# *A transformation group whose orbits are homeomorphic to a circle or a point*

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In his suggestive paper [2], D. Montgomery proved that a homeomorphism $T$ of a connected manifold $M$ is finitely periodic, if there is an integer $k=k(x)$ such that $T^k(x)=x$ for every point. This result can not be extended to the case of one parameter transformation groups, that is, a one parameter transformation group acting effectively on $M$ is not necessarily a circle group, even if every orbit of the group is homeomorphic to a circle $S^1$. A simple example of this fact can be made easily on a two-dimensional torus.

The topology, however, of the one parameter group is affected by the condition that every orbit is homeomorphic to $S^1$. The following theorem, which will be proved in this paper, shows a thing of this kind.

For convenience, by $M$ we mean a connected manifold with the second countability axiom and by $H(M)$ the group of all the homeomorphisms from $M$ onto $M$ with compact open topology. These notations are fixed throughout this paper.

THEOREM A. *Let $(L, \mathscr{T}_0)$ be a vector group of finite dimension, where $L$ is the underlying additive group and $\mathscr{T}_0$ is the topology for $L$. Let $\varphi$ be a non-trivial continuous homomorphism from $(L, \mathscr{T}_0)$ into $H(M)$. If every orbit of $\varphi(L)$ is homeomorphic to $S^1$ or a point, then $\varphi(L)$ is closed in $H(M)$.*

*More precisely, $\varphi(L)\cong(L', \mathscr{T}'_0)\times S^1$ or $(L', \mathscr{T}'_0)$ for some vector group $(L', \mathscr{T}'_0)$.*

Since $H(M)$ is a set of second category [1], the above theorem means that $\varphi$ is an open mapping from $(L, \mathscr{T}_0)$ onto $\varphi(L)$. Thus, $\varphi(L)$ is a Lie group under compact open topology.

Now, we consider the case where the above homomorphism is a monomorphism.

Let $\varphi$ be a continuous monomorphism from $(L, \mathscr{T}_0)$ into $H(M)$. If $\varphi(L)$ is not closed in $H(M)$, then the relative topology for $\varphi(L)$ in $H(M)$ introduces a new topology $\mathscr{T}$ for $L$ such that (i) $(L, \mathscr{T})$ satisfies the first countability axiom,

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(ii) $(L, \mathscr{T})$ satisfies Hausdorff's separation axiom, (iii) $\mathscr{T}$ is weaker than $\mathscr{T}_0$, (iv) $(L, \mathscr{T})$ is a topological additive group and (v) $(L, \mathscr{T}) \neq (L, \mathscr{T}_0)$.

For a fixed underlying group $L$, we denote by $T(L, \mathscr{T}_0)$ the collection of all the pairs of the fixed abstract group $L$ and a topology $\mathscr{T}$ for $L$ satisfying (i)~(iv) above.

For a subgroup $L'$ of $L$, $(L', \mathscr{T})$ means the subgroup $L'$ with the relative topology in $(L, \mathscr{T})$.

Under these notations, an element $(L, \mathscr{T}) \in T(L, \mathscr{T}_0)$ is said to be irreducible, if for any proper vector subgroup $L'$, $(L', \mathscr{T}) = (L', \mathscr{T}_0)$ but $(L, \mathscr{T}) \neq (L, \mathscr{T}_0)$.

Since $\dim L < \infty$, we see easily that if $(L, \mathscr{T}) \in T(L, \mathscr{T}_0)$ and $(L, \mathscr{T}) \neq (L, \mathscr{T}_0)$, there is a vector subgroup $(L', \mathscr{T})$ which is irreducible. We know in [4] that there is an example of topology $\mathscr{T}$ for two-dimensional vector group $L$ such that $(L, \mathscr{T}) \in T(L, \mathscr{T}_0)$ and $(L, \mathscr{T})$ is irreducible.

Now, in the case of monomorphic $\varphi$, Theorem A is obtained as an immediate consequence of the following Theorem *B*.

