The separation theorem on the classical system

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On a suitably constructed intuitionistic system we have the

Separation Theorem: *For a provable formula in the system, there is a proof in which are used only the axioms for implication and the axioms for the other logical symbols actually appearing in the formula.* (See e.g. Horn [2].)

Here we mean by an *axiom* also a rule of inference, regarding, for example, the modus ponens as an axiom for implication.

We shall call a system on which the above theorem holds as *separable* and the proof mentioned in the theorem as a *separated proof*. The usual classical system, which is obtained from an intuitionistic system by adding the *law of double negation* or the *law of excluded middle* as a new axiom, is not separable, since as is known, if A and B are propositional variables, Peirce's law

(1) $A\supset B\supset A\supset A$

is an example of those formulas which contain only implications as its logical symbols and which are classically true but cannot be proved without using the law of double negation or the law of excluded middle. (We assume the convention of association to the left.) So the separation theorem must suffer an alteration in order that it holds again on the usual classical system (see e.g. Kleene [3], Theorem 49).

So the investigations have been made if we can obtain a separable classical system from a separable intuitionistic system by adding some axioms. As the new axioms to be added, we expected formulas expressed by some schemata which contain only implication as its logical symbols (we call such an axiom as *implicational*), since (1) must be proved from the implicational axioms only. Moreover it was wanted, if possible, to avoid new axiom schemata containing other logical symbols than implications. LK of Gentzen [1] is a separable classical system, but it cannot be thought constructed from a separable intuitionistic system by merely adding some implicational axioms. So we did not think that it met our demand. (Since the above theorem does not use the word *sequent*, we assume the convention that a sequent can be thought as a formula regarding $\rightarrow B$ as B, and $A, \ldots, B\rightarrow C$ as $A\supset(\ldots\supset(B\supset C)\ldots)$, etc. Moreover, we regard the structural inference rules and the axiom schema $A\rightarrow A$ as implicational axioms.)

Now in this paper is shown an affirmative solution for the above problem.

And (1) is an example of axioms to be added to the intuitionistic system (for other examples, see **8**).

We shall call an intuitionistic system as *foundational* if we can construct a separable classical system from it by adding, for example, (1) as a new axiom schema. We shall take Gentzen's LJ as our foundational system in this paper, and prove that the separation theorem holds without any alterations on the system constructed from it. But the systems which can be taken as foundational are not restricted only to LJ (see **9**). As one of other examples of foundational systems, we can take the intuitionistic system of Kleene [3] if we modify it as will be mentioned in **9**.

We prove further that the separation theorem can be strengthened concerning the positive and the negative appearances of the logical symbols defined below. We say that the symbol &, $\supset$, $\neg$, $\vee$, $\forall$ or $\exists$ appears *positively* in $A\&B$, $A\supset B$, $\neg A$, $A\vee B$, $\forall xA(x)$ or $\exists xA(x)$, respectively. And the logical symbols which appear *positively* (*negatively*) in A, B or $A(x)$ of $A\&B$, $A\vee B$, $C\supset A$, $\forall xA(x)$ or $\exists xA(x)$ are said to appear *positively* (*negatively*) in $A\&B$, $A\vee B$, $C\supset A$, $\forall xA(x)$ or $\exists xA(x)$, respectively. And the logical symbols which appear *positively* (*negatively*) in C of $C\supset A$ or $\neg C$ are said to appear *negatively* (*positively*) in $C\supset A$ or $\neg C$, respectively. Now we state the separation theorem in the strengthened form. (The above separation theorem is also included in this form.) Since our system will be constructed from LJ, the theorem below is expressed by the terminology of LJ. (We assume some familiarity with LJ and LK of Gentzen [1]. And we borrow some notations of those calculi from Kleene [3].) But it can be easily translated into the usual terminology.

For a provable formula in the system, there is a separated proof in which $\rightarrow\&$, $\rightarrow\vee$ *or* $\rightarrow\exists$ ($\&\rightarrow$, $\vee\rightarrow$ *or* $\exists\rightarrow$) *is not used if* &, $\vee$ *or* $\exists$ *does not appear positively* (*negatively*) *in the formula, respectively, and in which* $\rightarrow\neg$ ($\rightarrow\forall$) *is not used if* $\neg$ ($\forall$) *does not appear positively in the formula.*

It will be proved that the separation theorem also holds in the modified system of Kleene [3] in the strengthened form.

Now we state the construction of a separable classical system and the proof of the strengthened separation theorem on it.

1. Definition of the new system. Our new classical system is obtained from LJ by adding new axioms of the form

$$(2)\quad A\cup B,\ A\supset C,\ B\supset C\rightarrow C.$$

Here we use $A\cup B$ as the abbreviation of $(A\supset B)\supset B$. It should be noted that this $\cup$ is not a primitive logical symbol, and that $A\cup B$ is classically equivalent to $A \vee B$.

