# *Kronecker's limit formulas and their applications*[*)]

By Koji KATAYAMA

Let $k$ and $K$ be algebraic number fields such that $k \subset K$. Let $h$ and $H$ be ideal class numbers of $k$ and $K$, respectively. Then "good" formula of the relative class number $H/h$ is obtained in every case of the following:

0) $k=\boldsymbol{Q}$, $K=$ the absolute abelian extension of $k$,

1) $k=\boldsymbol{Q}$, $K=$ imaginary quadratic extension of $k$,

2) $k=$ imaginary quadratic field,
   $K=$ Hilbert class field or ray class field over $k$,

3) $k=$ real quadratic field,
   $K=$ imaginary quadratic extension of $k$ (ramified or not).

Since $K/k$ is abelian, it is well-known that the computation of $H/h$ essentially reduces to get residues of zeta-functions of $k$ and $K$ at $s=1$ and the values of $L$-functions $L(s, \chi)$ of $k$ at $s=1$. In getting $L(1, \chi)$, Kronecker's limit formulas of the first and the second kinds play essential roles. The one of the first kind concerns with the cases 2) and 3) (unramified) and gives the residue at $s=1$ of "generic" zeta-function

$$\sum_{(m,n)}{}' \frac{y^s}{|m+nz|^{2s}}, \quad z=x+iy$$

of $k$.

The one of the second kind concerns with the cases 2) and 3) (ramified) and gives the value at $s=1$ of "generic" $L$-function

$$\sum_{(m,n)}{}' \frac{y^s e^{2\pi i(mu+nv)}}{|m+nz|^{2s}}, \quad u,\ v \in \boldsymbol{R},\ (u,\ v) \notin \boldsymbol{Z} \times \boldsymbol{Z}$$

of $k$.

Then in the first, there appears so-called Dedekind eta-function $\eta(z)$ and good formula for $H/h$ in 2) was obtained by the theory of complex multiplication of $\eta(z)$.

In the second, there appears a holomorphic function which is essentially the product of $\eta(z)$ and elliptic theta-function. In the case 3), the computations of $H/h$ were reduced to get periods of the abelian integral of the third kind which is the logarithm of $\eta(z)$ in unramified case and is the logarithm of the above holomorphic function in ramified case. (Hecke [2], Siegel [5]). Then "good"

* This work was partially supported by The Sakkokai Foundation.

formulas (elementary arithmetic function in the sense of Hecke [2]) for $H/h$ can be obtained by Riemann-Dedekind's "Grenzübergang".

Now, Hecke declaired the following as "Theorem":

Let $k$ be a totally real algebraic number field and $K=k(\sqrt{\delta})$ with totally negative number $\delta$ of $k$. Then $H/h$ can be written as elementary arithmetic function of $\delta$. (But he proved only in the case where $k$ is quadratic).

Recently, Siegel gave fine lectures at Tata Institute [5] concerning the problem of getting $H/h$ in the above cases under simplified calculations and casted new lights on the problem.

The present paper deals with some generalized cases 4), and 5) of 2), and 3), generalizing slightly Hecke-Siegel's methods but the results are so weak.

4) $k=$ imaginary quadratic extension of real quadratic field in 2),

5) $k=$ certain type of real biquadratic field (see conditions $(A-1)$ or $(A-\mathfrak{f})$).

In § 1, we shall recall several notions and facts from Siegel [5] for later uses. In § 2, we shall define the three kinds of Eisenstein series attached to a real quadratic field, which are to be regarded as "generic" zeta- or $L$-functions of fields of types 4), and 5) and compute the Kronecker's limit formulas for them. In § 3, we shall compute the number $H/h$ in the cases 4), following Siegel's methods. But functions appearing in the limit formulas are not holomorphic Hilbert modular forms but non-holomorphic modular forms of Hilbert type. Therefore we can not hope "good" formulas (there is no "theory of complex multiplication").

Another application of Kronecker's limit formulas is to solve Pell's equation. Also in § 3, we shall generalize (formally) this "solving Pell's equation".

In § 4, under the assumption $(A-\mathfrak{f})$ ($\mathfrak{f}=1$ or not) (the case 5)), we shall compute the constant terms of zeta-functions and $L$-functions at $s=1$ following Siegel's methods. In the final § 5, we shall show that the computations of $H/h$ are reduced to get periods of certain abelian integrals of the third kind.

The problem, being left, is to obtain the formulas of the above periods and transform them into elementary arithmetical forms.

## 1. Kronecker's limit formula for zeta-function of real quadratic field

**1.1.** In this and the next numero, we shall quote the results of Siegel [5] for later uses.

Let $z=x+iy$, $y>0$, and $s$ be complex numbers. We consider the following function:

$$f(z, s)=\sum_{(m,n)}{}' \frac{y^s}{|m+nz|^{2s}}, \tag{1.1.1}$$

where $\sum'$ denotes the sum over all couples $(m, n) \in \boldsymbol{Z} \times \boldsymbol{Z}$ except $(0, 0)$. The series converges absolutely for $Re(s)>1$ and can be continued analytically into $Re(s)>\frac{1}{2}$.

Let $F=\boldsymbol{Q}(\sqrt{d})$ be a real quadratic field. Let $\mathfrak{b}$ be an ideal in $F$, whose basis is $(1, \omega)$. Let $\omega'$ be the conjugate of $\omega$. We take a hyperbolic element $\xi$ of $SL(2, \boldsymbol{Z})$, whose fixed points are $\omega, \omega'$. We may assume $\omega>\omega'$. Let $\widehat{\omega'\omega}$ be the semi-circle, end points of whose diameter are $\omega'$ and $\omega$. For $z \in \widehat{\omega'\omega}$, we put

$$z=\frac{\omega u i+\omega'}{u i+1} . \tag{1.1.2}$$

Then $u$ is a positive real number. Let $\varepsilon$ be the fundamental unit of $F$. We may take $\varepsilon>1$. We put $u=\varepsilon^{2v}$ with a real parameter $v$. Then the transformation $z \to \xi(z)$ corresponds to the translation $v \to v+1$. Since $f(z, s)$ is invariant under $z \to \xi(z)$, $f(z, s)$, regarded as a function of $v$, has the following Fourier series expansion:

$$\left\{\begin{aligned} f(z, s) &= \sum_{k=-\infty}^{\infty} a_k \exp(2\pi i k v) , \\ a_k &= \int_0^1 f(z, s) \exp(-2\pi i k v) dv . \end{aligned}\right. \tag{1.1.3}$$

We define the Grössencharacter $\chi_k$ of $F$ by

$$\chi_k((\beta))=\left|\frac{\beta}{\beta'}\right|^{\frac{\pi i k}{\log \varepsilon}}, \qquad k \in \boldsymbol{Z},\ \beta \in F . \tag{1.1.4}$$

Let $B$ be the ideal class to whom $\mathfrak{b}^{-1}$ belongs. Then the zeta-function of $F$, with $\chi_k$, associated with $B$, is defined by

$$\zeta_F(s, \chi_k, B)=\sum_{(0) \neq \mathfrak{a} \in B} \chi_k(\mathfrak{a})|N\mathfrak{a}|^{-s} .$$

We have

$$\chi_k(\mathfrak{b}) \zeta_F(s, \chi_k, B)=(N\mathfrak{b})^s \sum_{(0) \neq \beta \in \mathfrak{b}}{}^{\circ} \chi_k((\beta))|N(\beta)|^{-s} , \tag{1.1.5}$$

where $\sum^\circ$ denotes the sum over all $\beta \in \mathfrak{b}$ but associates of $\beta$ appear only once in $\sum^\circ$. We write $\zeta_F(s, B)=\zeta_F(s, 1, B)$ for $\chi_k=\chi_0=1$.

Siegel discovered that the Fourier coefficient $a_k$ in (1.1.3) is essentially $\zeta_F(s, \chi_k, B)$ and gave the explicit formula for $a_k$:

$$a_k=\frac{d^{\frac{s}{2}} \Gamma\left(\frac{s}{2}-\frac{\pi i k}{2 \log \varepsilon}\right) \Gamma\left(\frac{s}{2}+\frac{\pi i k}{2 \log \varepsilon}\right)}{2 \log \varepsilon \cdot \Gamma(s)} \chi_k(\mathfrak{b}) \zeta_F(s, \chi_k, B) . \tag{1.1.6}$$

Now Kronecker's first limit formula for $f(z, s)$ is given by the following form:

$$\lim_{s\to 1}\left(f(z,s)-\frac{\pi}{s-1}\right)=2\pi(C-\log 2)-2\pi\log(\sqrt{y}\,|\eta(z)|^2)\,, \tag{1.1.7}$$

where $C$ is Euler's constant and $\eta(z)$ is the so-called Dedekind eta-function. Also in [5], Siegel obtained the limit formula for $\zeta_F(s, B)$ which comes from $a_0$, by integrating (1.1.7) with respect to $v$. The result is as follows:

$$\lim_{s\to 1}\left(\zeta_F(s,B)-\frac{2\log\varepsilon}{\sqrt{d}}\frac{1}{s-1}\right)=\frac{2\log\varepsilon}{\sqrt{d}}\left(2C-2\int_0^1\log(\sqrt{y}\,\sqrt[4]{d}\,|\eta(z)|^2)dv\right). \tag{1.1.8}$$

The limit formula for $\zeta_F(s, \hat{\chi}_k, B)$, coming from $a_k$, is not given in [5] for no use in the course of lectures, but it can be easily obtained by the same method. The result is as follows:

$$\zeta_F(1,\hat{\chi}_k,B)=\frac{-4\pi\log\varepsilon\cdot\hat{\chi}_k(\mathfrak{b})^{-1}\displaystyle\int_0^1\log(\sqrt{y}\,|\eta(z)|^2)e^{-2\pi ikv}dv}{\sqrt{d}\,\Gamma\left(\frac{1}{2}-\frac{\pi ik}{2\log\varepsilon}\right)\Gamma\left(\frac{1}{2}+\frac{\pi ik}{2\log\varepsilon}\right)}\,. \tag{1.1.9}$$

In the course of computation of (1.1.9), we see that the term corresponding to the pole $s=1$ of $f(z, s)$ vanishes, since the term involves the integration $\int_0^1\exp(-2\pi ikv)dv$. Thus it is automatically shown that $\zeta_F(s, \hat{\chi}_k, B)$ for $k\neq 0$, has no pole at $s=1$.

**1.2.** Let $K$ be an algebraic number field of finite degree. Let $\mathfrak{f}$ be an integral ideal in $K$. As usual, we denote by $\gamma_1\equiv\gamma_2 \pmod{^*\mathfrak{f}}$ the multiplicative congruence of $\gamma_1$ and $\gamma_2$ mod $\mathfrak{f}$. We define $\mathfrak{G}_\mathfrak{f}$ the group of ideals in $K$ whose denominators and numerators are integral ideals coprime to $\mathfrak{f}$. Put

$$\mathfrak{E}_\mathfrak{f}=\{(\alpha)\in\mathfrak{G}_\mathfrak{f}\,;\ \alpha>0\,,\ \alpha\equiv 1 \ (\mathrm{mod}^*\,f)\}\,.$$

Then $\mathfrak{G}_\mathfrak{f}/\mathfrak{E}_\mathfrak{f}$ is called the ray class group modulo $\mathfrak{f}$.

Let $\chi$ be a character of $\mathfrak{G}_\mathfrak{f}/\mathfrak{E}_\mathfrak{f}$. We define, for $Re(s)>1$,

$$L_K(s,\chi)=\sum_{(\mathfrak{a},\mathfrak{f})=1}\chi(\mathfrak{a})|N\mathfrak{a}|^{-s}\,.$$

It is known that $L_K(s, \chi)$ has Euler product formula;

$$L_K(s,\chi)=\prod_{(\mathfrak{p},\mathfrak{f})=1}(1-\chi(\mathfrak{p})|N\mathfrak{p}|^{-s})^{-1}$$

and can be continued meromorphically into the whole $s$-plane and satisfies the functional equation between $L_K(s, \chi)$ and $L_K(1-s, \hat{\chi})$ for proper $\chi$. If $\chi\neq 1$, $L_K(s, \chi)$ is an entire function of $s$.

For a ray class character $\chi$, we see that $\chi((\alpha))=\pm x((\beta))$ for non-zero integers

$\alpha$, $\beta$ in $K$ coprime to $\mathfrak{f}$ such that $\alpha\equiv\beta \pmod{\mathfrak{f}}$. We can define a character $v(\lambda)$ of signature, for $\lambda\in K^*$, associated with $\chi$ as follows. For $\lambda\in K$, there exists an integer $\alpha$ in $K$ such that $\alpha\equiv 1 \pmod{\mathfrak{f}}$ and $\alpha\lambda>0$. Then put $v(\lambda)=\chi((\alpha))$.

Every ray class character $\chi((\alpha)) \bmod \mathfrak{f}$ can be written as

$$\chi((\alpha))=v(\alpha)\chi(\alpha)\ , \tag{1.2.1}$$

where $\chi(\alpha)$ is a character of the group $G(\mathfrak{f})$ of prime residue classes mod $\mathfrak{f}$.

Now we can write

$$L_K(s,\chi)=\sum_{A}\sum_{\substack{\mathfrak{a}\in A^{-1}\\(\mathfrak{a},\mathfrak{f})=1}}\chi(\mathfrak{a})|N\mathfrak{a}|^{-s}\ , \tag{1.2.2}$$

where $A$ runs over all ideal classes and $\mathfrak{a}$ over all ideals belonging to $A$, coprime to $\mathfrak{f}$. Taking an ideal $\mathfrak{b}_A$ in $A^{-1}$ coprime to $\mathfrak{f}$ and $\mathfrak{a}\mathfrak{b}_A=(\beta)$ with an integer $\beta$ in $K$, we have

$$L_K(s,\chi)=\sum_{A}\bar{\chi}(\mathfrak{b}_A)N(\mathfrak{b}_A)^s\sum_{\substack{\mathfrak{b}_A|(\beta)\neq 0\\(\beta,\mathfrak{f})=1}}{}'\chi((\beta))|N(\beta)|^{-s}\ .$$

Extending $\chi(\beta)$ to all residue classes mod $\mathfrak{f}$ by setting $\chi(\alpha)=0$ for $\alpha$ not coprime to $\mathfrak{f}$, we may consider that the above inner sum is taken over all principal ideals $(\beta)\subset\mathfrak{b}_A$. Thus we have

$$L_K(s,\chi)=\sum_{A}\bar{\chi}(\mathfrak{b}_A)N(\mathfrak{b}_A)^s\sum_{\mathfrak{b}_A|(\beta)\neq 0}{}'v(\beta)\chi(\beta)|N(\beta)|^{-s}\ .$$

We define

$$T=\sum_{\lambda \bmod \mathfrak{f}}\bar{\chi}(\lambda)e^{2\pi i S(\lambda\gamma)}\ , \tag{1.2.3}$$

where $\gamma$ is chosen so that $(\gamma)\mathfrak{D}$ has exact denominator $\mathfrak{f}$, for different $\mathfrak{D}$. $\gamma$ is fixed once and for all. $\lambda$ runs over a full system of representatives mod $\mathfrak{f}$. It is shown that $T$ is independent of the choice of $\lambda$. For $\mathfrak{f}=(1)$, we have $T=1$. In general, $T=\sqrt{N(\mathfrak{f})}$. For a proper ray class character $\chi \bmod \mathfrak{f}$, we have the following expression (Siegel [5] p. 140):

$$L_K(s,\chi)=\frac{1}{T}\sum_{\lambda \bmod \mathfrak{f}}\bar{\chi}(\lambda)\sum_{A}\bar{\chi}(\mathfrak{b}_A)N(\mathfrak{b}_A)^s\sum_{(\beta)\subset\mathfrak{b}_A}v(\beta)e^{2\pi i S(\lambda\beta\gamma)}|N(\beta)|^{-s}\ . \tag{1.2.4}$$

where $\lambda$ runs over a full system of representatives of prime residue classes mod $\mathfrak{f}$, $A$ over representatives of the ideal classes of $K$ (in the wide sense) and $(\beta)$ over all principal ideals $\neq(0)$ divisible by $\mathfrak{b}_A$.

By the ray class field mod $\mathfrak{f}$ of $K$, we understand that the relative abelian extension $K_0$ of $K$, whose Galois group is isomorphic to $\mathfrak{G}_{\mathfrak{f}}/\mathfrak{E}_{\mathfrak{f}}$, such that the prime divisors of $\mathfrak{f}$ are the only prime ideals ramified in $K_0$. Denote by $\zeta_K(s)$ the Dedekind zeta-function of $K$. By the class field theory, it is known that

$$(1.2.5) \qquad \zeta_{K_0}(s)=\zeta_K(s)\prod_{\chi\neq1} L_K(s,\chi_0)\ ,$$

where $\chi$ runs over all the non-principal ray class characters mod $\mathfrak{f}$ and if $\mathfrak{f}_\chi$ is the conductor of $\chi$, then $\chi_0$ is the proper ray class character mod $\mathfrak{f}$, associated with $\chi$. It is also known that $\prod_\chi \mathfrak{f}_\chi$ is the relative discriminant of $K_0/K$.

## 2. Limit formulas for Eisenstein series attached to a real quadratic field

**2.0.** Let $F=\boldsymbol{Q}(\sqrt{d})$ be a real quadratic field. For $\alpha\in F$, we denote by $\alpha'$ the conjugate of $\alpha$. We assume that $F$ has the fundamental unit $\varepsilon$ of norm $-1$. Let $\mathfrak{a}$, $\mathfrak{b}$ be two ideals in $F$. We say that $(\alpha_1, \beta_1)$ and $(\alpha_2, \beta_2)$ in $F\times F$ are associated if there exists a totally positive unit $\mu$ of $F$ such that $\mu\alpha_1=\alpha_2$, $\mu\beta_1=\beta_2$. We consider the following three types of Eisenstein series 1°, 2°, 3°.

$$1°. \qquad \Phi_+(\tau,\tau';\mathfrak{a},\mathfrak{b};s_1,s_2)=\sum_{\substack{\alpha\in\mathfrak{a}\\ 0\ll\beta\in\mathfrak{b}}}{}^{\circ}\ |\alpha+\beta\tau|^{-2s_1}|\alpha'+\beta'\tau'|^{-2s_2}\ ,$$

where $\tau=x+iy$, $\tau'=x'+iy'$, $y>0$, $y'>0$, $s_1$, $s_2$ are complex variables and

$$(2.0.1) \qquad s_1=s+\frac{\pi ik}{\log\varepsilon}\ ,\qquad s_2=s-\frac{\pi ik}{\log\varepsilon}\ ,\qquad k\in\boldsymbol{Z}$$

with one complex variable $s$. $\beta\gg0$ means $\beta>0$ and $\beta'>0$; i.e., $\beta$ is totally positive. $\sum^\circ$ denotes the sum over all $\alpha\in\mathfrak{a}$, $\beta\in\mathfrak{b}$ but associates of $\beta$ appear only once.

$$2°. \qquad \Phi(\tau,\tau';\mathfrak{a},\mathfrak{b};s_1,s_2)=\sum_{\substack{\alpha\in\mathfrak{a}\\ \beta\in\mathfrak{b}\\ (\alpha,\beta)}}{}'\ \frac{y^{s_1}y'^{s_2}}{|\alpha+\beta\tau|^{2s_1}|\alpha'+\beta'\tau'|^{2s_2}}\ ,$$

where $\tau$, $\tau'$, $s_1$, $s_2$ are as in 1° and $\sum'$ denotes the sum over all $(\alpha,\beta)\in F\times F$, not being $(0, 0)$ and mutually non-associated.

$$3°. \qquad \Psi(\tau,\tau';\mathfrak{a},\mathfrak{b};s;(u),(v))=\sum_{\substack{\alpha\in\mathfrak{a}\\ \beta\in\mathfrak{b}\\ (\alpha,\beta)}}{}'\ \frac{(yy')^s e^{2\pi i(\alpha u+\beta v+\alpha'u'+\beta'v')}}{|\alpha+\beta\tau|^{2s}|\alpha'+\beta'\tau'|^{2s}}\ .$$

Here $(u)=(u, u')$, $(v)=(v, v')$ are couples of real numbers satisfying the following condition:

$$(2.0.2) \qquad \alpha u+\alpha'u'\notin\boldsymbol{Z} \text{ for all } \alpha\in\mathfrak{a} \text{ or } \beta v+\beta'v'\notin\boldsymbol{Z} \text{ for all } \beta\in\mathfrak{b}\ .$$

We know the following Poisson summation formula: Let $f(x_1, x_2)$ be a function continuous in $(x_1, x_2)$ and $\sum_{m_i=-\infty}^{\infty} f(x_1+m_1, x_2+m_2)$ converges absolutely and uniformly for $0 \leqq x_i < 1$. Further,

$$\varphi(x_1, x_2) = \iint_{-\infty}^{\infty} f(u_1, u_2) e^{2\pi i (u_1 x_1 + u_2 x_2)} du_1 du_2$$

exists and
$$\sum_{p,q=-\infty}^{\infty} e^{-2\pi i (p x_1 + q x_2)} \varphi(p, q)$$

converges. Then

$$\sum f(x_1 + m_1, x_2 + m_2) = \sum e^{-2\pi i (p x_1 + q x_2)} \cdot \varphi(p, q) . \tag{2.0.3}$$

**2.1. Limit formula for $\Phi_+$**

$\Phi_+(\tau, \tau'; \mathfrak{a}, \mathfrak{b}; s_1, s_2)$ converges absolutely for $Re(s_1) > 1$, $Re(s_2) > 1$, hence for $Re(s) > 1$. Let $\omega_1$, $\omega_2$ be a basis of $\mathfrak{a}$. Then every element $\alpha$ of $\mathfrak{a}$ is of the form $\alpha = m_1 \omega_1 + m_2 \omega_2$ with $m_i \in \boldsymbol{Z}$. We put

$$f(x_1, x_2) = |x_1 \omega_1 + x_2 \omega_2 + \beta \tau|^{-2s_1} |x_1 \omega_1' + x_2 \omega_2' + \beta' \tau'|^{-2s_2}$$

for $-\infty < x_i < \infty$. Then we see that the series

$$\sum_{m_i=-\infty}^{\infty} f(m_1, m_2)$$

converges absolutely and uniformly in every $-N_i \leqq x_i \leqq N_i$ for $Re(s) > \frac{1}{2}$. This is shown by the same method as in Siegel [5] p. 8. Also following the consideration of Siegel [5] pp. 9-12, we see that (2.0.3) is applicable to the present $f(x_1, x_2)$. Thus for $Re(s) > 1$, we obtain

$$\sum_{m_i=-\infty}^{\infty} f(m_1, m_2) = \sum_{p,q} \iint_{-\infty}^{\infty} \frac{e^{2\pi i (p u_1 + q u_2)} du_1 du_2}{|u_1 \omega_1 + u_2 \omega_2 + \beta \tau|^{2s_1} |u_1 \omega_1' + u_2 \omega_2' + \beta' \tau'|^{2s_2}} . \tag{2.1.1}$$

Put
$$\Delta = |\omega_1 \omega_2' - \omega_2 \omega_1'| = N(\mathfrak{a}) \sqrt{d} ,$$

$$\mu = p \omega_2 - q \omega_1 , \quad \mu' = p \omega_2' - q \omega_1' ,$$

$$v = u_1 \omega_1 + u_2 \omega_2 \quad \text{and} \quad v' = u_1 \omega_1' + u_2 \omega_2' .$$

Then substituting these in (2.1.1), we get for $Re(s) > 1$,

$$(2.1.1) = \frac{1}{\Delta} \sum_{\mu} \iint_{-\infty}^{\infty} \frac{e^{\frac{2\pi i (\mu' v - \mu v')}{\Delta}} dv dv'}{|v + \beta \tau|^{2s_1} |v' + \beta' \tau'|^{2s_2}} . \tag{2.1.2}$$

As in Siegel [6] pp. 97-98, we have the following formulas:

$$\int_{-\infty}^{\infty} \frac{e^{\frac{2\pi i \mu' v}{\Delta}} dv}{|v + \beta \tau|^{2s_1}} = \begin{cases} \dfrac{(2\pi)^{2s_1} \mu'^{2s_1 - 1}}{\Gamma(s_1)^2 \Delta^{2s_1 - 1}} e^{\frac{-2\pi i \mu' \beta \bar{\tau}}{\Delta}} h\left(s_1, s_1, \dfrac{4\pi \mu' \beta y}{\Delta}\right) & \text{if } \mu' > 0 , \\ \dfrac{(2\pi)^{2s_1} (-\mu')^{2s_1 - 1}}{\Gamma(s_1)^2 \Delta^{2s_1 - 1}} e^{\frac{-2\pi i \mu' \beta \tau}{\Delta}} h\left(s_1, s_1, \dfrac{-4\pi \mu' \beta y}{\Delta}\right) & \text{if } \mu' < 0 , \end{cases} \tag{2.1.3}$$

$$(2.1.4)\quad \int_{-\infty}^{\infty}\frac{e^{\frac{-2\pi i\mu v'}{\Delta}}dv'}{|v'+\beta'\tau'|^{2s_2}}=\begin{cases}\dfrac{(2\pi)^{2s_2}\mu^{2s_2-1}}{\Gamma(s_2)^2\Delta^{2s_2-1}}e^{\frac{2\pi i\mu\beta'\tau'}{\Delta}}h\left(s_2,\, s_2,\, \dfrac{4\pi u\beta' y'}{\Delta}\right) & \text{if } \mu>0\,, \\ \dfrac{(2\pi)^{2s_2}(-\mu)^{2s_2-1}}{\Gamma(s_2)^2\Delta^{2s_2-1}}e^{\frac{2\pi i\mu\beta'\bar{\tau}'}{\Delta}}h\left(s_2,\, s_2,\, \dfrac{-4\pi\mu\beta' y'}{\Delta}\right) & \text{if } \mu<0\,.\end{cases}$$

Here $h(\alpha, \beta, t)$ is the so-called confluent hypergeometric function defined by

$$h(\alpha, \beta, t)=\int_0^{\infty} w^{\alpha-1}(w+1)^{\beta-1}e^{-wt}dw\,,$$

with $t>0$, $Re(\alpha)>0$ and $Re(\beta)>0$.

