On the finiteness of perturbed eigenvalues

By Reiji KONNO and S. T. KURODA

§ 1. Introduction.

In connection with the study of partial differential operators, Birman [1] gave necessary and sufficient conditions in order that only a finite number of negative eigenvalues are created by perturbing a positive definite selfadjoint operator H_0. The setting of problems and the results obtained were given in terms of closed Hermitian forms and Friedrichs extensions. On the other hand Schwinger [6] considered the similar problem for the number of bound states of the three-dimensional Schrödinger operator and showed (partly by intuitive argument) that the total number of negative eigenvalues of the operator given formally by $-\Delta+q(x)\cdot$ in $L^2(R^3)$ is not greater than

$$\frac{1}{(4\pi)^2}\int_{R^3}\int_{R^3}\frac{|q(x)|\,|q(y)|}{|x-y|^2}dx\,dy.$$

This result, however, does not seem to be deduced from the condition and the estimate given by Birman.

The main purpose of the present paper is to give another sufficient condition for the finiteness of perturbed eigenvalues which includes Schwinger's condition and partly Birman's one. The setting of the problem is based on the technique of factoring the perturbation. This technique is currently investigated from several aspects of the perturbation theory[1]. In the present paper we shall exclusively follow the formulation given by Kato [3] in which the perturbed operator H_1 is given formally by $H_1\sim H_0+B^*A$[2].

Let the operator valued function $Q(z)$ be defined as $A(H_0-z)^{-1}B^*$ for any z belonging to the resolvent set $\rho(H_0)$ of H_0. It has been shown that the behavior of $Q(z)$ on or near the real axis has an intimate connection with the perturbation problems. Namely, its boundary values $Q(\lambda\pm i0)$ on the continuous spectrum of H_0, if they exist, have an important meaning in the "scattering theory" and the distribution of its "zeros" in $\rho(H_0)$ somehow determines the discrete part of the spectrum of H_1. Thus, the main intention of the present paper lies in adding another piece of information to such investigation of the perturba-

[1] See, e.g., Kato [3], Kuroda [5] and the literatures cited in them.

[2] A more general factorization referring to Cayley transforms, which was used in [5], seems not suitable for the present problem.

tion problem and not so much in the improvement of the condition from the practical viewpoint. Namely, our main result asserts roughly that, if $Q(z)$ is completely continuous for any $z \in \rho(H_0)$ and extendable continuously (in the norm topology) to an endpoint a of the continuous spectrum of H_0, then the eigenvalues of H_1 situated in $\rho(H_0)$ do not have a as a point of accumulation. This result will be formulated in § 2 and proved in § 3. An estimate of the number of perturbed eigenvalues is also given. In § 4 a brief account will be given on the application of the results to the cases considered by Birman and Schwinger.

§ 2. Assumptions and results.

Let $\mathfrak{H}$ be a Hilbert space. We denote by $\boldsymbol{B}$ the set of all bounded operators T with the domain $\mathfrak{D}(T)=\mathfrak{H}$ and the range $\mathfrak{R}(T)\subset\mathfrak{H}$. For any densely defined closed operator T in $\mathfrak{H}$, the resolvent set, the spectrum, and the point spectrum of T are denoted by $\rho(T)$, $\sigma(T)$, and $\sigma_p(T)$, respectively. When T is densely defined and bounded, we denote by $[T]\in\boldsymbol{B}$ the closure of T.

Now, let H_0 be a selfadjoint operator in $\mathfrak{H}$ and put for brevity $R_0(z)=(H_0-z)^{-1}$, $z\in\rho(H_0)$. Let A and B be densely defined closed operators which satisfy the following two assumptions.

ASSUMPTION 1. i) $\mathfrak{D}(H_0)\subset\mathfrak{D}(A)\cap\mathfrak{D}(B)$. ii) For some (or equivalently any) $z\in\rho(H_0)$, the densely defined operator $AR_0(z)B^*$ has a bounded extension $Q(z)\in\boldsymbol{B}$.

