, and $Z^{\sigma^i}\cap Z=1$ ($i=1, \cdots, 6$). Hence we get that the order of Z is two, which is impossible.

$p=3$: The order of $N_G(S_7)$ is 21. Let σ be a generator of S_7 and α be an element of M_σ. Then $G_\alpha\cap N=S_7$, and S_7 and S_7^τ have the trivial intersection for any τ of $S_7^\#$. Therefore there are four N-orbits consisting of 8 points, at least one of which is also a G-orbit if $p=3$. Hence there is an isotropy subgroup of order 21, so we may assume that $N_G(S_7)$ is the isotropy subgroup of α. On the other hand, the automorphism of S_2 induced by ρ of order 3 in $N_G(S_7)$ has at least one fixed point τ in $S_2^\#$, that is, $M_\rho=M_\tau$. Hence τ is contained in $N_G(S_7)$, which contradicts to $N_G(S_7)=21$.

$p=7$: S_7 is of order 7^2, so S_7 is abelian. S_7 is not a cyclic group, for M_{S_2} is empty. Hence S_7 is an elementary abelian group of order 7^2. Therefore in 48 elements of order 7 in S_7 there is at least one element σ, and σ induces an automorphism of S_2 with a fixed point. $\langle\sigma\rangle S_2$ is not an A_2-group and maximal normal in G.

Case 2. N is an A_3-group.

We can prove the lemma as in Case 1.

If G is a finite group of type k on a G-space M, $M=\bigcup_{i=1}^{r} M_i$ is the decomposition of M into G-orbits and for x_i in M_i ($i=1, \cdots, r$) G_i is an isotropy subgroup of x_i, then the following equation holds. (cf. [3])

$$\sum_{i=1}^{r} \frac{1}{|G_i|} = (r-k) + \frac{k}{|G|}, \qquad k < r < 2k .$$

Hence if G is an elementary abelian group of order 8 or of order 16, we obtain for the possible orders of $G_1, \cdots, G_r$, G the following table:

	$G_1,$						$\cdot \cdot \cdot$ G_r	: G	
$r=5$:	2,	4,	4,	4,	4			: 8	(α)
$r=6$:	2,	2,	2,	2,	4,	4		: 8	(β)
$r=7$:	2,	2,	2,	2,	2,	2,	2	: 8	(γ)
$r=5$:	4,	4,	4,	4,	4			: 16	

Conversely we can easily construct G-spaces M for the groups of the above table. We call G acting on M of type (α), of type (β) and of type (γ) respectively when G of order 8 has isotropy subgroups on the above table.

LEMMA 6. *Let G be a finite solvable group and M a G-space of type 4. If G is neither an A_2-group nor an A_3-group and if any maximal normal subgroup of G is not an A_1-group, then there is a G-orbit consisting of four points in M.*

PROOF. We shall prove this lemma by induction on the order n of G. If $t(G)=4$, $8 \leqq n$. The conclusion of the lemma is already proved for $n=8$. When $8<n$, let N be a maximal normal subgroup of G, then $(G:N)=p$. By the assumption and on the way of proving Lemma 5, N is of type 4 on M, and therefore by the assumption of induction there is an N-orbit A consisting of 4 points in M. If N is not an elementary abelian group of order 8 or of order 16, then by Lemma 3 a) A is the unique N-orbit consisting of 4 points, hence A is also a G-orbit in M. Therefore we have to consider the cases where N is an elementary abelian group of order 8 or of order 16.

Case 1. N is an elementary abelian group of order 8.

We have already known that N operates on M of type (α), (β) or (γ). If N operates on M of type (α), there is only one 4-points-N-orbit in M, and therefore this N-orbit is also a G-orbit. N operates on M of type (β). If $p \neq 2$, then at least one of four N-orbits consisting of 4 points is also a G-orbit. If $p=2$, M_N is empty, so two N-orbits consisting of 2 points from a G-orbit.(*)

Case 2. N is an elementary abelian group or order 16.

There are five N-orbits consisting of four points in M. Hence as in Case 1 of type (γ), we can choose a maximal normal subgroup different from N. Now we can determine the structure of a finite solvable group with $t(G)=4$.

THEOREM 1. *G is a finite solvable group with $t(G)=4$, if and only if G is one of the following groups.*

(1) *A_1-group, A_2-group or A_3-group.*

(2) *An elementary abelian group of order* 8 *or of order* 16.

