A remark on the Riemann-Roch-Weil Theorem

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A. Weil proved in [1] the generalized Riemann-Roch theorem for generalized matrix divisors in the algebraic function field k in one variable. There he considered the case where the Riemann surface $\mathfrak{R}$ of k contains a finite number of elliptic points but no parabolic points for a simply connected covering surface. Here we shall consider the case where $\mathfrak{R}$ contains also a finite number of parabolic points.

In § 1 we shall define a local divisor at a parabolic point. In § 2 we shall prove the Riemann-Roch-Weil theorem in our case. In § 3 we shall define the holomorphic form associated with a matrix representation $\mathfrak{M}$ of Γ. All the holomorphic forms associated with $\mathfrak{M}$, make a vector space over the complex number field. We shall calculate the dimension of this vector space by applying the Riemann-Roch-Weil theorem.

§ 1. Divisors of Riemann Surface.

Let k be a field of algebraic functions in one variable over the complex number field C, $\mathfrak{R}$ be its Riemann surface and g be its genus. We take a finite set $\{P_1, \cdots, P_s\}$ of points on $\mathfrak{R}$ and attach an integer n_λ to each point P_λ $(1 \leqq \lambda \leqq s_1)$ and ∞ to P_λ $(s_1+1 \leqq \lambda \leqq s)$. We can construct a simply connected covering Riemann surface $\mathfrak{R}$ over $\mathfrak{R}-\{P_{s_1+1}, \ldots, P_s\}$ whose covering transformation group Γ is generated by $\{\alpha_1, \cdots, \alpha_g, \beta_1, \cdots, \beta_g, \gamma_1, \cdots, \gamma_s\}$ and has (s_1+1)-fundamental relations,

$$\begin{cases} \alpha_1\beta_1\alpha_1^{-1}\beta_1^{-1} \cdots \alpha_g\beta_g\alpha_g^{-1}\beta_g^{-1}\gamma_1 \cdots \gamma_s=1 \\ \gamma_\lambda^{n_\lambda}=1 \qquad (1 \leqq \lambda \leqq s_1). \end{cases}$$

From now on we put the assumption

$$2g-2+\sum_{\lambda=1}^{s}\left(1-\frac{1}{n_\lambda}\right)>0.$$

From this assumption it follows that $\mathfrak{R}$ is isomorphic to the upper half plane $\mathfrak{H}$. Hence we regard $\mathfrak{R}$ and Γ as $\mathfrak{H}$ and a Fuchsian group of the first kind on $\mathfrak{H}$ respectively. Let $\mathfrak{P}$ be the set of all parabolic points with respect to Γ. Put $\mathfrak{H} \cup P=\mathfrak{H}^*$, then Γ operates on $\mathfrak{H}^*$. Let us consider the quotient space $\Gamma \backslash \mathfrak{H}^*$ and we denote the equivalence class of $z\,(z \in \mathfrak{H}^*)$ by $\Gamma(z)$. As usual we can make the quotient space $\Gamma\backslash\mathfrak{H}^*$ a compact Riemann surface by defining the local coordinate as follows.

(i) For an ordinary point z_0 of $\mathfrak{H}$, take the local coordinate $\tau=z-z_0$ at z_0 and the local coordinate $t=\tau$ at $\Gamma(z_0)$.

(ii) For an elliptic point z_0 of $\mathfrak{H}$ with the isotropy group Γ_{z_0} of order n_{z_0}, take the local coordinate $\tau=\frac{z-z_0}{z-\bar{z}_0}$ at z_0 and the local coordinate $t=\tau^n$ $(n=n_{z_0})$ at $\Gamma(z_0)$.

(iii) For a finite parabolic point z_0 with the isotropy group $\Gamma_{z_0}=\{\gamma^n|n\in z\}$ take the local coordinate $t=e^{2\pi i\tau}$ at $\Gamma(z_0)$, where $\tau=-1/h(z-z_0)$. Then $\gamma(\tau)=\tau+1$. It is well-known that $\Gamma\backslash\mathfrak{H}^*\approx\mathfrak{R}$.

Now, we define k_P and K_P for each point P of $\mathfrak{R}$ as follows.