THEOREM B. *Let $(L, \mathscr{T}) \in T(L, \mathscr{T}_0)$ be irreducible. Assume furthermore that there is a non-trivial continuous homomorphism $\varphi$ from $(L, \mathscr{T})$ into $H(M)$ such that every orbit $\varphi(L)(x)$ is homeomorphic to $S^1$ or a point. Then $\varphi(L)$ is isomorphic to $S^1$.*

COROLLARY. *Notations and assumptions being as in Theorem A, if $\varphi$ is monomorphic, then $\varphi(L)$ is closed in $H(M)$.*

The proof of Theorem B, which will be given later, is similar to that of the following well-known theorem.

THEOREM C. *Let $\varphi$ be a non-trivial homomorphism from a toroidal group $T$ into $H(M)$ such that every orbit $\varphi(T)(x)$ is homeomorphic to a circle or a point. Then $\varphi(T) \cong S^1$.*

The proof of this theorem consists of the following three steps, which correspond to those of the proof of our Theorem B.

a) It is well-known that the Pontryagin dual group Hom $(T, S^1)$ is a discrete group.

b) Let $T_x^0 (\subset T)$ be the connected component containing 0 of the full-inverse of the isotropy subgroup of $\varphi(T)$ at $x \in M$ and let $M'$ be the set of the points such that $\varphi(T)(x)$ is homeomorphic to a circle. Then $T/T_x^0 \cong S^1$ for $x \in M'$. Therefore, there is a homomorphism $\varphi_x$ from $T$ onto $S^1$ depending continuously on $x \in M'$. Thus, from a) we have that $\varphi_x$ is constant on every connected component of $M'$. In other words, $T$ operates as a circle group on each connected component of $M'$.

c) $M'$ is connected and dense in $M$, for the fixed point set of a compact Lie transformation group acting effectively on $M$ has no interior point of $M$.

Each step of the proof of Theorem B corresponds to each of a), b) and c), that is, § 1, § 2 and § 3 correspond a), b) and c) respectively. In our case, however, since the transformation group in question is a vector group, we have not to take a connected component of the isotropy group at $x$ but the isotropy group itself works well in our purpose.

Theorem A follows quite naturally from Theorems B and C. This will be seen in § 4.

1. As for an element $(L, \mathscr{T}) \in T(L, \mathscr{T}_0)$, we have the following lemmas whose proofs are seen in [3].

LEMMA 1. *If* $(L, \mathscr{T}) \in T(L, \mathscr{T}_0)$ *and* $(L, \mathscr{T}) \neq (L, \mathscr{T}_0)$, *then for any neighborhood* $U$ *of* 0 *in* $(L, \mathscr{T})$ *and for any positive number* $r$,

$$U \cap \{(x_1, \cdots, x_k) \in L ; \sum x_i^2 > r\} \neq \emptyset .$$

LEMMA 2. *Assumptions being as above, for any* $\varepsilon > 0$, *there is a neighborhood* $V$ *of* 0 *in* $(L, \mathscr{T})$ *such that the diameter of any connected component of* $V$ *is smaller than* $\varepsilon$, *where the metric on* $L$ *is the natural euclidean metric.*

Using these lemmas, we have the following lemma on $(L, \mathscr{T})$ which is irreducible.

LEMMA 3. *Let* $i$ *be an integer such that* $1 \leqq i \leqq k = \dim L$. If $(L, \mathscr{T}) \in T(L, \mathscr{T}_0)$ *is irreducible, then for any* $K > 0$ *and for any neighborhood* $U$ *of* 0 *in* $(L, \mathscr{T})$, *there is* $\boldsymbol{y} = (y_1, \cdots, y_k) \in U$ *such that* $|y_i| > K$.

PROOF. Assume that there are a neighborhood $U$ of 0 in $(L, \mathscr{T})$, a positive $K$ and an integer $j$, $1 \leqq j \leqq k = \dim L$ such that $|y_j| \leqq K$ for any point $(y_1, \cdots, y_k) \in U$. Without loss of generality, we assume that $j = 1$.