It is known that $h$ satisfies the following differential equation:

$$t\frac{d^2h}{dt^2}+(\alpha+\beta-t)\frac{dh}{dt}-\alpha h=0\,.$$

We divide (2.1.2) into five parts:

$$(2.1.2)=\frac{1}{\Delta}\left\{(\text{terms of } \mu=0)+\sum_{\substack{\mu>0\\ \mu'>0}}+\sum_{\substack{\mu>0\\ \mu'<0}}+\sum_{\substack{\mu<0\\ \mu'>0}}+\sum_{\substack{\mu<0\\ \mu'<0}}\right\}.$$

Using (2.1.3), (2.1.4) in (2.1.2) and transforming the last three parts of the sum decomposition of (2.1.2) into the sum over totally positive $\mu$'s, we get, with $c=\pi ik/\log\varepsilon$,

$$(2.1.5)\quad \begin{aligned}(2.1.2)=\frac{1}{\Delta}\Bigg[&N(\beta)^{1-2s}\chi(\beta)^{-2}(yy')^{1-2s}\left(\frac{y'}{y}\right)^{2c}\frac{\pi\Gamma\left(s-\frac{1}{2}+c\right)\Gamma\left(s-\frac{1}{2}-c\right)}{\Gamma(s+c)\Gamma(s-c)} \\ &+\frac{(2\pi)^{2s}}{\Delta^{4s-2}\Gamma(s+c)^2\Gamma(s-c)^2}\sum_{\mu\gg 0}N(\mu)^{2s-1}\chi(\mu)^2 \\ &\times\Bigg\{e^{\frac{2\pi i(\mu\beta'\tau'-\mu'\beta\bar{\tau})}{\Delta}}h\left(s_1,\, s_1,\, \frac{4\pi\mu'\beta y}{\Delta}\right)h\left(s_2,\, s_2,\, \frac{4\pi\mu\beta' y'}{\Delta}\right) \\ &+(-1)^{4s+2c}e^{\frac{2\pi i(\varepsilon\mu\beta'\tau'-\varepsilon'\mu'\beta\tau)}{\Delta}}h\left(s_1,\, s_1,\, \frac{-4\pi\varepsilon'\mu'\beta y}{\Delta}\right)h\left(s_2,\, s_2,\, \frac{4\pi\varepsilon\mu\beta' y'}{\Delta}\right) \\ &+(-1)^{8s-2c}e^{\frac{2\pi i(\varepsilon'\mu'\beta\bar{\tau}-\varepsilon\mu\beta'\bar{\tau}')}{\Delta}}h\left(s_1,\, s_1,\, \frac{-4\pi\varepsilon'\mu'\beta y}{\Delta}\right)h\left(s_2,\, s_2,\, \frac{4\pi\varepsilon\mu\beta' y'}{\Delta}\right) \\ &+(-1)^{8s}e^{\frac{2\pi i(\mu'\beta\tau-\mu\beta'\bar{\tau}')}{\Delta}}h\left(s_1,\, s_1,\, \frac{4\pi\mu'\beta y}{\Delta}\right)h\left(s_2,\, s_2,\, \frac{4\pi\mu\beta' y'}{\Delta}\right)\Bigg\}\Bigg].\end{aligned}$$

The part $\sum_{\mu\gg 0}$ has no pole at $s=1$. In order to get limit formula, we have further to take the sum $\sum^{\circ}_{0\ll\beta\in\mathfrak{b}}$. Then the part $\sum^{\circ}_{0\ll\beta\in\mathfrak{b}}\sum_{\mu\gg 0}$ converges uniformly in every compact set of $s$-plane. Hence it defines an entire function of $s$ and has no pole at $s=1$. Now the term $\sum^{\circ}_{0\ll\beta\in\mathfrak{b}}$ (term $\mu=0$) can be easily calculated. The result is as follows:

$$(2.1.6)\qquad \frac{1}{\Delta}\sum_{0\ll\beta\in\mathfrak{b}}^{\circ}(\text{term }\mu=0)$$
$$=\frac{\pi\Gamma\left(s-\frac{1}{2}+c\right)\Gamma\left(s-\frac{1}{2}-c\right)}{\Delta\Gamma(s+c)\Gamma(s-c)}(yy')^{1-2s}\left(\frac{y'}{y}\right)^{2c}\hat{\chi}^{-2}(\mathfrak{b})N(\mathfrak{b})^{1-2s}\zeta_F(2s-1,\hat{\chi}^{-2},B)\,.$$

Thus we see that $\Phi_+$ has a pole at $s=1$ for $k=0$, i.e., for $\hat{\chi}=1$, and the residue at $s=1$ is easily given by that of $\zeta_F(2s-1, B)$:

$$(2.1.7)\qquad \frac{1}{\Delta}\frac{\pi\Gamma\left(\frac{1}{2}\right)^2}{\Gamma(1)^2yy'N(\mathfrak{b})}\operatorname*{Res}_{s=1}\zeta_F(2s-1,B)=\frac{2\pi^2\log\varepsilon}{\Delta\sqrt{d}\,yy'N(\mathfrak{b})}\,,$$

(observe (1.1.8)). For $k\neq 0$, i.e., for $\hat{\chi}\neq 1$, $\Phi_+$ is regular at $s=1$. We can calculate the value at $s=1$ using (1.1.9).

Now we put

$$(2.1.8)\qquad F(k;\tau,\tau';\mathfrak{a},\mathfrak{b})$$
$$=\sum_{\mathfrak{b}\ni\beta\gg0}^{\circ}\sum_{\mu\gg0}N(\mu)\hat{\chi}(\mu)^2e^{\frac{2\pi i(\varepsilon\mu\beta'\tau'-\varepsilon'\mu'\beta\tau)}{\Delta}}h\left(1+c,1+c,\frac{-4\pi\varepsilon'\mu'\beta y}{\Delta}\right)$$
$$\times h\left(1-c,1-c,\frac{4\pi\varepsilon\mu\beta'y'}{\Delta}\right)$$

with $c=\pi ik/\log\varepsilon$. Then for $F(\tau,\tau';\mathfrak{a},\mathfrak{b})=F(0;\tau,\tau';\mathfrak{a},\mathfrak{b})$, we have

$$(2.1.9)\qquad F(\tau,\tau';\mathfrak{a},\mathfrak{b})=\frac{\Delta}{yy'(4\pi)^2}\sum_{\mathfrak{b}\ni\beta\gg0}^{\circ}\sum_{\mu\gg0}N(\beta)^{-1}e^{\frac{2\pi i(\varepsilon\mu\beta'\tau'-\varepsilon'\mu'\beta\tau)}{\Delta}}\,.$$

Further we put

$$(2.1.10)\qquad F^*(k;\tau,\tau',\mathfrak{a},\mathfrak{b})=F(k;-\varepsilon\bar{\tau},-\varepsilon'\tau';\mathfrak{a},\mathfrak{b})+(-1)^{2c}F(k;\tau,\tau';\mathfrak{a},\mathfrak{b})$$
$$+(-1)^{-2c}F(k;-\bar{\tau},-\bar{\tau}';\mathfrak{a},\mathfrak{b})+F(k;\varepsilon\tau,\varepsilon'\bar{\tau}';\mathfrak{a},\mathfrak{b})\,,$$
$$F^*(\tau,\tau';\mathfrak{a},\mathfrak{b})=F^*(0;\tau,\tau';\mathfrak{a},\mathfrak{b})\,.$$

By (2.1.5), (2.1.7), (2.1.9) and (2.1.10), we have the following.

PROPOSITION 1. (CASE: $k=0$).

$$\lim_{s\to1}\left[\Phi_+(\tau,\tau';\mathfrak{a},\mathfrak{b};s,s)-\frac{2\pi^2\log\varepsilon}{dyy'\,N(\mathfrak{a}\mathfrak{b})}\frac{1}{s-1}\right]$$
$$=\frac{2\pi^2\log\varepsilon}{dyy'N(\mathfrak{a}\mathfrak{b})}\left(2\log2-\log N(\mathfrak{b})-\log yy'+2C-2\int_0^1\log(\sqrt{y_z}\,\sqrt[4]{d}\,|\eta(z)|^2)dv\right)$$
$$+\frac{4}{d^3N(\mathfrak{a})^3}F^*(\tau,\tau';\mathfrak{a},\mathfrak{b})\,.$$

By (1.1.9), (2.1.5), (2.1.8) and (2.1.10), we have the following

PROPOSITION 2. (CASE: $k\neq 0$).

$$\Phi_+(\tau, \tau'; \mathfrak{a}, \mathfrak{b}, 1+c, 1-c)$$
$$=\frac{-4\pi^2 \log \varepsilon \int_0^1 \log(\sqrt{y_z}\,|\eta(z)|^2)\, e^{4\pi\sqrt{-1}kv}dv}{dN(\mathfrak{a}\mathfrak{b})y^{1+2c}y'^{1-2c}\Gamma(1+c)\Gamma(1-c)}$$
$$+\frac{4\pi}{d^3N(\mathfrak{a})^3\Gamma(1+c)^2\Gamma(1-c)^2}F^*(k;\tau,\tau';\mathfrak{a},\mathfrak{b}),$$

*with* $c=\pi\sqrt{-1}k/\log\varepsilon$, $k\in \boldsymbol{Z}$.

### 2.2. Transformation formula

We consider only the case $k=0$. Put

$$F_0(\tau, \tau'; \mathfrak{a}, \mathfrak{b})=\frac{2\sqrt{d}\,N(\mathfrak{b})}{\pi^2 \log\varepsilon\,(4\pi)^2}\sum_{\mathfrak{b}\ni\beta\gg 0}^{\circ}\sum_{\mu\gg 0}N(\beta)^{-1}e^{\frac{2\pi i(\varepsilon\mu\beta'\tau'-\varepsilon'\mu'\beta\tau)}{\Delta}} \tag{2.2.1}$$

and

$$\varphi_0(\tau, \tau'; \mathfrak{a}, \mathfrak{b})=\exp\{F_0(-\varepsilon\bar{\tau}, -\varepsilon'\tau')+F_0(\tau, \tau')+F_0(-\bar{\tau}, -\bar{\tau}')+F_0(\varepsilon\tau, \varepsilon'\bar{\tau}')\}\,. \tag{2.2.2}$$

(for brevity, $\mathfrak{a}$, $\mathfrak{b}$ in $F_0$ are omitted)

Take $\xi=\begin{pmatrix}\nu, & \mu\\ \gamma, & \delta\end{pmatrix}$ satisfying the following condition:

(2.2.3) $\nu$, $\mu$, $\gamma$, $\delta$ are integers in $F$, $\mu\in\mathfrak{a}$, $\gamma\in\mathfrak{b}$, and
det $\xi$=totally positive unit in $F$.

Then we see

$$\Phi_+(\xi(\tau), \xi'(\tau'); \mathfrak{a}, \mathfrak{b}, s, s)=\Phi_+(\tau, \tau'; \mathfrak{a}, \mathfrak{b}; s, s)|\gamma\tau+\delta|^{2s}|\gamma'\tau'+\delta'|^{2s}\,.$$

Computing the limit formula for the above left hand side and comparing it with that of $\Phi_+$, we get the following.

PROPOSITION 3. ($\mathfrak{a}$, $\mathfrak{b}$ being omitted in $\varphi_0$)

$$\varphi_0(\xi(\tau), \xi'(\tau'))=\varphi_0(\tau, \tau')\,|\gamma\tau+\delta|^{-2}|\gamma'\tau'+\delta'|^{-2}$$

*holds for* $\xi=\begin{pmatrix}\nu, & \mu\\ \gamma, & \delta\end{pmatrix}$ *with condition* (2.2.3).

$\varphi_0(\tau, \tau')$ is non-holomorphic with respect to $\tau$, $\tau'$. On direct computation, we observe that $\varphi_0$ is a potential function of $x$, $y$, $x'$ and $y'$. Hence it is real analytic with respect to $x$, $y$, $x'$, $y'$. As for the case $k\neq 0$, we can not speak of the transformation formula of Hilbert modular type for $F^*(k;\tau,\tau')$ or $\exp(F^*(k;\tau,\tau'))$. The reason comes from the existence of exponent 1 of $y^{1+2c}y'^{1-2c}$ instead of 2 in the denominator of the right hand side of Prop. 2. Also, we can not speak of the existence of simple partial differential equation (for example, analogue to that treated by Siegel [6] and Maass [4]) satisfied by $F^*(k;\tau,\tau')$ or $\exp(F^*(k;\tau,\tau'))$. The reason comes from the existence of $y, y'$ in the exponential function of (2.1.8). If the function $h$ in (2.1.8) is reduced to exponential function, then our $F^*(k;\tau,\tau')$ becomes a type of function treated by Hecke in §6 [3].

Also Hecke gave the almost explicit limit formula and transformation formula for the case $k=0$ in [1] and investigated the analytic nature of analogous function to $F_0(\tau,\tau')$ in detail in [3].

REMARK. Let $K$ be a totally real algebraic number field of finite degree $n$. Let $\varepsilon_1,\cdots,\varepsilon_{n-1}$ be a system of fundamental units of $K$. Let $c_1,\cdots,c_n$ be a solution of the following simultaneous linear equation:

$$c_1\log|\varepsilon_i^{(1)}|+c_2\log|\varepsilon_i^{(2)}|+\cdots+c_n\log|\varepsilon_i^{(n)}|=2\pi\nu_i\sqrt{-1}\qquad i=1,\cdots,n-1\,,$$

with $\nu_i\in\boldsymbol{Z}$. With one complex variable $s$, put

$$s_\lambda=s+c_\lambda\qquad \lambda=1,\cdots,n\,.$$

We define, for ideals $\mathfrak{a}$, $\mathfrak{b}$ in $K$,

$$\Phi_+^{(n)}(\tau^{(1)},\cdots,\tau^{(n)};\mathfrak{a},\mathfrak{b};s_1,\cdots,s_n)=\sum_{\substack{\alpha\in\mathfrak{a}\\ 0\ll\beta\in\mathfrak{b}}}{}^{\circ}\prod_{\lambda=1}^{n}|\alpha^{(\lambda)}+\beta^{(\lambda)}\tau^{(\lambda)}|^{-2s}$$

where every $\tau^{(\lambda)}$ is a complex number whose imaginary part is positive. We can derive the limit formula for $\Phi_+^{(n)}$ by the same method as for $\Phi_+$. For $(c_1,\cdots,c_n)=(0,0\cdots,0)$, there appears non-holomorphic modular form of Hilbert type, analogous to $\varphi_0(\tau,\tau')$, which is a potential function with respect to real and imaginary parts of $\tau^{(\lambda)}$. For $(c_1,\cdots,c_n)\neq(0,\cdots,0)$, there appears an analogous function to $F(k;\tau,\tau')$ with Grössencharacter $\hat{\chi}$ of $K$ defined by

$$\hat{\chi}((\beta))=|\beta^{(1)}|^{c_1}\cdots|\beta^{(n)}|^{c_n}\,,\quad \beta\in K.$$

### 2.3. Limit formula for $\Phi$

In this section we shall compute the limit formula for

$$\Phi(\tau,\tau';\mathfrak{a},\mathfrak{b};s_1,s_2)=\sum_{\substack{\alpha\in\mathfrak{a}\\ \beta\in\mathfrak{b}\\ (\alpha,\beta)}}{}'\frac{y^{s_1}y'^{s_2}}{|\alpha+\beta\tau|^{2s_1}|\alpha'+\beta'\tau'|^{2s_2}}\,.$$

We divide $\Phi$ into five parts:

$$(2.3.1)\qquad \Phi(\tau,\tau';\mathfrak{a},\mathfrak{b};s_1,s_2)=\sum_{\alpha\in\mathfrak{a}}{}^{\circ}(\text{term }\beta=0)+\sum_{\substack{\alpha\in\mathfrak{a}\\ 0\ll\beta\in\mathfrak{b}}}{}^{\circ}+\sum_{\substack{\alpha\in\mathfrak{a}\\ \beta>0,\beta'<0}}{}^{\circ}+\sum_{\substack{\alpha\in\mathfrak{a}\\ \beta<0,\beta'>0}}{}^{\circ}+\sum_{\substack{\alpha\in\mathfrak{a}\\ \beta\ll0}}{}^{\circ}\,,$$

where $\sum^{\circ}$ have the same meaning as in 2.0. Further we divide the term $\sum^{\circ}(\text{term }\beta=0)$ into four parts:

$$\sum^{\circ}(\text{term } \beta=0)=\sum_{\substack{\alpha\in\mathfrak{a}\\ \alpha\gg 0}}^{\circ}+\sum_{\substack{\alpha\in\mathfrak{a}\\ \alpha>0,\alpha'<0}}^{\circ}+\sum_{\substack{\alpha\in\mathfrak{a}\\ \alpha<0,\alpha'>0}}^{\circ}+\sum_{\substack{\alpha\in\mathfrak{a}\\ \alpha\ll 0}}^{\circ}.$$

We can make the last three sums into the sums over totally positive $\alpha$'s and we get, for example,

$$\sum_{\substack{\alpha\in\mathfrak{a}\\ \alpha>0\\ \alpha'<0}}^{\circ}\frac{y^{s_1}y'^{s_2}}{|\varepsilon|^{2s_1}|\varepsilon'|^{2s_2}}=\sum_{\substack{\alpha\in\mathfrak{a}\\ \alpha\gg 0}}^{\circ}\frac{y^{s_1}y'^{s_2}}{|\varepsilon\alpha|^{2s_1}|\varepsilon'\alpha'|^{2s_2}}=\frac{y^{s_1}y'^{s_2}}{|\varepsilon|^{2s_1}|\varepsilon'|^{2s_2}}\zeta_F(2s,\hat{\chi}^{-2},A),$$

where $A$ is the ideal class to whom $\mathfrak{a}^{-1}$ belongs.

Therefore, we get

$$\sum_{\alpha\in\mathfrak{a}}^{\circ}(\text{term } \beta=0)=2y^{s_1}y'^{s_2}\left(1+\frac{1}{|\varepsilon|^{2s_1}|\varepsilon'|^{2s_2}}\right)\zeta_F(2s,\hat{\chi}^{-2},A).$$

As for the last three terms of (2.3.1), we see that they are essentially equal to $\Phi_+(\tau,\tau';\mathfrak{a},\mathfrak{b};s_1,s_2)$. For example, we have

$$\sum_{\substack{\alpha\in\mathfrak{a}\\ \beta\in\mathfrak{b}\\ \beta>0,\beta'<0}}^{\circ}\frac{y^{s_1}y'^{s_2}}{|\alpha+\beta\tau|^{2s_1}|\alpha'+\beta'\tau'|^{2s_2}}=\sum_{\substack{\alpha\in\mathfrak{a}\\ 0\ll\beta\in\mathfrak{b}}}^{\circ}\frac{y^{s_1}y'^{s_2}}{|\alpha+\varepsilon\beta\tau|^{2s_1}|\alpha'+\varepsilon'\beta'\tau'|^{2s_2}}$$

$$=\sum_{\substack{\alpha\in\mathfrak{a}\\ 0\ll\beta\in\mathfrak{b}}}^{\circ}\frac{y^{s_1}y'^{s_2}}{|\varepsilon\alpha+\varepsilon\beta\tau|^{2s_1}|\varepsilon'\alpha'+\varepsilon'\beta'\tau'|^{2s_2}}=\frac{y^{s_1}y'^{s_2}}{|\varepsilon|^{2s_1}|\varepsilon'|^{2s_2}}\Phi_+(\tau,\tau';\mathfrak{a},\mathfrak{b};s_1,s_2).$$

Thus we obtain

$$\begin{aligned}(2.3.2)\qquad \Phi(\tau,\tau';\mathfrak{a},\mathfrak{b};s_1,s_2)=2y^{s_1}y'^{s_2}\Big[&\zeta_F(2s,\hat{\chi}^{-2},A)+\frac{\zeta_F(2s,\hat{\chi}^{-2},A)}{|\varepsilon|^{2s_1}|\varepsilon'|^{2s_2}}\\ &+\Phi_+(\tau,\tau';\mathfrak{a},\mathfrak{b};s_1,s_2)+\frac{\Phi_+(\tau,\tau';\mathfrak{a},\mathfrak{b};s_1,s_2)}{|\varepsilon|^{2s_1}|\varepsilon'|^{2s_2}}\Big].\end{aligned}$$

Now $\Phi_+$ converges absolutely for $Re(s)>1$ and can be continued analytically into $Re(s)>\frac{1}{2}$. $\zeta_F(2s,\hat{\chi}^{-2},A)$ converges for $Re(s)>\frac{1}{2}$ and is, in particular, regular at $s=1$. For $k=0$, i.e., $\hat{\chi}=1$, $\Phi_+$, and so $\Phi$, has a pole at $s=1$. (Residues at $s=1$ of four terms of $\Phi$ do not cancel with together.) For $k\neq 0$, $\Phi$ is regular at $s=1$. By (2.3.2), we see that the limit formula for $\Phi$ is obtained by multiplying $yy'$ to that of $\Phi_+$, roughly speaking. Define

$$(2.3.3)\qquad G(\tau,\tau';\mathfrak{a},\mathfrak{b})=-4\pi^2 dN\mathfrak{a}^2\zeta_F(2,A)\tau\tau'+\sum_{\mathfrak{b}>\beta\ni\gg 0}^{\circ}\sum_{\mu\gg 0}N(\beta)^{-1}e^{\frac{2\pi\sqrt{-1}(\varepsilon\mu\beta'\tau'-\varepsilon'\mu'\beta\tau)}{\Delta}}$$

and

$$\begin{aligned}(2.3.4)\qquad G^*(\tau,\tau';\mathfrak{a},\mathfrak{b})=&G(-\varepsilon\bar{\tau},-\varepsilon'\tau';\mathfrak{a},\mathfrak{b})+G(\tau,\tau';\mathfrak{a},\mathfrak{b})+G(-\bar{\tau},-\bar{\tau};\mathfrak{a},\mathfrak{b})\\ &+G(\varepsilon\tau,\varepsilon'\bar{\tau}';\mathfrak{a},\mathfrak{b}).\end{aligned}$$

Observing $yy'=-(\tau\tau'+\tau\bar{\tau}'+\bar{\tau}\tau'+\bar{\tau}\bar{\tau}')/4$, we obtain the following.

