On the class number and the unit group of certain algebraic number fields

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Let $k_1, k_2, \cdots, k_m$ be m quadratic extensions of the rational number field $\boldsymbol{Q}$. The fields $k_1, k_2, \cdots, k_m$ are called *independent* if the compositum $K=k_1\cdot k_2\cdots k_m$ is of degree 2^m (over $\boldsymbol{Q}$). We shall suppose $k_1, k_2, \cdots, k_m$ as independent. Then K is a Galois extension of $\boldsymbol{Q}$ with the Galois group of type $(2, 2, \cdots, 2)$ and there are exactly $t=2^m-1$ different quadratic fields between $\boldsymbol{Q}$ and K including $k_1, k_2, \cdots, k_m$. We shall denote them with k_i, $i=1, 2, \cdots, t$. Let h_i and e_i be the class number and the unit group of k_i, and H, E the class number and the unit group of K. Then it is known (cf. [3]) that

$$H=\frac{1}{2^v}\left[E:\prod_{i=1}^{t} e_i\right]\cdot\prod_{i=1}^{t} h_i\,, \tag{1}$$

where

$$v=\begin{cases} m(2^{m-1}-1) & \text{when } K \text{ is real,}\\ (m-1)(2^{m-2}-1)+2^{m-1}-1 & \text{when } K \text{ is imaginary.}\end{cases}$$

Supposing h_i and e_i as known (in fact h_i can be calculated by the "class number formula" from the value of the discriminant of k_i, and a generator of e_i (a fundamental unit of k_i) can also be found by the classical method,) the problem of calculating H is thus reduced to that of determining E. In [2], Kubota has given a method to find out a system of fundamental units, i.e. a system of generators of E in case $m=2$. In § I we shall generalize this method and give an algorism to obtain a system of generators of E, for any $m\geqq 2$. Thus E and H can be found in concrete cases. We shall show this in examples in § 2, and list our results on H and E for certain K's with Galois group of type (2, 2, 2) in a table given in § 3.

§ 1. A principle of reduction

Let $K=k_1\cdot k_2\cdots k_m$ be a field of the type described above and σ, ρ be two (different) generators of the Galois group of $K/\boldsymbol{Q}$. Put $\sigma\rho=\tau$. Let K_1, K_2, K_3 be the invariant subfields of K by σ, ρ, τ respectively, and E_i be the unit group of K_i. Then clearly $E_1\cdot E_2\cdot E_3$ are subgroup of the unit group E of K. Put furthermore $F=K_1\cap K_2\cap K_3$, so that K/F has the Galois group of type (2, 2).

$$\begin{tikzcd} & K \arrow[dl, dash, "\sigma"'] \arrow[d, dash, "\rho"] \arrow[dr, dash, "\tau"] & \\ K_1 \arrow[dr, dash] & K_2 \arrow[d, dash] & K_3 \arrow[dl, dash] \\ & F & \end{tikzcd}$$

Let ε be any unit in K. Then we have obviously (cf. [2])

$$\varepsilon^2=\frac{(\varepsilon\cdot\varepsilon^{\sigma})\cdot(\varepsilon\cdot\varepsilon^{\rho})}{(\varepsilon\cdot\varepsilon^{\tau})^{\sigma}}\in E_1\cdot E_2\cdot E_3\ .$$

From this simple formula, we can conclude the following.

Assume the group E_1, E_2, E_3 as known, i.e. assume as known some systems of generators of these groups. Then a system of E can be obtained in the following way. Let $\varepsilon_{i1}, \cdots, \varepsilon_{ir_i}$ $(i=1, 2, 3)$ be a system of generators of E_i. Then pick out the numbers which are perfect squares in K from among the $2^{r_1+r_2+r_3}-1$ numbers

$$* \qquad \varepsilon_{11}^{\alpha}\cdots\varepsilon_{1r_1}^{\gamma}\cdot\varepsilon_{21}^{\delta}\cdots\varepsilon_{2r_2}^{\eta}\cdot\varepsilon_{31}^{\theta}\cdots\varepsilon_{3r_3}^{\mu}\quad(\neq 1)\ ,$$

where $\alpha, \cdots, \mu=0$ or 1. We denote them by $\nu_1, \nu_2, \cdots, \nu_k$. Then E is generated by

$$** \qquad \varepsilon_{11}, \cdots, \varepsilon_{1r_1}, \varepsilon_{21}, \cdots, \varepsilon_{2r_2}, \varepsilon_{31}, \cdots, \varepsilon_{3r_3} \text{ and } \sqrt{\nu_1}, \sqrt{\nu_2}, \cdots, \sqrt{\nu_k}\ .$$

Thus the existence of an algorism for the determination of a system of E will be shown, if we prove the following theorem.