ASSUMPTION 2. There exists $z\in\rho(H_0)$ such that

$$-1\in\rho(Q(z))\,. \tag{2.1}$$

In our later arguments, we use the following propositions and equalities which are easy consequences of Assumption 1[3].

$$\begin{cases} \text{i)} & \text{The densely defined operator } R_0(z)B^* \text{ is bounded;} \\ \text{ii)} & [R_0(z_1)B^*]-[R_0(z_2)B^*]=(z_1-z_2)R_0(z_1)[R_0(z_2)B^*]\,; \\ \text{iii)} & \mathfrak{R}([R_0(z)B^*])\subset\mathfrak{D}(A)\,; \\ \text{iv)} & Q(z_1)-Q(z_2)=(z_1-z_2)AR_0(z_1)[R_0(z_2)B^*]\,. \end{cases} \tag{2.2}$$

For any $z\in\rho(H_0)$ satisfying (2.1), the operator $R_1(z)\in\boldsymbol{B}$ is defined by

$$R_1(z)=R_0(z)-[R_0(z)B^*](1+Q(z))^{-1}AR_0(z)\,. \tag{2.3}$$

As was shown by Kato[3], $R_1(z)$ is the resolvent of a closed operator H_1, which is an extension of H_0+B^*A. This is the definition of our perturbed operator H_1. Its selfadjointness will be deduced later on the basis of Assumptions 3 and 4.

Since our theory is in the category of the so-called large perturbation, we need the following additional assumption.

[3] For the proof, see Kato [3; § 2].

ASSUMPTION 3. The operator $Q(z)$ is completely continuous for every $z \in \rho(H_0)$.

Under Assumptions 1, 2 and 3, there is an intimate relation between the discrete part of the spectrum of H_1 and that of $Q(z)$. This fact, which plays a fundamental role in our argument, is formulated as in the following lemma.

LEMMA 1. *Suppose that* $z \in \rho(H_0)$. *Then:* i) $z \in \sigma_p(H_1)$ *if and only if* $-1 \in \sigma_p(Q(z))$; *and* ii) $z \in \rho(H_1)$ *if and only if* $-1 \in \rho(Q(z))$. *Moreover, in the case of* i), *the (geometric) multiplicity of* z *as an eigenvalue of* H_1 *is equal to that of* -1 *as an eigenvalue of* $Q(z)$.

Although this lemma can be deduced essentially from Remark 7.1 of [5], a direct proof will be given in § 3 for the sake of completeness.

In order to make H_1 selfadjoint, we follow the argument given in Kato [3] and introduce the following assumption.

ASSUMPTION 4. For each $x, y \in \mathfrak{D}(A) \cap \mathfrak{D}(B)$ we have $(Ax, By) = (Bx, Ay)$.

THEOREM 1. *Suppose that Assumptions 1-4 are satisfied. Then, the operator* H_1 *defined as above is selfadjoint. Let* I *be an open (possibly infinite) interval contained in* $\rho(H_0)$. *Then,* $\sigma(H_1) \cap I$ *consists only of isolated eigenvalues of finite multiplicity. Furthermore, let* a *be one of the end points of* I ($a = \pm\infty$ *is permitted) and suppose that* $Q(\lambda)$, $\lambda \in I$, *tends to some operator* $Q(a) \in \boldsymbol{B}$ *in* $\boldsymbol{B}$ *as* $\lambda \to a$. *Then* a *is not a point of accumulation*[4] *of* $\sigma_p(H_1) \cap I$.

COROLLARY. *In addition to the hypotheses of Theorem 1, suppose that* $Q(\lambda)$ *has a limit at each end point of* I. *Then,* $\sigma(H_1) \cap I$ *consists only of finite number of eigenvalues of finite multiplicity.*

REMARK 1. If $I = (-\infty, a)$ (or (a, ∞)) and if $B = WA$ where $W \in \boldsymbol{B}$ is such that $W^{-1} \in \boldsymbol{B}$, then $\|Q(\lambda)\| \to 0$ as $\lambda \to -\infty$ (or ∞). (For the proof, see § 3). Therefore $\sigma_p(H_1) \cap I$ is bounded and hence, for assuring the finiteness, we have only to assume that $Q(\lambda) \to Q(a)$ as $\lambda \to a$.