THEOREM 1. (i) CASE: $k=0$.

$$\lim_{s\to 1}\left[\Phi(\tau, \tau'; \mathfrak{a}, \mathfrak{b}; s, s)-\frac{8\pi^2\log\varepsilon}{dN(\mathfrak{a}\mathfrak{b})}\frac{1}{s-1}\right]$$
$$=\frac{8\pi^2\log\varepsilon}{dN(\mathfrak{a}\mathfrak{b})}\left(2\log 2-\log N(\mathfrak{b})-\log yy'+2C-2\int_0^1\log(\sqrt{y_z}\sqrt[4]{d}\,|\eta(z)|^2)dv\right)$$
$$+\frac{1}{4\pi^2dN(\mathfrak{a})^2}G^*(\tau, \tau'; \mathfrak{a}, \mathfrak{b}).$$

(ii) CASE: $k\neq 0$.

$$\Phi(\tau, \tau'; \mathfrak{a}, \mathfrak{b}; 1+c, 1-c)=4y^{1+c}y'^{1-c}\zeta_F(2, \hat{\chi}^{-2}, A)$$
$$+\frac{-16\pi^2\log\varepsilon\int_0^1\log(\sqrt{y_z}|\eta(z)|^2)e^{4\pi\sqrt{-1}kv}dv}{dN(\mathfrak{a}\mathfrak{b})y^c y'^{-c}\Gamma(1+c)\Gamma(1-c)}$$
$$+\frac{16\pi^2y^{1+c}y'^{1-c}}{d^3N(\mathfrak{a})^3\Gamma(1+c)^2\Gamma(1-c)^2}F^*(k; \tau, \tau'; \mathfrak{a}, \mathfrak{b}).$$

As for the transformation formula in the case $k=0$, put

$$\log H(\tau, \tau'; \mathfrak{a}, \mathfrak{b})=\frac{N(\mathfrak{b})G^*(\tau, \tau'; \mathfrak{a}, \mathfrak{b})}{32\pi^4\log\varepsilon N(\mathfrak{a})}$$

Then as in 2.2, we have the following.

THEOREM 3.

$$H(\xi(\tau), \xi'(\tau'); \mathfrak{a}, \mathfrak{b})=H(\tau, \tau'; \mathfrak{a}, \mathfrak{b})|\gamma\tau+\delta|^{-2}|\gamma'\tau'+\delta'|^{-2}$$

*holds for* $\xi$ *satisfying* (2.2.3).

### 2.4. Limit formula for $\Psi$

We shall consider the function $\Psi$ defined in 2.0-3° and compute the limit formula for it.

For ideals $\mathfrak{a}$, $\mathfrak{b}$ in $F$, we define ideals $\delta_{\mathfrak{a}}$, $\delta_{\mathfrak{b}}$ by

$$\delta_{\mathfrak{a}}^{-1}=\{u\in F;\ Tr(\alpha u)\in \boldsymbol{Z} \text{ for all } \alpha\in\mathfrak{a}\}$$
$$\delta_{\mathfrak{b}}^{-1}=\{v\in F;\ Tr(\beta v)\in \boldsymbol{Z} \text{ for all } \beta\in\mathfrak{b}\}.$$

For given basis $[\omega_1, \omega_2]$, $[\omega_{1\mathfrak{b}}, \omega_{2\mathfrak{b}}]$ of $\mathfrak{a}$, $\mathfrak{b}$, respectively, basis of $\delta_{\mathfrak{a}}^{-1}$, $\delta_{\mathfrak{b}}^{-1}$ are given by

$$\left[\frac{\omega'_2}{\Delta}, \frac{\omega'_1}{\Delta}\right], \left[\frac{\omega'_{2\mathfrak{b}}}{\Delta_{\mathfrak{b}}}, \frac{\omega'_{1\mathfrak{b}}}{\Delta_{\mathfrak{b}}}\right]$$

respectively, where

$$\Delta = |\omega_1\omega'_2 - \omega_2\omega'_1|$$

$$\Delta_{\mathfrak{b}} = |\omega_{1\mathfrak{b}}\omega'_{2\mathfrak{b}} - \omega_{2\mathfrak{b}}\omega'_{1\mathfrak{b}}| \,.$$

We write

$$\text{(2.4.1)} \qquad \begin{cases} u = \dfrac{a\omega'_2 + b\omega'_1}{\Delta}, & u' = \dfrac{a'\omega_2 + b'\omega_1}{\Delta} \\ v = \dfrac{a_{\mathfrak{b}}\omega'_{2\mathfrak{b}} + b_{\mathfrak{b}}\omega'_{1\mathfrak{b}}}{\Delta_{\mathfrak{b}}}, & v' = \dfrac{a'_{\mathfrak{b}}\omega_{2\mathfrak{b}} + b_{\mathfrak{b}}\omega_{1\mathfrak{b}}}{\Delta_{\mathfrak{b}}} \end{cases}$$

with $a, b, a', b', a_{\mathfrak{b}}, b_{\mathfrak{b}}, a'_{\mathfrak{b}}, b'_{\mathfrak{b}} \in \boldsymbol{R}$. Then the condition (2.0.2) is equivalent to

$$\text{(2.4.2)} \qquad (a, b, a', b') \notin \boldsymbol{Z}^4 \quad \text{or} \quad (a_{\mathfrak{b}}, b_{\mathfrak{b}}, a'_{\mathfrak{b}}, b'_{\mathfrak{b}}) \notin \boldsymbol{Z}^4 \,.$$

We can write

$$\Psi(\tau, \tau'; \mathfrak{a}, \mathfrak{b}; s; (u), (v))$$
$$= \sum \frac{(yy')^s e^{2\pi i S((m_1\omega_1 + m_2\omega_2)u + (n_1\omega_{1\mathfrak{b}} + n_2\omega_{2\mathfrak{b}})v)}}{|m_1\omega_1 + m_2\omega_2 + (n_1\omega_{1\mathfrak{b}} + n_2\omega_{2\mathfrak{b}})\tau|^{2s} |m_1\omega'_1 + m_2\omega'_2 + (n_1\omega'_{1\mathfrak{b}} + n_2\omega'_{2\mathfrak{b}})\tau'|^{2s}}$$

where $S(\mu u + \nu v) = \mu u + \mu' u' + \nu v + \nu' v'$. Thus, up to the factor $(yy')^s$, $\Psi$ is the Epstein zeta-function of four variables $(m_1, m_2, n_1, n_2)$, without spherical function, defined in Siegel [5] p. 61. It converges absolutely for $Re(s) > 1$ (here, our $2s$ stands for $s$ in Siegel [5] p. 61) uniformly in every half-plane $Re(s) \geqslant 1 + \kappa (\kappa > 0)$. By Th.3 in Siegel [5], $\Psi(\tau, \tau'; \mathfrak{a}, \mathfrak{b}; s; (u), (v))$ has an analytic continuation, which is an entire function of $s$, into the whole $s$-plane.

Now we shall consider the value of $\Psi$ at $s = 1$, which leads us to the limit formula for $\Psi$. The computation follows the course in 2.1, through Poisson summation formula (2.0.3), and (2.1.3-4). We have

$$\Psi(\tau, \tau'; \mathfrak{a}, \mathfrak{b}; s; (u), (v))$$
$$= y^s y'^s \Bigg[ 4 \sum_{0 \ll \alpha \in \mathfrak{a}}^{\circ} \frac{e^{2\pi i S(\alpha u)}}{|\alpha|^{2s} |\alpha'|^{2s}}$$
$$+ \sum_{\substack{\alpha \in \mathfrak{a} \\ 0 \ll \beta \in \mathfrak{b}}}^{\circ} \left\{ \frac{e^{2\pi i S(\alpha u + \beta v)}}{|\alpha + \beta\tau|^{2s} |\alpha' + \beta'\tau'|^{2s}} + \frac{e^{2\pi i S(\alpha u + \varepsilon\beta v)}}{|\alpha + \varepsilon\beta\tau|^{2s} |\alpha' + \varepsilon'\beta'\tau'|^{2s}} \right.$$
$$\left. + \frac{e^{2\pi i S(\alpha u - \varepsilon\beta v)}}{|\alpha - \varepsilon\beta\tau|^{2s} |\alpha' - \varepsilon'\beta'\tau'|^{2s}} + \frac{e^{2\pi i S(\alpha u - \beta v)}}{|\alpha - \beta\tau|^{2s} |\alpha' - \beta'\tau'|^{2s}} \right\} \Bigg] .$$

Applying Poisson summation formula for the first term in { }, we have the following result.

The first term in { } at $s=1$

$$
\begin{aligned}
&=\frac{\pi^2}{\Delta}\sum_{0\ll\beta\in\mathfrak{b}}^{\circ}\frac{1}{N(\beta)}\,e^{2\pi i[\beta(v-u\bar{\tau})+\beta'(v'-u'\bar{\tau}')]}\\
&\quad+\frac{\pi^2}{\Delta}\sum_{0\ll\beta\in\mathfrak{b}}^{\circ}\frac{1}{N(\beta)}\,e^{2\pi i S(\beta v)}\Bigg\{\sum^{*}_{\substack{u+\frac{\mu'}{\Delta}>0\\ u'+\frac{\mu}{\Delta}>0}} e^{-2\pi i\left(\frac{\mu'}{\Delta}+u\right)\beta\bar{\tau}-2\pi i\left(-\frac{\mu}{\Delta}+u'\right)\beta'\bar{\tau}'}\\
&\qquad\qquad+\sum^{*}_{\substack{>0\\ <0}} e^{-2\pi i\left(\frac{\mu'}{\Delta}+u\right)\beta\bar{\tau}-2\pi i\left(-\frac{\mu}{\Delta}+u'\right)\beta'\tau'}\\
&\qquad\qquad+\sum^{*}_{\substack{<0\\ >0}} e^{-2\pi i\left(\frac{\mu'}{\Delta}+u\right)\beta\tau-2\pi i\left(-\frac{\mu}{\Delta}+u'\right)\beta'\bar{\tau}'}\\
&\qquad\qquad+\sum^{*}_{\substack{<0\\ <0}} e^{-2\pi i\left(\frac{\mu'}{\Delta}+u\right)\beta\tau-2\pi i\left(-\frac{\mu}{\Delta}+u'\right)\beta'\tau'}\Bigg\}\,,
\end{aligned}
$$

where $\sum^*$ denotes the omission of $\mu=0$.

The second, third and last terms in { } at $s=1$ can be calculated in the same way and are of the forms analogous to the above. Then the term involving only $\bar{\tau}$, $\bar{\tau}'$ in $\Psi$ is given by

$$
\begin{aligned}
(2.4.3)\qquad &\frac{\pi^2}{\Delta}\sum_{0\ll\beta\in\mathfrak{b}}^{\circ}\frac{1}{N(\beta)}\Bigg\{e^{2\pi i[\beta(v-u\bar{\tau})+\beta'(v'-u'\bar{\tau}')]}\\
&\quad+e^{2\pi i S(\beta v)}\sum^{*}_{\substack{u+\frac{\mu'}{\Delta}>0\\ u'+\frac{\mu'}{\Delta}>0}} e^{-2\pi i\left(\frac{\mu'}{\Delta}+u\right)\beta\bar{\tau}-2\pi i\left(-\frac{\mu}{\Delta}+u'\right)\beta'\bar{\tau}'}\\
&\quad+e^{2\pi i S(\varepsilon\beta v)}\sum^{*}_{\substack{>0\\ <0}} e^{-2\pi i\left(\frac{\mu'}{\Delta}+u\right)\varepsilon\beta\bar{\tau}-2\pi i\left(-\frac{\mu}{\Delta}+u'\right)\varepsilon'\beta'\bar{\tau}'}\\
&\quad+e^{2\pi i S(-\varepsilon\beta v)}\sum^{*}_{\substack{<0\\ >0}} e^{2\pi i\left(\frac{\mu'}{\Delta}+u\right)\varepsilon\beta\bar{\tau}+2\pi i\left(-\frac{\mu}{\Delta}+u'\right)\varepsilon'\beta'\bar{\tau}}\\
&\quad+e^{2\pi i S(-\beta v)}\sum^{*}_{\substack{<0\\ <0}} e^{2\pi i\left(\frac{\mu'}{\Delta}+u\right)\beta\bar{\tau}+2\pi i\left(-\frac{\mu}{\Delta}+u'\right)\beta'\bar{\tau}'}\Bigg\}\,.
\end{aligned}
$$

We can transform the summation conditions on $\frac{\mu'}{\Delta}+u$, $-\frac{\mu}{\Delta}+u'$ into that on only $\mu$, $\mu'$. This is shown by elementary, case by case considerations: For example, assume that $\omega_1>\omega_2$, $\omega_1'>\omega_2'$. There exist $i, j\in \boldsymbol{Z}$ such that

$$
(i+1)|\omega_2|>|\omega_1|>i|\omega_2|\,,
$$
$$
(j+1)|\omega_2'|>\omega_1'>j|\omega_2'|\,.
$$

$u$, $u'$ being as in (2.4.1), we take $b=b'=0$ and $0<a<1$, $0<a'<1$, namely

$$
u=\frac{a|\omega_2'|}{\Delta}\,,\quad u'=\frac{a'|\omega_2|}{\Delta}\,,\quad 0<a<1\,,\quad 0<a'<1\,. \tag{2.4.4}
$$

Suppose that

$$(2.4.5) \qquad 0<u<\mathrm{Min}\left(\frac{|\omega_2'|}{\Delta},\ \frac{|\omega_1'|-j|\omega_2'|}{\Delta}\right).$$

$$0<u'<\mathrm{Min}\left(\frac{|\omega_2|}{\Delta},\ \frac{|\omega_1|-i|\omega_2|}{\Delta}\right).$$

Then we see that

$$-\mu>0 \Longleftrightarrow -\frac{u}{\Delta}+u'>0\,,$$

$$\mu'>0 \Longleftrightarrow \frac{\mu'}{\Delta}+u>0\,.$$

Hence under the supposition (2.4.5) the summation conditions on $-\frac{\mu}{\Delta}+u'$, $\frac{\mu'}{\Delta}+u$ can be transformed into that on $\mu$, $\mu'$. But the expression thus obtained is valid for all $u$, $u'$ of (2.4.4). This is also shown by elementary, case by case considerations. In fact, for example, let us change the summation conditions

$$\left(\frac{\mu'}{\Delta}+u>0,\ -\frac{\mu}{\Delta}+u'>0\right),\ \left(\frac{\mu'}{\Delta}+u>0,\ -\frac{\mu}{\Delta}+u'<0\right),$$

$$\left(\frac{\mu'}{\Delta}+u<0,\ -\frac{\mu}{\Delta}+u'>0\right),\ \left(\frac{\mu'}{\Delta}+u<0,\ -\frac{\mu}{\Delta}+u'<0\right)$$

to $(\mu'>0,\ -\mu>0)$, $(\mu'>0,\ -\mu<0)$, $(\mu'<0,\ -\mu>0)$, $(\mu'<0,\ -\mu<0)$. Suppose, say, that the number of terms in $(\mu'>0,\ -\mu>0)$ decreases than the number of terms in

$$\left(\frac{\mu'}{\Delta}+u>0,\ -\frac{\mu}{\Delta}+u'>0\right).$$

Then the number of terms in $(\mu'<0,\ -\mu<0)$ increases than the number of terms in

$$\left(\frac{\mu'}{\Delta}+u<0,\ -\frac{\mu}{\Delta}+u'<0\right).$$

And increasing terms in $(\mu'<0,\ -\mu<0)$ and decreasing terms in $(\mu'>0,\ -\mu>0)$ cancel out. Further in this case, there is no change in the second and third terms of double sum in (2.4.3).

Analogous expressions hold for the rest parts involving only $(\tau, \bar{\tau}')$, $(\tau, \tau')$ and $(\bar{\tau}, \tau')$, respectively.

We put

$$\Theta_1(\tau, \tau'; \mathfrak{a}, \mathfrak{b}; (u), (v)) = \frac{-\Delta}{\pi^2} \sum_{0\ll\alpha\in\mathfrak{a}}^{\circ} \frac{e^{2\pi\sqrt{-1}\,S(\alpha u)}}{N(\alpha)^2}\tau\tau'$$

$$+\sum_{0\ll\beta\in\mathfrak{b}}^{\circ} \frac{1}{N(\beta)} e^{-2\pi\sqrt{-1}\,[\beta(v-u\tau)+\beta'(v'-u'\tau')]}$$

$$+\sum_{0\ll\beta\in\mathfrak{b}}^{\circ}\frac{1}{N(\beta)}\Big\{e^{-2\pi\sqrt{-1}\,S(\beta v)}\sum_{\substack{\mu'>0\\-\mu>0}}^{*}e^{2\pi\sqrt{-1}\left(\frac{\mu'}{\Delta}+u\right)\beta\tau+2\pi\sqrt{-1}\left(-\frac{\mu}{\Delta}+u'\right)\beta'\tau'}$$

$$+e^{-2\pi\sqrt{-1}\,S(\varepsilon\beta v)}\sum_{\substack{\mu'>0\\-\mu<0}}^{*}e^{2\pi\sqrt{-1}\left(\frac{\mu'}{\Delta}+u\right)\varepsilon\beta\tau+2\pi\sqrt{-1}\left(-\frac{\mu}{\Delta}+u'\right)\varepsilon'\beta'\tau'}$$

$$+e^{2\pi\sqrt{-1}\,S(\varepsilon\beta v)}\sum_{\substack{\mu'<0\\-\mu>0}}^{*}e^{-2\pi\sqrt{-1}\left(\frac{\mu'}{\Delta}+u\right)\varepsilon\beta\tau-2\pi\sqrt{-1}\left(-\frac{\mu}{\Delta}+u'\right)\varepsilon'\beta'\tau'}$$

$$+e^{2\pi\sqrt{-1}\,S(\beta v)}\sum_{\substack{\mu'<0\\-\mu<0}}^{*}e^{-2\pi\sqrt{-1}\left(\frac{\mu'}{\Delta}+u\right)\beta\tau-2\pi\sqrt{-1}\left(-\frac{\mu}{\Delta}+u'\right)\beta'\tau'}\Big\}$$

and

$$\Theta_2(\tau,\tau';\mathfrak{a},\mathfrak{b};(u),(v))=\frac{-\Delta}{\pi^2}\sum_{0\ll\alpha\in\mathfrak{a}}^{\circ}\frac{e^{2\pi\sqrt{-1}\,S(\alpha u)}}{N(\alpha)^2}\tau\tau'$$

$$+\sum_{0\ll\beta\in\mathfrak{b}}^{\circ}\frac{1}{N(\beta)}e^{-2\pi\sqrt{-1}\,[\varepsilon\beta(v-u\tau)-\varepsilon'\beta'(v'-u'\tau')]}$$

$$+\sum_{0\ll\beta\in\mathfrak{b}}^{\circ}\frac{1}{N(\beta)}\Big\{e^{-2\pi\sqrt{-1}\,S(\varepsilon\beta v)}\sum_{\substack{\mu'>0\\-\mu>0}}^{*}e^{2\pi\sqrt{-1}\left(\frac{\mu'}{\Delta}+u\right)\varepsilon\beta\tau+2\pi\sqrt{-1}\left(-\frac{\mu}{\Delta}+u'\right)\varepsilon'\beta'\tau'}$$

$$+e^{-2\pi\sqrt{-1}\,S(\beta v)}\sum_{\substack{\mu'>0\\-\mu<0}}^{*}e^{2\pi\sqrt{-1}\left(\frac{\mu'}{\Delta}+u\right)\beta\tau+2\pi\sqrt{-1}\left(-\frac{\mu}{\Delta}+u'\right)\beta'\tau'}$$

$$+e^{2\pi\sqrt{-1}\,S(\beta v)}\sum_{\substack{\mu'<0\\-\mu>0}}^{*}e^{-2\pi\sqrt{-1}\left(\frac{\mu'}{\Delta}+u\right)\beta\tau-2\pi\sqrt{-1}\left(-\frac{\mu}{\Delta}+u'\right)\beta'\tau'}$$

$$+e^{2\pi\sqrt{-1}\,S(\varepsilon\beta v)}\sum_{\substack{\mu'<0\\-\mu<0}}^{*}e^{-2\pi\sqrt{-1}\left(\frac{\mu'}{\Delta}+u\right)\varepsilon\beta\tau-2\pi\sqrt{-1}\left(-\frac{\mu}{\Delta}+u'\right)\varepsilon'\beta'\tau'}\Big\}.$$

Then we have

$$\begin{aligned}(2.4.6)\qquad &\Psi(\tau,\tau';\mathfrak{a},\mathfrak{b};1;(u),(v))\\ &=\frac{\pi^2}{\Delta}\{\Theta_1(\tau,\tau';\mathfrak{a},\mathfrak{b};(u),(v))+\Theta_1(-\bar{\tau},-\bar{\tau}';\mathfrak{a},\mathfrak{b};(u),(-v))\\ &\qquad+\Theta_2(\tau,\bar{\tau}';\mathfrak{a},\mathfrak{b};(u),(v))+\Theta_2(-\bar{\tau},-\tau';\mathfrak{a},\mathfrak{b};(u),(-v))\}\end{aligned}$$

under the condition (2.4.4) on $u$, $u'$.

We see that the expression (2.4.6) is valid for all $u$, $u'$ satisfying (2.0.2), hence (2.4.2). To prove this, it is sufficient to show that the right hand side of (2.4.6) is invariant under $a\to a+1$, $a'\to a'+1$, since $\Psi$ is invariant under the translations. (As for $a$, $a'$ see (2.4.2)). We consider the terms involving $(\tau,\tau')$, $(\tau,\bar{\tau}')$, $(\bar{\tau},\tau')$, $(\bar{\tau},\bar{\tau}')$ separately. The term involving $(\bar{\tau},\bar{\tau})$ was given by changing the sum conditions on $\frac{\mu'}{\Delta}+u$, $-\frac{\mu}{\Delta}+u'$ to the respective conditions on $\mu'$, $\mu$. The invariance under the above translations will be shown by elementary, case by case considerations. For example, assume that there occur changes in the second term. Then, changes in the first, second and fifth terms cancel out. And in this case, there are no changes in the third and fourth terms.

