A lemma on open convex cones

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§0. Let V be an n-dimensional real vector space with a positive definite inner product $<,>$, and Ω be an open convex cone which does not completely contain any straight line. We assume the vertex of the cone Ω is the origin of V. We denote by Ω^* the subset $\{x\in V|<x, u>>0$ for $^{\forall}u\in\bar{\Omega}-\{0\}\}$, where $\bar{\Omega}$ is the topological closure of Ω in V. Then the following theorem is well known [6];

THEOREM A. (i) *$\Omega^*\neq\phi$.*

(ii) *Ω^* is also an open convex cone which does not completely contain any straight line, and whose vertex is the origin of V. (Therefore $(\Omega^*)^*$ has a meaning.)*

(iii) $(\Omega^*)^*=\Omega$

Using $\Omega^*\neq\phi$ we shall prove

LEMMA B. $\Omega\cap\Omega^*\neq\phi$.

The following theorems C and D are corollaries to Lemma B.

THEOREM C. *Suppose Ω is homogeneous, then Ω is self-dual if and only if the automorphism group $G(\Omega)$ of Ω is self-adjoint with respect to the inner product $<,>$.*

The precise definitions will be given in the section 1. Combining the theorem C and a theorem of G. D. Mostow, we shall prove

THEOREM D. *Suppose Ω is homogeneous, then Ω is self-dual if and only if the automorphism group $G(\Omega)$ of Ω is completely reducible (for some inner product of V).*

Theorem D was mentioned in Rothaus [5]. After the preparation of this paper, we have heard he has a proof.

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§1. For any element v of V, we denote by P_v the hyperplane $\{x\in V|<x, v>=0\}$. We define the subset P_v^+ (resp. P_v^-) of V by $P_v^+=\{x\in V|<x, v>>0\}$, $P_v^-=\{x\in V|<x, v><0\}$. Then from the definition we have $v\in\Omega^*$ if and only if $P_v^+\supset\Omega-\{0\}$. The following Lemma 1 will be proved in the next section.

LEMMA 1. (i) *For any element $v\in\Omega^*$, the intersection of Ω and any hyper-*

plane parallel with P_v is bounded.

(ii) *For any $v_0 \in \Omega^*$ such that $v_0 \notin \bar{\Omega}$, there exists $v_1 \in P_{v_0}$ such that $v_1 \in \Omega^*$.*

We shall prove the next Lemma B by the induction on the dimension of the cone Ω.

LEMMA B. *The intersection of Ω and Ω^* is not empty.*

PROOF. We do not prove Theorem A but use it here. When the dimension of Ω is 1, Lemma B is obvious. Let us assume that we can prove it provided the dimension of Ω is less than n. Now suppose the dimension of Ω is n. Take any element v_0 of Ω^* and fix it once for all. If $v_0 \in \bar{\Omega}$, since Ω is open, we can easily see $\Omega \cap \Omega^* \neq \phi$. Therefore we assume $v_0 \notin \bar{\Omega}$. By Lemma 1, there exists an element $v_1 \in \Omega^*$ such that $<v_0, v_1>=0$. We may assume that v_0 and v_1 are unit vectors. We decompose V in the following two ways: (a) $V=P_{v_1} \oplus \{v_1\}$, and (b) $V=P_{v_0} \oplus \{v_0\}$ where $\{v_\alpha\}$ is the line spanned by v_α. We denote by p_1 the projection of V onto P_{v_1} according to the decomposition (a). Let Ω_1 be the image of Ω by the projection p_1. It is easy to see that Ω_1 is an open convex cone in P_{v_1}, (the vertex being the origin of P_{v_1}). Now we shall verify that Ω_1 does not completely contain any straight line. For any element $r \in V$, we denote $p_1(r)$ by r_1. Then we have $r=r_1+\lambda v_1$ where $\lambda=<r_1, v_1>$. For any $r \in \Omega$, since $v_0 \in \Omega^*$ and $<v_0, v_1>=0$, we have $0<<v_0, r>=<v_0, r_1>+\lambda<v_0, v_1>=<v_0, r_1>$. Therefore $\Omega_1 \cap P_{v_0}=\phi$. Now suppose Ω_1 thatcontained a straight line l. Since $\Omega_1 \cap P_{v_0}=\phi$, l must be parallel with P_{v_0}. Therefore l is contained in some hyperplane $P^c_{v_0}=\{v \in V \mid <v, v_0>=c\}$ parallel with P_{v_0}. For any element $x \in p_1^{-1}(l) \cap \Omega$ we have $<x, v_0>=<x_1+<x, v_1> v_1, v_0>=<x_1, v_0>=c$. Hence we have $p_1^{-1}(l) \cap \Omega \subset P^c_{v_0} \cap \Omega$. By Lemma 1, $p_1^{-1}(l) \cap \Omega$ is bounded, contrary to the assumption that l is a straight line. Thus Ω_1 does not completely contain a straight line.