(3) *A Frobenius group with the kernel N which is abelian and of order* $m\equiv 1$ (mod. 4), *such that $G=N\cdot H$ (semi-direct product), where $H=\langle 1,\sigma,\sigma^2,\sigma^3\rangle$ is cyclic of order* 4 *and the order of $\sigma\tau$ and $\sigma^3\tau$ for any τ in N is* 4, *while the order of $\sigma^2\tau$ is* 2.

PROOF. It is clear by lemmas in this section.

§ 2. On the structure of finite solvable groups with $t(G)=5$

We can deal with the case $t(G)=5$ by a method similar to that used in [4] for the case $t(G)=3$. We may assume that G is a finite solvable group with non-trivial partition. The results and method in [3], [4] are used in this section, but we shall repeat the necessary definitions so as to make our main theorem understandable independently from [3], [4]. We shall denote by $F(G)$, the Fitting subgroup of G. G is a $(\mathfrak{P}_P)$-group if the following conditions are satisfied;

(1) There is a normal subgroup N of index p in G such that $N\neq\{1\}$

(2) For any a in $G-N$, $a^p=1$

(3) For any a in $G-N$,

$$|C_G(a)|=\begin{cases} p, & \text{if } |N|\not\equiv 0 \pmod{p} \\ p^2, & \text{if } |N|\equiv 0 \pmod{p}\,. \end{cases}$$

Then a $(\mathfrak{P}_P)$-group which is not a Frobenius group will be called $(\mathfrak{P}_P)'$-group.

We shall first prove:

LEMMA 7. *Let G be a finite solvable group and M a G-space of type p which is an add prime. Then G is a Frobenius group or a $(\mathfrak{P}_P)'$-group.*

PROOF. It is sufficient to prove that G is a $(\mathfrak{P}_P)'$-group when it is not a Frobenius group. By Baer [1], [2] and Kegel [7], G has a Normal subgroup K such that $(G:K)=q$ for some prime, $|K|\equiv 0$ (mod. q) and $\sigma^q=1$ for any $\sigma\in G-K$. By Kegel [6] K is nilpotent. Let $M=\bigcup_{i=1}^{t} M_i$ be the decomposition of M into G-orbits. If K is not a q-group then M_K is a G-orbit consisting of p-points, and therefore $p=q$. We may assume that $M_K=M_1$. Let $x_i\in M_i$ and denote by G_i the isotropy groups of x_i. Then for $i\geqq 2$, $G_i\cap K=(1)$, namely G_i is of order p. By the property of p-groups, we have $|C_G(\sigma)|\geqq p^2$, and by Lemma 1 $|C_G(\sigma)|\leqq p^2$. Hence G is a $(\mathfrak{P}_P)'$-groop. If K is a q-group then G is a q-group and therefore $p=q$. Hence there is a non-identity central element τ and M_τ is a G-orbit consisting of p-points. For $x\in M_\tau$, $(G:G_x)=p$ and so $G\triangleright G_x$. The result follows immediately.

REMARK. Using this lemma we can conclude that a group G of order 24 does not admit any G-space of type 5.

LEMMA 8. *Let G be a generalized dihedral group of order $2p^n$ and p an odd prime. Then G does not admit any G-space of type p. Particularly a group G of order* 50 *does not admit any G-space of type* 5.

PROOF. If G does not have an elementary abelian normal subgroup of order p^n, then by Iwahori and Kondo [4] G admits a G-space of type $2k$ where k is a positive integer. Hence we may assume that G has an elementary abelian normal subgroup of order p^n. Therefore there exist $(1+p+\cdots+p^{n-1})$-different normal subgroups of order p, say $\langle\sigma_i\rangle$ $i=1,2,\cdots,1+p+\cdots+p^{n-1}$, and all involutions are conjugate in G. If M is a G-space of type p, M_{σ_i} is a G-orbit consisting of p-points and for $x\in M_{\sigma_i}$ G_x is of order $2p^{n-1}$. Then for any involution ρ we have $|M_\rho\cap M_{\sigma_i}|\geqq 1$, $i=1,\cdots,1+p+\cdots+p^{n-1}$ and so $|M_\rho|\geqq p+1$, which is impossible. If G is of order 50 and $t(G)>0$ then it is easily verified that G is a generalized dihedral group.