Let $\rho$ be the metric on $L$ defined by

$$\rho(\boldsymbol{x}, \boldsymbol{y})^2 = \sum (x_i - y_i)^2 .$$

From the condition that $(L, \mathscr{T})$ satisfies the first countability axiom and from Lemma 2, there is a basis $\{V_i\}$ of the neighborhoods of the identity 0 in $(L, \mathscr{T})$ satisfying a) $U \supset V_i$, b) $V_i \supset 2V_{i+1}$, c) $-V_i = V_i$, d) the diameter of each connected component of $V_i$ is less than $K$.

Let $E_q = \prod_{i=1}^{K} [-qK, qK]$ be a cube in $L$ containing 0 in the center of $E_q$ and let $F_q = L - E_q$. By Lemma 1, we see that $V_i \cap F_q \neq 0$ for any $i$ and $q$, because $(L, \mathscr{T}) \neq (L, \mathscr{T}_0)$.

Let $V_i^{(q)}$ be the union of the connected components of $V_i$ which intersect $F_q$. From the condition *d*) of $\{V_i\}$, we have $V_i^{(q)} \cap F_{q-1} = \emptyset$ for every $i$ and $q$.

Considering the projection Pr from $L$ onto $R$ (real number field) defined by

$$\Pr(y_1, \cdots, y_k) = y_1 ,$$

we see that $\Pr(V_i^{(q)}) \subset [-K, K]$ for all $i$ and $q$. Thus, there is $\hat{y} \in [-K, K]$ such that $\hat{y} \in \bigcap_q (\bigcap_i \mathrm{Cl}(\Pr(V_i^{(q)}))$, where $\mathrm{Cl}(A)$ is the closure of $A$ in $[-K, K]$. This implies that for any $\varepsilon$, $q$ and $i$, there is $\boldsymbol{y} \in V_i^{(q)}$ satisfying

$$|\Pr(\boldsymbol{y}) - \hat{y}| < \varepsilon .$$

It follows that $(\boldsymbol{y} + V(\varepsilon)) \cap H(\hat{y}) \neq \emptyset$, where $V(\varepsilon)$ is an $\varepsilon$-neighborhood of 0 under the metric $\rho$ and $H(\hat{y})$ is the hyperplane defined by $y_1 = \hat{y}$. On the other hand, since the identity mapping from $(L, \mathscr{T}_0)$ onto $(L, \mathscr{T})$ is continuous and $V(\varepsilon)$ is connected, we can choose sufficiently small $\varepsilon_i$ such that $V(\varepsilon_i)$ is contained in the connected component of $V_i$ containing 0. Thus, we have

$$\boldsymbol{y} + V(\varepsilon_i) \subset V_i^{(q)} + V(\varepsilon_i) \subset V_{i-1}^{(q-1)} .$$

It follows that $V_i^{(q)} \cap H(\hat{y}) \neq \emptyset$ for all $i$ and $q$, because $(\boldsymbol{y} + V(\varepsilon_i)) \cap H(\hat{y}) \neq \emptyset$. Therefore, there are $\boldsymbol{y}, \boldsymbol{y}' \in V_i \cap H(\hat{y})$ such that $\rho(\boldsymbol{y}, \boldsymbol{y}') \geqq N$ for any positive number $N$.

Let $L'$ be the vector subspace defined by $y_1 = 0$. As for $\boldsymbol{y}, \boldsymbol{y}'$ above, we see that $\boldsymbol{y} - \boldsymbol{y}' \in L'$, $\boldsymbol{y} - \boldsymbol{y}' \in 2V_i \subset V_{i-1}$ and $\rho(\boldsymbol{y} - \boldsymbol{y}', 0) \geqq N$. This implies that

$$L' \cap F_q \cap V_i \neq \emptyset$$

for all $i$ and $q$. It follows that $(L', \mathscr{T}) \neq (L', \mathscr{T}_0)$, contradicting the assumption that $(L, \mathscr{T})$ is irreducible.