THEOREM. *Let K and k_i be as above, and let d_i denote the discriminants of the quadratic fields k_i $(1\leq i\leq t)$. Let A be an element in K. Then there exists an algorism to determine wheather A is a perfect square in K or not. If A is a perfect square, we can calculate explicitly its square root in the form $a_0+\sum_{i=1}^{t} a_i\sqrt{d_i}$ where a_i are rational numbers.*

PROOF. We prove by induction on the degree of K.

1) When $m=0$, i.e. $K=\boldsymbol{Q}$, then this is clear by the fundamental theorem of arithmetic.

2) When $m=1$, i.e. $K=\boldsymbol{Q}(\sqrt{d})$ with a rational integer d, then an element $A=a+b\sqrt{d}$, $(a, b\in \boldsymbol{Q})$ is a perfect square in K if and only if there exist two rational numbers x and y such that $a+b\sqrt{d}=(x+y\sqrt{d})^2$, namely if and only if the equation

$$\begin{cases} a=x^2+dy^2 \\ b=2xy \end{cases}$$

has a solution in $\boldsymbol{Q}$. Now this equation is explicitly solved in the form:

$$x=\pm\sqrt{\frac{a\pm\sqrt{a^2-db^2}}{2}}\,, \qquad y=\frac{b}{2x}\,.$$

Our problem is thus reduced to the case 1).

3) When $m\geq 2$, we assume that the assertion in the theorem is true for K_1, K_2, K_3 and F. If A is a perfect square in K, then so should be $N_{K/K_1}A=AA^\sigma$ in K_1, $N_{K/K_2}A=AA^\rho$ in K_2, $(N_{K/K_3}A)^\sigma=A^\sigma A^\rho$ in K_3, and it can be decided by the algorism in K_1, K_2, K_3 if this is the case. If these are perfect squares, we may consider by assumption the square roots $B_1=\sqrt{AA^\sigma}$, $B_2=\sqrt{AA^\rho}$ and $B_3=\sqrt{A^\sigma A^\rho}$ as explicitly given. As

$$B_1\cdot B_2\cdot B_3=AB_3^2\,, \tag{2}$$

A is a perfect square if and only if $B_1\cdot B_2\cdot B_3$ is so. Put now

$$\begin{aligned} &b=N_{K_i/F}B_i=\sqrt{N_{K/F}A} \qquad (\in F)\,,\\ &\xi=B_1\cdot B_2\cdot B_3+bB_1+bB_2+bB_3\,,\\ &C=\xi+\xi^\sigma+\xi^\rho+\xi^\tau \qquad (=S_{K/F}\xi)\,. \end{aligned}$$

Then we obtain

$$B_1\cdot B_2\cdot B_3\cdot C=\xi^2\,. \tag{3}$$

In fact

$$\begin{aligned} B_1\cdot B_2\cdot B_3\cdot C=&B_1\cdot B_2\cdot B_3\cdot\xi+B_1\cdot B_2\cdot B_3\cdot(B_1\cdot B_2^\sigma\cdot B_3^\sigma+bB_1+bB_2^\sigma+bB_3^\sigma)\\ &+B_1\cdot B_2\cdot B_3\cdot(B_1^\rho\cdot B_2\cdot B_3^\rho+bB_1^\rho+bB_2+bB_3^\rho)\\ &+B_1\cdot B_2\cdot B_3\cdot(B_1^\tau\cdot B_2^\tau\cdot B_3+bB_1^\tau+bB_2^\tau+bB_3)\\ =&B_1\cdot B_2\cdot B_3\cdot\xi+bB_1\xi+bB_2\xi+bB_3\xi\\ =&\xi^2\,. \end{aligned}$$