Now we can change the sum conditions in $\Theta$ into that of totally positive $\mu$'s. Thus we have

$$(2.4.7)\qquad \Theta_1(\tau,\tau';\mathfrak{a},\mathfrak{b};(u),(v))=\frac{-\Delta}{\pi^2}\sum_{0\ll\alpha\in\mathfrak{a}}^{\circ}\frac{e^{2\pi\sqrt{-1}\,S(\alpha u)}}{N(\alpha)^2}\tau\tau'$$
$$+\sum_{0\ll\beta\in\mathfrak{b}}^{\circ}\frac{1}{N(\beta)}e^{-2\pi\sqrt{-1}\,[\beta(v-u\tau)+\beta'(v'-u'\tau')]}$$
$$+\sum_{0\ll\beta\in\mathfrak{b}}^{\circ}\frac{1}{N(\beta)}\Big\{e^{-2\pi\sqrt{-1}\,S(\beta v)}\sum_{\mu\gg0}^{*}e^{2\pi\sqrt{-1}\left[\left(-\frac{\varepsilon'\mu'}{\Delta}+u\right)\beta\tau+\left(\frac{\varepsilon\mu}{\Delta}+u'\right)\beta'\tau'\right]}$$
$$+e^{-2\pi\sqrt{-1}\,S(\varepsilon\beta v)}\sum_{\mu\gg0}^{*}e^{2\pi\sqrt{-1}\left[\left(\frac{\mu'}{\Delta}+u\right)\varepsilon\beta\tau+\left(-\frac{\mu}{\Delta}+u'\right)\varepsilon'\beta'\tau'\right]}$$
$$+e^{2\pi\sqrt{-1}\,S(\varepsilon\beta v)}\sum_{\mu\gg0}^{*}e^{-2\pi\sqrt{-1}\left[\left(-\frac{\mu'}{\Delta}+u\right)\varepsilon\beta\tau+\left(\frac{\mu}{\Delta}+u'\right)\varepsilon'\beta'\tau'\right]}$$
$$+e^{2\pi\sqrt{-1}\,S(\beta v)}\sum_{\mu\gg0}^{*}e^{-2\pi\sqrt{-1}\left[\left(\frac{\varepsilon'\mu'}{\Delta}+u\right)\beta\tau+\left(-\frac{\varepsilon\mu}{\Delta}+u'\right)\beta'\tau'\right]}\Big\}$$

and

$$(2.4.8)\qquad \Theta_2(\tau,\tau';\mathfrak{a},\mathfrak{b};(u),(v))=\frac{-\Delta}{\pi^2}\sum_{0\ll\alpha\in\mathfrak{a}}^{\circ}\frac{e^{2\pi\sqrt{-1}\,S(\alpha u)}}{N(\alpha)}\tau\tau'$$
$$+\sum_{0\ll\mu\in\mathfrak{b}}^{\circ}\frac{1}{N(\beta)}e^{-2\pi\sqrt{-1}\,[\varepsilon\beta(v-u\tau)-\varepsilon'\beta'(v'-u'\tau')]}$$
$$+\sum_{0\ll\beta\in\mathfrak{b}}^{\circ}\frac{1}{N(\beta)}\Big\{e^{-2\pi\sqrt{-1}\,S(\varepsilon\beta v)}\sum_{\mu\gg0}^{*}e^{2\pi\sqrt{-1}\left[\left(-\frac{\varepsilon'\mu'}{\Delta}+u\right)\varepsilon\beta\tau+\left(\frac{\varepsilon\mu}{\Delta}+u'\right)\varepsilon'\beta'\tau'\right]}$$
$$+e^{-2\pi\sqrt{-1}\,S(\beta v)}\sum_{\mu\gg0}^{*}e^{2\pi\sqrt{-1}\left[\left(\frac{\mu'}{\Delta}+u\right)\beta\tau+\left(-\frac{\mu}{\Delta}+u'\right)\beta'\tau'\right]}$$
$$+e^{2\pi\sqrt{-1}\,S(\beta v)}\sum_{\mu\gg0}^{*}e^{-2\pi\sqrt{-1}\left[\left(\frac{\mu'}{\Delta}+u\right)\beta\tau+\left(\frac{\mu}{\Delta}+u'\right)\beta'\tau'\right]}$$
$$+e^{2\pi\sqrt{-1}\,S(\varepsilon\beta v)}\sum_{\mu\gg0}^{*}e^{-2\pi\sqrt{-1}\left[\left(\frac{\varepsilon'\mu'}{\Delta}+u\right)\varepsilon\beta\tau+\left(-\frac{\varepsilon\mu}{\Delta}+u'\right)\varepsilon'\beta'\tau'\right]}\Big\}.$$

Further we define

$$(2.4.9)\qquad \log\Delta(\tau,\tau';(u),(v))$$
$$=\sqrt{d}\,\{\Theta_1(\tau,\tau';\mathfrak{a},\mathfrak{b};(u),(v))+\Theta_1(-\bar{\tau},-\bar{\tau}';\mathfrak{a},\mathfrak{b};(u),(-v))$$
$$+\Theta_2(\tau,\bar{\tau}';\mathfrak{a},\mathfrak{b};(u),(v))+\Theta_2(-\bar{\tau},-\tau';\mathfrak{a},\mathfrak{b};(u),(-v))\}.$$

We may summarize the above results in

THEOREM 4. *Let* $\mathfrak{a}=[\omega_1, \omega_2]$, $\mathfrak{b}=[\omega_{1\mathfrak{b}}, \omega_{2\mathfrak{b}}]$ *be ideals in* $F$. *Let* $u, u', v, v'$ *be real numbers satisfying*

$$(a, b, a', b')\notin Z^4 \quad \text{or} \quad (a, b, a', b')\notin Z^4$$

*in writing as*

$$u=(a\omega_2'+b\omega_1')/\Delta, \quad u'=(a'\omega_2+b'\omega_1)/\Delta,$$

$$v=(a_{\mathfrak{b}}\omega_{2\mathfrak{b}}'+b_{\mathfrak{b}}\omega_{1\mathfrak{b}}')/\Delta_{\mathfrak{b}}, \quad v'=(a'_{\mathfrak{b}}\omega_{2\mathfrak{b}}+b'_{\mathfrak{b}}\omega_{1\mathfrak{b}})/\Delta_{\mathfrak{b}}.$$

*Then*

$$\Psi(\tau, \tau'; \mathfrak{a}, \mathfrak{b}; 1; (u), (v))=\frac{\pi^2}{\Delta\sqrt{d}}\log A(\tau, \tau'; (u), (v)),$$

*where* $A$ *is defined in* (2.4.9).

*In particular, if* $(a, b, a', b')\in Z^4$, *then the first term in* $\Theta_i$ *is given by*

$$\frac{-\Delta\sqrt{d}}{\pi^2}\tau\tau'\zeta_F(2, A)$$

*where* $A$ *is the ideal class to whom* $\mathfrak{a}^{-1}$ *belongs.*

Let $\xi=\begin{pmatrix}\nu & \mu\\ \gamma & \delta\end{pmatrix}$ be as in (2.2.3). Put $(u^*)=(\gamma u+\delta)$, $(v^*)=(\nu v+\mu)$. Then

$$\Psi(\xi(\tau), \xi'(\tau'); \mathfrak{a}, \mathfrak{b}; (u^*), (v^*))=\Psi(\tau, \tau'; \mathfrak{a}, \mathfrak{b}; (u), (v)).$$

Denote $u=u(a, b)$, $v=v(a_{\mathfrak{b}}, b_{\mathfrak{b}})$ as in (2.4.1). Then we see easily the following transformation formulas for $A$.

(2.4.10)
$$\begin{aligned}&A(\xi(\tau), \xi'(\tau'); (u^*), (v^*))=A(\tau, \tau'; (u), (v)),\\ &A(\tau, \tau'; (u(a+1, b)), (v))=A(\tau, \tau'; (u(a, b+1)), (v))\\ &=A(\tau, \tau'; (u), (v(a_{\mathfrak{b}}+1, b_{\mathfrak{b}})))=A(\tau, \tau'; (u), (v(a_{\mathfrak{b}}, b_{\mathfrak{b}}+1)))\\ &=A(\tau, \tau'; (u), (v)).\end{aligned}$$

## 3. Limit formula for zeta-function of a certain biquadratic field

**3.1.** Let $F=Q(\sqrt{d})$ be a real quadratic field with the fundamental unit $\varepsilon$ of norm $-1$, as above. For a totally negative non-quadratic number $\mu$ of $F$, we define a biquadratic field $K=F(\sqrt{\mu})=Q(\sqrt{d}, \sqrt{\mu})$. We shall consider the zeta-function of $K$ and compute the limit formula for it.

Let $\mathscr{A}$ be an ideal class in $K$ and $\mathfrak{C}_{\mathscr{A}}$ an ideal belonging to $\mathscr{A}^{-1}$. There exist elements $\Omega_1$, $\Omega_2$ of $K$ and an ideal $\mathfrak{a}_{\mathscr{A}}$ in $F$ such that $\mathfrak{C}_{\mathscr{A}}=\mathfrak{a}_{\mathscr{A}}\Omega_1+\mathfrak{o}\Omega_2$, where $\mathfrak{o}$ is the ring of integers in $F$. We may take Im $\Omega_1^{-1}\Omega_2>0$. Put $\tau_{\mathscr{A}}=\Omega_1^{-1}\Omega_2$ and $\tau_{\mathscr{A}}=x_{\mathscr{A}}+iy_{\mathscr{A}}$.

Let $\varepsilon_K$ be the fundamental unit of $K$. We take that $|\varepsilon_K|>1$. There are two cases:

$$1)\quad |\varepsilon_K|=\varepsilon\,,$$

$$2)\quad |\varepsilon_K|=\sqrt{\varepsilon}\,.$$

Firstly we consider the case 1). We have, with $N_{K/Q}(\tilde{\beta})>0$,

$$\zeta_K(s,\mathscr{A})=\frac{(N_{K/Q}(\mathfrak{C}_{\mathscr{A}}))^s}{w}\sum_{0\neq\tilde{\beta}\in\mathfrak{C}_{\mathscr{A}}}^{\circ} N_{K/Q}(\tilde{\beta})^{-s}$$

$$=\frac{(N_{K/Q}(\mathfrak{C}_{\mathscr{A}}))^s}{4w}\sum_{\substack{\alpha\in\mathfrak{a}_{\mathscr{A}}\\ \beta\in\mathfrak{d},\ (\alpha,\beta)}}' N_{K/Q}(\alpha\Omega_1+\beta\Omega_2)^{-s}\,,$$

where $w$ is the number of roots of unity in $K$. Now we have

$$N_{K/Q}(\alpha\Omega_1+\beta\Omega_2)^{-s}=N_{F/Q}((\alpha\Omega_1+\beta\Omega_2)(\alpha\bar{\Omega}_1+\beta\bar{\Omega}_2))^{-s}=N_{F/Q}(\Omega_1\bar{\Omega}_1|\alpha+\beta\tau_{\mathscr{A}}|^2)^{-s}$$

$$=N_{F/Q}\left(\frac{\sqrt{D_{\mathscr{A}}}}{y_{\mathscr{A}}}|\alpha+\beta\tau_{\mathscr{A}}|^2\right)^{-s}\,,$$

where $D_{\mathscr{A}}$ is the discriminant of positive definite quadratic form $(\alpha\Omega_1+\beta\Omega_2)(\alpha\bar{\Omega}_1+\beta\bar{\Omega}_2)$, hence $D_{\mathscr{A}}=\dfrac{(\Omega_1\bar{\Omega}_2-\bar{\Omega}_1\Omega_2)^2}{4}$. Thus we have

$$\zeta_K(s,\mathscr{A})=\frac{N_{K/Q}(\mathfrak{C}_{\mathscr{A}})^s}{4w(\sqrt{D_{\mathscr{A}}D'_{\mathscr{A}}})^s}\Phi(\tau_{\mathscr{A}},\tau'_{\mathscr{A}};\mathfrak{a}_{\mathscr{A}},\mathfrak{d};s,s)\,.$$

By Th.1 (i), we obtain the limit formula for $\zeta_K(s,\mathscr{A})$:

$$\zeta_K(s,\mathscr{A})=\frac{\pi^2PN_{K/Q}(\mathfrak{C}_{\mathscr{A}})}{4w\sqrt{D_{\mathscr{A}}D'_{\mathscr{A}}}N(\mathfrak{a}_{\mathscr{A}})}\left(\frac{1}{s-1}+M+\log\frac{N_{K/Q}(\mathfrak{C}_{\mathscr{A}})H(\tau_{\mathscr{A}},\tau'_{\mathscr{A}};\mathfrak{a}_{\mathscr{A}},\mathfrak{d})}{\sqrt{D_{\mathscr{A}}D'_{\mathscr{A}}}y_{\mathscr{A}}y'_{\mathscr{A}}}\right.$$
$$\left.+*(s-1)\cdots\right),$$

where we put

$$P=8\log\varepsilon/d\,,\quad M=2\log 2+2C-2\int_0^1\log(\sqrt{y_z}\sqrt[4]{d}\,|\eta(z)|^2)dv\,. \tag{3.1.2}$$

We can set

$$H(\mathscr{A})=\frac{N_{K/Q}(\mathfrak{C}_{\mathscr{A}})H(\tau_{\mathscr{A}},\tau'_{\mathscr{A}};\mathfrak{a}_{\mathscr{A}},\mathfrak{D})}{\sqrt{D_{\mathscr{A}}D'_{\mathscr{A}}}y_{\mathscr{A}}y'_{\mathscr{A}}} \tag{3.1.3}$$

since it can be easily seen that the right hand side is a class invariant.

We see

$$\frac{N_{K/Q}(\mathfrak{C}_{\mathscr{A}})}{4\sqrt{D_{\mathscr{A}}D'_{\mathscr{A}}}N(\mathfrak{a}_{\mathscr{A}})}=\frac{1}{\sqrt{N(\mathfrak{d})}} \tag{3.1.4}$$

with the relative discriminant $\mathfrak{d}$ of $K/F$. Since in this case the regulator $R$ of $K$ equals $2\log|\varepsilon_K|=2\log\varepsilon$, we have

$$\zeta_K(s, \mathscr{A})=\frac{4\pi^2 Z}{wd\sqrt{N(\mathfrak{d})}}\left(\frac{1}{s-1}+M+\log H(\mathscr{A})+*(s-1)\cdots\right).$$

Secondly, we consider the case 2). In this case, the summation condition in (3.1.1) must be changed to that over non-associated $(\alpha, \beta)$ with units in $K$. Hence the sum $\sum'$ in (3.1.1) is multiplied by $\frac{1}{2}$. The regulator $R$ equals $2\log|\varepsilon_K|=\log\varepsilon$. Thus we have the limit formula of the same form as in case 1).

THEOREM 5. *Let $K$ be biquadratic field obtained by adjunction, to $F=\boldsymbol{Q}(\sqrt{d})$, of a totally negative number in $F$. Let $\zeta_K(s, \mathscr{A})$ be the zeta-function of $K$ corresponding to an ideal class $\mathscr{A}$. Then the limit formula for $\zeta_K(s, \mathscr{A})$ is given by*

$$\lim_{s\to 1}\left(\zeta_K(s, \mathscr{A})-\frac{4\pi^2 R}{wd\sqrt{N(\mathfrak{d})}}\right)=\frac{4\pi^2 R}{wd\sqrt{N(\mathfrak{d})}}(M+\log H(\mathscr{A})),$$

*where $w$ is the number of roots of unity in $K$, $M$ is the constant, not depending on $\mathscr{A}$, defined in* (3.1.2), *$R$ is the regulator of $K$ and $H(\mathscr{A})$ is the class invariant defined by* (3.1.3).

**3.2.** Let $k$ be an arbitrary algebraic number field of finite degree. Let $K=k(\sqrt{\mu})$ be relative quadratic extension of $k$ by adjunction of non-quadratic integer $\mu$ in $k$. Let $\mathfrak{p}$ be a prime ideal in $k$. We define, following Hilbert [7], the symbol $\left(\frac{\mu}{\mathfrak{p}}\right)$ as follows:

$$\left(\frac{\mu}{\mathfrak{p}}\right)=1 \quad \text{if } \mathfrak{p} \text{ decomposes into different prime ideals in } K,$$

$$\left(\frac{\mu}{\mathfrak{p}}\right)=-1 \quad \text{if } \mathfrak{p} \text{ remains prime in } K,$$

and

$$\left(\frac{\mu}{\mathfrak{p}}\right)=0 \quad \text{if } \mathfrak{p} \text{ is a quadrat of a prime ideal in } K.$$

Let $\mathfrak{d}$ be the relative discriminant of $K/k$. Then Satz 6 of Hilbert [7] asserts that

$$\left(\frac{\mu}{\mathfrak{p}}\right)=0 \quad \text{if and only if } \mathfrak{p}|\mathfrak{d}. \tag{3.2.1}$$

For $\mathfrak{p}$ coprime to 2 and $\mu$, Satz 7 in Hilbert [7] asserts

(3.2.2) $\left(\frac{\mu}{\mathfrak{p}}\right)=1$ if and only if $\mu$ is quadratic residue mod $\mathfrak{p}$,

(3.2.3) $\left(\frac{\mu}{\mathfrak{p}}\right)=-1$ if and only if $\mu$ is quadratic non-residue mod $\mathfrak{p}$.

Here we quote Satz 4, 5 in [7] in the following forms:

(3.2.4) Let $\mathfrak{p}$ be a prime ideal coprime to 2. We let $\mathfrak{p}^a$ exactly divide $\mu$. Then $\mathfrak{p}|\mathfrak{d}$ for odd $a$ and $(\mathfrak{p}, \mathfrak{d})=1$ for even $a$.

(3.2.5) Let $\mathfrak{l}$ be a prime ideal such that $\mathfrak{l}^l$ exactly divide 2. Further we let $\mathfrak{l}^a$ exactly divide $\mu$. Then $(\mathfrak{l}, \mathfrak{d})=1$ if and only if $\mu$ is quadratic residue mod $\mathfrak{l}^{2l+a}$.

(3.2.6) When $\mu$ is coprime to 2, $(\mathfrak{d}, 2)=1$ if and only if $\mu$ is quadratic residue mod 4.

Thus for a prime factor $\mathfrak{l}$ of 2, we have $\left(\frac{\mu}{\mathfrak{l}}\right)=0$ if $\mathfrak{l}^a$ divides exactly $\mu$ with odd exponent $a$. If this $a$ is even, take an integer $\mu^*$ in $k$ such that

(3.2.7) $$\mu\equiv\lambda^a\mu^* \pmod{\mathfrak{l}^{2l+a+1}} \text{ for } \lambda \text{ in } k,\ \mathfrak{l}|\lambda \text{ but } \mathfrak{l}^2\nmid\lambda.$$

Then we have $\left(\frac{\mu}{\mathfrak{l}}\right)=0$ if $\mu^*$ is not quadratic mod $\mathfrak{l}^{2l}$, $\left(\frac{\mu}{\mathfrak{l}}\right)=1$ if $\mu^*$ is quadratic mod $\mathfrak{l}^{2l+1}$ and $\left(\frac{\mu}{\mathfrak{l}}\right)=1$ if $\mu^*$ is not quadratic mod $\mathfrak{l}^{2l+1}$. We extend the symbol $\left(\frac{\mu}{*}\right)$ to an arbitrary integral ideal in $k$ multiplicatively.

Put $\mu=\mu_1\mu_2$ with coprime $\mu_1$, $\mu_2$ in $k$ and $K_1=k(\sqrt{\mu_1})$, $K_2=k(\sqrt{\mu_2})$. Let $\mathfrak{d}_1$, $\mathfrak{d}_2$ be the relative discriminants of $K_1$, $K_2$ respectively. For a prime factor $\mathfrak{p}$ of $\mathfrak{d}$ not dividing 2, we see, by (3.2.4), that $\mathfrak{p}$ divides only $\mathfrak{d}_1$ or only $\mathfrak{d}_2$. (Here $(\mu_1, \mu_2)=1$ is unneccessary). We consider a prime factor $\mathfrak{l}$ of 2. If $(\mu, 2)=1$, we have $(\mu_1, 2)=(\mu_2, 2)=1$. Then by (3.2.6) it can be easily seen that if $(\mathfrak{d}, 2)=1$ then $(\mathfrak{d}_1, 2)=(\mathfrak{d}_2, 2)=1$ or $(\mathfrak{d}_1, 2)\neq1$, $(\mathfrak{d}_2, 2)\neq1$. In the first case, we can deduce that $\left(\frac{\mu}{\mathfrak{l}}\right)=\left(\frac{\mu_1}{\mathfrak{l}}\right)\left(\frac{\mu_2}{\mathfrak{l}}\right)$. In the second case, $\left(\frac{\mu}{\mathfrak{l}}\right)\neq0$ but $\left(\frac{\mu_1}{\mathfrak{l}}\right)=\left(\frac{\mu_2}{\mathfrak{l}}\right)=0$. Also by (3.2.6), if $(\mathfrak{d}, 2)\neq1$, then $(\mathfrak{d}_i, 2)\neq1$ and $(\mathfrak{d}_j, 2)=1$ for $i, j=1, 2$. In this case, we have $\left(\frac{\mu}{\mathfrak{l}}\right)=\left(\frac{\mu_1}{\mathfrak{l}}\right)\left(\frac{\mu_2}{\mathfrak{l}}\right)=0$.

Now assume that

(3.2.8) $\mathfrak{d}_1$ and $\mathfrak{d}_2$ are coprime mutually.

Then we have

(3.2.9) $$\left(\frac{\mu}{\mathfrak{p}}\right)=\left(\frac{\mu_1}{\mathfrak{p}}\right)\left(\frac{\mu_2}{\mathfrak{p}}\right) \text{ for } \mathfrak{p}\nmid 2.$$

Since there does not appear the case $(\mathfrak{d}, 2)=1$, $(\mathfrak{d}_1, 2)\neq1$, $(\mathfrak{d}_2, 2)\neq1$, we have

(3.2.10) if $(\mu, 2)=1$, then $\left(\frac{\mu}{\mathfrak{l}}\right)=\left(\frac{\mu_1}{\mathfrak{l}}\right)\left(\frac{\mu_2}{\mathfrak{l}}\right)$ for $\mathfrak{l}|2$.

We consider the case $(\mu, 2)\neq 1$ under (3.2.7). Let $l$, $a$, $\lambda$, $\mu^*$ be as in (3.2.5) and (3.2.7).

If $a$ is odd, then $\left(\frac{\mu}{\mathfrak{l}}\right)=0$ and since $\mathfrak{l}^a$ exactly divides $\mu_1$ or $\mathfrak{l}^a$ exactly divides $\mu_2$, we have $\left(\frac{\mu_1}{\mathfrak{l}}\right)=0$ or $\left(\frac{\mu_2}{\mathfrak{l}}\right)=0$. Let $a$ be even. If $\left(\frac{\mu}{\mathfrak{l}}\right)=0$, then $\mu^*$ is not quadratic mod $\mathfrak{l}^{2l}$. We denote by $\mu_1^*$, $\mu_2^*$ numbers for $\mu_1$, $\mu_2$ analogous to $\mu^*$ for $\mu$ as in (3.2.7). Then $\mu_1^*$ or $\mu_2^*$ must be non-quadratic mod $\mathfrak{l}^{2l}$; namely $\left(\frac{\mu_1}{\mathfrak{l}}\right)=0$ or $\left(\frac{\mu_2}{\mathfrak{l}}\right)=0$. If $\left(\frac{\mu}{\mathfrak{l}}\right)\neq 0$, then $(\mathfrak{l}, \mathfrak{d})=1$. Since $\mu_1$, $\mu_2$ are coprime, $\mathfrak{l}^a$ exactly divides $\mu_1$ or $\mu_2$. This is naturally satisfied without the condition $(\mu_1, \mu_2)=1$, if $k$ is quadratic. By this reason, we can deduce that if $(\mathfrak{l}, \mathfrak{d})=1$, then $(\mathfrak{l}, \mathfrak{d}_1)=(\mathfrak{l},\mathfrak{d}_2)=1$ or $(\mathfrak{l}, \mathfrak{d}_1)\neq 1$, $(\mathfrak{l}, \mathfrak{d}_2)\neq 1$. But by (3.2.8), the second case does not occur. In the first case, it can be easily seen that if $\left(\frac{\mu}{\mathfrak{l}}\right)=1$, then $\left(\frac{\mu_1}{\mathfrak{l}}\right)=\left(\frac{\mu_2}{\mathfrak{l}}\right)$ and if $\left(\frac{\mu}{\mathfrak{l}}\right)=-1$, then $\left(\frac{\mu_1}{\mathfrak{l}}\right)=-\left(\frac{\mu_2}{\mathfrak{l}}\right)$. Summing up, we have the following

LEMMA 1. *Let $\mu$ be a non-quadratic number of $k$. Put $\mu=\mu_1, \mu_2$ with non-quadratic integers $\mu_1$, $\mu_2$ of $k$. Let $\mathfrak{d}$, $\mathfrak{d}_1$, $\mathfrak{d}_2$ be the relative discriminants of $k(\sqrt{\mu_1})$, $k(\sqrt{\mu_1})$, $k(\sqrt{\mu_2})$, respectively.*

(1) *Assume that $(\mathfrak{d}_1, \mathfrak{d}_2)=1$ and $(\mu_1, \mu_2)=1$. Then*

i) *for any prime ideal $\mathfrak{p}$ in $k$, $\left(\frac{\mu}{\mathfrak{p}}\right)=\left(\frac{\mu_1}{\mathfrak{p}}\right)\left(\frac{\mu_2}{\mathfrak{p}}\right)$.*

ii) *Only prime factor of $\mathfrak{d}$ divides $\mathfrak{d}_1$ or $\mathfrak{d}_2$. Conversely, only prime factor of $\mathfrak{d}_i$ $(i=1, 2)$ divides $\mathfrak{d}$.*

2) *If $k$ is a quadratic field, then $(\mu_1, \mu_2)=1$ follows from $(\mathfrak{d}_1, \mathfrak{d}_2)=1$.*

**3.3.** Let $k$ and $K=k(\sqrt{\mu})$ be as in 3.2. As usual, we denote by $r_1$, $2r_2$ the number of real, complex conjugates of $k$, over $\boldsymbol{Q}$, respectively. It is well-known, after Dirichlet and Dedekind, that

$$\operatorname*{Res}_{s=1} \zeta_k(s)=\frac{2^{r_1}(2\pi)^{r_2}R_k h_k}{w_k\sqrt{|\Delta_k|}}, \tag{3.3.1}$$

where $h_k$ is the class number of $k$, $R_k$ is the regulator of $k$, $\Delta_k$ is the discriminant of $k$ over $\boldsymbol{Q}$ and $w_k$ is the number of roots of unity in $k$.