Let $\langle \boldsymbol{x}, \boldsymbol{y} \rangle$ be the ordinary inner product in $L$ i.e. $\langle \boldsymbol{x}, \boldsymbol{y} \rangle = \sum x_i y_i$. Then, as a consequence of *Lemma* 3, we have the following:

COROLLARY. *Let $(L, \mathscr{T}) \in T(L, \mathscr{T}_0)$ be irreducible. If $x \in L$ satisfies $|\langle \boldsymbol{x}, \boldsymbol{y} \rangle| < \delta$ (bounded) for any $\boldsymbol{y}$ of some neighborhood $U$ of 0 in $(L, \mathscr{T})$, then* $\boldsymbol{x} = 0$.

Let $S^1 = \{e^{i\theta}\}$ be the unit circle with the natural topology and let $\mathrm{Hom}((L, \mathscr{T}_0), S^1)$ be the set of the continuous homomorphisms from $(L, \mathscr{T}_0)$ into $S^1$ with compact open topology. For any $(L, \mathscr{T}) \in T(L, \mathscr{T}_0)$, a homomorphism $\varphi$ from $(L, \mathscr{T})$ into $S^1$ can be considered as a homomorphism from $(L, \mathscr{T}_0)$ into $S^1$. By $\mathrm{Hom}((L, \mathscr{T}), S^1)$ we mean the set of the continuous homomorphisms from $(L, \mathscr{T})$ into $S^1$ with relative topology in $\mathrm{Hom}((L, \mathscr{T}_0), S^1)$.

It is well-known that $(L, \mathscr{T}_0)$ is isomorphic to $\mathrm{Hom}((L, \mathscr{T}_0), S^1)$. The isomorphism $\eta$ is given by $\eta(\boldsymbol{x})(\boldsymbol{y}) = e^{i\langle x, y \rangle}$.

For a neighborhood $U$ of 0 in $(L, \mathscr{T})$, $0 < \varepsilon < \frac{\pi}{2}$ and $\varepsilon$-neighborhood $V(\varepsilon)$ of

0 in $S^1$, we denote

$$\mathscr{S}(U, \varepsilon)=\{\varphi \in \text{Hom}((L, \mathscr{T}), S^1) ; \varphi(U)\subset V(\varepsilon)\} .$$

PROPOSITION 1. *Notations being as above, if $(L, \mathscr{T})$ is irreducible, then $\mathscr{S}(U, \varepsilon)$ is totally disconnected.*

PROOF. Let $W_\varphi$ be the connected component of $\mathscr{S}(U, \varepsilon)$ containing $\varphi$. For every $\varphi' \in W_\varphi$, $\varphi'(\boldsymbol{x})=e^{i\langle x, \eta^{-1}\varphi'\rangle}$. Thus, if $\boldsymbol{x}\in U$, then $e^{i\langle x, \eta^{-1}\varphi'\rangle}\in V(\varepsilon)$. Since $\varepsilon<\frac{\pi}{2}$, there is an integer $m_x(\varphi')$ such that

$$|<\boldsymbol{x}, \eta^{-1}\varphi'>-2\pi m_x(\varphi')|<\varepsilon .$$

It follows that $m_x(\varphi')$ is constant on $W_\varphi$ for every $\boldsymbol{x}\in U$. Therefore

$$|<\boldsymbol{x}, \eta^{-1}\varphi'-\eta^{-1}\varphi>|<2\varepsilon$$

for every $\boldsymbol{x}\in U$. Since $(L, \mathscr{T})$ is irreducible, by Corollary to Lemma 3 we have $\varphi=\varphi'$.

As an application of the Proposition 1 to transformation groups, we consider $(L, \mathscr{T})$ operating continuously on a metric space $X$. The operation is denoted by $f$. Assume furthermore that there is continuous operation $\tilde{f}$ of $S^1$ on $X$ such that if $\tilde{f}(s, x)=x$ for a point $x\in X$, then $s=0$, and that there is a mapping $\Psi: (L, \mathscr{T})\times X\to S^1\times X$ satisfying (1) $\tilde{f}\Psi=f$ (2) $\Psi(l, x)=(\Psi_x(l), x)$ and $\Psi_x$ is a homomorphism from $(L, \mathscr{T})$ onto $S^1$.