If C is a perfect square in K, $F(\sqrt{C})$ is contained in K. Hence $F(\sqrt{C})$ is F itself or there is some quadratic field $k_i=Q(\sqrt{d_i})$ in K such that $F(\sqrt{C})=F\cdot k_i=F(\sqrt{d_i})$. In the latter case, as $F(\sqrt{C})/F$ is kummerian of degree 2, there is some element g_i in F such that $C=d_i\cdot g_i^2$. Hence by (3), $B_1\cdot B_2\cdot B_3$ is a perfect square in K if and only if C or any one of $C\cdot d_i$ $(1\leq i\leq t)$ is a perfect square in F. By our assumption, we can decide if this is a case by the algorism in F.

From (2), (3) we obtain

$$\sqrt{A}=\frac{\sqrt{B_1\cdot B_2\cdot B_3}}{B_3}=\frac{\xi}{\sqrt{C}\,B_3}\,,$$

and if $\sqrt{C}\in F$ or $\sqrt{Cd_i}\in F$, then $\xi, \sqrt{C}, B_3$ can be expressed in the form $a_0+\sum a_i\sqrt{d_i}$ by our assumption. So $\sqrt{A}$ can also be expressed in this form. Q.E.D.

REMARK. Now if we put,

$$\begin{aligned}
\xi_1 &= B_1\cdot B_2\cdot B_3+bB_1-bB_2-bB_3\ ,\\
\xi_2 &= B_1\cdot B_2\cdot B_3-bB_1+bB_2-bB_3\ ,\\
\xi_3 &= B_1\cdot B_2\cdot B_3-bB_1-bB_2+bB_3\ ,\\
C_j &= \xi_j+\xi_j^{\sigma}+\xi_j^{\rho}+\xi_j^{\tau}\ , \qquad j=1, 2, 3,
\end{aligned}$$

we have obviously

$$B_1\cdot B_2\cdot B_3=\frac{1}{4}(\xi+\xi_1+\xi_2+\xi_3)\ , \tag{4}$$

and we obtain

$$B_1\cdot B_2\cdot B_3\cdot C_j=\xi_j^2 \qquad j=1, 2, 3, \tag{5}$$

in a similar way as we obtained (3) above.

From (4), (5) we have

$$\sqrt{B_1\cdot B_2\cdot B_3}=\frac{1}{4}(\sqrt{C}+\sqrt{C_1}+\sqrt{C_2}+\sqrt{C_3})\ . \tag{6}$$

This formula is more convenient to calculate explicitly $\sqrt{A}$.

When $N_{K_1/F}B_1=1$ and $B_2=B_3=1$, we get from (6) (cf. [2])

$$\sqrt{B_1}=\frac{\sqrt{S_{K_1/F}(B_1+1)}+\sqrt{S_{K_1/F}(B_1-1)}}{2}\ . \tag{7}$$

As a system of generators of E can be found, a system of fundamental units of K can be obtained using the free group theory.

§ 2. Examples

I. $K=\boldsymbol{Q}(\sqrt{2}, \sqrt{3})$.

Put $k_1=\boldsymbol{Q}(\sqrt{2})$, $k_2=\boldsymbol{Q}(\sqrt{3})$, $k_3=\boldsymbol{Q}(\sqrt{6})$. $k_i, i=1, 2, 3$ are real quadratic fields, so each k_i has a fundamental unit ε_i. ε_i, $i=1, 2, 3$ are given by: $\varepsilon_1=1+\sqrt{2}$, $\varepsilon_2=2+\sqrt{3}$, $\varepsilon_3=5+2\sqrt{6}$.

$$\begin{tikzcd}
 & K \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & \\
\boldsymbol{Q}(\sqrt{2}) \arrow[dr, dash] & \boldsymbol{Q}(\sqrt{3}) \arrow[d, dash] & \boldsymbol{Q}(\sqrt{6}) \arrow[dl, dash] \\
 & \boldsymbol{Q} &
\end{tikzcd}$$