The finiteness of the eigenvalues being established by this corollary, it is desirable to have some estimate of their number. This becomes possible if we add one more condition on $Q(z)$. To simplify the exposition we first introduce the notion of the *total multiplicity* of $\sigma_p(H) \cap I$ for a selfadjoint operator H and an interval I. This is by definition the sum of the multiplicities of the eigenvalues of H_1 lying in I. Furthermore, for a completely continuous operator T we denote by $S(T; c, d)$ the sum of algebraic multiplicities of the eigenvalues of T lying in the interval (c, d) not containing 0.

THEOREM 2. *Let* $I = (a, b)$ ($a = -\infty$ *or* $b = \infty$ *is permitted). In addition to the*

[4] Here and in what follows, we call ∞ (or $-\infty$) a point of accumulation of a set ω of real numbers, if ω is not bounded from above (or from below).

hypotheses of the corollary to Theorem 1, we assume that for any τ, $0<\tau\leqslant 1$, *there is at least one* $z\in\rho(H_0)$ *satisfying* $-1\in\rho(\tau Q(z))$. *Then the total multiplicity of* $\sigma_p(H_1)\cap(a, b)$ *does not exceed* $S(Q(a); -\infty, -1)+S(Q(b); -\infty, -1)$.

COROLLARY. *If, in particular,* $I=(-\infty, a)$ *and* $A=\pm B$, *then the total multiplicity of* $\sigma_p(H_1)\cap I$ *is equal to that of* $\sigma_p(Q(a))\cap(-\infty, -1)$ (*note that* $Q(a)$ *is now selfadjoint*). *The same is true for* (a, ∞) *in place of* $(-\infty, a)$.

§ 3. Proof of theorems.

PROOF OF LEMMA 1. For proving i) we fix $z_0\in\rho(H_0)$ in such a way that $-1\in\rho(Q(z_0))$ and note that $H_1u=zu$ if and only if $u=(z-z_0)R_1(z_0)u$. Therefore, we see by (2.3) that $z\in\sigma_p(H_1)$ if and only if there exists $u\neq 0$ satisfying

$$(H_0-z)R_0(z_0)u=-(z-z_0)[R_0(z_0)B^*](1+Q(z_0))^{-1}AR_0(z_0)u\ . \tag{3.1}$$

Suppose now that $u\neq 0$ satisfies (3.1) and put

$$v=(1+Q(z_0))^{-1}AR_0(z_0)u\ . \tag{3.2}$$

By applying $(1+Q(z_0))^{-1}AR_0(z)$ to both sides of (3.1), we then get

$$v=-(z-z_0)(1+Q(z_0))^{-1}AR_0(z)[R_0(z_0)B^*]v\ . \tag{3.3}$$

By virtue of iv) of (2.2) the right member is equal to

$$-(1+Q(z_0))^{-1}(Q(z)-Q(z_0))v=-(1+Q(z_0))^{-1}(Q(z)+1)v+v\ .$$

Hence, we get $Q(z)v=-v$ by (3.3). If $v=0$, then $AR_0(z_0)u=0$ by (3.2) and hence $(H_0-z)R_0(z_0)u=0$ by (3.1). Since this is impossible, we see $v\neq 0$ and hence $-1\in\sigma_p(Q(z))$. Conversely, suppose that $Q(z)v=-v$, $v\neq 0$. Then, following the previous manipulation in the reverse order, we see that v satisfies (3.3). Here, the order of $R_0(z)$ and $R_0(z_0)$ on the right side of (3.3) can be reversed. Therefore, applying $[R_0(z_0)B^*]$ to both sides of (3.3) we see that

$$u=[R_0(z_0)B^*]v \tag{3.4}$$

satisfies (3.1). Furthermore $u=0$ implies $v=0$ by (3.3) and (3.4). Hence $z\in\sigma_p(H_1)$. The assertion ii) follows immediately from i) and the definition of H_1. As we have seen, each of (3.2) and (3.4) gives one-to-one mapping from one of the eigenspace of H_1 and that of $Q(z)$ into the other. Furthermore, the eigenspace of $Q(z)$ is finite-dimensional. The last assertion of the lemma follows from this immediately.