Since $f, \tilde{f}$ are continuous, so is $\Psi$. In fact, if $\lim (l_n, x_n)=(l_0, x_0)$, then by compactness of $S^1$, there is a subsequence $(l_{n'}, x_{n'})$ such that

$$\lim (\Psi_{x_{n'}}(l_{n'}), x_{n'})=(s_0, x_0) .$$

On the other hand,

$$\tilde{f}(s_0, x_0)=\lim \tilde{f}(\Psi_{x_{n'}}(l_{n'}), x_{n'})=\lim f(l_{n'}, x_{n'})=f(l_0, x_0)=\tilde{f}(\Psi_{x_0}(l_0), x_0) .$$

Therefore $\Psi_{x_0}(l_0)=s_0$.

From the continuity of $\Psi$, we see that for an $\varepsilon$-neighborhood $V(\varepsilon)$ of 0 in $S^1$ there are a neighborhood $U$ of 0 in $(L, \mathscr{T})$ and an open set $Y$ of $X$ such that $\Psi(U, Y)\subset(V(\varepsilon), Y)$. This means that the mapping $x\to\Psi_x$ is continuous from $Y$ into $\mathscr{S}(U, \varepsilon)$. Thus, we have

COROLLARY. *If $(L, \mathscr{T})$ is irreducible, then the mapping $x\to\Psi_x$ is constant on every connected component of $Y$. Moreover, if $X$ is locally connected and connected, then $x\to\Psi_x$ is constant on $X$. In other words, $(L, \mathscr{T})$ operates on $X$ as a circle group.*

2. Now we consider $(L, \mathscr{T})\in T(L, \mathscr{T}_0)$ acting effectively and continuously on a manifold $M$ as a transformation group. Assume that every orbit is homeo-

morphic to a circle or a point. Clearly, the subset of $M$ consisting of all the points $x$ such that the orbit of $x$ is homeomorphic to $S^1$ is $L$-invariant and is acted on by $(L, \mathscr{T})$ as a transformation group. Since the identity mapping from $(L, \mathscr{T}_0)$ onto $(L, \mathscr{T})$ is continuous, $(L, \mathscr{T}_0)$ acts naturally on $M$. Thus, we assume from the beginning, to simplify the argument below, that $(L, \mathscr{T}_0)$ acts effectively and continuously on $M$ itself as a transformation group and that every orbit is homeomorphic to $S^1$.

Denote by $L_x^0$ the connected component containing the identity of the isotropy subgroup $L_x$ of $L$ at $x$.

Clearly $L_x^0$ is continuous, that is, if $x_n \to x_0$, then

$$\lim L_{x_n}^0 = \{\lim k_n ; k_n \in L_{x_n}^0\} = L_{x_0}^0 .$$

Since $M$ is locally simply connected, there is a connected open set $M'$ on which the unit vector $n(x)$ orthogonal to $L_x^0$ can be chosen in such a way that it is continuous with respect to the variable $x$. Since $L_x^0$ is constant on every orbit, we may assume that $M'$ is an $L$-invariant open connected subset of $M$.

LEMMA 4. *Let $\lambda(x) = \min\{\lambda > 0 ; \lambda n(x) \in L_x\}$. Then $\lambda(x)$ is lower semi-continuous, the points of continuity are open and dense in $M'$, and $\lambda(x)$ is $L$-invariant.*

PROOF. It is easy to see that $\lambda(x)$ is $L$-invariant and lower semi-continuous.