To find a system of generators of the unit group E of K, we have to pick up the numbers which are perfect squares in K from among 7 numbers

$$\varepsilon_1^\alpha \varepsilon_2^\beta \varepsilon_3^\gamma \quad (\neq 1) \qquad \alpha, \beta, \gamma = 0 \quad \text{or} \quad 1 .$$

For the value of (α, β, γ) other than $(0,1,0)$, $(0,0,1)$, $(0,1,1)$, $\varepsilon_1^\alpha \varepsilon_2^\beta \varepsilon_3^\gamma$ does not become totally positive, so that these three are only possible ones. When $(\alpha, \beta, \gamma) = (0,1,0)$, then

$$\xi = 1\cdot\varepsilon_2\cdot 1 + 1 + \varepsilon_2 + 1 = 2(\varepsilon_2 + 1) = 2(3+\sqrt{3}) ,$$
$$C = S_{K/Q}\xi = 24 .$$

As $\sqrt{C}$ $(=2\sqrt{6})$ is an element of K, ε_2 is a perfect square and its square root is

$$\sqrt{\varepsilon_2} = \frac{\xi}{\sqrt{C}} = \frac{1}{2}(\sqrt{2}+\sqrt{6}) .$$

In the same way $\sqrt{\varepsilon_3} = \sqrt{2}+\sqrt{3}$. Thus a system of fundamental units is $\{\varepsilon_1, \sqrt{\varepsilon_2}, \sqrt{\varepsilon_3}\}$. From (1) the class number of K is

$$H = \frac{1}{2^2}\cdot[E : \prod e_i]\cdot \prod h_i = \frac{1}{4}\cdot 4\cdot 1 = 1 .$$

II. $K = \boldsymbol{Q}(\sqrt{2}, \sqrt{3}, \sqrt{11})$

Put $k_1 = \boldsymbol{Q}(\sqrt{2})$, $k_2 = \boldsymbol{Q}(\sqrt{3})$, $k_3 = \boldsymbol{Q}(\sqrt{6})$, $k_4 = \boldsymbol{Q}(\sqrt{11})$, $k_5 = \boldsymbol{Q}(\sqrt{22})$, $k_6 = \boldsymbol{Q}(\sqrt{33})$, $k_7 = \boldsymbol{Q}(\sqrt{66})$, $K_1 = \boldsymbol{Q}(\sqrt{2}, \sqrt{3})$, $K_2 = \boldsymbol{Q}(\sqrt{2}, \sqrt{11})$, $K_3 = \boldsymbol{Q}(\sqrt{2}, \sqrt{33})$.

$$\begin{tikzcd}
 & \boldsymbol{Q}(\sqrt{2}, \sqrt{3}, \sqrt{11}) \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & \\
\boldsymbol{Q}(\sqrt{2}, \sqrt{3}) \arrow[dr, dash] & \boldsymbol{Q}(\sqrt{2}, \sqrt{11}) \arrow[d, dash] & \boldsymbol{Q}(2, \sqrt{33}) \arrow[dl, dash] \\
 & \boldsymbol{Q}(\sqrt{2}) &
\end{tikzcd}$$

Let ε_i be a fundamental unit of k_i, then from I a system of fundamental units of K_1 is $\{\varepsilon_1, \sqrt{\varepsilon_2}, \sqrt{\varepsilon_3}\}$. In the same way $\{\varepsilon_1, \sqrt{\varepsilon_4} = \frac{1}{2}(3\sqrt{2}+\sqrt{22}), \sqrt{\varepsilon_5} = 7\sqrt{2} + 3\sqrt{11}\}$ and $\{\varepsilon_1, \varepsilon_6 = 23+4\sqrt{33}, \sqrt{\varepsilon_7} = 4\sqrt{2}+\sqrt{33}\}$ are systems of fundamental units of K_2, K_3 respectively. As $\sqrt{\varepsilon_6}$ $(=2\sqrt{3}+\sqrt{11})$ is contained in K, we shall pick up totally positive units from among 63 numbers

$$\varepsilon_1^\alpha \sqrt{\varepsilon_2}^{\,\beta} \sqrt{\varepsilon_3}^{\,\gamma} \sqrt{\varepsilon_4}^{\,\delta} \sqrt{\varepsilon_5}^{\,\eta} \sqrt{\varepsilon_7}^{\,\theta} \quad (\neq 1), \qquad \alpha, \cdots, \theta = 0 \text{ or } 1 .$$