We define

$$L_\mu(s)=\sum_{\mathfrak{a}}\left(\frac{\mu}{\mathfrak{a}}\right)N\mathfrak{a}^{-s}.$$

It has Euler product formula:

$$L_\mu(s)=\prod_{\mathfrak{p}}\left(1-\left(\frac{\mu}{\mathfrak{p}}\right)N\mathfrak{p}^{-s}\right)^{-1} .$$

On direct computation, we have

$$\zeta_K(s)=\zeta_k(s)L_\mu(s) .$$

Then by (3.3.1), we have

$$L_\mu(1)=\frac{2^{r_1'-r_1}(2\pi)^{r_2'-r_2}\sqrt{|\varDelta_k|}\,R_K\cdot h_K\cdot w_k}{\sqrt{|\varDelta_K|}w_K R_k h_k} . \tag{3.3.2}$$

In particular, when $k=F=\boldsymbol{Q}(\sqrt{d})$, $K=F(\sqrt{\mu_1})$ with totally positive $\mu_1$, then

$$L_{\mu_1}(1)=\frac{4\sqrt{d}\,R_1 h_1}{\sqrt{|\varDelta_1|}\,R_k h_k} , \tag{3.3.3}$$

where we put $R_1=R_K$, $h_1=h_K$ and $\varDelta_1=\varDelta_K$.
When $K=F(\sqrt{\mu_2})$ with totally negative $\mu_2$, we have

$$L_{\mu_2}(1)=\frac{2\pi^2\cdot\sqrt{d}\cdot R_2\cdot h_2}{\sqrt{|\varDelta_2|}\cdot w_2\cdot R_k\cdot h_k} , \tag{3.3.4}$$

where we put $R_2=R_K$, $h_2=h_K$, $\varDelta_2=\varDelta_K$ and $w_2=w_K$.

**3.4.** Let $k=F=\boldsymbol{Q}(\sqrt{d})$ and $K=F(\sqrt{\mu})$ with totally negative $\mu$ in $F$. Put $\mu=\mu_1\mu_2$ with totally positive $\mu_1$ and totally negative $\mu_2$ in $F$. Let $\mathfrak{d}_1$, $\mathfrak{d}_2$ be the relative discriminants of $K_1=F(\sqrt{\mu_1})$, $K_2=F(\sqrt{\mu_2})$, respectively. Assume that $(\mathfrak{d}_1, \mathfrak{d}_2)=1$. By Lemma 1, (2), $(\mu_1, \mu_2)=1$. For such a decomposition of $\mu$, and for any prime ideal $\mathfrak{P}$, in $K$, coprime to $\mathfrak{d}$, we define

$$\psi(\mathfrak{P})=\left(\frac{\mu_1}{N_{K/F}\mathfrak{P}}\right) ,$$

where the symbol $\left(\frac{*}{*}\right)$ is defined in 3.2. Then we have

$$\psi(\mathfrak{P})=\left(\frac{\mu_2}{N_{K/F}\mathfrak{P}}\right) .$$

This can be shown by Lemma 1 as in Siegel [5] p. 78. When $\mathfrak{P}|\mathfrak{d}$, only $\left(\frac{\mu_1}{N_{K/F}\mathfrak{P}}\right)$ is zero or only $\left(\frac{\mu_2}{N_{K/F}\mathfrak{P}}\right)$ is zero. Then we define $\psi$ to be the non-zero value. We extend $\psi$ to all ideals $\mathfrak{A}$ in $K$. It is proved that $\psi(\mathfrak{A})=\psi(\mathfrak{B})$ if $\mathfrak{A}$ is equivalent to $\mathfrak{B}$ (in the narrow sense). $\psi$ is the so-called genus character of $K$.

LEMMA 2. *$\psi(\mathfrak{A})=\psi(\mathfrak{B})$ holds if $\mathfrak{A}$ and $\mathfrak{B}$ are equivalent (in the narrow sense).*

Consider $L$-series, for $Re(s)>1$,

$$L_K(s,\ \psi)=\sum \psi(\mathfrak{A})N\mathfrak{A}^{-s}=\prod_{\mathfrak{p}}\prod_{\mathfrak{P}|\mathfrak{p}}(1-\psi(\mathfrak{P})N\mathfrak{P}^{-s})^{-1}\ .$$

As in Siegel [5] pp. 79-80, we obtain

$$L_K(s,\ \psi)=L_{\mu_1}(s)L_{\mu_2}(s)\ .$$

Then Lemma 2 can be proved following Hecke's method (see, Siegel [5] pp. 84-85).

Let $\mathfrak{d}_i$ be the relative discriminants of $F(\sqrt{\mu_i})/F$. Then (3.3.3), (3.3.4) are rewritten as

$$L_{\mu_1}(1)=4R_1h_1/\sqrt{d}\sqrt{N\mathfrak{d}_1}\cdot R_k\cdot h_k\ ,$$

$$L_{\mu_2}(1)=2\pi^2R_2h_2/\sqrt{d}\sqrt{N\mathfrak{d}_2}w_2R_kh_k\ .$$

We know by Th.5 that for non-principal ideal class character $\chi$ of $K$,

$$L_K(1,\ \chi)=\frac{4\pi^2R_K}{w_K\cdot d\sqrt{N(\mathfrak{d})}}\sum_{\mathscr{A}}\chi(\mathscr{A})\log H(\mathscr{A})$$

holds. Thus we have the following

PROPOSITION 4. *Let $\mu$ be a totally negative number of $F=\boldsymbol{Q}(\sqrt{d})$, $d>0$. Put $K=F(\sqrt{\mu})$. Decompose $\mu$ as $\mu=\mu_1\mu_2$ with totally positive $\mu_1$ and totally negative $\mu_2$ in $F$. Denote by $\mathfrak{d}$, $\mathfrak{d}_1$, $\mathfrak{d}_2$ the relative discriminants of $K$, $F(\sqrt{\mu_1})$, $F(\sqrt{\mu_2})$ over $F$, respectively. Assume that $(\mathfrak{d}_1, \mathfrak{d}_2)=1$. Then for a genus character $\psi$ corresponding to the decomposition $\mu=\mu_1\mu_2$, the following formula holds:*

$$\frac{R_K}{w_K\sqrt{N(\mathfrak{d})}}\sum_{\mathscr{A}}\psi(\mathscr{A})\log H(\mathscr{A})=\frac{2R_1h_1R_2h_2}{R_k^2h_k^2w_2\sqrt{N(\mathfrak{d}_1\mathfrak{d}_2)}}\ .$$

This is a generalization of "Kronecker's solution of Pell's equation".

**3.5.** Class number of the absolute class field of $\boldsymbol{Q}(\sqrt{d},\ \sqrt{\mu})$.

$$(d>0,\ \mu\ll 0\quad\text{in}\quad \boldsymbol{Q}(\sqrt{d}))$$

Let $F=\boldsymbol{Q}(\sqrt{d})$ be as in 3.1. Let $K_0$ be the absolute class field of $K=F(\sqrt{\mu})$ with totally negative $\mu$. We have, by class field theory,

$$\zeta_{K_0}(s)=\prod_{\chi}L_K(s,\ \chi)\ ,$$

where $\chi$ runs over all ideal class characters in $K$. For $\chi=1$, we have $L_K(s,1)=\zeta_K(s)$. Hence

$$\zeta_{K_0}(s)=\prod_{\chi\neq 1}{}' L_K(s,\ \chi)\zeta_K(s)\ .$$

$K_0$ is totally imaginary of degree $4h$ ($h=h_K$). Since $K_0$ is unramified over $K$, we have the discriminant of $K_0/\boldsymbol{Q}$ equals $(d^2N(\mathfrak{d}))^h$, where $\mathfrak{d}$ is the relative discriminant of $K/F$. On comparing the residues at $s=1$, we obtain, by (3.3.1) (or Th.5 for $\zeta_K(s)$), the following

PROPOSITION 5. *Let $K$ be a imaginary quadratic extension of a real quadratic field. Let $K_0$ be the absolute class field of $K$. Then*

$$\frac{R_{K_0}\cdot h_{K_0}}{w_{K_0}}=\frac{R_K^{h_K}\cdot h_K}{w_K^{h_K}}\prod_{\chi\neq 1}\sum_{\mathscr{A}}\chi(\mathscr{A})\log H(\mathscr{A}),$$

*where $\mathscr{A}$ runs over all ideal classes in $K$ and $\chi$ over all non-principal class characters in $K$.*

**3.6.** Ray class field over $K=\boldsymbol{Q}(\sqrt{d},\sqrt{\mu})$ $(d>0,\ \mu<0)$. Let $K$, $F$ be as in 3.5. Let $\mathfrak{f}\neq(1)$ be an integral ideal in $K$. We consider the ray class group $\mathfrak{g}_\mathfrak{f}/\mathfrak{c}_\mathfrak{f}$ modula $\mathfrak{f}$. Let $\chi$ be a character of $\mathfrak{g}_\mathfrak{f}/\mathfrak{c}_\mathfrak{f}$.

We start from (1.2.4) for $L$-series of $K$:

$$L_K(s,\chi)=T^{-1}\sum_{\lambda \bmod \mathfrak{f}}\bar{\chi}(\lambda)\sum_{\mathscr{A}}\bar{\chi}(\mathfrak{C}_{\mathscr{A}})(N\mathfrak{C}_{\mathscr{A}})^s\sum_{\mathfrak{C}_{\mathscr{A}}|(\tilde{\beta})}v(\tilde{\beta})e^{2\pi\sqrt{-1}S(\tilde{\beta}\lambda)}|N(\tilde{\beta})|^{-s}$$

In the present case, $v(\tilde{\beta})=1$ identically.

Let $w$ and $w_\mathfrak{f}$ be the number of roots of 1 and of roots of unit $\varepsilon$ satisfying $\varepsilon\equiv 1 \pmod{\mathfrak{f}}$. We have

$$\sum_{\mathfrak{C}_{\mathscr{A}}|(\tilde{\beta})\neq 0}=w^{-1}\sum_{\mathfrak{C}_{\mathscr{A}}|(\tilde{\beta})_u\neq 0}$$

where $(\tilde{\beta})_u$ denotes that it runs over non-associated (with respect to units not being roots of 1) $\tilde{\beta}$. As $\lambda$ runs over a full system of representatives of $G(\mathfrak{f})$ and $\mathfrak{C}_{\mathscr{A}}$ over a complete set of representatives of the classes, then $(\lambda)\mathfrak{C}_{\mathscr{A}}$ covers exactly $w/w_\mathfrak{f}$ times, a complete system of representatives of $\mathfrak{G}_\mathfrak{f}/\mathfrak{E}_\mathfrak{f}$. Thus we have, for $Re(s)>1$,

$$L_K(s,\chi)=(Tw_\mathfrak{f})^{-1}\sum_{\mathscr{B}}\bar{\chi}(\mathfrak{C}_{\mathscr{B}})(N\mathfrak{C}_{\mathscr{B}})^s\sum_{\mathfrak{C}_{\mathscr{B}}|(\tilde{\beta})_u\neq 0}e^{2\pi\sqrt{-1}S(\tilde{\beta}\gamma)}|N\tilde{\beta}|^{-s},$$

where $\mathscr{B}$ runs over $\mathfrak{G}_\mathfrak{f}/\mathfrak{E}_\mathfrak{f}$ and $\mathfrak{C}_{\mathscr{B}}$ is a fixed integral ideal in $\mathscr{B}$ coprime to $\mathfrak{f}$.

Write $\qquad \mathfrak{C}_{\mathscr{B}}=\mathfrak{a}_{\mathscr{B}}\Omega_1+\mathfrak{o}\Omega_2 \quad$ and $\quad \tau_{\mathscr{B}}=\Omega_1^{-1}\Omega_2$

and put

$$u_{\mathscr{B}}=\Omega_1\gamma+\bar{\Omega}_1\bar{\gamma},\quad v_{\mathscr{B}}=\Omega_2\gamma+\bar{\Omega}_2\bar{\gamma}.$$

Then $(u_{\mathscr{B}})=(u_{\mathscr{B}},u'_{\mathscr{B}})$ and $(v_{\mathscr{B}})=(v_{\mathscr{B}},v'_{\mathscr{B}})$ satisfy (2.4.2), by our choice of $\gamma$, $\mathfrak{C}_{\mathscr{B}}$ and $\mathfrak{f}$.

Put
$$D_{\mathscr{B}}=(\Omega_1\bar{\Omega}_2-\Omega_2\bar{\Omega}_1)^2/4,$$
$$D'_{\mathscr{B}}=(\Omega'_1\bar{\Omega}'_2-\Omega'_2\bar{\Omega}'_1)^2/4.$$

Then for case 1) in § 3.1, we have, as in § 3.1,

$$L_K(s, \chi)=\frac{4\pi^2}{Tw_{\mathfrak{f}}}\sum_{\mathscr{B}}\bar{\chi}(\mathfrak{C}_{\mathscr{B}})N(\mathfrak{C}_{\mathscr{B}})^s/4\sqrt{d}\,\Delta\sqrt{D_{\mathscr{B}}D'_{\mathscr{B}}}^s\log\Lambda(\tau_{\mathscr{B}}, \tau'_{\mathscr{B}}:(u_{\mathscr{B}}), (v_{\mathscr{B}}))\,.$$

With the relative discriminant $\mathfrak{d}$ of $K/F$,

$$N(\mathfrak{C}_{\mathscr{B}})/4\sqrt{D_{\mathscr{B}}D'_{\mathscr{B}}}N\mathfrak{a}_{\mathscr{B}}=1/\sqrt{N\mathfrak{d}}$$

holds.

For case 2) in § 3.1, the right hand side of $L_K(s, \chi)$ is multiplied by $\frac{1}{2}$.

By Th.4, and the above, we obtain the following limit formula for $L_K(s, \chi)$.

THEOREM 5. *Let $F$ be a real quadratic field of discriminant $d$ with fundamental unit $\varepsilon$ of norm $-1$. Let $K$ be a imaginary quadratic extension of $F$ with the relative discriminant $\mathfrak{d}$ over $F$. Define $\iota=1$ if $\varepsilon_K$ with $|\varepsilon_K|=\varepsilon$ is fundamental unit in $K$, $\iota=\frac{1}{2}$ if $\varepsilon_K$ with $|\varepsilon_K|=\sqrt{\varepsilon}$ is fundamental unit in $K$. Let $\mathfrak{f}$ be an integral ideal in $K$, $\chi$ be a ray class character* mod $\mathfrak{f}$ *and $w_{\mathfrak{f}}$ the number of roots of unity congruent to* 1 mod $\mathfrak{f}$ *in $K$. Then*

$$L_K(1, \chi)=\frac{4\iota\pi^2}{d\sqrt{N\mathfrak{d}}\,w_{\mathfrak{f}}T}\sum_{\mathscr{B}}\bar{\chi}(\mathfrak{C}_{\mathscr{B}})\log\Lambda(\tau_{\mathscr{B}}, \tau'_{\mathscr{B}}.(u_{\mathscr{B}}), (v_{\mathscr{B}}))\,,$$

*where $\mathscr{B}$ runs over $\mathfrak{G}_{\mathfrak{f}}/\mathfrak{E}_{\mathfrak{f}}$ and $T$ is defined in* § 1.2.

Let $K_r$ be the ray class field modulo $\mathfrak{f}$ of $K=F(\sqrt{\mu})$, $\mu\ll 0$. We denote $\Delta_r=\Delta_{K_r}$, $R_r=R_{K_r}$, $w_r=w_{K_r}$.

Let $g$ be the order of Galois group of $K_r/K$ and $\mathfrak{d}_r=\mathfrak{d}_{K_r}$ the relative discriminant of $K_r/K$. $K_r$ is totally imaginary over $\boldsymbol{Q}$ of degree $4g$. Let $\chi_0$, $\mathfrak{f}_\chi$ be as in § 1.2. Then $|\Delta_r|=N(\mathfrak{d}_r)(\sqrt{N\mathfrak{d}}\cdot d)^g$ and $\mathfrak{d}_r=\Pi\mathfrak{f}_\chi$. By Th. 5, we have

$$L_K(1, \chi_0)=\frac{4\cdot\iota\cdot\pi^2}{w_{\mathfrak{f}_\chi}T_0\sqrt{|\Delta_K|}}\sum_{\mathscr{B}_0}\bar{\chi}_0(\mathfrak{C}_{\mathscr{B}_0})\log\Lambda(\tau_{\mathscr{B}_0}, \tau'_{\mathscr{B}_0}; (u_{\mathscr{B}_0}), (v_{\mathscr{B}_0}))\,,$$

where
$$T_0=T(\chi_0)=\sum_{\lambda \bmod \mathfrak{f}_\chi}\bar{\chi}_0(\lambda)e^{2\pi\sqrt{-1}\,S(\lambda\gamma_0)}$$

and $\gamma_0$ is chosen so that $(\gamma_0)\sqrt{\mathfrak{d}_K}$ has an exact denominator $\mathfrak{f}_\chi$. Multiplying $(s-1)$ on both side of (1.2.5) and tending $s$ to 1, we have the following

THEOREM 6. *Let $K$, $K_r$, $\mathfrak{d}$, $\mathfrak{d}_r$, $\mathfrak{f}$, $\chi$, $\chi_0$, $\mathfrak{f}_\chi$, $w_{\mathfrak{f}_\chi}$, $\iota$, $R_K$, $R_r$ be as above. Let $h_r$ be the ideal class number of $K_r$. Then*

$$\frac{h_r}{h_K}=\frac{R_Kw_r}{R_rw_K}\prod_{\chi\neq 1}\frac{\iota_\chi\sqrt{N_{\mathfrak{f}_\chi}}}{w_{\mathfrak{f}_\chi}T_0}\sum_{\mathscr{B}_0}\bar{\chi}_0(\mathfrak{C}_{\mathscr{B}_0})\log\Lambda(\tau_{\mathscr{B}_0}, \tau'_{\mathscr{B}_0}; (u_{\mathscr{B}_0}), (v_{\mathscr{B}_0}))$$

*where $\iota=\iota_\chi$ for all $\chi$ and $\mathscr{B}_0$ runs over all $\mathfrak{G}_{\mathfrak{f}_\chi}/\mathfrak{E}_{\mathfrak{f}_\chi}$.*

## 4. Limit formula for zeta-function of a real biquadratic field

**4.1.** We call a biquadratic field, whose Galois group is "Vierergpuppe", a Viererkorper or a $V$-field, (c.f., Kuroda [8])*. Let $K$ be a real $V$-field. Then $K$ contains exactly three real quadratic fields $\boldsymbol{Q}(\sqrt{m_1})$, $\boldsymbol{Q}(\sqrt{m_2})$ and $\boldsymbol{Q}(\sqrt{m_3})$ with rational integers $m_i$ such that $m_1m_2m_3$ is non-quadratic. Let $\varepsilon_1$, $\varepsilon_2$, $\varepsilon_3$ be three fundamental units of $\boldsymbol{Q}(\sqrt{m_1})$, $\boldsymbol{Q}(\sqrt{m_2})$ and $\boldsymbol{Q}(\sqrt{m_3})$ respectively. Then S. Kuroda gave the following table of possibilities of a system of fundamental units in $K$ and proved that every case really occurs.

1. $\varepsilon_1,\ \varepsilon_2,\ \varepsilon_3,$
2. $\sqrt{\varepsilon_1},\ \varepsilon_2,\ \varepsilon_3, \qquad N\varepsilon_1=1\,.$
3. $\sqrt{\varepsilon_1},\ \sqrt{\varepsilon_2},\ \varepsilon_3, \qquad N\varepsilon_1=N\varepsilon_2=1\,,$
4. $\sqrt{\varepsilon_1\varepsilon_2},\ \varepsilon_2,\ \varepsilon_3, \qquad N\varepsilon_1=N\varepsilon_2=1\,,$
5. $\sqrt{\varepsilon_1\varepsilon_2},\ \sqrt{\varepsilon_3},\ \varepsilon_2, \qquad N\varepsilon_1=N\varepsilon_2=N\varepsilon_3=1\,,$
6. $\sqrt{\varepsilon_1\varepsilon_2},\ \sqrt{\varepsilon_2\varepsilon_3},\ \sqrt{\varepsilon_3\varepsilon_1},\ N\varepsilon_1=N\varepsilon_2=N\varepsilon_3=1\,,$
7. $\sqrt{\varepsilon_1\varepsilon_2\varepsilon_3},\ \varepsilon_2,\ \varepsilon_3\,.$

**4.2.** Let $F=\boldsymbol{Q}(\sqrt{d})$ be a real quadratic field of discriminant $d$ with the fundamental unit $\varepsilon$ of norm $-1$. Let $\mathfrak{o}$ be the ring of integers in $F$. Let $\begin{pmatrix} a & b \\ c & d \end{pmatrix}=\xi$ be a hyperbolic element with $a, b, c, d\in\mathfrak{o}$, $b\equiv 0 \pmod{\mathfrak{a}}$ and $c\equiv 0 \pmod{\mathfrak{b}}$. Let $K$ be the totally real field of degree 4 obtained by adding fixed points of $\xi$ to $F$.

Let $\sigma_i(i=0,\cdots 3)$ be four distinct isomorphisms of $K$ into $\boldsymbol{R}$. We take as follows: $\sigma_0=1$, $\sigma_2$ fixes $F$ and $\sigma_1$ induces $'$ on $F$. We also write $a'=a^{\sigma_1}$ for $a\in K$.

For an element $a$ of $K$, we denote $a^{\sigma_i}=a^{(i)}$. Hence $a=a^{(0)}$. By the notation, fixed points of $\xi$ are denoted by $\omega^{(0)}$, $\omega^{(2)}$ and that of $\xi'$ by $\omega^{(1)}$, $\omega^{(3)}$. We put one more assumption:

(A.1) *Let* $\varepsilon_1$, $\varepsilon_2$ *be relative units of* $K/F$ (i.e., $N_{K/F}(\varepsilon_i)=\pm 1$, *for* $i=1, 2$). *Then* $\varepsilon$, $\varepsilon_1$, $\varepsilon_2$ *make a system of fundamental units in* $K$.

REMARK 1. For any given real $V$-field $K=\boldsymbol{Q}(\sqrt{d}, \sqrt{m})$, we can construct $K$ by adding hyperbolic fiixed points to $F=\boldsymbol{Q}(\sqrt{d})$.

REMARK 2. We consider $F$ with $\varepsilon$ of norm $-1$. Hence for $V$-field $K$, in Kuroda's table, cases 1, 2, 3, 4, 7 can occur. Cases 1, 2, 3, 4, 7 really occur, putting $\varepsilon=\varepsilon_3$ in the table, see Kuroda's examples [8] p.p. 397-398.