2. Proof that our system is equivalent to LK. As is easily seen, the new axiom (2) is provable in the classical system LK. So our system is a subsystem of LK. And conversely, LK is a subsystem of our system, since $\neg\neg A \to A$ is obtained in our system if we take A, a contradiction, A for A, B, C, respectively in (2). So our system is equivalent to LK.

3. Some implicational theorems. It is easily seen that the sequents of the form

(3) $A \to A \cup B$,

(4) $B \to A \cup B$,

(5) $A \cup (B \cup C) \to (A \cup B) \cup C$,

(6) $A \cup B \to B \cup A$

and

(7) $A \cup A \to A$

are proved in our system by only using the implicational axioms.

Since these sequents hold good, expressions like $A \cup B \cup \ldots\ldots \cup C$ can be allowed in our system. And formulas changed in the appearing order of $A, B, \ldots\ldots, C$ in $A \cup B \cup \ldots\ldots \cup C$ are all equivalent to one another. Moreover, this fact is ascertained by only using the implicational axioms.

4. Some admissible inferences. We define, in our system, $\Gamma \to A, \ldots\ldots, B$ by $\Gamma \to A \cup \ldots\ldots \cup B$. Then, by those obtained in **3**, the inferences

$$(\to T^*)\ \frac{\Gamma \to \Theta}{\Gamma \to \Theta,\ A,}$$

$$(\to C^*)\ \frac{\Gamma \to \Theta,\ A,\ A}{\Gamma \to \Theta,\ A,}$$

$$(\to I^*)\ \frac{\Gamma \to \Theta,\ A,\ B,\ \Lambda}{\Gamma \to \Theta,\ B,\ A,\ \Lambda}$$

and

$$(cut^*)\ \frac{\Gamma \to \Theta,\ A \qquad A,\ \Gamma \to \Theta}{\Gamma \to \Theta}$$

are ascertained in our system. The first three are immediate from **3**. If Θ is void, *cut** is *cut* of LJ. If non-void, it can be ascertained as follows, putting the formula which Θ represents as C.

$$\dfrac{\dfrac{A,\ \Gamma \to C}{\Gamma \to A \supset C}\to\supset \qquad \dfrac{\Gamma \to C \cup A \qquad \dfrac{\dfrac{C \to C}{\to C \supset C}\to\supset \qquad \begin{matrix}\text{By (2)}\\ C \cup A,\ C \supset C,\ A \supset C \to C\end{matrix}}{C \cup A,\ A \supset C \to C}\,cut}{A \supset C,\ \Gamma \to C}\,cut}{\Gamma \to C.}\,cut$$

Hereafter we shall regard these inferences to be proper to our system, and discard $\rightarrow T$ and *cut* since they are included in the above inferences as the cases of void Θ. Further, we shall cite them without putting an asterisk since there occurs no confusion. So the structural inferences of our system are these ($\rightarrow T$, $\rightarrow C$, $\rightarrow I$, *cut*) and $T\rightarrow$, $C\rightarrow$ and $I\rightarrow$. It should be noted that our system is not made larger nor smaller by this modification.

5. Some theorems in *LJ*. The following (8)–(16) are provable in LJ (so also in our system) by using at most the implicational axioms and those shown in the parentheses right to the sequents.

(8)	$A,\ A\supset B\rightarrow B$	
(9)	$B\rightarrow A\supset B$	
(10)	$\rightarrow A,\ A\supset B$	
(11)	$A,\ \neg A\rightarrow$	$(\neg\rightarrow)$
(12)	$\rightarrow A,\ \neg A$	(both $\neg\rightarrow$ and $\rightarrow\neg$)
(13)	$A,\ B\rightarrow A\& B$	$(\rightarrow\&)$
(14)	$A\rightarrow A\vee B$ and $B\rightarrow A\vee B$	$(\rightarrow\vee)$
(15)	$A(t)\rightarrow\exists x A(x)$	$(\rightarrow\exists)$
(16)	$C\supset\forall x A(x)\rightarrow\forall x(C\supset A(x))$	(both $\forall\rightarrow$ and $\rightarrow\forall$)

The proof of (10) is as follows.

$$\dfrac{\dfrac{\dfrac{A\rightarrow A \qquad \dfrac{A\rightarrow A \quad B\rightarrow B}{A,\ A\supset B\rightarrow B}\ \supset\rightarrow}{A,\ A\supset(A\supset B)\rightarrow B}\ \supset\rightarrow}{A\supset(A\supset B)\rightarrow A\supset B}\ \rightarrow\supset}{\rightarrow A\cup(A\supset B).}\ \rightarrow\supset$$

The proof of (12) is as follows.

$$\dfrac{\dfrac{\dfrac{A\rightarrow A \qquad \dfrac{A\rightarrow A}{\neg A,\ A\rightarrow}\ \neg\rightarrow}{A,\ A\supset\neg A\rightarrow}\ \supset\rightarrow}{A\supset\neg A\rightarrow\neg A}\ \rightarrow\neg}{\rightarrow A\cup\neg A.}\ \supset\rightarrow$$

Others are easily proved.