They are $\sqrt{\varepsilon_3}\cdot\sqrt{\varepsilon_5}\cdot\sqrt{\varepsilon_7}$, $\sqrt{\varepsilon_2}\cdot\sqrt{\varepsilon_4}\cdot\sqrt{\varepsilon_7}$, $\sqrt{\varepsilon_2}\cdot\sqrt{\varepsilon_3}\cdot\sqrt{\varepsilon_4}\cdot\sqrt{\varepsilon_5}$, $\varepsilon_1\cdot\sqrt{\varepsilon_2}\cdot\sqrt{\varepsilon_3}$,

$\varepsilon_1\cdot\sqrt{\varepsilon_4}\cdot\sqrt{\varepsilon_5}$, $\varepsilon_1\cdot\sqrt{\varepsilon_3}\cdot\sqrt{\varepsilon_4}\cdot\sqrt{\varepsilon_7}$ and $\varepsilon_1\sqrt{\varepsilon_2}\cdot\sqrt{\varepsilon_5}\cdot\sqrt{\varepsilon_7}$. The unit $\sqrt{\varepsilon_3}\cdot\sqrt{\varepsilon_5}\cdot\sqrt{\varepsilon_7}$ is a perfect square, because

$C=S_{K/k_1}(\xi)=S_{K/k_1}(\sqrt{\varepsilon_3}\cdot\sqrt{\varepsilon_5}\cdot\sqrt{\varepsilon_7}-\sqrt{\varepsilon_3}-\sqrt{\varepsilon_5}-\sqrt{\varepsilon_7})=44\cdot(1+2\sqrt{2})^2$ is a perfect square in K. As $C_1=132\cdot(1+\sqrt{2})^2$, $C_2=36\cdot(3+\sqrt{2})^2$ and $C_3=12\cdot(5+2\sqrt{2})^2$, the unit $\sqrt{\varepsilon_3}\cdot\sqrt{\varepsilon_5}\cdot\sqrt{\varepsilon_7}$ is the square of

$$\begin{aligned}\sqrt[4]{\varepsilon_3\cdot\varepsilon_5\cdot\varepsilon_7}&=\frac{1}{4}(\sqrt{C}+\sqrt{C_1}+\sqrt{C_2}+\sqrt{C_3})\\&=\frac{1}{2}(9+3\sqrt{2}+5\sqrt{3}+2\sqrt{6}+\sqrt{11}+2\sqrt{22}+\sqrt{33}+\sqrt{66}).\end{aligned}$$

In the same way it is assured that $\sqrt{\varepsilon_2}\cdot\sqrt{\varepsilon_4}\cdot\sqrt{\varepsilon_7}$ and $\sqrt{\varepsilon_2}\cdot\sqrt{\varepsilon_3}\cdot\sqrt{\varepsilon_4}\cdot\sqrt{\varepsilon_5}$ are perfect squares, and that the four others are not so in K. Hence ε_1, $\sqrt{\varepsilon_2}$, $\sqrt{\varepsilon_3}$, $\sqrt{\varepsilon_4}$, $\sqrt{\varepsilon_6}$, $\sqrt[4]{\varepsilon_2\varepsilon_4\varepsilon_7}$ and $\sqrt[4]{\varepsilon_3\varepsilon_5\varepsilon_7}$ constitute a system of fundamental units of K. The group index $\left[E:\prod_{i=1}^{7}e_i\right]$ is obviously 2^8. From (1), we obtain $H=\frac{1}{2^9}\cdot 2^8\cdot\prod_{i=1}^{7}h_i=1$.

§ 3. A table of E and H

We shall give a table of E and H for all the fields of type $K=k_1\cdot k_2\cdot k_3$ with independent k_i such that, if $K\supset Q(\sqrt{d_i})$ $1\le i\le 7$, the following conditions hold:

1) All $|m_i|<100$, $1\le i\le 7$, where $m_i=d_i$ when $d_i\equiv 1 \pmod 4$, $m_i=\frac{1}{4}d_i$ when $d_i\equiv 0 \pmod 4$.