We may take $\varepsilon_1^{(0)}>1$, $\varepsilon_1^{(1)}<1$, $|\varepsilon_2^{(0)}|>1$ and $|\varepsilon_2^{(1)}|<1$. Put

---

* I thank Prof. M. Ishida, who has kindly informed me the existence of this paper.

$$u=|\varepsilon_1^{(0)}/\varepsilon_1^{(2)}|^{x_1}|\varepsilon_2^{(0)}/\varepsilon_2^{(2)}|^{x_2}\,, \tag{4.2.0}$$
$$u'=|\varepsilon_1^{(1)}/\varepsilon_1^{(3)}|^{x_1}|\varepsilon_2^{(1)}/\varepsilon_2^{(3)}|^{x_2}\,.$$

with two real parameters $x_1$, $x_2$. Put for an ideal $\mathfrak{a}$ in $F$, $\mathfrak{C}=\mathfrak{a}+\mathfrak{d}\omega$. $(\omega=\omega^{(0)})$, Then $\mathfrak{C}$ is an ideal in $K$. We consider $\Phi(\tau, \tau'; \mathfrak{a}, \mathfrak{d}, s_1, s_2)$ in 2.0, 2.3, and $\tau$, $\tau'$ under

$$\tau=\frac{u\sqrt{-1}\omega^{(0)}+\omega^{(2)}}{u\sqrt{-1}+1}\,,\quad \tau'=\frac{u'\sqrt{-1}\omega^{(1)}+\omega^{(3)}}{u'\sqrt{-1}+1}\,. \tag{4.2.1}$$

Then we have

$$y=(\omega^{(0)}-\omega^{(2)})\frac{u}{u^2+1}\,,\quad y'=(\omega^{(1)}-\omega^{(3)})\frac{u'}{u'^2+1} \tag{4.2.2}$$

and

$$\mu^{(0)}=\alpha+\beta\omega^{(0)},\ \mu^{(2)}=\alpha+\beta\omega^{(2)},\ \mu^{(1)}=\alpha'+\beta'\omega^{(1)},\ \mu^{(3)}=\alpha'+\beta'\omega^{(3)}\,.$$

Then we have

$$|\alpha+\beta\tau|^2=\frac{\mu^{(0)2}u^2+\mu^{(2)2}}{u^2+1}\,,\quad |\alpha'+\beta'\tau'|^2=\frac{\mu^{(1)2}u'^2+\mu^{(3)2}}{u'^2+1}\,. \tag{4.2.3}$$

Remark that $\mu$ is an element of $\mathfrak{C}$ but comes from only non-associated $(\alpha, \beta)$ in $F\times F$. We can rewrite

$$\Phi(\tau, \tau'; \mathfrak{a}, \mathfrak{d}; s_1, s_2)$$
$$=\Big(\sum_{0\ll\alpha\in\mathfrak{a}}{}^{\circ}+\sum_{\substack{\alpha\in\mathfrak{a}\\0\ll\beta\in\mathfrak{d}}}{}^{\circ}\Big)+\Big(\sum_{\substack{\alpha\in\mathfrak{a}\\\alpha>0\\\alpha'<0}}{}^{\circ}+\sum_{\substack{\alpha\in\mathfrak{a}\\\beta\in\mathfrak{d}\\\beta>0,\beta'<0}}{}^{\circ}\Big)+\Big(\sum_{\substack{\alpha\in\mathfrak{a}\\\alpha<0,\alpha'>0}}{}^{\circ}+\sum_{\substack{\alpha\in\mathfrak{a}\\\beta\in\mathfrak{d}\\\beta<0,\beta'>0}}{}^{\circ}\Big)+\Big(\sum_{\substack{\alpha\in\mathfrak{a}\\\alpha\ll0}}{}^{\circ}+\sum_{\substack{\alpha\in\mathfrak{a}\\\beta\in\mathfrak{d}\\\beta\ll0}}{}^{\circ}\Big)$$
$$=\Big(\sum_{0\ll\alpha\in\mathfrak{a}}{}^{\circ}+\sum_{\substack{\alpha\in\mathfrak{a}\\0\ll\beta\in\mathfrak{d}}}{}^{\circ}\Big)+\frac{1}{|\varepsilon|^{2s_1}|\varepsilon'|^{2s_2}}\Big(\sum_{0\ll\alpha\in\mathfrak{a}}{}^{\circ}+\sum_{\substack{\alpha\in\mathfrak{a}\\0\ll\beta\in\mathfrak{d}}}{}^{\circ}\Big)+\frac{1}{|\varepsilon|^{2s_1}|\varepsilon'|^{2s_2}}\Big(\sum_{0\ll\alpha\in\mathfrak{a}}{}^{\circ}+\sum_{\substack{\alpha\in\mathfrak{a}\\0\ll\beta\in\mathfrak{d}}}{}^{\circ}\Big)$$
$$+\Big(\sum_{0\ll\alpha\in\mathfrak{a}}{}^{\circ}+\sum_{\substack{\alpha\in\mathfrak{a}\\0\ll\beta\in\mathfrak{d}}}{}^{\circ}\Big)\,.$$

Inserting (4.2.3) in the above last formula, we see that there only appear non-$F$-associated $\mu$'s of $\mathfrak{C}$ in the sums; namely, in the sums, every element of the form $\varepsilon^n\mu$, $n\in\boldsymbol{Z}$, appears only once. Thus we have

$$\Phi(\tau, \tau'; \mathfrak{a}, \mathfrak{d}; s_1, s_2) \tag{4.2.4}$$
$$=2\Big(1+\frac{1}{|\varepsilon|^{2s_1}|\varepsilon'|^{2s_2}}\Big)(\omega^{(0)}-\omega^{(2)})^{s_1}(\omega^{(1)}-\omega^{(3)})^{s_2}$$
$$\times\sum_{0\neq(\mu)_F\subset\mathfrak{C}}\frac{u^{s_1}u'^{s_2}}{(\mu^{(0)2}u^2+\mu^{(2)2})^{s_1}(\mu^{(1)2}u'^2+\mu^{(3)2})^{s_2}}$$

where $\sum_{0\neq(\mu)F\subset\mathfrak{C}}$ means the sum over all non-zero non-$F$-associated $\mu$ in $\mathfrak{C}$.

**4.3.** $\Phi$ is invariant under $\tau\to\xi(\tau)$. This transformation gives the translation $x_1\to x_1+1$, $x_2\to x_2+1$. We see that $\Phi$, regarding as a function of $x_1$, $x_2$, has period 1 and Fourier series expansion. The Fourier coefficients are given by

$$a_{k_1k_2}=\int_{x_i=0}^{1}\int \Phi e^{-2\pi\sqrt{-1}\sum_{i=1}^{2}x_ik_i}\,dx_1dx_2\,.$$

We shall show, as in Siegel [5], that every $a_{k_1k_2}$ is nothing but the zeta-function with Grössencharacter of $K$ up to $\Gamma$-factors.

Since $\Phi$, as function of $\tau$, $\tau'$, is uniformly convergent on (4.2.1), we can change $\sum$ and $\iint$. Changing variables $x_1$, $x_2$ to $u$, $u'$, we have

$$a_{k_1k_2}=2\left(1+\frac{1}{|\varepsilon|^{2s_1}|\varepsilon'|^{2s_2}}\right)(\omega^{(0)}-\omega^{(2)})^{s_1}(\omega^{(1)}-\omega^{(3)})^{s_2}\frac{1}{D} \tag{4.3.1}$$
$$\times\sum_{0\neq(\mu)F\subset\mathfrak{C}}\iint_G\frac{u^{s_1}u'^{s_2}u^{-2\pi\sqrt{-1}\frac{A}{D}}u'^{-2\pi\sqrt{-1}\frac{B}{D}}}{(\mu^{(0)2}u^2+\mu^{(2)2})^{s_1}(\mu^{(1)2}u'^2+\mu^{(3)2})^{s_2}}\frac{du\,du'}{uu'}\,.$$

Here the notation is follows:

$D$ is the functional determinant $\dfrac{\partial(\log u,\log u')}{\partial(x_1x_2)}=\begin{vmatrix}\log\varepsilon_1{}^{(0)2} & \log\varepsilon_2{}^{(0)2}\\ \log\varepsilon_1{}^{(1)2} & \log\varepsilon_2{}^{(1)2}\end{vmatrix}$,

$$A=k_1\log\varepsilon_2{}^{(1)2}-k_2\log\varepsilon_1{}^{(0)2}\,,$$
$$B=-k_1\log\varepsilon_2{}^{(0)2}+k_2\log\varepsilon_1{}^{(0)2}\,.$$

and

$G$ is the image of the unit square, whose vertices are (0, 0), (1, 0), (1, 1), (0, 1), in $(x_1, x_2)$-plane by the map (4.2.0). Changing $\left|\dfrac{\mu^{(0)}}{\mu^{(2)}}\right|u$ to $u$ and $\left|\dfrac{\mu^{(1)}}{\mu^{(3)}}\right|u'$ to $u'$, we have

$$(4.3.1)=2\left(1+\frac{1}{|\varepsilon|^{2s_1}|\varepsilon'|^{2s_2}}\right)(\omega^{(0)}-\omega^{(2)})^{s_1}(\omega^{(1)}-\omega^{(3)})^{s_2}\frac{1}{D}$$
$$\times\sum_{0\neq(\mu)F\subset\mathfrak{C}}\left|\frac{\mu^{(0)}}{\mu^{(2)}}\right|^{2\pi\sqrt{-1}\frac{A}{D}}\left|\frac{\mu^{(1)}}{\mu^{(4)}}\right|^{2\pi\sqrt{-1}\frac{B}{D}}N_1(\mu^{(0)})^{-s_1}N_1(\mu^{(0)})'^{-s_2}$$
$$\times\iint_{G_{0,0}}\frac{u^{s_1-2\pi\sqrt{-1}\frac{A}{D}}u'^{s_2-2\pi\sqrt{-1}\frac{B}{D}}}{(u^2+1)^{s_1}(u'^2+1)^{s_2}}\frac{du\,du'}{uu'}\,,$$

where $G_{n,m}$ is the image of the square, whose vertices $(n, m)$, $(n+1, m)$, $(n+1, m+1)$, $(n, m+1)$, in $(x_1, x_2)$-plane by the map

$$\left|\frac{\mu^{(0)}}{\mu^{(2)}}\right|^{-1} u=\left|\frac{\varepsilon_1^{(0)}}{\varepsilon_1^{(2)}}\right|^{x_1}\left|\frac{\varepsilon_2^{(0)}}{\varepsilon_2^{(2)}}\right|^{x_2}$$

$$\left|\frac{\mu^{(1)}}{\mu^{(3)}}\right|^{-1} u'=\left|\frac{\varepsilon_1^{(1)}}{\varepsilon_1^{(3)}}\right|^{x_1}\left|\frac{\varepsilon_2^{(1)}}{\varepsilon_2^{(3)}}\right|^{x_2}$$

We have to make change the sum $\sum_{(\mu)_F\subset\mathfrak{C}}$ to the sum $\sum_{(\mu)\subset\mathfrak{C}}$. To do this, we remark that if $(\beta)_F=(\gamma)_F$, then $\gamma=\pm\beta\varepsilon_1^n\varepsilon_2^m$ for $n, m\in \boldsymbol{Z}$. Observing sign distributions of $\gamma$, we have

$$\sum_{(\mu)\subset\mathfrak{C}}(*)\iint_{G_{0,0}}=4\sum_{(\mu)\subset\mathfrak{C}}(*)\sum_{m,n=-\infty}^{\infty}\iint_{G_{n,m}}. \tag{4.3.2}$$

In the above, $\sum_{m,n=-\infty}^{\infty} G_{n,m}$ covers the whole first quaters of $(u, u')$-plane without gaps and overlaps. Therefore, we have

$$=4\sum_{(\mu)\subset\mathfrak{C}}(*)\int_0^\infty\int_0^\infty\frac{u^{s_1-2\pi\sqrt{-1}\frac{A}{D}}u'^{s_2-2\pi\sqrt{-1}\frac{B}{D}}}{(u^2+1)^{s_1}(u'^2+1)^{s_2}}\frac{du\,du'}{uu'} \tag{4.3.2}$$

Computing

$$2\int_0^\infty\frac{u^{s-2\pi\sqrt{-1}t}}{(u^2+1)^s}\frac{du}{u}=\frac{\Gamma\left(\frac{s}{2}-\pi\sqrt{-1}t\right)\Gamma\left(\frac{s}{2}+\pi\sqrt{-1}t\right)}{\Gamma(s)},$$

we obtain

$$\begin{aligned}a_{k_1k_2}=&2\left(1+\frac{1}{|\varepsilon|^{2s_1}|\varepsilon'|^{2s_2}}\right)(\omega^{(0)}-\omega^{(2)})^{s_1}(\omega^{(1)}-\omega^{(3)})^{s_2}\frac{1}{D}\\&\times\sum_{(\mu)\subset\mathfrak{C}}\left|\frac{\mu^{(0)}}{\mu^{(2)}}\right|^{2\pi\sqrt{-1}\frac{A}{D}}\left|\frac{\mu^{(1)}}{\mu^{(3)}}\right|^{2\pi\sqrt{-1}\frac{B}{D}}N_1(\mu^{(0)})^{-s_1}N_1(\mu^{(0)})'^{-s_2}\\&\times\frac{\Gamma\left(\frac{s_1}{2}-\frac{\pi\sqrt{-1}A}{D}\right)\Gamma\left(\frac{s_1}{2}+\frac{\pi\sqrt{-1}A}{D}\right)\Gamma\left(\frac{s_2}{2}-\frac{\pi\sqrt{-1}B}{D}\right)\Gamma\left(\frac{s_2}{2}+\frac{\pi\sqrt{-1}B}{D}\right)}{\Gamma(s_1)\Gamma(s_2)}.\end{aligned} \tag{4.3.3}$$

For $c=\pi\sqrt{-1}k_0/\log\varepsilon$, $k_0\in\boldsymbol{Z}$, we put

$$\hat{\chi}_{k_0k_1k_2}((\mu))=|\mu^{(0)}|^{2\pi\sqrt{-1}\frac{A}{D}-c}\,|\mu^{(1)}|^{+2\pi\sqrt{-1}\frac{B}{D}+c}\,|\mu^{(2)}|^{-2\pi\sqrt{-1}\frac{A}{D}-c}\,|\mu^{(3)}|^{-2\pi\sqrt{-1}\frac{B}{D}+c},$$

which is a Grössencharacter of $K$. Thus defining Grössencharacter $\hat{\chi}_{k_0}((\beta))=|\beta/\beta'|^c$ of $F$, and observing

$$N(\mathfrak{C})^{-s}\hat{\chi}_{k_0k_1k_2}(\mathfrak{C})\zeta_k(s,\hat{\chi}_{k_0k_1k_2},\mathscr{C})=\sum_{(\mu)\subset\mathfrak{C}}\hat{\chi}_{k_0k_1k_2}((\mu))N(\mu)^{-s}$$

for the zeta-function of $K$ associated with Grössencharacter $\hat{\chi}_{k_0k_1k_2}$ and the ideal class $\mathscr{C}$ of $\mathfrak{C}^{-1}$, we may summarize the above in

$$(4.3.4) \qquad a_{k_0k_1k_2}=(a_{k_1k_2} \text{ with } c=\pi\sqrt{-1}k_0/\log\varepsilon)$$

$$=2\left(1+\frac{1}{|\varepsilon|^{2s_1}|\varepsilon'|^{2s_2}}\right)N(\mathfrak{C})^{-s}((\omega^{(0)}-\omega^{(2)})(\omega^{(1)}-\omega^{(3)}))^s\left(\frac{\omega^{(0)}-\omega^{(2)}}{\omega^{(1)}-\omega^{(3)}}\right)^c\frac{1}{D}$$

$$\times\frac{\Gamma\left(\frac{s_1}{2}-\frac{\pi\sqrt{-1}A}{D}\right)\Gamma\left(\frac{s_2}{2}-\frac{\pi\sqrt{-1}B}{D}\right)\Gamma\left(\frac{s_1}{2}+\frac{\pi\sqrt{-1}A}{D}\right)\Gamma\left(\frac{s_2}{2}+\frac{\pi\sqrt{-1}B}{D}\right)}{\Gamma(s_1)\Gamma(s_2)}$$

$$\times\hat{\chi}_{k_0k_1k_2}(\mathfrak{C})\zeta_K(s,\hat{\chi}_{k_0k_1k_2},\mathscr{C}).$$

**4.4.** We computed, in §2, limit formula for $\Phi(\tau,\tau';\mathfrak{a},\mathfrak{d};s_1,s_2)$ with $c=0$ or $c=\pi\sqrt{-1}k/\log\varepsilon$. Therefore, by integration with respect to $u, u'$, we may obtain the limit formula for $\zeta_K(s,\hat{\chi}_{k_0k_1k_2},\mathscr{C})$. Observe that by our chice of $\varepsilon$, $\varepsilon_1$, $\varepsilon_2$, we have $D\log\varepsilon=4R_K$ and $N_{K/Q}(\mathfrak{C})/N\mathfrak{a}(\omega^{(0)}-\omega^{(2)})(\omega^{(1)}-\omega^{(3)})=1/\sqrt{N\mathfrak{d}}$ with relative discriminant $\mathfrak{d}$ of $K/F$.

THEOREM 7. *Let $K$ be a real biquadratic field over $F=\boldsymbol{Q}(\sqrt{d})$. Let $\mathfrak{d}$ be the relative discriminant of $K/F$ and $R_K$ the regulator of $K$. Under the assumption* (A.1), *the following limit formulas hold. In* (i), (ii), *$H(\tau,\tau';\mathfrak{a},\mathfrak{d})$ is the function defined in* 2.3.

(i) CASE: $k_0=k_1=k_2=0$

$$\lim_{s\to1}\left(\zeta_K(s,\mathscr{C})-\frac{a_{-1}}{s-1}\right)=a_{-1}\left[\log\frac{16N\mathfrak{a}}{\sqrt{N\mathfrak{d}}}+\frac{8\pi^2\log\varepsilon\cdot P}{d\cdot N\mathfrak{a}}\right.$$
$$\left.+\iint_{x_i=0}^{1}\log\left(\frac{H(\tau,\tau';\mathfrak{a},\mathfrak{d})}{yy'}\right)dx_1dx_2\right],$$

*where* $a_{-1}=16R_K/w\cdot d\cdot\sqrt{N\mathfrak{d}}$, *and*

$$P=2\log2+2C-2\int_0^1\log(\sqrt{y_z}\cdot\sqrt[4]{d}\,|\eta(z)|^2)dv.$$

(ii) CASE: $k_0=0$, $(k_1,k_2)\neq(0,0)$

$$\zeta_K(1,\hat{\chi}_{0k_1k_2},\mathscr{C})$$
$$=\frac{\hat{\chi}_{0k_1k_2}^{-1}(\mathfrak{C})\cdot8\cdot R_k\int_0^1\int_0^1\log\left(\frac{H(\tau,\tau';\mathfrak{a},\mathfrak{d})}{yy'}\right)e^{-2\pi\sqrt{-1}\sum_{i=1}^{2}k_ix_i}dx_1dx_2}{d\sqrt{N\mathfrak{d}}\cdot\gamma(A,B,D)}$$

where

$$\gamma(A,B,D)$$
$$=\Gamma\left(\frac{1}{2}-\frac{\pi\sqrt{-1}A}{D}\right)\Gamma\left(\frac{1}{2}+\frac{\pi\sqrt{-1}A}{D}\right)\Gamma\left(\frac{1}{2}-\frac{\pi\sqrt{-1}B}{D}\right)\Gamma\left(\frac{1}{2}+\frac{\pi\sqrt{-1}B}{D}\right),$$

$$A=k_1\log\varepsilon_2^{(1)2}-k_2\log\varepsilon_1^{(1)2}$$

$$B=-k_1\log\varepsilon_2^{(0)2}+k_2\log\varepsilon_1^{(0)2}$$

$$D=\begin{vmatrix} \log \varepsilon_1^{(0)2} & \log \varepsilon_2^{(0)2} \\ \log \varepsilon_1^{(1)2} & \log \varepsilon_2^{(1)2} \end{vmatrix}$$

*and* $\hat{\chi}_{0k_1k_2}$ *comes from the Grössencharacter defined by*

$$\hat{\chi}_{k_0k_1k_2}((\mu))=|\mu^{(0)}|^{2\pi\sqrt{-1}\frac{A}{D}-c}\,|\mu^{(1)}|^{2\pi\sqrt{-1}\frac{B}{D}+c}\,|\mu^{(2)}|^{-2\pi\sqrt{-1}\frac{A}{D}-c}\,|\mu^{(3)}|^{-2\pi\sqrt{-1}\frac{B}{D}+c}$$

*with* $c=\pi\sqrt{-1}k_0/\log\varepsilon$ .

(iii) CASE: $k_0\neq 0$

$$\zeta_K(1,\hat{\chi}_{k_0k_1k_2},\mathscr{C})=\frac{\hat{\chi}_{k_0k_1k_2}^{-1}(\mathfrak{C})\cdot D\hat{\chi}_{k_0}(N_{K/F}(\mathfrak{C})^{-1}\mathfrak{a}\cdot\mathfrak{d}^{-\frac{1}{2}})N\mathfrak{a}\cdot\gamma(c)}{4\sqrt{N\mathfrak{d}}\cdot\gamma(A,B,D)}$$

$$\times\Bigg[4\zeta_F(2,\hat{\chi}_{k_0}^{-2},A)\int_0^1\!\!\int_0^1 y^{1+c}y'^{1-c}\,e(dx_1dx_2)$$

$$-\frac{16\pi^2\log\varepsilon\int_0^1\log(\sqrt{y_z}\,|\eta(z)|^2)e^{-4\pi\sqrt{-1}k_0v}\,dv\int_0^1\!\!\int_0^1 y^{-c}y'^{c}\,e(dx_1dx_2)}{dN\mathfrak{a}\cdot\gamma(c)}$$

$$+\frac{16\pi^2}{\gamma(c)^2\sqrt{d}^{\,3}N\mathfrak{a}^3}\int_0^1\!\!\int_0^1 y^{1+c}y'^{1-c}F^*(k_0;\mathfrak{a},\mathfrak{o};\tau,\tau')\,e(dx_1dx_2)\Bigg],$$

*where*
$$e(dx_1dx_2)=e^{-2\pi\sqrt{-1}\sum_{i=1}^{2}k_ix_i}\,dx_1dx_2\,,$$

$$\hat{\chi}_{k_0}((\beta))=|\beta/\beta'|^c\quad\textit{with}\quad c=\pi\sqrt{-1}k_0/\log\varepsilon$$

*and*
$$\gamma(c)=\Gamma(1+c)\Gamma(1-c)\,.$$

## 5. Ray class fields of a real biquadratic field

**5.1.** Let $F$ be a real quadratic field of discriminant $d$ with fundamental unit $\varepsilon$ of norm $-1$. Let $K$ be a real biquadratic field containing $F$, $\mathscr{A}$ an ideal class in $K$. $\mathfrak{C}_{\mathscr{A}}$ being an ideal in $\mathscr{A}$, we write

$$\mathfrak{C}_{\mathscr{A}}=\mathfrak{a}_{\mathscr{A}}\Omega_1+\mathfrak{o}\Omega_2$$

with an ideal $\mathfrak{a}_{\mathscr{A}}$ in $F$. Let $\sigma_i$ be as in § 4. We have

$$(\Omega_1^{(0)}\Omega_2^{(2)}-\Omega_1^{(2)}\Omega_2^{(0)})(\Omega_1^{(1)}\Omega_2^{(3)}-\Omega_1^{(3)}\Omega_2^{(1)})=N(\mathfrak{C}_{\mathscr{A}})\sqrt{N\mathfrak{d}}/N(\mathfrak{a}_{\mathscr{A}})\,. \tag{5.1.1}$$

Let $\mathfrak{f}$ be an integral ideal in $K$. We take $\gamma\in K$ so that $(\gamma)\sqrt{\mathfrak{d}}$ has an exact denominator $\mathfrak{f}$, as in § 2.2. We start from

$$L_K(s,\chi)=T^{-1}\sum_{\lambda \bmod \mathfrak{f}}\bar{\chi}(\lambda)\sum_{\mathscr{A}}\bar{\chi}(\mathfrak{C}_{\mathscr{A}})N(\mathfrak{C}_{\mathscr{A}})^s\sum_{\mathfrak{C}_{\mathscr{A}}\,|\,(\tilde{\beta})\neq 0}v(\tilde{\beta})e^{2\pi\sqrt{-1}S((\lambda\tilde{\beta}\gamma))}|N\tilde{\beta}|^{-s} \tag{5.1.2}$$

with $v(\tilde{\beta})\equiv 1$ identically.