6. Transformation rule. Now we give a transformation rule by which an LK-proof is transformed into a proof of our system. First, a sequent of the form $\Gamma\rightarrow A,\ \ldots,\ B$ in the LK-proof is transformed into $\Gamma\rightarrow A\cup\ \ldots\ \cup B$. (In our system, this sequent can be written again as $\Gamma\to A, \ldots, B$ as defined in **4**.) Next we shall define the transformation of each inferences of the so changed proof into the deductions in our system below, then it will be easily seen that the definition of the transformation rule of an *LK*-proof into a proof of our system is completed. For the structural inferences and antecedent rules for &, $\vee$, $\forall$ and $\exists$, we define no transformation since those can be regarded as to be of our system.

$\supset\to$ is transformed into the form

$$\frac{\dfrac{\Gamma\to\Theta,\ A \quad \overset{(8)}{A,\ A\supset B\to B}}{A\supset B,\ \Gamma\to\Theta,\ B} \qquad B,\ \Gamma\to\Theta}{A\supset B,\ \Gamma\to\Theta.}$$

We omitted to show the structural inferences but *cuts* which could easily be complemented. So the inferences shown are all *cuts*.

$\to\supset$ is into

$$\frac{\overset{(10)}{\to A,\ A\supset B} \qquad \dfrac{A,\ \Gamma\to\Theta,\ B \quad \overset{(9)}{B\to A\supset B}}{A,\ \Gamma\to\Theta,\ A\supset B}}{\Gamma\to\Theta,\ A\supset B.}$$

$\neg\to$ is into

$$\frac{\Gamma\to\Theta,\ A \qquad \overset{(11)}{A,\ \neg A\to}}{\neg A,\ \Gamma\to\Theta.}$$

$\to\neg$ is into

$$\frac{\overset{(12)}{\to A,\ \neg A} \qquad A,\ \Gamma\to\Theta}{\Gamma\to\Theta,\ \neg A.}$$

$\to\&$ is into

$$\frac{\Gamma\to\Theta,\ B \qquad \dfrac{\Gamma\to\Theta,\ A \quad \overset{(13)}{A,\ B\to A\& B}}{B,\ \Gamma\to\Theta,\ A\& B}}{\Gamma\to\Theta,\ A\& B.}$$

$\to\vee$ is into

$$\frac{\Gamma\to\Theta,\ A \quad \overset{(14)}{A\to A\vee B}}{\Gamma\to\Theta,\ A\vee B} \quad \text{or} \quad \frac{\Gamma\to\Theta,\ B \quad \overset{(14)}{B\to A\vee B}}{\Gamma\to\Theta,\ A\vee B.}$$

$\to\exists$ is into

$$\frac{\Gamma\to\Theta,\ A(t) \qquad \overset{(15)}{A(t)\to\exists x A(x)}}{\Gamma\to\Theta,\ \exists x A(x).}$$

$\rightarrow\forall$ has been transformed by the first step into $\rightarrow\forall$ of our system or into the inference of the form

$$\frac{\Gamma\rightarrow C\cup A(a)}{\Gamma\rightarrow C\cup\forall xA(x).}$$

This is transformed into the form

$$\dfrac{\dfrac{\dfrac{C\supset A(a), \quad \dfrac{\Gamma\rightarrow C\cup A(a)}{\Gamma\rightarrow A(a)}\ \supset\rightarrow,\ cut}{C\supset\forall xA(x), \quad \Gamma\rightarrow A(a)}\ \forall\rightarrow,\ (16),\ cut}{C\supset\forall xA(x), \quad \Gamma\rightarrow\forall xA(x)}\ \rightarrow\forall}{\Gamma\rightarrow C\cup\forall xA(x).}\ \rightarrow\supset$$

Now every *LK*-proof can be transformed into that of our system.