2) When K is imaginary the group $E^2=\{\varepsilon^2;\varepsilon\in E\}$ is not contained in $\prod_{i=1}^{7}e_i$. (When E^2 is contained in $\prod_{i=1}^{7}e_i$, we can easily find a system of fundamental units. (Cf. [1], [2]).)

The field $K=Q(\sqrt{m_1},\sqrt{m_2},\sqrt{m_3})$ has 7 quadratic subfields. We shall put $k_1=Q(\sqrt{m_1})$, $k_2=Q(\sqrt{m_2})$, $k_3=Q(\sqrt{m_1m_2})$, $k_4=Q(\sqrt{m_3})$, $k_5=Q(\sqrt{m_1m_3})$, $k_6=Q(\sqrt{m_2m_3})$, $k_7=Q(\sqrt{m_1m_2m_3})$. ε_i in the table means a fundamental unit of k_i.

1) $K=Q(\sqrt{2},\sqrt{3},\sqrt{5})$

$\varepsilon_1=1+\sqrt{2}$, $\sqrt{\varepsilon_2}=\frac{1}{2}(\sqrt{2}+\sqrt{6})$,

$\sqrt{\varepsilon_3}=\sqrt{2}+\sqrt{3}$, $\varepsilon_4=\frac{1}{2}(1+\sqrt{5})$,

$\sqrt{\varepsilon_1\varepsilon_4\varepsilon_5}=\frac{1}{2}(3+\sqrt{2}+\sqrt{5}+\sqrt{10})$, $\sqrt{\varepsilon_6}=\frac{1}{2}(\sqrt{6}+\sqrt{10})$,

$\sqrt[4]{\varepsilon_1^2\varepsilon_2\varepsilon_4^2\varepsilon_6\varepsilon_7}=\frac{1}{4}(6+5\sqrt{2}+2\sqrt{3}+\sqrt{6}+2\sqrt{5}+\sqrt{10}+2\sqrt{15}+\sqrt{30})$,

$H=1$.

2) $K=\boldsymbol{Q}(\sqrt{2}, \sqrt{3}, \sqrt{7})$

$\varepsilon_1=1+\sqrt{2}$, $\sqrt{\varepsilon_2}=\frac{1}{2}(\sqrt{2}+\sqrt{6})$,

$\sqrt{\varepsilon_3}=\sqrt{2}+\sqrt{3}$, $\sqrt{\varepsilon_6}=\frac{1}{2}(\sqrt{3}+\sqrt{7})$,

$\sqrt{\varepsilon_7}=\sqrt{6}+\sqrt{7}$,

$\sqrt[4]{\varepsilon_1^2\varepsilon_5\varepsilon_6\varepsilon_7}=\frac{1}{4}(6+5\sqrt{2}+2\sqrt{3}+3\sqrt{6}+2\sqrt{7}+\sqrt{14}+2\sqrt{21}+\sqrt{42})$,

$\sqrt[4]{\varepsilon_2\varepsilon_3\varepsilon_4\varepsilon_6\varepsilon_7}=\frac{1}{4}(6+7\sqrt{2}+6\sqrt{3}+3\sqrt{6}+2\sqrt{7}+3\sqrt{14}+2\sqrt{21}+\sqrt{42})$,

$H=1$.

3) $K=\boldsymbol{Q}(\sqrt{2}, \sqrt{3}, \sqrt{11})$

$\varepsilon_1=1+\sqrt{2}$, $\sqrt{\varepsilon_2}=\frac{1}{2}(\sqrt{2}+\sqrt{6})$,

$\sqrt{\varepsilon_3}=\sqrt{2}+\sqrt{3}$, $\sqrt{\varepsilon_4}=\frac{1}{2}(3\sqrt{2}+\sqrt{22})$,

$\sqrt{\varepsilon_6}=2\sqrt{3}+\sqrt{11}$,

$\sqrt[4]{\varepsilon_2\varepsilon_4\varepsilon_7}=\frac{1}{4}(8+\sqrt{2}+2\sqrt{3}+3\sqrt{6}+2\sqrt{11}+\sqrt{22}+\sqrt{66})$,

$\sqrt[4]{\varepsilon_3\varepsilon_5\varepsilon_7}=\frac{1}{2}(9+3\sqrt{2}+5\sqrt{3}+2\sqrt{6}+\sqrt{11}+2\sqrt{22}+\sqrt{33}+\sqrt{66})$,

$H=1$.