(i) For $P=\Gamma(z_0)$, $z_0\in\mathfrak{H}$ we denote by k_P the set of all meromorphic functions $f(t)$ at $t=0$ and by K_P the set of all meromorphic functions $f(\tau)$ at $\tau=0$.

(ii) For $P=\Gamma(z_0)$, $z_0\in\mathfrak{P}$ we denote by k_P the set of all meromorphic functions $f(t)$ at $t=0$, by $k_P^{(n)}$ the set of all meromorphic functions $f(t_n)$ $(t_n=t^{1/n})$, at $t_n=0$ and put $K_P=\bigcup_{n=1}^{\infty}k_P^{(n)}$. Then K_P is the algebraic closure of k_P. Each element $f\in K_P$ can be expressed as $f=t^{a_0}(\alpha_0+\alpha_1 t^{a_1}+\cdots)$ where $\alpha_0\neq 0$ and each a_i $(i=0, 1, \cdots)$ is a rational number. We denote the local index of f at P by $i_P(f)=a_0$ and $f\succ_P 0$ when $i_P(f)\geqq 0$.

Let $F=(f_{ij})$ be a function matrix of K_P, namely a square matrix of degree r, with entries $f_{ij}\in K_P$. We define the local index of F at P by $i_P(F)=i_P(\det(F))$ and denote $F\succ_P 0$ if all $f_{ij}\succ_P 0$ $(i, j=1, 2, \cdots, r)$. It follows immediately $i_P(F_1F_2)=i_P(F_1)+i_P(F_2)$.

A local divisor Θ_P of degree r at $P\in\mathfrak{R}$ is defined as follows(*). Put $U_P=\{F\in GL(r, K_P)|F\succ_P 0,\ F^{-1}\succ_P 0\}$, which is a subgroup of $GL(r, K_P)$ invariant under each transformation of the isotropy group Γ_{z_0}. Then a local divisor Θ_P is defined as a left coset $U_P\theta_P$ $(\theta_P\in GL(r, K_P))$ which is invariant under each transformation of Γ_{z_0}. We define the local index of Θ_P by $i_P(\Theta_P)=i_P(\theta_P)$. This is independent of the choice of the representative θ_P.

PROPOSITION 1. *Let Θ_P be a local divisor of degree r at an elliptic or a parabolic point P, then θ_P can be chosen in the form*

$$\theta_P=\begin{pmatrix} t^{d_1} & & 0 \\ & \ddots & \\ 0 & & t^{d_r}\end{pmatrix}\cdot\theta_{0P}(t)$$

(*) This definition is given by A. Weil [1] in case P is an ordinary point or an elliptic point.

where $\theta_{0P}(t)$ belongs to $GL(r, k_P)$, d_i $(1\leqq i\leqq r)$ is a rational number such that $0\leqq d_i<1$, and d_i is a multiple of $1/n_P$ when P is elliptic.

PROOF. This was proved by A. Weil [1] in case P is elliptic. Hence we consider the parabolic case. By the definition of Θ_P, there exists $V\in U_P$ such that $\theta_P^{\gamma}=V\cdot\theta_P$. It is easy to see $\theta_P^{\gamma^\nu}=V^{\gamma^{\nu-1}}\cdot\ \cdots\ \cdot V^{\gamma}\cdot V\cdot\theta_P$. If we take another representative θ'_P of Θ_P, there exists $U\in U_P$ such that $\theta'_P=U\theta_P$. Then $\theta_P'^{\gamma}=(U^{\gamma}VU^{-1})$ θ'_P. Put $V'=U^{\gamma}VU^{-1}$, then we have $V'(0)=U(0)V(0)U^{-1}(0)$ for $t=0$. Since there exists an integer l such that $\theta_P^{\gamma^l}=\theta_P$ by the definition of Θ_P, we have $V^{\gamma^{l-1}}\ V^{\gamma^{l-2}}$ $\cdots\ V^{\gamma}V=E$ (unit matrix) and hence $V^l(0)=E$ for $t=0$. We know that there exists $M\in GL(r, C)$ such that $V(0)=M^{-1}DM$ where