7. Proof of the strengthened separation theorem on our system. Suppose a formula D is provable in our system. Then there is an *LK*-proof for the sequent $\rightarrow D$ since our system is equivalent to *LK*. Moreover, we can think, owing to Gentzen's Hauptsatz on *LK*, that the *LK*-proof for the sequent contains no *cut*. Then we transform this proof into that of our system as defined in **6**. If the endsequent has at most only one formula in the succedent, the endsequent is not changed by the transformation. So our endsequent $\rightarrow D$ remains the same. It is also easily seen from the definition of the transformation that in the transformation of each inference are used only implicational axioms and axioms for the logical symbol which concerns the inference in question. And since in the *LK*-proof without *cut* only those axioms for the symbols which actually appear in the endsequent are used besides structural axioms, the proof of our system which is obtained now uses only implicational axioms and those for the symbols which appear in D. The property concerning the positive and the negative appearances of logical symbols are easily seen from **5** and **6**. So the theorem is proved in our system.

8. Examples of axioms to be added. We have added (2) to *LJ* in constructing a separable classical system. But (2) is not the only possibility which gives the property of separation to the classical system. If we allow in *LJ* the rule of substitution that we can substitute any formulas for the propositional variables, it is obvious from above that any *LK*-provable sequents which contain only implication as its logical symbols and of which we can prove (2) by using only the implicational axioms of *LJ* can serve our purpose. Some examples of substitutes for (2) are (1) and (6). (6) is equivalent to $B\supset A,\ A\supset B\supset B\rightarrow A$. And if we take a contradiction for B, we obtain the law of double negation from this.

As is mentioned in **3**, (6) is provable from (2). (1) can be proved from (6) by using only implicational axioms as follows: Substituting $A\supset B$ for B in (6), we have $A\cup(A\supset B)\rightarrow(A\supset B)\cup A$, but by (10), we have $\rightarrow A\cup(A\supset B)$ in LJ, so (1) is obtained. The proof of (2) from (1) is as follows.

$$\dfrac{\dfrac{\dfrac{\dfrac{A\supset C,\ C\supset B\rightarrow A\supset B \qquad B\rightarrow B}{A\cup B,\ A\supset C,\ C\supset B\rightarrow B}\supset\rightarrow \qquad C\rightarrow C}{A\cup B,\ A\supset C,\ C\supset B,\ B\supset C\rightarrow C}\supset\rightarrow}{A\cup B,\ A\supset C,\ B\supset C\rightarrow C\supset B\supset C}\rightarrow\supset \qquad \underset{\text{By (1)}}{C\supset B\supset C\rightarrow C}}{A\cup B,\ A\supset C,\ B\supset C\rightarrow C.}\ cut$$

Here we used only implicational axioms.

9. Other foundational systems. Examples of the foundational systems are not restricted to LJ. As Gentzen showed in [1], LJ is equivalent to other intuitionistic systems. So we can define a transformation rule of an LJ-proof into another intuitionistic system. It is easily seen that if we can give such a transformation rule for a given intuitionistic system as satisfies the following (a) and (b), then we can take the given system as foundational. Moreover, if (c) is satisfied, the strengthened separation theorem holds on the classical system obtained from the given system.

(a) Each inferences of the LJ-proof are transformed into deductions of the given system.

(b) In the transformation of each inferences in the LJ-proof are used at most the implicational axioms and those axioms for the logical symbol which concerns the inference.

(c) In the transformation of a succedent (antecedent) inference of &, $\vee$ or $\exists$ are used at most the implicational axioms and the axioms for the logical symbol called positive (negative), and in the transformation of an antecedent inference of $\neg(\forall)$ are used at most the implicational axioms and the axioms for $\neg(\forall)$ called negative.

As an example, let us take the intuitionistic system of Kleene [3]. Since it is an intuitionistic system, the axiom schema 8 ($\neg\neg A\supset A$) is replaced by

$$8^{I}.\quad \neg A\supset(A\supset B).$$

Further we replace his

$$9.\quad \frac{C\supset A(x)}{C\supset\forall x A(x)}$$

by

$$9'.\quad \frac{C\supset(D\supset A(x))}{C\supset(D\supset\forall x A(x)).}$$

Note that this modification is introduced to obtain a *separable intuitionistic system.*

It is assured that a transformation rule satisfying (a) and (b) can be defined concerning this system.

As usual, the axioms for the each logical symbols, except implication, in this system are divided into those introducing the symbol and those eliminating the symbol. So let us call the former as positive and the latter as negative. (3, 5a, 5b, 7, 9′ and 11 are positive. 4a, 4b, 6, 8′, 10 and 12 are negative.) Then we can define a transformation rule satisfying all (a), (b) and (c). So we can construct a classical system on which the strengthened separation theorem holds from this system by adding, for example, (1) as a new axiom schema.

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After this paper was submitted, the author obtained two further results as follows.

(1) The separation theorem on the classical propositional system can be proved algebraically as Horn did on the intuitionistic system.

(2) The intermediate system LC obtained by adding $((A \supset B) \supset C) \supset (((B \supset A) \supset C) \supset C)$ as a new axiom schema to the intuitionistic system is separable. (For this system, see [4] and [5].)

The details will be given out later.

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