REMARK. If $p=2$, the conclusion of Lemma 7 does not hold. For example let G be a symmetric group on 4 letters. Then G is a solvable group and $t(G)=2$. But G is neither a Frobenius group nor a $(\mathfrak{P}_P)'$-group. Also if $p=2$, the conclusion of Lemma 8 does not hold. Since G is a dihedral group of order 2^{n+1}, G admits a G-space of type 2. C.f. Iwahori [3].

LEMMA 9. *Let G be a finite solvable group and M a G-space of type* 5. *If there exist two G-orbits A, B in M such that $|A|=|B|=5$, $A\cap B=\phi$, then G is isomorphic to an elementary abelian group of order* 25.

PROOF. Let $M=\bigcup_{i=1}^{r} M_i$ be the decomposition of M into G-orbits such that $M_r=A$, $M_{r-1}=B$. The action of G on A defines a homomorphism $\varphi\colon G\to\mathfrak{S}_5$. Let $b\in B$. Since $G_b\cap\ker\varphi=(1)$ and $(G\colon G_b)=5$, $\ker\varphi$ is of order 1 or 5. Since G is solvable and G acts transitively on A, we may assume that $|G|\leqq 100$ and $|G|\equiv 0$ (mod. 5). When G is a $(\mathfrak{P}_5)'$-group, it is easily seen that G is isomorphic to an elementary abelian group of order 25. Thus by Lemmas 7 and 8 we may assume that G is a Frobenius group of order 100. Since $|F(G)|=25$, G has a normal subgroup K of order 50. Then M_K is G-stable and $|M_K|=5,3,2$ or 0. Clearly $|M_K|\neq 5,3$. If $|M_K|=0$, K is of type 5 on M which is impossible by Lemma 8. If $|M_K|=2$, K is of type 3 on $M-M_K$. Then by Iwahori and Kondo [4] G is a $(\mathfrak{P}_3)$-group, which is also impossible.

LEMMA 10. *Let G be a finite solvable group and M a G-space of type* 5. *If $|G|\geqq 25$, then there is a G-orbit in M consisting of* 5-*points.*

PROOF. We shall prove our assertion by the induction on $|G|$. If $|G|=25$

then $G\cong Z_5\times Z_5$ and it is easy to verify that the pure part P of M consists of 6 G-orbits M_i $(i=1, 2, \cdots, 6)$ with $|M_i|=5$. Now let $n=|G|>25$ and assume that our assertion is valid for finite solvable groups of order $<n$. Let H be a maximal normal subgroup of G and $(G: H)=p$. Then we have $|M_H|=5, 3, 2, 0$ and M_H is G-stable.

Case 1. $|M_H|=0$

H is of type 5 on M. If $|H|\leqq 24$, H is isomorphic to $\mathfrak{A}_4$ or a generalized dihedral group D of order 18, and G is of order $12p$ or $18p$ respectively. By Lemma 7 we may assume that G is a Frobenius group. Since H is not nilpotent in both cases, we have $H\supsetneqq F(G)$. Then we may assume that $H\cong D$ and $F(G)\cong Z_3\times Z_3$. Since for any σ in $F(G)^{\#}$, $C_G(\sigma)\subseteqq F(G)$, we have $p=2$. Hence by Iwahori [3] G admits a G-space of type 4, namely $t(G)\neq 5$. Therefore we may assume that $|H|\geqq 25$ and by our inductive-assumption there is an H-orbit A in M consisting of 5 points. If $H\not\cong Z_5\times Z_5$, such a set A is unique by Lemma 9, then A is a G-orbit and $|A|=5$. If $H\cong Z_5\times Z_5$, we may assume that $p>5$ and G is a Frobenius group by Lemmas 7 and 8. Since $|\mathrm{Aut}\, H|=2^5\cdot 3\cdot 5$, G has a non-trivial central element and this is impossible.

Case 2. $|M_H|=2$.

H is abelian and is of type 3 on $M-M_H$. Then $H\cong Z_3\times Z_3$. This contradicts to our assumption.

Case 3. $|M_H|=3$.

H is of type 2 on $M-M_H$. If $H\cong\mathfrak{S}_4, \mathfrak{A}_4$, or $\mathfrak{A}_5$, then sylow 3-subgroup of G is of order 9 and so G is a $(\mathfrak{P}_3)$-group. Thus we may assume that H is isomorphic to a generalized dihedral group and $|H|\equiv 0$ (mod. 3^2). H has an abelian normal subgroup N such that (H: N)$=2$ and $\tau^2=1$ for any τ in $H-N$. By our assumption $|M_N|=5$. Since for x in $M-M_H$, $G_x\supsetneqq N$ we have $(G: G_x)=3$ or 6. If $(G: G_x)=3$, $G_x\supseteqq H$ or $G_x\cap H=N$, which is impossible. If $(G: G_x)=6$, the action of G on G/G_x defines homomorphism $\varphi; G\to\mathfrak{S}_6$ and clearly $\ker\varphi=1$. But in this case $|G|\equiv 0$ (mod. $2\cdot 3^3$) and $|\mathfrak{S}_6|=2^4\cdot 3^2\cdot 5$. Then we get contradiction.