Let $x$ be a point of continuity. Then there is an open neighborhood $V_x$ such that $|\lambda(x) - \lambda(y)| < \varepsilon$ for any $y \in V_x$. If there is a sequence $\{y_n\}$ in $V_x$ converging to $y$ in $V_x$ and $\lim \lambda(y_n) \neq \lambda(y)$, then we have

$$\lim \lambda(y_n) \geqq 2\lambda(y) .$$

Thus, for sufficiently large $n$, we have

$$\lambda(y_n) - \lambda(y) \geqq \lambda(y) - \varepsilon \geqq \lambda(x) - 2\varepsilon .$$

On the other hand, $|\lambda(y_n) - \lambda(y)| < 2\varepsilon$. It follows that if $\varepsilon < \frac{1}{4}\lambda(x)$, then $\lim \lambda(y_n) = \lambda(y)$. Thus, the point of continuity of $\lambda(x)$ are open.

This argument shows that if $\lambda(x)$ is bounded on an open set $U$ and $\lambda_0 = \sup\{\lambda(x) ; x \in U\}$, then for a sufficiently small $\varepsilon > 0$, a point $x \in U$ satisfying $\lambda(x) \geqq \lambda_0 - \varepsilon$ is a point of continuity. Since every open set in $M'$ is a set of second category, a category argument gives that the points of continuity is dense in $M'$.

Let $M''$ be an open $L$-invariant subset of $M'$ on which $\lambda(x)$ is continuous. Let $\Psi$ be a mapping from $(L, \mathscr{T}_0) \times M''$ into $S^1 \times M''$ defined by

$$\Psi(l, x) = (e^{i\langle 2\pi\lambda(x)^{-1} n(x), l\rangle}, x) ,$$

and $\tilde{f}$ be a mapping from $S^1 \times M''$ into $M''$ defined by

$$\tilde{f}(e^{is}, x) = f\left(\frac{1}{2\pi} s\lambda(x)n(x), x\right),$$

where $f$ is the continuous operation of $(L, \mathscr{T}_0)$ on $M$. It is easy to see that $\tilde{f}$ is a continuous operation of $S^1$ on $M''$ with $f = \tilde{f}\Psi$ and if $\tilde{f}(s, x) = x$ for some $x \in M''$, then $s = 0$.

Now, let $(L, \mathscr{T})$ be irreducible and act on $M$ as a transformation group. Assume that every orbit is homeomorphic to $S^1$. By the same argument as above, there is an $L$-invariant open subset $M''$ on which $\lambda(x)$ is continuous and the continuous mappings $\tilde{f}$ and $\Psi$ above are defined. Thus, we have from Corollary to Proposition 1 the following

LEMMA 5. *$(L, \mathscr{T})$ operates as a circle group on every connected component of $M''$. More precisely, let $A$ be a connected $L$-invariant subset of $M'$ on which $\lambda(x)$ is continuous. Then $(L, \mathscr{T})$ operates on $A$ as a circle group.*

3. In this section, it will be proved that $M'' = M'$. The fundamental fact used in proving this is that if a compact Lie group $G$ acts effectively and continuously on a connected manifold and if the fixed point set of $G$ contains an interior point, then $G = \{e\}$.

Let $K$ be the collection of points $x \in M'$ such that on every neighborhood of $x$, $\lambda(x)$ has an infinite least upper bound. Then $K$ is a closed and $L$-invariant subset of $M'$ and is nowhere dense.

LEMMA 6. *On every connected component $R$ of $M' - K$, the function $\lambda(x)n(x)$ is constant, that is, $(L, \mathscr{T})$ operates on $R$ as a circle group.*

PROOF. Clearly $R$ is an $L$-invariant open subset. Since the points of continuity is dense and open, there is a connected open set $H$ in $R$ such that $\lambda(x)n(x)$ is constant on $H$. If $H$ is not all of $R$, let $b$ be a point of $R$ on the boundary of $H$. There are an open neighborhood $U$ of $b$ in $R$ and a positive number $m$ such that $\sup\{\lambda(x); x \in U\} = m$, because $\lambda(x)$ has a finite least upper bound at $b$. There is a point $y \in U$ such that $\lambda(y) \geqq m - \varepsilon$ and we see easily that for sufficiently small $\varepsilon$, such $y$ is a point of continuity. It follows that there exists an open connected subset $V$ of $H \cup U$ on which $\lambda(x)n(x)$ is constant. Since $\lambda(x)n(x)$ is $L$-invariant, the set $V$ can be assumed to be $L$-invariant.