For $\tilde{\beta}\in\mathfrak{C}_{\mathscr{A}}$, we write $\tilde{\beta}=\alpha\Omega_1+\beta\Omega_2$ with $\alpha\in\mathfrak{a}_{\mathscr{A}}$, $\beta\in\mathfrak{o}$. Putting $u_{\mathscr{A}}=S_{K/F}(\lambda\Omega_1\gamma)$ and $v_{\mathscr{A}}=S_{K/F}(\lambda\Omega_2\gamma)$, we have

$$S(\lambda\tilde{\beta}\gamma)=S_{F/Q}(\alpha u_{\mathscr{A}}+\beta v_{\mathscr{A}})\,.$$

We consider $\Psi(\tau,\tau';\mathfrak{a}_{\mathscr{A}},\mathfrak{o};s;(u_{\mathscr{A}}),(v_{\mathscr{A}}))$, which is denoted by $\Psi_{\mathscr{A}}(\tau,\lambda)$ for brevity. Let $\Gamma$ be the group of all units of $K$, $\Gamma_{\mathfrak{f}}$ the group of $\varepsilon\in\Gamma$ with $\varepsilon\equiv 1 \bmod \mathfrak{f}$, and $\Gamma_{\mathfrak{f}}{}^*$ the group of totally positive units in $\Gamma_{\mathfrak{f}}$. We set the following assumption $(A-\mathfrak{f})$ instead of $(A-1)$:

$(A-\mathfrak{f})$ There are three units $\eta$, $\eta_1$, $\eta_2$ in $\Gamma_{\mathfrak{f}}{}^*$, such that they generate $\Gamma_{\mathfrak{f}}{}^*$ and $\eta_1$, $\eta_2$ are relative units of $K/F$.

Put $$\omega^{(0)}=\Omega_2^{(0)}/\Omega_1^{(0)} \quad\text{and}\quad \omega^{(1)}=\Omega_2^{(1)}/\Omega_1^{(1)}\,.$$

Consider a hyperbolic transformation whose fixed points are $\omega^{(0)}$, $\omega^{(2)}$, $\omega^{(1)}$, $\omega^{(3)}$. As in §4, we introduce variables

$$p=\left(\frac{\eta_1^{(0)}}{\eta_1^{(2)}}\right)^{x_1}\left(\frac{\eta_2^{(0)}}{\eta_2^{(2)}}\right)^{x_2}$$

$$p'=\left(\frac{\eta_1^{(1)}}{\eta_1^{(3)}}\right)^{x_1}\left(\frac{\eta_2^{(1)}}{\eta_2^{(3)}}\right)^{x_2}$$

with real parameters $x_1$, $x_2$ and consider

$$\tau=\frac{p\sqrt{-1}\,\omega^{(0)}+\omega^{(2)}}{p\sqrt{-1}+1}\,,\quad \tau'=\frac{p'\sqrt{-1}\,\omega^{(1)}+\omega^{(3)}}{p'\sqrt{-1}+1}\,. \tag{5.1.3}$$

Writing $\tilde{\mu}^{(0)}=\alpha+\beta\omega^{(0)}$, $\tilde{\mu}^{(1)}=\alpha'\beta'\omega^{(1)}$, we have, as in §4,

$$\begin{aligned}\Xi(\mathscr{A},\lambda,s)&=(N(\mathfrak{a}_{\mathscr{A}}))^s\int_0^1\!\!\int_0^1\Psi_{\mathscr{A}}(\tau,\lambda)dx_1dx_2 \\ &=\frac{N(\mathfrak{a}_{\mathscr{A}})^s}{D}\sum_{\tilde{\mu}}e^{2\pi\sqrt{-1}S(\lambda\tilde{\beta}\gamma)}(\omega^{(0)}-\omega^{(2)})^s(\omega^{(1)}-\omega^{(3)})^s \\ &\quad\iint_G\frac{p^sp'^s}{(p^2\tilde{\mu}^{(0)2}+\tilde{\mu}^{(2)2})^s(p'^2\tilde{\mu}^{(1)2}+\tilde{\mu}^{(3)2})^s}\frac{dpdp'}{pp'}\end{aligned} \tag{5.1.4}$$

where $D$ is the functional determinant $(\partial\log p\,\partial\log p')/(\partial x_1\,\partial x_2)$ and $G$ has the same meaning for $\eta_1$, $\eta_2$ as in §4. By (5.1.2) and changing variables

$$p\to\left(\frac{\tilde{\mu}^{(2)}}{\tilde{\mu}^{(0)}}\right)p\,,\qquad p'\to\left(\frac{\tilde{\mu}^{(3)}}{\tilde{\mu}^{(1)}}\right)p'\,,$$

we have

$$(5.1.4)=\frac{N(\mathfrak{C}_{\mathscr{A}})^s\sqrt{N\mathfrak{d}}^s}{D}\sum_{\tilde{\beta}}\frac{e^{2\pi\sqrt{-1}S(\lambda\tilde{\beta}\gamma)}}{|N\tilde{\beta}|^s}\iint_{G_{0,0}}\frac{p^sp'^s}{(p^2+1)^s(p'^2+1)^s}\frac{dpdp'}{pp'} \tag{5.1.5}$$

where $G_{n,m}$ is the domain defined for $\eta_1$, $\eta_2$ instead for $\varepsilon_1$, $\varepsilon_2$ in §4.

Since $\eta_i\equiv 1 \bmod \mathfrak{f}$ and $N\eta_i=1$,

$$e^{2\pi\sqrt{-1}S(\lambda\tilde{\beta}\eta_1^n\eta_2^m\gamma)}=e^{2\pi\sqrt{-1}S(\lambda\tilde{\beta}\gamma)}$$

and $\qquad |N\tilde{\beta}\eta^k\eta_1^n\eta_2^m|=|N\tilde{\beta}|$ hold.

Changing $\tilde{\beta}\to\pm\tilde{\beta}_1^n\eta_2^m$ and noting the sign distributions, we have by $(A-\mathfrak{f})$,

$$(5.1.6)\qquad (5.1.5)=\frac{4N(\mathfrak{C}_{\mathscr{A}})^s\sqrt{N\mathfrak{d}^s}}{D}\sum_{(\tilde{\beta})\Gamma_{\mathfrak{f}}^*}\frac{e^{2\pi\sqrt{-1}S(\lambda\tilde{\beta}\gamma)}}{|N(\tilde{\beta})|^s}\sum_{m,n}\iint_{G_{m,n}}\frac{p^sp'^s\dfrac{dpdp'}{pp'}}{(p^2+1)^s(p'^2+1)^s}$$

where $\sum_{(\tilde{\beta})\Gamma_{\mathfrak{f}}^*}$ denotes the sum over all $\tilde{\beta}$, not being $\Gamma_{\mathfrak{f}}^*$-associated. Since $\sum_{n,m}G_{n,m}$ covers the whole "first quarter" of $(p, p')$-plane without gaps and overlappings, we have $\sum_{n,m}\iint_{G_{m,n}}=\int_0^\infty\int_0^\infty$. Thus we have

$$(5.1.7)\qquad (5.1.6)=\frac{N(\mathfrak{C}_{\mathscr{A}})^s\sqrt{N\mathfrak{d}^s}}{D}\frac{\Gamma\left(\dfrac{s}{2}\right)^4}{\Gamma(s)^2}\sum_{(\tilde{\beta})\Gamma_{\mathfrak{f}}^*}e^{2\pi\sqrt{-1}S(\lambda\tilde{\beta}\gamma)}|N(\tilde{\beta})|^{-s}.$$

Let $\{\rho_i\}$ be a complete set of units incongruent mod $\mathfrak{f}$, $\{\varepsilon_j\}$ a complete set of representatives of $\Gamma_{\mathfrak{f}}/\Gamma_{\mathfrak{f}}^*$. We see that $\{\rho_i\varepsilon_j\}$ is a full set of representatives of $\Gamma/\Gamma_{\mathfrak{f}}^*$. Hence,

$$(5.1.7)=\frac{N(\mathfrak{C}_{\mathscr{A}})^s\sqrt{N\mathfrak{d}^s}}{D}\frac{\Gamma\left(\dfrac{1}{2}s\right)^4}{\Gamma(s)^2}\sum_{\varepsilon_j,\rho_i}\sum_{(\tilde{\beta})\subset\mathfrak{C}_{\mathscr{A}}}\frac{e^{2\pi\sqrt{-1}S(\lambda\rho_i\varepsilon_j\tilde{\beta}\gamma)}}{|N(\tilde{\beta})|^s}.$$

Now there exists a set $\{\mu_l\}$ of integers in $K$ such that $\mu_l\equiv1$ mod and $\{\varepsilon_j\mu_l\}$ runs over all $2^4$ possible signatures. Let $E(\mathfrak{f})$ be the set of elements of $G(\mathfrak{f})$ containing at least one unit in $K$, $\{\lambda_k\}$ be a complete set of $G(\mathfrak{f})/E(\mathfrak{f})$. Then $\{\lambda_k\rho_i\}$ runs a complete set of representatives of prime residue classes mod $\mathfrak{f}$ and $\{(\lambda_k\mu_l)\mathfrak{C}_{\mathscr{A}}\}$ covers a full set of representatives of $\mathfrak{G}_{\mathfrak{f}}/\mathfrak{E}_{\mathfrak{f}}$, (Siegel's proof of Proposition 15 in [5] works well in the present case). Hence we have

$$(5.1.8)\qquad \sum_{\lambda_k,\mu_l,\mathscr{A}}\bar{\chi}(\lambda_k\mu_l)\bar{\chi}(\mathfrak{C}_{\mathscr{A}})\Xi_{\mathscr{A}}(\tau,\lambda_k\mu_l)$$

$$=\frac{\sqrt{N\mathfrak{d}^s}\,\Gamma\left(\dfrac{1}{2}s\right)^4}{D\Gamma(s)^2}\sum_{\lambda_k,\mu_l,\mathscr{A}}N(\mathfrak{C}_{\mathscr{A}})^s\bar{\chi}(\lambda_k\mu_l)\bar{\chi}(\mathfrak{C}_{\mathscr{A}})$$

$$\times\sum_{\rho_i,\varepsilon_j,(\tilde{\beta})\subset\mathfrak{C}_{\mathscr{A}}}\frac{e^{2\pi\sqrt{-1}S(\lambda_k\mu_l\rho_i\varepsilon_j\tilde{\beta}\gamma)}}{|N(\tilde{\beta})|^s}$$

$$=\frac{\sqrt{N\mathfrak{d}^s}\Gamma\left(\frac{1}{2}s\right)^4}{D\Gamma(s)^2}\sum_{\lambda_k,\rho_i,\mathscr{A}}\bar{\chi}(\lambda_k)\bar{\chi}(\mathfrak{C}_{\mathscr{A}})N(\mathfrak{C}_{\mathscr{A}})^s\sum_{\varepsilon_j,\mu_l}\bar{\chi}(\mu_l)\sum_{(\tilde{\beta})\subset\mathfrak{C}_{\mathscr{A}}}\frac{e^{2\pi\sqrt{-1}S(\lambda_k\mu_l\rho_i\varepsilon_j\tilde{\beta}\gamma)}}{|N(\tilde{\beta})|^s}$$

$$=\frac{2^4\sqrt{N\mathfrak{d}^s}\Gamma\left(\frac{1}{2}s\right)^4}{D\Gamma(s)^2}\sum_{\lambda_k,\rho_i,\mathscr{A}}\bar{\chi}(\lambda_k\rho_i)\bar{\chi}(\mathfrak{C}_{\mathscr{A}})N(\mathfrak{C}_{\mathscr{A}})^s\sum_{(\tilde{\beta})\subset\mathfrak{C}_{\mathscr{A}}}\frac{e^{2\pi\sqrt{-1}S(\lambda_k\rho_i\tilde{\beta}\gamma)}}{|N(\tilde{\beta})|^s}$$

observing that

$$\bar{\chi}(\lambda_k)=\bar{\chi}(\lambda_k\rho_i)\ ,$$

$$e^{2\pi\sqrt{-1}S(\lambda_k\mu_l\rho_j\varepsilon_j\tilde{\beta}\gamma)}=e^{2\pi\sqrt{-1}S(\lambda_k\rho_i\tilde{\beta}\gamma)}\quad(\text{since } \varepsilon_j\mu_l\equiv 1 \bmod \mathfrak{f})$$

and $$\chi(\mu_l)=1\quad(\text{since } \mu_l\equiv 1 \bmod \mathfrak{f})\ .$$

Write $(\lambda_k\mu_l)\mathfrak{C}_{\mathscr{A}}=\mathfrak{C}_{\mathscr{B}}$ for some $\mathscr{B}\in\mathfrak{G}_{\mathfrak{f}}/\mathfrak{E}_{\mathfrak{f}}$. Then $\mathfrak{C}_{\mathscr{B}}$ runs over all $\mathfrak{G}_{\mathfrak{f}}/\mathfrak{E}_{\mathfrak{f}}$. Putting $\Omega_1^*=\lambda_k\mu_l\Omega_1$ and $\Omega_2^*=\lambda_k\mu_l\Omega_2$, we can write

$$\mathfrak{C}_{\mathscr{B}}=\mathfrak{a}_{\mathscr{B}}\Omega_1^*+\mathfrak{b}\Omega_2^*\quad\text{with}\quad \mathfrak{a}_{\mathscr{B}}=\mathfrak{a}_{\mathscr{A}}\ .$$

We set for brevity

$$\Psi_{\mathscr{B}}(\tau,\lambda)=y^sy'^s\sum\frac{e^{2\pi\sqrt{-1}S((\alpha\Omega_1^*+\beta\Omega_2^*)\gamma)}}{|\alpha+\beta\tau|^{2s}|\alpha'+\beta'\tau'|^{2s}}$$

and $$\Xi(\mathscr{B},s)=N\mathfrak{a}_{\mathscr{B}}{}^s\iint_{(\tau_0,\tau_0')}^{(\tau_0^*,\tau_0'^*)}\Psi_{\mathscr{B}}(\tau,\lambda)\frac{dpdp'}{pp'}\ ,$$

which depends only on the ray class $\mathscr{B}$ mod $\mathfrak{f}$ and $\gamma$.

Since $\{\lambda\}=\{\lambda_k\rho_i\}$ runs a complete set of prime residue classes mod $\mathfrak{f}$, we have

$$L_K(s,\chi)=\frac{D\Gamma(s)^2}{2^4T\Gamma\left(\frac{1}{2}s\right)^4\sqrt{N\mathfrak{d}^s}}\sum_{\mathscr{B}}\bar{\chi}(\mathfrak{C}_{\mathscr{B}})\Xi(\mathscr{B},s)\ .$$

Now we have

$$\frac{dpdp'}{pp'}=\frac{d\sqrt{N\mathfrak{d}}d\tau d\tau'}{A_{\mathscr{B}}(\tau,\tau')}\ ,$$

where

$$A_{\mathscr{B}}(\tau,\tau')=\frac{N\mathfrak{a}_{\mathscr{B}}}{N\mathfrak{C}_{\mathscr{B}}}(\Omega_2^{(0)}-\Omega_1^{(0)}\tau)(\Omega_2^{(2)}-\Omega_1^{(2)}\tau)(\Omega_2^{(1)}-\Omega_1^{(1)}\tau')(\Omega_2^{(3)}-\Omega_1^{(3)}\tau')\ .$$

Tending $s$ to 1, we have the following

THEOREM 8. *Let $K$ be a real bi-quadratic field satisfying $(A-\mathfrak{f})$. Let $\chi$ be* *a non-principal proper ray class character* mod $\mathfrak{f}$ *in* $K$ *with associated character of signature* $v(\lambda)\equiv 1$. *Then*

$$L_K(1, \chi)=\frac{D}{2^4T}\sum \bar{\chi}(\mathfrak{C}_{\mathscr{B}})\iint \log \Lambda(\tau, \tau'; (u_{\mathscr{B}}), (v_{\mathscr{B}}))\frac{d\tau d\tau'}{\Lambda_{\mathscr{B}}(\tau, \tau')}$$

*where* $D$ *is defined in* Th. 7 *for* $\eta_1$, $\eta_2$ *instead for* $\varepsilon_1$, $\varepsilon_2$, $\mathscr{B}$ *runs over all representatives of* $\mathfrak{G}_{\mathfrak{f}}/\mathfrak{C}_{\mathfrak{f}}$, $T$ *is defined in* (1.2.3) $\Lambda$ *is defined in* (2.4.9) *and* $\Lambda_{\mathscr{B}}$ *is defined just above.*

**5.2.** We shall consider the case where $v(\lambda)$, associated with given ray class character $\chi$, is of type $v(\lambda)=N\lambda/|N\lambda|$ for $\lambda\neq 0$. In this case, every unit congruent to 1 mod $\mathfrak{f}$ should be of norm 1.

Let $K$, $F$, $\mathscr{A}$, $\mathfrak{C}_{\mathscr{A}}$, $\Omega_i$, $\omega^{(i)}$, $p$, $p'$ be as in 5.1. Also we start, under $(A-\mathfrak{f})$, from

$$L_K(s, \chi)=T^{-1}\sum_{\lambda \bmod \mathfrak{f}}\bar{\chi}(\lambda)\sum_{\mathscr{A}}\bar{\chi}(\mathfrak{C}_{\mathscr{A}})N(\mathfrak{C}_{\mathscr{A}})^s\sum_{\mathfrak{C}_{\mathscr{A}}|(\tilde{\beta})}\frac{N(\tilde{\beta})e^{2\pi\sqrt{-1}S(\lambda\tilde{\beta}\gamma)}}{|N(\tilde{\beta})|\,|N(\tilde{\beta})|^s}$$

and

$$\Psi(\tau, \tau'; \mathfrak{a}_{\mathscr{A}}, \mathfrak{d}; s; (u_{\mathscr{A}}), (v_{\mathscr{A}}))=\sum{}'\frac{y^sy'^se^{2\pi\sqrt{-1}S(\lambda\tilde{\beta}\gamma)}}{|\alpha+\beta\tau|^{2s}|\alpha'+\beta'\tau'|^{2s}}$$

which, for brevity, is denoted by $\Psi_{\mathscr{A}}(\tau, \lambda)$.

We have

$$\frac{\partial}{\partial\tau'}\frac{\partial}{\partial\tau}\Psi_{\mathscr{A}}(\tau, \lambda)=(s/2\sqrt{-1})^2(yy')^{s-1} \tag{5.1.2}$$

$$\sum\frac{e^{2\pi\sqrt{-1}S(\lambda\tilde{\beta}\gamma)}}{|\alpha+\beta\tau|^{2s-2}(\alpha+\beta\tau)^2|\alpha'+\beta'\tau'|^{2s-2}(\alpha'+\beta'\tau')^2}$$

Now consider the integral

$$\iint_{(\tau_0,\tau_0')}^{(\tau_0^*,\tau_0'^*)}\frac{\partial^2\Psi_{\mathscr{A}}(\tau, \lambda)}{\partial\tau'\partial\tau}d\tau d\tau'$$

extended over $\tau$, $\tau'$ in (5.1.3). By the uniformity of convergence of (5.2.1) on (5.1.3), we have

$$\iint\frac{\partial^2\Psi_{\mathscr{A}}(\tau, \lambda)}{\partial\tau'\partial\tau}d\tau d\tau'=-(s^2/4)\sum\iint\frac{(yy')^{s-1}e^{2\pi\sqrt{-1}S(\lambda\tilde{\beta}\gamma)}d\tau d\tau'}{|\alpha+\beta\tau|^{2s-2}|\alpha'+\beta'\tau'|^{2s-2}(\alpha+\beta\tau)^2(\alpha'+\beta'\tau')^2}$$

$$=(s^2/4)\left(\frac{N\mathfrak{C}_{\mathscr{A}}}{N\mathfrak{a}_{\mathscr{A}}}\right)\frac{\sqrt{N\mathfrak{d}^s}}{D}\sum|N\Omega_1|^{-s}\cdot e^{2\pi\sqrt{-1}S(\lambda\tilde{\beta}\gamma)}$$

$$\iint_{(p_0,p_0')}^{(p_0^*,p_0'^*)}\frac{(pp')^{s-1}\,dpdp'}{(p^2\tilde{\mu}^{(0)2}+\tilde{\mu}^{(2)2})^{s-1}(p'^2\tilde{\mu}^{(1)2}+\mu^{(3)2})^{s-1}(p\mu^{(0)}i+\mu^{(2)})^2(p'\mu^{(1)}i+\mu^{(3)})^2}$$

Changing variables as in (5.1.5), we see

$$\iint (*) dp dp' = N(\tilde{\mu})^{-s} \iint_{G_{0,0}} \frac{(pp')^{s-1} dp dp'}{(p^2+1)^{s-1}(p'^2+1)^{s-1}(p\sqrt{-1}\xi+1)^2(p'\sqrt{-1}\xi'+1)^2}$$

where $\quad \xi = (\tilde{\mu}^{(0)}/\tilde{\mu}^{(2)})/|\tilde{\mu}^{(0)}/\tilde{\mu}^{(2)}| = N_{K/F}(\tilde{\mu})/|N_{K/F}(\tilde{\mu})| = \dfrac{N_{K/F}\tilde{\beta}}{|N_{K/F}\tilde{\beta}|} \dfrac{N_{K/F}\Omega_1}{|N_{K/F}\Omega_1|}$

and so

$$\xi\xi' = v(\tilde{\beta})v(\Omega_1) . \tag{5.2.3}$$

Further $\quad (p\sqrt{-1}\,\xi+1)^2 = (p\sqrt{-1}+\xi)^2 .$

Since $\quad v\left(\dfrac{\eta_1^{(0)}}{\eta_1^{(2)}}\right) = v\left(\dfrac{\eta_1^{(1)}}{\eta_1^{(3)}}\right) = 1 ,$

the values of $\xi$, $\xi'$ are the same for $\tilde{\beta}$, $\tilde{\beta}\eta_1^m\eta_2^n\,(m, n \in \boldsymbol{Z})$ and $e^{2\pi\sqrt{-1}S(\lambda\tilde{\beta}\gamma)} = e^{2\pi\sqrt{-1}S(\lambda\tilde{\beta}\eta_1^m\eta_2^n\gamma)}$ holds. Then we have, as in §4, §5.1,

$$\iint \frac{\partial^2 \Psi_{\mathscr{A}}(\tau, \lambda)}{\partial\tau\partial\tau'} d\tau d\tau' = \frac{s^2}{4D}\left(\frac{N\mathfrak{C}_{\mathscr{A}}\sqrt{N\mathfrak{d}}}{N\mathfrak{a}_{\mathscr{A}}}\right)^s \tag{5.2.4}$$

$$\sum_{(\tilde{\beta})} |N(\tilde{\beta})|^{-s} e^{2\pi\sqrt{-1}S(\tilde{\beta}\gamma)} \iint \frac{p^{s-1}p'^{s-1}dp dp'}{(p^2+1)^{s-1}(pi+\xi)^2(p'^2+1)^{2'}(p'i+\xi')^2}$$

$$\sum (*) \iint = \sum_{(\tilde{\beta})} \sum_{m,n} (*) \iint_{G_{m,n}} = \sum_{(\tilde{\beta})_{\Gamma_{\mathfrak{f}}^*}} \int_0^\infty \int_0^\infty . \tag{5.2.4'}$$