PROOF OF THEOREM 1. For the proof we need to introduce a family of operators H_τ depending on a complex parameter τ. Let Ω be the set of all complex numbers τ which satisfy the following condition:

(3.5) there exists $z \in \rho(H_0)$ such that $-1 \in \rho(\tau Q(z))$.

Since $Q(z)$ is completely continuous, the complement of Ω in the complex plane is at most denumerable and has no point of accumulation. For any $\tau \in \Omega$ and any z such that $-1 \in \rho(\tau Q(z))$ we put

(3.6) $$R_\tau(z) = R_0(z) - \tau[R_0(z)B^*](1+\tau Q(z))^{-1}AR_0(z) .$$

Again $R_\tau(z)$, τ being fixed, is the resolvent of a closed operator H_τ. As is easily seen, H_τ is regular in Ω in the sense that, if $z \in \rho(H_\tau)$, $\tau \in \Omega$, then $z \in \rho(H_{\tau'})$ for any τ' belonging to a sufficiently small neighbourhood of τ and that resolvent $R_{\tau'}(z)$ is regular there. In particular, Assumption 2 implies that H_τ is defined and regular in a neighbourhood of $\tau=1$. (Note that H_τ coincides with the original H_1 when $\tau=1$).

We first verify that H_1 is selfadjoint. To do so, we have only to prove the following lemma.

LEMMA 2. *If* $-1 \in \rho(Q(z))$, *then* $-1 \in \rho(Q(\bar{z}))$ *and* $R_1(z) = R_1(\bar{z})^*$.

PROOF. If $|\tau| < \min(\|Q(z)\|^{-1}, \|Q(\bar{z})\|^{-1})$, it has been shown by Kato (cf. [3; § 4]) that $R_\tau(z)$ and $R_\tau(\bar{z})^*$ exist and equal to each other. Considering that $Q(z)$ is completely continuous, both $R_\tau(z)$ and $R_\tau(\bar{z})^*$ are analytic functions of τ having only isolated singularities. Therefore the process of analytic continuation shows that $R_1(z) = \lim_{\tau\to 1} R_\tau(\bar{z})^* \equiv R$. By the resolvent equation, however, we see that R is actually equal to $R_1(\bar{z})^*$ and the lemma is established.

Next we show that, when $Q(a) = \lim_{\lambda\to a} Q(\lambda)$ exists, a is not a point of accumulation of $\sigma_p(H_1) \cap I$. The other assertion of the theorem is essentially well-known. Without loss of generality we suppose that a is the upper end point of the interval I. Let an interval $(\sigma, 1]$ of τ be contained in Ω and let Λ be the set of all pairs $\{\tau, \lambda\}$ such that $\sigma < \tau \leqslant 1$ and $\lambda \in \sigma_p(H_\tau) \cap I$. Furthermore, put[5] $F = \{\tau; \{\tau, a\}$ is a point of accumulation of $\Lambda\}$. We first show that F is a finite set. For any $\tau \in F$, one can choose a sequence $\{\{\tau_n, \lambda_n\}\} \subset \Lambda$ converging to $\{\tau, a\}$. In virtue of Lemma 1, there exists $u_n \in \mathfrak{H}$ with $\|u_n\|=1$ such that $-u_n = \tau_n Q(\lambda_n)u_n$. The last relation is rewritten as $-u_n = [\tau_n Q(\lambda_n) - \tau Q(a)]u_n + \tau Q(a)u_n$. Therefore, since $\tau_n Q(\lambda_n) \to \tau Q(a)$ and $Q(a)$ is completely continuous, a subsequence $\{u_{n'}\}$ of $\{u_n\}$ converges to some vector, say, $-u$. This yields that $-u = \tau Q(a)u$, $\|u\|=1$. Hence $F \subset \{\tau; -\tau^{-1} \in \sigma_p(Q(a))\}$ and F is a finite set. Let λ_0 be an arbitrary point of I. Since the set $F_{\lambda_0} = \{\tau; \{\tau, \lambda_0\} \in \Lambda\} = \{\tau; -\tau^{-1} \in \sigma_p(Q(\lambda_0)), \sigma < \tau \leqslant 1\}$ is a finite set, one can choose τ_0, $\sigma \leqslant \tau_0 < 1$, in such a way that no point of F_{λ_0} and no point of F are contained in the interval $(\tau_0, 1)$. Now it is clear that for any ε,