Assume furthermore that $V$ is a maximal open connected subset on which $\lambda(x)n(x)$ is constant and equal to $\lambda(b')n(b')$ for a point $b' \in V$. From Lemma 4, we have that $\lambda(x) < \lambda(b')$ on the boundary point of $V$ in $H \cup U$. It follows that $\lambda(x) = \frac{1}{k}\lambda(b')$ for some integer $k = k(x) \geqq 2$. Since the boundary $B$ of $V$ is closed in $H \cup U$ and then a set of second category, we see by a category argument that there is an $L$-invariant open subset $W$ in $H\cup U$ such that $\lambda(x)$ is constant and equal to $\frac{1}{k}\lambda(b')$ on $W\cap B$ for some integer $k\geqq 2$. Then, an operation of $Z_k=\{e^{2\pi i\frac{l}{k}}\}$ on $V\cup W$ is defined as follows:

$$g(e^{2\pi i\frac{l}{k}}, x)=\begin{cases} f\left(\frac{l}{k}\lambda(b')n(b'), x\right) & \text{if } x\in V \\ x & \text{if } x\in V\cup W-V, \end{cases}$$

where $f$ is the operation of $(L, \mathscr{T})$ on $M$ and $l$ is an integer. It is easy to see that $g: Z_k\times V\cup W\to V\cup W$ is continuous.

Since $V\cup W$ is a connected manifold and $V\cup W-V$ has an interior point, $Z_k$ operates trivially on $V\cup W$. This contradicts the definition of $\lambda(x)$. Thus, we have $V=R-H$.

PROPOSITION 2. *Let $(L, \mathscr{T})\in T(L, \mathscr{T}_0)$ be irreducible and act on a connected manifold $M$. If every orbit is homeomorphic to $S^1$, then $(L, \mathscr{T})$ acts on $M$ as a circle group.*

PROOF. Notations being as above, it is easy to see that for every point $x\in M$ there is an open, connected and $L$-invariant subset $M'$ containing $x$ on which $n(y)$ is continuous. We have only to show that $\lambda(x)n(x)$ is constant on $M'$. If $\lambda(x)$ is bounded on $M'$, then by Lemma 6 we have $\lambda(x)n(x)$ is constant. The proposition will now be proved by the method of contradiction. Assume that $\lambda(x)$ is unbounded on $M'$. On the basis of this assumption the lemma above shows that $M'-K$ is not connected and therefore that the closed set $K$ is not vacuous.

Let $\lambda(x|K)$ denote $\lambda(x)$ restricted to $K$; $\lambda(x|K)$ is lower semi-continuous on K. Since $K$ is a set of second category, we have by the same argument as in Lemma 4 that the set of continuity of $\lambda(x|K)$ is open and dense. Thus, there is an $L$-invariant connected open subset $U$ in $M'$ such that $\lambda(x|K)$ is continuous on $U\cap K\neq\emptyset$. Let $R$ be any connected component of $M'-K$ and $\lambda(x)=\lambda_0$ on $R$. The boundary $B_R$ of $R$ is contained in $K$ and $\lambda(x|K)=\frac{1}{k}\lambda_0$ on $B_R$ for some integer $k=k(x)$. Since $\lambda(x|K)$ is continuous on $B_R\cap U$, we see that $\lambda(x|K)$ is constant and equal to $\lambda_0$ on $B_R\cap U$ by the same reason as in Lemma 6 because the set of the points $x$ where $\lambda(x|K)=\frac{1}{k}\lambda_0$ is open in $B_R\cap U$ for every fixed $k$.