$$D=\begin{pmatrix} \xi_l^{d_1} & & 0 \\ & \ddots & \\ 0 & & \xi_l^{d_r}\end{pmatrix},$$

d_i's are rational integers such that $0\leqq d_1\leqq d_2\leqq\ \cdots\ \leqq d_r\leqq l-1$, ξ_l is a l-th primitive root of 1. Put $\theta'_P=M\theta_P$, then $\theta_P'^{\gamma}=V'\theta'_P$ where $V'(0)=M\cdot V(0)M^{-1}=D$. Hence we can assume that $\theta_P^{\gamma}=V\cdot\theta_P$, $V(0)=D$, without loss of generality. Put $\theta_P^{\gamma^\nu}=V_\nu\theta_P$ and $\bar{\theta}_P=\sum_{\nu=0}^{l-1}D^{-\nu}\theta_P^{\gamma^\nu}=\sum_{\nu=0}^{l-1}(D^{-\nu}\cdot V_\nu)\theta_P$, then $\sum_{\nu=0}^{l-1}D^{-\nu}V_\nu\succ_P 0$, $V_\nu(0)=D^\nu$ and $(\sum_{\nu=0}^{l-1}D^{-\nu}V_\nu)(0)$ $=l\cdot E$. Hence $\sum_{\nu=0}^{l-1}D^{-\nu}V_\nu$ belongs to U_P and we have $\Theta_P=U_P\bar{\theta}_P$. It is easy to see that $\bar{\theta}_P^{\gamma}=D\cdot\bar{\theta}_P$. Put

$$\Delta=\begin{pmatrix} t^{d_1/l} & & 0 \\ & \ddots & \\ 0 & & t^{d_r/l}\end{pmatrix}$$

then $(\Delta^{-1}\bar{\theta}_P)^{\gamma}=\Delta^{-1}\bar{\theta}_P$ and there exists $\theta_{0P}(t)\in GL(r, k_P)$ such that $\bar{\theta}_P=\Delta\cdot\theta_{0P}$

(q.e.d.)

DEFINITION. As usual, a divisor Θ of degree r on $\mathfrak{R}$ is defined as a system of local divisors $\{\Theta_P\}$ $(P\in\mathfrak{R})$ such that $\Theta_P=E$ for almost all P. We define the total index of Θ by $I(\Theta)=\sum_{P\in\mathfrak{R}}i_P(\Theta_P)$.

§ 2. The Riemann-Roch-Weil Theorem.

Let a divisor Θ of degree r and a divisor Θ' of degree r' be given. Put $L(\Theta, \Theta')=\{\Phi\in M(r\times r', k)\,|\,\Theta\Phi\Theta'^{-1}\succ_P 0$ for all $P\in\mathfrak{R}\}$, then $L(\Theta, \Theta')$ is a vector space over C. By Proposition 1 we can assume that

(1) $\Theta_P=\Delta_P\Theta_{0P}$, $\Delta_P=(\delta_{ij}t^{d_i})$ $(0\leqq d_i<1)$, $\Theta_{0P}=(\theta_{ij})\in GL(r, k_P)$, $\Theta'_P=\Delta'_P\Theta'_{0P}$, $\Delta'_P=(\delta_{ij}t^{d'_j})$ $(0\leqq d'_i<1)$, $\Theta'_{0P}=(\theta'_{ij})\in GL(r', k_P)$ where $d_h=d'_k=0$ when P is an ordinary point, and

d_h, d'_k are multiples of $1/n_P$ when P is an elliptic point. We denote $\nu_P=\sum_{h=1}^{r}\sum_{k=1}^{r'}$ $\langle d_h-d'_k\rangle$ where $\langle x\rangle=x-[x]$.

Let dj be a differential of k. We call $(f_{ij})dj$ for $f_{ij}\in k$, a differential matrix of k. Let $\mathcal{D}$ be the set of all $r'\times r$ differential matrices of k such that $\Theta'\frac{dI}{d\tau}\Theta^{-1}\succ_P 0$ for all $P\in\mathfrak{R}$. Let $\mathcal{D}_0$ be the subset of $\mathcal{D}$ consisting of all dI's such that at each parabolic point P, $S_{kh}(t)\succ_P 0$ holds for each pair (k, h) with $d_h=d'_k$ at P where $(S_{kh}(t))=\Theta'_0\frac{dI}{dt}\Theta_0^{-1}$. It is easy to see that $\mathcal{D}$ is a vector space over C and $\mathcal{D}_0$ is a subspace of $\mathcal{D}$. We denote $l(\Theta, \Theta')$=dimension of $L(\Theta, \Theta')$ and d_0=dimension of $\mathcal{D}_0$.