Since the dimension of Ω_1 is $n-1$, using the hypothesis of the induction, we can find an element $u_1 \in \Omega_1$ such that $<u_1, x_1>>0$ for any $x_1 \in \bar{\Omega}_1-\{0\}$. (Here the inner product in P_{v_1} is the one induced from $<\ ,\ >$.). Take any $u \in \Omega$ such that $p_1(u)=u_1$ and fix u once for all. For any $x \in \bar{\Omega}-\{0\}$, $x_1=p_1(x)$ is not 0. In fact, if otherwise x will be contained in $\{v_1\}$. Therefore $<v_0, x>=0$ contrary to the assumption that $<v_0, x>>0$ for any $x \in \bar{\Omega}-\{0\}$. For any $x \in \bar{\Omega}-\{0\}$, $x_1=p_1(x) \in p_1(\Omega) \subset \overline{p_1(\Omega)}=\bar{\Omega}_1$. Thus we have $<u_1, x_1>>0$. Then $<u, x>=<u_1, x_1>+\lambda\mu<v_1, v_1>$ when $\lambda=<u, v_1>$ and $\mu=<x, v_1>$. Since $u, x \in \bar{\Omega}$ and $v_1 \in \Omega^*$, we have $0<(v_1, u)=\lambda$ and $0<(v_1, x)=\mu$. Therefore $\lambda>0$ and $\mu>0$. Consequently we get $<u, x>>0$ for any $x \in \bar{\Omega}-\{0\}$. This means that $u \in \Omega^*$. Thus we have proved $\Omega \cap \Omega^* \neq \phi$. Q.E.D.

Let $G(\Omega)$ be the identity component of the group $\{s \in GL(V) | s(\Omega)=\Omega\}$. $G(\Omega)$

is called the *automorphism group of* Ω.

We denote by a^* the transpose-inverse ${}^t a^{-1}$ of a with respect to $<\,,\,>$. Let $G(\Omega)^*$ denote $\{s^*|s\in G(\Omega)\}$. It is easy to see $G(\Omega^*)=G(\Omega)^*$, by Theorem A. We say that Ω is *homogeneous* if $G(\Omega)$ is transitive on Ω. It is well known that Ω^* is homogeneous if Ω is so [6]. When $\Omega=\Omega^*$, we call Ω a *self-dual cone*. A subgroup G of $GL(V)$ is called *self-adjoint* if $G^*=G$.

Suppose now $\Omega=\Omega^*$. Then $G(\Omega)=G(\Omega^*)=G(\Omega)^*$. Therefore we see that $G(\Omega)$ is self-adjoint if Ω is self-dual. Conversely suppose $G(\Omega)=G(\Omega)^*$ then $G(\Omega)=G(\Omega^*)$. Since $\Omega=G(\Omega)\cdot p$ and $\Omega^*=G(\Omega^*)\cdot p$ for any $p\in\Omega\cap\Omega^*$ ($\neq\phi$ from Lemma B). Thus Theorem C is proved.

PROOF OF THEOREM D. If Ω is self-dual with respect to the inner product $<\,,\,>$, then by Theorem C, $G(\Omega)=G(\Omega)^*$. Therefore $G(\Omega)$ is completely reducible. $G(\Omega)$ is the identity component of an algebraic group $\widetilde{G}$ ([6], Note that $\widetilde{G}$ is not only a subgroup of the affine group of V but that of $GL(V)$.) Now suppose that $G(\Omega)$ is completely reducible. Then so is $\widetilde{G}$. (See e.g. G. D. Mostow, Amer. J. Math. 78(1956), 200–221, Lemma 3.1). By Mostow's theorem ([4] or [1] p. 492), there exists $g\in GL(V)$ such that $g\widetilde{G}g^{-1}$ is stable under $x\to x^*$. Therefore $\widetilde{G}$, in particular $G(\Omega)$, self-adjoint with respect to a suitable inner product. By Theorem C, Ω is self-dual with respect to that inner product. Q.E.D.

§ **2.** In this section we shall prove Lemma 1.

PROOF OF (i). We have to show that $\Omega\cap P_v^a$ is bounded for any real number a. Suppose that for any integer $n>0$, there existed $\alpha_n\in\Omega^*$ such that $||\alpha_n||>n$ and $<\alpha_n, v>=a$. Taking a subsequence if necessary we may assume that the sequence $\alpha_n/||\alpha_n||$ converges to some $\beta\in\Omega^*$. Since $||\beta||=1$, we have $\beta\neq 0$. Then $<\beta, v>=\lim\limits_n \alpha/||\alpha_n||=0$. Since $\Omega=(\Omega^*)^*$, $\beta\in\Omega^*-\{0\}$, and $v\in\Omega$, we must have $<v, \beta>>0$. This is a contradiction.

PROOF OF (ii). We write P for the hyperplane $P_{v_0}^{<v_0, v_0>}=\{x\in V|<x-v_0, v_0>=0\}$. $P\cap\Omega$ is not empty. In fact $<x, v_0>$ is positive for any $x\in\Omega$ so that $\Omega=\{cy|y\in P\cap\Omega,\ c>0\}$. Since $P\cap\bar{\Omega}$ is compact by (i) and $v_0\notin\bar{\Omega}$ by our hypothesis, we have $v_0\notin P\cap\bar{\Omega}$. Therefore the distance between v_0 and $P\cap\bar{\Omega}$ is strictly positive. Let s be a point in $P\cap\bar{\Omega}$ such that the length $||s-v_0||$ equals that distance. We put $v_1=s-v_0$. Now v_1 is what we want to have; i.e. $<v_1, x>>0$ for any $x\in\bar{\Omega}-\{0\}$. We may assume that x is in $P\cap\bar{\Omega}$. Since $P\cap\bar{\Omega}$ is convex, $P\cap\bar{\Omega}$ contains $tx+(1-t)s$ for any t in the internal [0, 1]. By the choice of s, we have $||s-v_0||^2\leqq||tx+(1-t)s-v_0||^2=||t(x-v_0)+(1-t)v_1||^2$. The first derivative of the right hand side must be non-negative at $t=0$. Hence we have $2<x-v_0-v_1,$

$v_1>\geqq 0$. In view of $<v_0, v_1>=0$, this gives $<x, v_1>\geqq||v_1||^2>0$ and (ii) is proved.

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