Case 4. $|M_H|=5$

M_H is a G-orbit consisting of 5-points.

Now we can prove the following theorem.

THEOREM 2. *Let G be a finite solvable group. Then $t(G)=5$ if and only if G is a $(\mathfrak{P}_5)$-group and G is not a Frobenius group of order 80 whose $F(G)$ is an elementary abelian group.*

PROOF. Let M be a G-space of type 5 and M_0 a G-orbit consisting of 5-

points. The action of G on M_0 defines a homomorphism $\varphi: G \to \mathfrak{S}_5$ and $G/\ker\varphi$ is isomorphic to a solvable subgroup of $\mathfrak{S}_5$. Since G acts transitively on M we have $(G: \ker\varphi) = 5, 10$ or 20. Put $\ker\varphi = H_0$, $M_0 = \{x, y, z, u, v\}$ and $G_0 = G_v$.

Case 1. $(G: H_0) = 5$

We have $\sigma^5 = 1$ for σ in $G - H_0$ and $G_a \cap H_0 = (1)$ for any a in $M - M_0$, that is, G_a is of order 5. If ξ in $G_0 \cap C_G(\sigma)$, $\xi^5 = 1$. Thus $|G_G(\sigma)|$ is a power of 5. Hence if $|G_0| \not\equiv 0 \pmod{5}$ we have $|G_G(\sigma)| = 5$. If $|G_0| \equiv 0 \pmod{5}$ we have $|C_G(\sigma)| \geqq 5^2$. The case can be treated in a manner similar to that used in the proof of Lemma 7 and G is a $(\mathfrak{P}_5)$-group.

Case 2. $(G: H_0) = 10$

There exist $\sigma, \tau \in G$ such that

$$\sigma x = y,\ \sigma y = z,\ \sigma z = u,\ \sigma u = v,\ \sigma v = x,\ \tau x = u,\ \tau y = z,\ \tau u = x,\ \tau z = y,\ \tau v = v,\ \sigma^5 = \tau^2 = 1\ .$$

For any $\xi \in H_0$, $\xi^\tau = \xi^{-1}$. Hence H_0 is an abelian group. Put $\rho = \tau^{\sigma^{-1}}$. Then

$$\rho x = x,\ \rho y = v,\ \rho z = u,\ \rho u = z,\ \rho v = y,\ \rho^2 = 1,\ \xi^{\rho^{-1}} = \xi^{-1}, \qquad (\tau^{-1}\rho)\xi = \xi(\tau^{-1}\rho)\ .$$

Thus $\tau^{-1}\rho$ commutes with every element of H_0. Therefore

$$K = H_0 \cup H_0(\tau^{-1}\rho) \cup H_0(\tau^{-1}\rho)^2 \cup H_0(\tau^{-1}\rho)^3 \cup H_0(\tau^{-1}\rho)^4$$

is an abelian subgroup of G. Obviously we have $M_K = \phi$. Thus K is of type 5 on M and so $K \cong Z_5 \times Z_5$. Now since $(G: H_0) = 10$ we have $|G| = 50$ which is impossible.

Case 3. $(G: H_0) = 20$

By the same way as in Case 2 we have $|G| = 100$, which is impossible. Thus we have finished the proof of this theorem.

From this theorem follows immediately the following corollary, which verifies partially the result of § 1.

COROLLARY. *Let G be a $(\mathfrak{P}_5)$-group and $t(G) = 4$. Then G is a Frobenius group of order 80 and $F(G)$ is an elementary abelian group.*

Continuation of (*):

If N operates on M of type (γ), there are seven N-orbits consisting of four points in M. If $p \neq 7$, then at least one of seven N-orbits is also a G-orbit. Also if $p = 7$, then S_7 is normal in G (because G is not an A_2-group). Hence there is a maximal normal subgroup N_1 of order 28 in G. Therefore we can choose N_1 as a maximal normal subgroup instead of N.

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