[5] In what follows a is supposed to be finite. The case when $a = \infty$ can be handled quite similarly with slight changes, say, in the definition of Λ.

$0<\varepsilon<(1-\tau_0)/2$, there is a positive number δ such that $\sigma_p(H_\tau)\cap(\lambda_0, a)\subset(\lambda_0, a-\delta)$ for every $\tau\in[\tau_0+\varepsilon, 1-\varepsilon]$. (Note that every eigenvalue of H_τ depends regularly and a fortiori continuously on τ by virtue of the regularity of H_τ). Hence one can draw a closed rectifiable curve Γ in $\rho(H_0)$ in such a way that, for any $\tau\in[\tau_0+\varepsilon, 1-\varepsilon]$, the interior of Γ contains $\sigma_p(H_\tau)\cap(\lambda_0, a)$ and no other points of $\sigma_p(H_\tau)$. Then we get

$$E_\tau(a-0)-E_\tau(\lambda_0)=-\frac{1}{2\pi i}\int_\Gamma R_\tau(z)dz \tag{3.7}$$

where $E_\tau(\mu)$ is the right continuous resolution of the identity with respect to H_τ. The right hand side of (3.7) is regular and hence continuous in τ in the sense of the norm of $\boldsymbol{B}$. Therefore, the dimension of the range of $E_\tau(a-0)-E_\tau(\lambda_0)$, that is, the total multiplicity of $\sigma_p(H_\tau)\cap(\lambda_0, a)$ is constant in $[\tau_0+\varepsilon, 1-\varepsilon]$. Since ε is arbitrary, the total multiplicity is constant throughout $(\tau_0, 1)$. Moreover this constant, say, m is clearly finite, because $\sigma_p(H_\tau)\cap(\lambda_0, a)$ has no point of accumulation if $\tau\notin F$. If there are an infinite number of eigenvalues of H_1 in the neighbourhood of a, one can enclose an arbitrary number of them with a closed curve. By carrying out the same integration as (3.7), it follows that there exists $\tau\in(\tau_0, 1)$ for which the total multiplicity of $\sigma_p(H_\tau)\cap(\lambda_0, a)$ is larger than m. This contradicts the previous result and Theorem 1 is established.

In order to prove the statement mentioned in Remark 1, we need the following lemma.

LEMMA 3.[6] *Let T, T_n, $n=1, 2, \cdots$, be in $\boldsymbol{B}$ and let S be completely continuous. If T_n converges strongly to T then T_nS converges to TS in $\boldsymbol{B}$.*