It will be shown below that $\lambda(x)$ is continuous on $U$. Let $\{x_n\}$ be a sequence converging to a point $x_0$ in $U$. If $x_0\notin K$, then $\lim\lambda(x_n)=\lambda(x_0)$ because any connected component of $M'-K$ is an open subset. Assume $x_0\in K\cap U$. There is an arc $C:[0, 1]\to U$ such that $C(t_n)=x_n$, $C(1)=x_0$ and $\lim_{t\to 1} C(t)=x_0$. For every $t_n$ there is $t_{n'}$ such that $t_n\leqq t_{n'}$, $C(t_{n'})\in K\cap U$ and $C([t_n, t_{n'}])$ is contained in the closure of a connected component of $M'-K$. Since $\lambda(x|K)$ is continuous on $U\cap K$, we see $\lim \lambda(C(t_{n'}))=\lambda(x_0)$. From this and the fact that $\lambda(x_n)=\lambda(C(t_{n'}))$, we have $\lim \lambda(x_n)=\lambda(x_0)$. Thus, $\lambda(x)$ is continuous on $U$ and then constant on $U$. This contradicts the definition of the set $K$. It follows that $K$ is vacuous. Then $\lambda(x)n(x)$ is constant on $M'$ and then on $M$. This means that $(L, \mathscr{T})$ acts as a circle group on $M$.

PROOF OF THEOREM B.

Let $K$ be the set of points such that $\varphi(L)(x)$ is a point. Clearly $K$ is a closed subset. By Proposition 2, we see that $\lambda(x)$ is constant on a connected component $M'$ of $M-K$. That is, $(L, \mathscr{T})$ acts as a circle group on $M'$. Define an operation $f'$ of $(L, \mathscr{T})$ as follows:

$$f'(l, x)=\begin{cases} f(l, x) & \text{if } x\in M' \\ x & \text{if } x\in M-M', \end{cases}$$

where $f$ is the operation of $(L, \mathscr{T})$ on $M$. Clearly $f'$ is an operation of $(L, \mathscr{T})$ on $M$ as a circle group. Therefore $M-M'$ contains no interior point. This means $f=f'$, completing the proof.

## 4. PROOF OF THEOREM A.

Let $\varphi$ be a homomorphism from $(L, \mathscr{T}_0)$ into $H(M)$ and $K$ be the kernel of $\varphi$. The factor group $(L, \mathscr{T}_0)/K$ is isomorphic to $(L', \mathscr{T}_0)\times T$ where $T$ is a toroidal group. Naturally, there is a monomorphism $\tilde{\varphi}$ from $(L', \mathscr{T}_0)\times T$ into $H(M)$ such that $\tilde{\varphi}\circ\pi=\varphi$ where $\pi$ is the natural projection from $(L, \mathscr{T}_0)$ onto $(L', \mathscr{T}_0)\times T$. Assume furthermore that every orbit $\varphi(L)(x)$ is homeomorphic to a circle or a point. Then, we see that every orbit $\tilde{\varphi}(T)(x)$ is homeomorphic to a circle or a point. Thus, from Theorem $C$ we have $T=S^1$.

We have only to show that $\tilde{\varphi}(L')$ is closed in $H(M)$. Assume that $\tilde{\varphi}(L')$ is not closed in $H(M)$. Then the relative topology for $\tilde{\varphi}(L')$ introduces a topology $\mathscr{T}$ for $L'$ such that $(L', \mathscr{T})\in T(L', \mathscr{T}_0)$ and $(L', \mathscr{T})\neq(L', \mathscr{T}_0)$. It follows that there is a vector subspace $L''$ of $L'$ such that $(L'', \mathscr{T})$ is irreducible. From the irreducibility, we see that every orbit $\tilde{\varphi}(L'')$ is homeomorphic to a circle or a point. It follows by Theorew $B$ that $\tilde{\varphi}(L'')=S^1$, contradicting the fact that $\tilde{\varphi}$ is isomorphic. Thus, we see that $\varphi(L)$ is closed and isomorphic to $(L', \mathscr{T}_0)\times S^1$.

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