4) $K=\boldsymbol{Q}(\sqrt{2}, \sqrt{3}, \sqrt{13})$

$\varepsilon_1=1+\sqrt{2}$, $\sqrt{\varepsilon_2}=\frac{1}{2}(\sqrt{2}+\sqrt{6})$,

$\sqrt{\varepsilon_3}=\sqrt{2}+\sqrt{3}$, $\varepsilon_4=\frac{1}{2}(3+\sqrt{13})$,

$\sqrt{\varepsilon_1\varepsilon_4\varepsilon_5}=\frac{1}{4}(5+3\sqrt{2}+\sqrt{13}+\sqrt{26})$, $\sqrt{\varepsilon_6}=2\sqrt{3}+\sqrt{13}$,

$\sqrt[4]{\varepsilon_3\varepsilon_4^2\varepsilon_6\varepsilon_7}=\frac{1}{2}(8+5\sqrt{2}+3\sqrt{3}+3\sqrt{6}+2\sqrt{13}+\sqrt{26}+\sqrt{39}+\sqrt{78})$,

$H=1$.

5) $K=\boldsymbol{Q}(\sqrt{2}, \sqrt{5}, \sqrt{7})$

$\varepsilon_1=1+\sqrt{2}$, $\varepsilon_2=\frac{1}{2}(1+\sqrt{5})$,

$\sqrt{\varepsilon_1\varepsilon_2\varepsilon_3}=\frac{1}{4}(3+\sqrt{2}+\sqrt{5}+\sqrt{10})$, $\sqrt{\varepsilon_4}=\frac{1}{2}(3\sqrt{2}+\sqrt{14})$,

$\sqrt{\varepsilon_5}=2\sqrt{2}+\sqrt{7}$, $\sqrt{\varepsilon_6}=\frac{1}{2}(\sqrt{10}+\sqrt{14})$,

$\sqrt[4]{\varepsilon_1^2\varepsilon_4\varepsilon_6\varepsilon_7}=\frac{1}{4}(8+9\sqrt{2}+8\sqrt{5}+5\sqrt{10}+6\sqrt{7}+5\sqrt{14}+2\sqrt{35}+\sqrt{70})$,

$H=1$.

6) $K=\boldsymbol{Q}(\sqrt{2}, \sqrt{15}, \sqrt{21})$

$\varepsilon_1=1+\sqrt{2}$, $\varepsilon_2=4+\sqrt{15}$,

$\sqrt{\varepsilon_2\varepsilon_3}=\frac{1}{2}(6+5\sqrt{2}+2\sqrt{15}+\sqrt{30})$, $\varepsilon_4=\frac{1}{2}(5+\sqrt{21})$,

$\sqrt{\varepsilon_4\varepsilon_5}=\frac{1}{2}(7+3\sqrt{2}+\sqrt{21}+\sqrt{42})$, $\varepsilon_6=6+\sqrt{35}$,

$\sqrt{\varepsilon_6\varepsilon_7}=\frac{1}{2}(42+25\sqrt{2}+6\sqrt{35}+5\sqrt{70})$,

$H=2$.

7) $K=\boldsymbol{Q}(\sqrt{2}, \sqrt{3}, i)$ where $i=\sqrt{-1}$

$$\varepsilon_1=1+\sqrt{2}, \qquad \sqrt{\varepsilon_2}=\tfrac{1}{2}(\sqrt{2}+\sqrt{6}),$$

$$\sqrt[4]{i\varepsilon_1^2\varepsilon_2\varepsilon_3}=\tfrac{1}{4}(4+3\sqrt{2}+2\sqrt{3}+\sqrt{6}+2\sqrt{-1}+\sqrt{-2}+\sqrt{-6}),$$

$H=1$.