Without loss of generality we assume $I=(-\infty, a)$. Noting that $R_0(\lambda)^{\frac{1}{2}}A^*\subset[AR_0(\lambda)^{\frac{1}{2}}]^*$ for $\lambda\in I$ and that the densely defined operator $AR_0(\lambda)^{\frac{1}{2}}R_0(\lambda)^{\frac{1}{2}}A^*$ has an extension $Q(\lambda)W^{*-1}\in\boldsymbol{B}$, it is easily seen that $AR_0(\lambda)^{\frac{1}{2}}\equiv T(\lambda)$ belongs to $\boldsymbol{B}$. As a matter of fact, $T(\lambda)$ is completely continuous since $Q(\lambda)W^{*-1}=T(\lambda)T(\lambda)^*$ is completely continuous. On the other hand, we have $Q(\lambda)=T(\lambda_0)K(\lambda)T(\lambda_0)^*W^*$ where $K(\lambda)=(H_0-\lambda_0)R_0(\lambda)\in\boldsymbol{B}$. Since $K(\lambda)\to 0$ strongly as $\lambda\to-\infty$, Lemma 3 therefore shows that $\|Q(\lambda)\|\to 0$ as $\lambda\to-\infty$.

PROOF OF THEOREM 2. The additional assumption of the theorem shows that H_τ exists and regular in a region containing $0\leqslant\tau\leqslant 1$. Similarly as in the proof of Theorem 1, let $\Lambda=\{\{\tau, \lambda\}; 0<\tau\leqslant 1, \lambda\in\sigma_p(H_\tau)\cap I\}$. Let $F_a=\{\tau; \{\tau, a\}$ is a point of accumulation of $\Lambda\}$. The set F_b is defined analogously with $\{\tau, a\}$ replaced by $\{\tau, b\}$. Again F_a and F_b are finite sets. Let $F_a\cup F_b-\{1\}=\{\tau_1, \tau_2, \cdots, \tau_{N-1}\}$,

[6] For the proof, approximate S by a sequence of operators with finite-dimensional range.

$\tau_0=0$, and $\tau_N=1$. Using the same argument as bofore, one can conclude without difficulty that the total multiplicity of $\sigma_p(H_\tau)\cap I$ is equal to a finite constant m_k throughout the interval (τ_{k-1}, τ_k), $k=1, 2, \cdots$. Note that every eigenvalue of H_τ lying in I depends regularly on τ unless it comes in contact with a or b[7].

Now, we estimate the difference $n_k=m_{k+1}-m_k$, $k=1, 2, \cdots, N-1$, and first consider the case $\tau_k\in F_a-F_b$. According to the theory of regular perturbation of selfadjoint operators[7], the eigenvalues of H_τ in I are given by a set of m_k regular functions $\lambda_1(\tau), \cdots, \lambda_{m_k}(\tau)$ of τ, each eigenvalue being repeated according to its multiplicity. Suppose that p_k functions among λ_i's, say, $\lambda_1(\tau), \cdots, \lambda_{p_k}(\tau)$ tend to a as $\tau\uparrow\tau_k$ and the others do not. The number q_k is determined similarly referring to the interval (τ_k, τ_{k+1}), so that there are exactly q_k eigenvalues $\mu_j(\tau)$ tending to a as $\tau\downarrow\tau_k$. Since $\tau_k\in F_a-F_b$, it follows that $n_k=q_k-p_k$ and hence $n_k\leqslant p_k+q_k$. Let ε be such that $\varepsilon<\min(\tau_{k+1}-\tau_k, \tau_k-\tau_{k-1})$. Then, if $\lambda_0\in I$ is sufficiently close to a, each $\lambda_i(\tau)$, $i=1, 2, \cdots, p_k$, takes the value λ_0 at least once in $(\tau_k-\varepsilon, \tau_k)$. The similar statement holds for μ_j in place of λ_i. Now, put $\delta_k^{\pm}=(\tau_k\pm\varepsilon)^{-1}$. Then it follows from the above fact and Lemma 1 that $p_k+q_k\leqslant S(Q(\lambda_0); -\delta_k^-, -\delta_k^+)$. On the other hand, $S(Q(\lambda_0); -\delta_k^-, -\delta_k^+)$ is the rank of the projection operator