Computing above integral, we obtain, for $Re(s)>1$,

$$(N\mathfrak{a}_{\mathscr{A}})^s v(\Omega_1) \iint \frac{\partial^2 \Psi_{\mathscr{A}}(\tau, \lambda)}{\partial\tau\partial\tau'} d\tau d\tau'$$

$$= -\frac{\Gamma\left((s+1)\dfrac{1}{2}\right)^4 (N\mathfrak{C}_{\mathscr{A}}\sqrt{N\mathfrak{d}})^s}{4D\Gamma(s)^2} \sum_{(\tilde{\beta})\subset\Gamma_{\mathfrak{f}}^*} |N(\tilde{\beta})|^{-s} v(\tilde{\beta})\, e^{2\pi\sqrt{-1}S(\lambda\tilde{\beta}\gamma)} .$$

Let $\rho_i$, $\varepsilon_j$ be as in 5.1. We have, as in 5.1, the right hand side of the above formula

$$= -\Gamma\left(\frac{1}{2}(s+1)\right)^4 \frac{(N\mathfrak{C}_{\mathscr{A}}\sqrt{N\mathfrak{d}})^s}{4\Gamma(s)^2 D} \sum_{\rho_i,\varepsilon_j,(\tilde{\beta})} v(\tilde{\beta}\rho_i\varepsilon_j) e^{2\pi\sqrt{-1}S(\lambda\rho_i\varepsilon_j\tilde{\beta}\gamma)} |N(\tilde{\beta})|^{-s} \tag{5.2.5}$$

Let $\{\lambda_k\}$ $\{\mu_l\}$ be as in 5.1. Summing over all ideal classes $\mathscr{A}$ and elements of $\{\lambda_k\}$, $\{\mu_l\}$, we obtain, by (5.2.5),

$$\sum_{\lambda_k,\mu_l,\mathscr{A}} \bar{\chi}(\lambda_k\mu_l)\bar{\chi}(\mathfrak{C}_{\mathscr{A}})v(\Omega_1) \iint \frac{\partial^2 \Psi_{\mathscr{A}}(\tau, \lambda)}{\partial\tau\partial\tau} d\tau d\tau' \tag{5.2.6}$$

$$=-\frac{\Gamma\left(\frac{1}{2}(s+1)\right)^4(\sqrt{N\mathfrak{d}})^s}{4\Gamma(s)^2 D}\sum_{\lambda_k,\mu_l,\mathscr{A}}\bar{\chi}(\lambda_k\mu_l)\bar{\chi}(\mathfrak{C}_{\mathscr{A}})N(\mathfrak{C}_{\mathscr{A}})^s$$
$$\sum_{\rho_i,\varepsilon_j,(\tilde{\beta})} v(\tilde{\beta}\rho_i\varepsilon_j)e^{2\pi\sqrt{-1}S(\lambda_k\mu_l\tilde{\beta}\rho_i\varepsilon_j\gamma)}|N(\beta)|^{-s}.$$

Now we have

$$v(\rho_i\varepsilon_j)=\chi((\rho_i\varepsilon_j))\bar{\chi}(\rho_i\varepsilon_j)=\bar{\chi}(\rho_i\varepsilon_j)=\bar{\chi}(\rho_i),$$
$$e^{2\pi\sqrt{-1}S(\lambda_k\mu_l\tilde{\beta}\rho_i\varepsilon_j\gamma)}=e^{2\pi\sqrt{-1}S(\lambda_k\rho_i\tilde{\beta}\gamma)}.$$

Therefore, as in 5.1, we obtain

The right hand side of (5.2.6)

$$=\frac{16\Gamma\left(\frac{1}{2}(s+1)\right)^4\sqrt{N\mathfrak{d}}^s}{4D\Gamma(s)^2}\sum_{\lambda \bmod \mathfrak{f}}\bar{\chi}(\lambda)\sum_{\mathscr{A}}\bar{\chi}(\mathfrak{C}_{\mathscr{A}})N(\mathfrak{C}_{\mathscr{A}})^s\sum_{\mathfrak{C}_{\mathscr{A}}|(\tilde{\beta})}v(\tilde{\beta})e^{2\pi\sqrt{-1}S(\lambda\tilde{\beta}\gamma)}|N(\beta)|^{-s}$$
$$=-\frac{4\Gamma\left(\frac{1}{2}(s+1)\right)^4}{D\Gamma(s)^2}\sqrt{N\mathfrak{d}}^sTL_K(s,\chi).$$

Observing that $\bar{\chi}(\lambda_k\mu_l)=\bar{\chi}((\lambda_k\mu_l))v(\lambda_k\mu_l)$ holds and $\mathfrak{C}_{\mathscr{B}}=(\lambda_k\mu_l)\mathfrak{C}_{\mathscr{A}}$ runs over a full system of representatives of $\mathfrak{G}_\mathfrak{f}/\mathfrak{E}_\mathfrak{f}$, we can write the left hand side of (5.2.6) as follows: put $\Omega_1^*=\lambda_k\mu_l\Omega_1$ and $\Omega_2^*=\lambda_k\mu_l\Omega_2$. Then $\mathfrak{C}_{\mathscr{B}}=\mathfrak{a}_{\mathscr{B}}\Omega_1^*+\mathfrak{b}\Omega_2^*$ with $\mathfrak{a}_{\mathscr{B}}=\mathfrak{a}_{\mathscr{A}}$. Further put $u_{\mathscr{B}}=S_{K/F}(\Omega_1^*\gamma)$, $v_{\mathscr{B}}=S_{K/F}(\Omega_2^*\gamma)$ and denote

$$(yy')^s\sum\frac{e^{2\pi\sqrt{-1}S(\alpha u_{\mathscr{B}}+\beta v_{\mathscr{B}})}}{|\alpha+\beta\tau|^{2s}|\alpha'+\beta'\tau'|^{2s}}$$

by $\Psi_{\mathscr{B}}(\tau,s)$, which may depend on $\mathscr{B}$, the choice of $\mathfrak{C}_{\mathscr{B}}$ and $\Omega_1^*$, $\Omega_2^*$. But for the hyperbolic transformation with fixed points $\omega^{(0)}$, $\omega^{(2)}$, $\omega^{(1)}$, $\omega^{(3)}$, $\Psi_{\mathscr{B}}(\tau,s)$ is invariant. Thus

The left hand side of (5.2.6)

$$=\sum\bar{\chi}(\mathfrak{C}_{\mathscr{B}})v(\Omega_1)\iint\frac{\Psi_{\mathscr{B}}(\tau,s)}{\partial\tau'\partial\tau}d\tau d\tau'.$$

We have

$$L_K(s,\chi)=-\frac{D\Gamma(s)^2}{4\sqrt{N\mathfrak{d}}^s\Gamma\left(\frac{1}{2}(s+1)\right)^4T}\sum\bar{\chi}(\mathfrak{C}_{\mathscr{B}})v(\Omega_1)\iint\frac{\partial^2\Psi_{\mathscr{B}}(\tau,s)}{\partial\tau'\partial\tau}\partial\tau d\tau'.$$

Put
$$\Xi(\mathscr{B},s)=-\frac{D}{4\pi^4}v(\Omega_1)\iint\frac{\partial^2\Psi_{\mathscr{B}}(\tau,s)}{\partial\tau'\partial\tau}d\tau d\tau'.$$

Then by (5.2.5), we see that $\Xi(\mathscr{B},s)$ depends only on $\mathscr{B}$ (of course on $\gamma$).

Writing $\bar{\chi}(\mathscr{B})=\bar{\chi}(\mathfrak{C}_{\mathscr{B}})$, we obtain

$$L_K(s,\chi)=\frac{\pi^4\Gamma(s)^2}{\sqrt{N\mathfrak{d}^s}\,T\Gamma\left(\frac{1}{2}(s+1)\right)^4}\sum\bar{\chi}(\mathscr{B})\Xi(\mathscr{B},s). \tag{5.2.7}$$

$\dfrac{\partial^2\Psi_{\mathscr{B}}(\tau,s)}{\partial\tau'\partial\tau}$ is essentially an Epstein zeta-function, so $\displaystyle\iint\frac{\partial^2\Psi_{\mathscr{B}}(\tau,s)}{\partial\tau'\sigma\tau}\,d\tau d\tau'$ is an entire function of $s$. Then (5.2.7) gives the analytic continuation of $L_K(s,\chi)$ into the whole $s$-plane.

We shall consider the value of $L_K(1,\chi)$.

(i) $\mathfrak{f}\neq(1)$. In this case, $(u_{\mathscr{B}})$, $(v_{\mathscr{B}})$ satisfy (2.4.2) for $\mathfrak{C}_{\mathscr{B}}=\mathfrak{a}_{\mathscr{B}}\Omega_1^*+\mathfrak{d}\Omega_2^*$. By Th. 4, we have

$$\Psi_{\mathscr{B}}(\tau,s)=\frac{\pi^2}{\Delta\sqrt{d}}\log\Lambda(\tau,\tau';(u_{\mathscr{B}}),(v_{\mathscr{B}}))+*(s-1)+\cdots.$$

Applying $\dfrac{\partial^2}{\partial\tau'\partial\tau}$ to the above, we have

$$\frac{\partial^2}{\partial\tau'\partial\tau}\Psi_{\mathscr{B}}(\tau,s)=\frac{\pi^2}{\Delta}\frac{\partial^2}{\partial\tau'\partial\tau}[\Theta_1((\tau);(u_{\mathscr{B}}),(v_{\mathscr{B}}))]$$

where $\Theta_1$ is defined in (2.4.7) and for brevity, we write

$$\Theta_1((\tau);(u_{\mathscr{B}}),(v_{\mathscr{B}}))=\Theta_1(\tau,\tau';\mathfrak{a}_{\mathscr{B}},\mathfrak{d};(u_{\mathscr{B}}),(v_{\mathscr{B}})).$$

Since $\Theta_1$ is absolutely uniformly convergent, we have

$$\begin{aligned}\frac{\partial^2}{\partial\tau'\partial\tau}[(*)]=-\frac{\partial\tau}{\partial\tau}\frac{\partial\tau'}{\partial\tau'}\sum\frac{e^{2\pi\sqrt{-1}S(\alpha u_{\mathscr{B}})}}{N(\alpha)^2}\frac{\Delta}{\pi^2}&\\ +\sum_{\beta}{}^{\circ}\frac{1}{N(\beta)}\frac{\partial e^{-2\pi\sqrt{-1}\beta(v_{\mathscr{B}}-u_{\mathscr{B}}\tau)}}{\partial\tau}\frac{\partial e^{-2\pi\sqrt{-1}\beta'(v'_{\mathscr{B}}-u'_{\mathscr{B}}\tau')}}{\partial\tau'}&\\ +\sum_{\beta}{}^{\circ}\frac{1}{N(\beta)}\left\{e^{-2\pi\sqrt{-1}S(\beta v_{\mathscr{B}})}\sum_{\mu\gg0}{}^{*}\frac{\partial e^{2\pi\sqrt{-1}\left(-\frac{\varepsilon'\mu'}{\Delta}+u_{\mathscr{B}}\right)\beta\tau}}{\partial\tau}\frac{\partial e^{2\pi\sqrt{-1}\left(\frac{\varepsilon\mu}{\Delta}+u'_{\mathscr{B}}\right)\beta'\tau'}}{\partial\tau'}+\cdots\right\}&.\end{aligned}$$

$e^{\Theta_1}$ is regular and non-zero with respect to $\tau$, $\tau'$ in $\operatorname{Im}\tau>0$, $\operatorname{Im}\tau'>0$. Hence we can choose fixed branch of $\Theta_1$ and $\dfrac{\partial}{\partial\tau}$, $\dfrac{\partial}{\partial\tau'}$ in the above right hand side are $\dfrac{d}{d\tau}$, $\dfrac{d}{d\tau'}$, respectively. Thus

$$\iint\frac{\partial^2}{\partial\tau'\partial\tau}\Theta_1\,d\tau d\tau'=\Big[\Theta_1((\tau),(u_{\mathscr{B}}),(v_{\mathscr{B}}))\Big]_{(\tau_0,\tau_0')}^{(\tau_0^*,\tau_0'^*)}.$$

Letting $s$ tend to 1, we have

(5.2.8) $$\Xi(\mathscr{B}, 1) = -Dv(\Omega_1)/4\Delta\Big[\Theta_1((\tau), (u_{\mathscr{B}}), (v_{\mathscr{B}}))\Big]_{(\tau_0, \tau_0')}^{(\tau_0^*, \tau_0'^*)},$$

Putting

(5.2.8') $$\Xi(\mathscr{B}) = d\Xi(\mathscr{B}, 1),$$

we have, by (5.2.7),

$$L_K(1, \chi) = \frac{\pi^4}{T\sqrt{|\Delta_K|}} \sum \bar{\chi}(\mathscr{B})\Xi(\mathscr{B}),$$

where $\Delta_K$ is the discriminant of $K/\boldsymbol{Q}$.

(ii) $\mathfrak{f}=(1)$. In this case, for all $\mathscr{B}$, we see that

$$\Psi_{\mathscr{B}}(\tau, s) = \Phi(\tau, \tau'; \mathfrak{a}_{\mathscr{B}}, \mathfrak{d}; s, s).$$

By Th. 1, we have

$$\Psi_{\mathscr{B}}(\tau, s) = \frac{8\pi^2 \log \varepsilon}{dN\mathfrak{a}_{\mathscr{B}}} \frac{1}{s-1} + \frac{8\pi^2 \log \varepsilon}{dN\mathfrak{a}_{\mathscr{B}}} \log\left(\frac{H(\tau, \tau'; \mathfrak{a}_{\mathscr{B}}, \mathfrak{d})}{yy'}\right) + M + *(s-1) + \cdots$$

with obvious constant $M$. Since

$$\frac{\partial}{\partial \tau} \log y = 1/(\tau - \bar{\tau}), \ \frac{\partial}{\partial \tau'} \log y' = 1/(\tau' - \bar{\tau}'),$$

we have

$$\frac{\partial^2 \Psi_{\mathscr{B}}(\tau, s)}{\partial \tau' \partial \tau} = \frac{8\pi^2 \log \varepsilon}{dN\mathfrak{a}_{\mathscr{B}}}\left[\frac{1}{\tau - \bar{\tau}} + \frac{1}{\tau' - \bar{\tau}'} + \frac{\partial^2}{\partial \tau' \partial \tau} \log H(\tau, \tau'; \mathfrak{a}_{\mathscr{B}}, \mathfrak{d})\right] + *(s-1) + \cdots.$$

When $\tau$, $\tau'$ are given by (5.1.3), then

$$1/(\tau - \bar{\tau}) = \frac{d}{d\tau} \log \sqrt{(\tau - \omega^{(0)})(\tau - \omega^{(2)})}$$

and $$1/(\tau' - \bar{\tau}') = \frac{d}{d\tau'} \log \sqrt{(\tau' - \omega^{(1)})(\tau' - \omega^{(3)})}.$$

Hence for $\tau$, $\tau'$ in (5.1.3), we have

$$\begin{aligned} \frac{\partial^2 \Psi_{\mathscr{B}}(\tau, s)}{\partial \tau' \partial \tau} = \frac{8\pi^2 \log \varepsilon}{dN\mathfrak{a}_{\mathscr{B}}}\Big[&\frac{d}{d\tau} \log \sqrt{(\tau - \omega^{(0)})(\tau - \omega^{(2)})} \\ &+ \frac{d}{d\tau'} \log \sqrt{(\tau' - \omega^{(1)})(\tau' - \omega^{(3)})} \\ &+ \frac{\partial^2}{\partial \tau' \partial \tau} \log H(\tau, \tau'; \mathfrak{a}_{\mathscr{B}}, \mathfrak{d})\Big] + *(s-1) + \cdots. \end{aligned}$$

Using the series expansion of $\log H(\tau, \tau'; \mathfrak{a}_{\mathscr{B}}, \mathfrak{d})$ (see the formula preceeding to

Th. 3, (2.1.2), (2.3.3) and (2.3.4), we have

$$(5.2.9)\qquad \iint \frac{\partial^2 \Psi_{\mathscr{B}}(\tau, s)}{\partial\tau'\partial\tau} d\tau d\tau' = \frac{8\pi^2 \log\varepsilon}{dN\mathfrak{a}_{\mathscr{B}}}\left[\log\sqrt{(\tau-\omega^{(0)})(\tau-\omega^{(2)})(\tau'-\omega^{(1)})(\tau'-\omega^{(3)})} + \frac{G(\tau, \tau'; \mathfrak{a}_{\mathscr{B}}, \mathfrak{o})}{32\pi^4 \log\varepsilon\cdot N(\mathfrak{a}_{\mathscr{B}})}\right]_{(\tau_0, \tau_0')}^{(\tau_0^*, \tau_0'^*)} + *(s-1)+\cdots .$$

Put

$$(5.2.9')\qquad \Xi(\mathscr{B}) = d\cdot\Xi(\mathscr{B}, 1) = -\frac{dD}{4\pi^4}v(\Omega_1)\iint \frac{\partial^2 \Psi_{\mathscr{B}}(\tau, 1)}{\partial\tau\partial\tau'} d\tau d\tau' .$$

Tending $s$ to 1 and observing the value of $\Xi(\mathscr{B}, 1)$, we have, by (5.2.7),

$$L_K(1, \chi) = \frac{\pi^4}{\sqrt{|\Delta_K|}\cdot T}\sum_{\mathscr{B}}\bar{\chi}(\mathscr{B})\Xi(\mathscr{B}) .$$

We summarize the above in

THEOREM 9. *Let $K$ be a real biquadratic field satisfying $(A-\mathfrak{f})$. Let $\chi$ be a non-principal ray class proper character* mod $\mathfrak{f}$ *in $K$ and $v(\lambda)=N(\lambda)/|N(\lambda)|$, where $v(\lambda)$ is the character of signature associated with $\chi$. Then we have*

$$L_K(1, \chi) = \frac{\pi^4}{T\sqrt{|\Delta_K|}}\sum_{\mathscr{B}}\bar{\chi}(\mathscr{B})\Xi(\mathscr{B}) ,$$

*where $\mathscr{B}$ runs over all elements of $\mathfrak{G}_{\mathfrak{f}}/\mathfrak{E}_{\mathfrak{f}}$ and*

$$\Xi(\mathscr{B}) = \begin{cases} (5.2.8),\ (5.2.8') & \text{for } \mathfrak{f}\neq(1) \\ (5.2.9),\ (5.2.9') & \text{for } \mathfrak{f}=(1) \end{cases}$$

**5.3.** $F$, $K$ being as above, we denote by $K_0$ a relative abelian extension of $K$. Let $\mathfrak{f}$ be the conductor of $K_0$ over $K$. Then we have

$$\zeta_{K_0}(s) = \zeta_K(s)\prod_{\chi\neq 1} L_K(s, \chi) ,$$

where $\chi$ runs over a complete set of non-principal ray class characters modulo $\mathfrak{f}_\chi$ with conductor $\mathfrak{f}_\chi|\mathfrak{f}$. From this we have

$$\frac{2^{r_{1,0}}(2\pi)^{r_{2,0}}R_0 h_0}{w_0\sqrt{|\Delta_0|}} = \frac{4 r_K h_K}{w_K\sqrt{|\Delta_K|}}\prod_{\chi\neq 1} L_K(1, \chi) ,$$

where $r_{1,0}=r_{1,K_0}$, $r_{2,0}=r_{2,K_0}$, $R_0=R_{K_0}$, $h_0=h_{K_0}$, $w_0=w_{K_0}$, $\Delta_0=\Delta_{K_0}$ and $w_K=2$.

Let $v(\lambda)=N\lambda/|N\lambda|$ be the associated character of signature with $\chi$. We consider $K_0$ such that only character $\chi$ associated the above $v$ appears. Thus $K_0$ is quadratic over $K$, and so

$$\zeta_{K_0}(s) = \zeta_K(s) L_K(s, \chi) .$$

Let $K_0$ be $K(\sqrt{\mu})$ with non-quadratic $\mu$ in $K$ and $\mathfrak{d}$ the relative discriminant of $K_0/K$. Let $\mathfrak{P}$ be a prime ideal in $K$ coprime to $\mathfrak{d}$. There exists $c \in K$ such that $\mu^* = \mu c^2$ is coprime to $\mathfrak{P}$. We consider the ray class character $\psi(\mathfrak{P}) = \left(\frac{\mu^*}{\mathfrak{P}}\right) \bmod \mathfrak{d}$ defined in 3.2, which is equal to $\chi(\mathfrak{P})$ for $\mathfrak{P} \nmid \mathfrak{d}$. Hence $\chi$ and $\psi$ coincide on $\mathfrak{G}_{\mathfrak{d}}/\mathfrak{E}_{\mathfrak{d}}$, $\chi$ is a ray class character mod $\mathfrak{d}$ and $\mathfrak{f}_\chi | \mathfrak{d}$. (in fact $\mathfrak{f}_\chi = \mathfrak{d}$.) Now by the quadratic reciprocity law, we see that if $\mu$ totally negative, then $\chi((\lambda)) = \psi(\lambda) v(\lambda)$. Thus we are in the situation of 5.2 and obtain, by Th. 9,

$$\frac{2(\pi)^4 R_0 h_0}{w_0\sqrt{|\Delta_0|}} = \frac{2^4 r_K h_K}{w_K\sqrt{|\Delta_K|}} L_K(1, \chi)$$

$$= \frac{2^4 r_K h_K}{w_K\sqrt{|\Delta_K|}} \frac{\pi^4}{T\sqrt{|\Delta_K|}} \sum_{\mathscr{B}} \bar{\chi}(\mathscr{B}) \Xi(\mathscr{B}) .$$

Hecke [2] explicitly gave the value of $T$. In our notation,

$$T = \sqrt{N\mathfrak{d}}\, v(\gamma\sqrt{\mathfrak{q}}) \bar{\chi}(\mathfrak{q})$$

where $(\gamma)\sqrt{\mathfrak{d}} = \mathfrak{q}/\mathfrak{f}$ with integral ideal in $K$. Since we may assume $(\mu) = \mathfrak{d}\cdot\mathfrak{c}^2$ for some ideal $\mathfrak{c}$ and $\chi$ is a real character, we obtain

$$T = \sqrt{N\mathfrak{d}}\, \chi(\mu\gamma\sqrt{\mathfrak{d}}) .$$

THEOREM 10. *Let $K_0$ be an imaginary quadratic extension of a real biquadratic field $K$, ($K_0 = K(\sqrt{\mu})$), with the relative discriminant $\mathfrak{d}$. Then*

$$h_0 = h_K \frac{R_K w_0}{R_0 w_K} \chi(\mu\gamma\sqrt{\mathfrak{d}}) \sum_{\mathscr{B}} \bar{\chi}(\mathscr{B}) \Xi(\mathscr{B}) ,$$

*where $\mathscr{B}$ runs over all the ray classes* mod $\mathfrak{d}$ *in $K$, $\Xi(\mathscr{B})$ is given in* Th. 9 *and $\chi$ is the non-principal ray class character* mod $\mathfrak{d}$ *associated with $v(\lambda) = N\lambda/|N\lambda|$.*

Tsuda College

## Bibliography


[1] E. Hecke: Über die Konstruktion der Klassenkörper reeller quadratischer Körper mit Hilfe von automorphen Funktionen, Nach. zu Göttingen Math. phys. Kl. 1910, S. 619-623 (Werke 64-68).

[2] ———: Bestimung der Klassenzahl einer neuen Reihe von algebraischen Zahlkörpern, Werke 290-312.

[3] ———: Analytische Funktionen und algebraische Zahlen I, Werke 336-360, II, 381-404.

[4] H. Maass: Die Differentialgleichungen in der Theorie der elliptischen Modulfunktionen, Math. Ann. 125 (1953), 235-263.

[5] C. L. Siegel: Lectures on advanced analytic number theory, Tata Inst. Fund. Res. Bombay (1961).

[ 6 ] ______ : Quadratische Formen, Vorles, in Sommersem. 1955, an der Univ. Göttingen.
[ 7 ] D. Hilbert: Über die Theorie des relativquadratischen Zahlkörpers. M.A. (1899) 1-127.
[ 8 ] S. Kuroda: Über den Dirichletschen Körper, J. Fac. Sci. Univ. Tokyo, 4 (1943), 382-406.

(Received December 18, 1965)