8) $K=\boldsymbol{Q}(\sqrt{2}, \sqrt{7}, i)$

$$\varepsilon_1=1+\sqrt{2}, \qquad \sqrt{\varepsilon_2}=\tfrac{1}{2}(3\sqrt{2}+\sqrt{14}),$$

$$\sqrt[4]{i\varepsilon_2\varepsilon_3}=\tfrac{1}{4}(4+3\sqrt{2}+2\sqrt{7}+\sqrt{14}+2\sqrt{-1}+\sqrt{-2}+\sqrt{-14}),$$

$H=2$.

9) $K=\boldsymbol{Q}(\sqrt{2}, \sqrt{11}, i)$

$$\varepsilon_1=1+\sqrt{2}, \qquad \sqrt{\varepsilon_2}=\tfrac{1}{2}(3\sqrt{2}+\sqrt{22}),$$

$$\sqrt[4]{i\varepsilon_1^2\varepsilon_2\varepsilon_3}=\tfrac{1}{4}(14+9\sqrt{2}+4\sqrt{11}+3\sqrt{22}$$
$$+4\sqrt{-1}+5\sqrt{-2}+2\sqrt{-11}+\sqrt{-22}),$$

$H=1$.

10) $K=\boldsymbol{Q}(\sqrt{2}, \sqrt{19}, i)$

$$\varepsilon_1=1+\sqrt{2}, \qquad \sqrt{\varepsilon_2}=\tfrac{1}{2}(13\sqrt{2}+3\sqrt{38}),$$

$$\sqrt[4]{i\varepsilon_1^2\varepsilon_2\varepsilon_3}=\tfrac{1}{4}(18+13\sqrt{2}+4\sqrt{19}+3\sqrt{38}$$
$$+8\sqrt{-1}+5\sqrt{-2}+2\sqrt{-19}+\sqrt{-38}),$$

$H=3$.

11) $K=\boldsymbol{Q}(\sqrt{2}, \sqrt{23}, i)$

$$\varepsilon_1=1+\sqrt{2}, \qquad \sqrt{\varepsilon_2}=\tfrac{1}{2}(5\sqrt{2}+\sqrt{46}),$$

$$\sqrt[4]{i\varepsilon_2\varepsilon_3}=\tfrac{1}{4}(34+27\sqrt{2}+8\sqrt{23}+5\sqrt{46}$$
$$+20\sqrt{-1}+7\sqrt{-2}+2\sqrt{-23}+3\sqrt{-46}),$$

$H=6$.

12) $K=\boldsymbol{Q}(\sqrt{2}, \sqrt{31}, i)$

$$\varepsilon_1=1+\sqrt{2}, \qquad \sqrt{\varepsilon_2}=\tfrac{1}{2}(39\sqrt{2}+7\sqrt{62}),$$

$$\sqrt[4]{i\varepsilon_2\varepsilon_3}=\tfrac{1}{4}(22+17\sqrt{2}+4\sqrt{31}+3\sqrt{62}$$
$$+12\sqrt{-1}+5\sqrt{-2}+2\sqrt{-31}+\sqrt{-62}),$$

$H=12$.

13) $K=\boldsymbol{Q}(\sqrt{2}, \sqrt{43}, i)$

$\varepsilon_1=1+\sqrt{2}$, $\qquad \sqrt{\varepsilon_2}=\frac{1}{2}(59\sqrt{2}+9\sqrt{86})$,

$\sqrt[4]{i\varepsilon_1^2\varepsilon_2\varepsilon_3}=\frac{1}{4}(158+111\sqrt{2}+24\sqrt{43}$

$+17\sqrt{86}+64\sqrt{-1}+47\sqrt{-2}+10\sqrt{-43}+7\sqrt{-86})$,

$H=5$.

14) $K=\boldsymbol{Q}(\sqrt{2}, \sqrt{47}, i)$

$\varepsilon_1=1+\sqrt{2}$, $\qquad \sqrt{\varepsilon_2}=\frac{1}{2}(7\sqrt{2}+\sqrt{94})$,

$\sqrt[4]{i\varepsilon_2\varepsilon_3}=\frac{1}{4}(126+97\sqrt{2}+20\sqrt{47}$

$+13\sqrt{94}+68\sqrt{-1}+29\sqrt{-2}+6\sqrt{-47}+7\sqrt{-98})$,

$H=20$.

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