$$-\frac{1}{2\pi i}\int_C (Q(\lambda_0)-\zeta)^{-1}d\zeta\equiv P_{\lambda_0},$$

where C is a properly chosen closed curve enclosing the interval $(-\delta_k^-, -\delta_k^+)$. On account of the continuity of $Q(\lambda)$ at a, it is clear that $P_{\lambda_0}\to P_a$ as $\lambda_0\to a$. Hence its rank, $S(Q(\lambda_0); -\delta_k^-, -\delta_k^+)$, is constant and coincides with the algebraic multiplicity of the eigenvalue $-\tau_k^{-1}$ of $Q(a)$. The case $\tau_k\in F_b-F_a$ or $\tau_k\in F_a\cap F_b$ can be treated in a completely analogous way. Thus, we see that the total multiplicity of $\sigma_p(H_1)\cap I$, which is equal to $\sum_{k=1}^{N-1} n_k$, is majorized by

$$S(Q(a); -\infty, -1)+S(Q(b); -\infty, -1)$$

and the theorem is proved.

PROOF OF COROLLARY. Let $I=(-\infty, a)$. Since $\|Q(\lambda)\|\to 0$ as $\lambda\to-\infty$ by virtue of Remark 1, $F_{-\infty}$ is empty. Moreover, each regular function, say, $\lambda(\tau)$ representing an eigenvalue of H_τ is (strictly) monotone decreasing. Indeed, in the contrary case $\lambda(\tau)$ has an extremum $\lambda_0\in I$. Therefore, its inverse function $\theta(\lambda)$ and hence $-1/\theta(\lambda)$ have a branch point at λ_0. But this does not occur, because the analytic function $-1/\theta(\lambda)$ represents an eigenvalue of the regular function $Q(\lambda)$ of selfadjoint operators[7]. Thus we get $p_k=0$ and $q_k=n_k$, from which the assertion of the corollary follows at once.

[7] See Kato [2; §2].

§ 4. Applications.

Let $J_0[u, v]$ be a closed Hermitian form in $\mathfrak{H}$ with the dense domain $\mathfrak{D}$. Suppose that $J_0[u]=J_0[u, u]\geq 0$ for any $u\in\mathfrak{D}$ and that $J_0[u]=0$ if and only if $u=0$. Then, $J_0[u]^{\frac{1}{2}}$ difines a norm in $\mathfrak{D}$. Let $\mathfrak{H}_0$ be the completion of $\mathfrak{D}$ with respect to this norm. The following result is due to Birman [1].

Let $J'[u, v]$ be a Hermitian form with the domain $\mathfrak{D}$ such that $J'[u]\geq 0$, $u\in\mathfrak{D}$. Furthermore, suppose that J' admits an extension $\tilde{J}'$ to $\mathfrak{H}_0$ which is completely continuous in $\mathfrak{H}_0$.[8] *Then, the Hermitian form $J_1=J_0-J'$ is closed and bounded below. Let H_j be the selfadjoint operator in $\mathfrak{H}$ associated with J_j in the sense of Friedrichs. Then, $\sigma(H_1)\cap(-\infty, 0)$ consists only of a finite number of eigenvalues, each being of finite multiplicity.* Actually, Birman formulated the result with additional assumptions. But his method of proof can be carried over to the general case stated above.

For this problem Birman's method is simpler, and more powerful in the sense that it yields some sort of converse proposition. Nevertheless, it may not be worthless to see briefly how our results are applied to this problem. Unfortunately, the pair of H_0 and H_1 itself does not seem to satisfy the assumption of our theorems and we have to consider their resolvent. We shall give only an outline, omitting the details.

Let $\mathfrak{H}_\gamma$, $\gamma<0$, be the Hilbert space $\mathfrak{D}$ equipped with the inner product $(J_0-\gamma)[u, v]$ and let T_γ be the non-negative completely continuous operator associated with J' in $\mathfrak{H}_\gamma$. The fact that J_1 is closed and bounded below follows easily from the complete continuity of T_γ. Without loss of generality we assume that $H_j\geq\eta>-1$ and $(1-T_{-1})^{-1}$ exists. Now, instead of H_j consider $K_j=(H_j+1)^{-1}$, $j=0, 1$. Then, it can be shown that

$$K_1=K_0+T_{-1}(1-T_{-1})^{-1}K_0\,. \tag{4.1}$$

In fact, this is verified by a calculation similar to the one used in the proof of Proposition 5.3 of [4]. The second term on the right of (4.1) can be factored as B^*XA' with

$$B^*=T_{-1}^{\frac{1}{2}}K_0^{\frac{1}{2}},\quad X=K_0^{-\frac{1}{2}}(1-T_{-1})^{-1}K_0^{\frac{1}{2}},\quad\text{and}\quad A'=K_0^{-\frac{1}{2}}T_{-1}^{\frac{1}{2}}K_0\,.$$

Using this factorization $(A=XA')$, the function Q for the pair K_0 and K_1 can be computed as

---

[8] A form $\tilde{J}'$ is completely continuous in $\mathfrak{H}_0$ if $\tilde{J}'$ is bounded in $\mathfrak{H}_0$ and the operator associated with $\tilde{J}'$ in the usual sense is completely continuous in $\mathfrak{H}_0$. This is equivalent to say that any bounded sequence $\{u_n\}$ in $\mathfrak{H}_0$ contains a subsequence $\{u_{n'}\}$ such that $\tilde{J}'[u_{n'}-u_{m'}]\rightarrow 0$, m', $n'\rightarrow\infty$.

$$\begin{aligned} Q(\mu) &= XA'(K_0-\mu)^{-1}B^* \\ &= -\mu^{-1}XF(\lambda)(H_0-\lambda)^{-\frac{1}{2}}T_{-1}^{\frac{1}{2}}K_0^{\frac{1}{2}} \\ &= -\mu^{-1}XF(\lambda)K_0F(\lambda)^*, \end{aligned}$$

where we put $\lambda=(1-\mu)/\mu$ and

$$F(\lambda)=K_0^{-\frac{1}{2}}T_{-1}^{\frac{1}{2}}(H_0-\lambda)^{-\frac{1}{2}}, \quad \lambda<0,$$

which is in $\boldsymbol{B}$. Since $\sigma(K_0)\subset[0, 1]$ and $\lambda\uparrow 0$ when $\mu\downarrow 1$, it suffices to show that $F(\lambda)$ is completely continuous and converges as $\lambda\uparrow 0$. On the basis of the assumed complete continuity of $\tilde{J}'$ in $\mathfrak{H}_0$, however, it can be shown that

$$F(0)\equiv[K_0^{-\frac{1}{2}}T_{-1}^{\frac{1}{2}}H_0^{-\frac{1}{2}}]$$

is completely continuous in $\mathfrak{H}$. Then, the desired conclusion follows from the obvious relation

$$F(\lambda)=F(0)H_0^{\frac{1}{2}}(H_0-\lambda)^{-\frac{1}{2}}$$

and Lemma 3.

Schwinger's result concerning the Schrödinger operator with non-positive potential function is deduced from the corollary to Theorem 2. This can easily be seen from the argument given in Kato [3; §6] and the details are omitted.

Senshu University and University of Tokyo

References

[1] Birman, M. Š., On the spectrum of singular boundary-value problems (Russian), Mat. Sb. (N. S.) **55** (**97**) (1961), 125-174.
[2] Kato, T., On the convergence of the perturbation method, J. Fac. Sci., Univ. of Tokyo, Sect. I, **6** (1951), 145-226.
[3] Kato, T., Wave operators and similarity for some non-selfadjoint operators, Math. Ann., **162** (1966), 258-279.
[4] Kuroda, S. T., Perturbation of continuous spectra by unbounded operators, II, J. Math. Soc. Japan, **12** (1960), 243-257.
[5] Kuroda, S. T., An abstract stationary approach to perturbation of continuous spectra and scattering theory, to appear.
[6] Schwinger, J., On the bound states of a given potential, Proc. Nat. Acad. U.S.A., **47** (1961), 122-129.

(Received May 12, 1966)