THEOREM (Riemann-Roch-Weil).

$$l(\Theta, \Theta')=r'I(\Theta)-rI(\Theta')-rr'(g-1)-\sum_{P\in\mathfrak{R}}\nu_P+d_0\,.$$

PROOF. We can prove this theorem by the same method as Weil [1]. But it is necessary to consider here the parabolic points which did not appear in his case. Let $\Phi=(\varphi_{ij})\in L(\Theta, \Theta')$. Then $\Theta\Phi\Theta'^{-1}=\Delta\Psi\Delta'^{-1}\succ_P 0$ for all $P\in\mathfrak{R}$, where $\Psi=(\psi_{hk})$ $=\Theta_0\Phi\Theta_0'^{-1}$. Since $-1<d_h-d'_k<1$, we obtain $\Psi\succ_P 0$ for all $P\in\mathfrak{R}$. Put $\Theta_0^{-1}=(\vartheta_{ij})$ and $\Theta_0'^{-1}=(\vartheta'_{ij})$.

Let us attach a non-negative integer $b(P)$ to each point $P\in\mathfrak{R}$, such that

$$-b(P)\leqq\min_{j,k}(i_P(\vartheta_{jk}))+\min_{j,k}(i_P(\theta'_{jk}))\,.$$

where $b(P)=0$ in case $\Theta_P=E_r$ and $\Theta'_P=E_{r'}$. Then φ_{ij} belongs to $L(\prod_{P\in\mathfrak{R}}P^{b(P)})$. If we take $\{b(P)\}$ such that $\sum_{P\in\mathfrak{R}}b(P)>2g-2$, then by the Riemann-Roch theorem

$$l(\prod_{P\in\mathfrak{R}}P^{b(P)})=\dim L(\prod_{P\in\mathfrak{R}}P^{b(P)})=\sum_{P\in\mathfrak{R}}b(P)-g+1\,.$$

Let us attach a non-negative integer $a(P)$ to each $P\in\mathfrak{R}$ such that $-a(P)\leqq i_P(\theta_{ij})$ for $1\leqq i, j\leqq r$ and $-a(P)\leqq i_P(\vartheta'_{kl})$ for $1\leqq k, l\leqq r'$, where $a(P)=0$ in case $\Theta_P=E_r$ and $\Theta'_P=E_{r'}$. ψ_{hk} has a pole at each $P\in\mathfrak{R}$ at most of order $2a(P)+b(P)$ and satisfies the following condition.

(2) $\qquad \psi_{hk}\succ_P 0$ for all $P\in\mathfrak{R}$ and $\psi_{hk}(0)=0$ for (h, k) with $d_h<d'_k$.

(2) is equivalent to

$$\mathrm{Res}_P(t^{\rho}\cdot\psi_{hk}(t))=0 \text{ at each point } P \text{ on } \mathfrak{R}$$

for $0\leqq\rho\leqq 2a(P)+b(P)-1$, and for $\rho=-1$ for the pair (h, k) with $d_h<d'_k$ at P. This condition can be described by Φ as follows.

(3) $$\mathrm{Res}_P\{t^{\rho}(\sum_{i,j}\theta_{hi}\varphi_{ij}\vartheta'_{jk})\}=0 \text{ at each } P\in\mathfrak{R}$$

for $1\leqq h\leqq r$, $1\leqq k\leqq r'$, $0\leqq\rho\leqq 2a(P)+b(P)$ and moreover for $\rho=-1$ for the pair (h, k) with $d_h<d'_k$.

Now, put $\mathfrak{M}(P)=\{(h, k);\ d_h<d'_k\}$ and $m(P)=$the number of the element of $\mathfrak{M}(P)$. Since there are $rr'(2a(P)+b(P))+m(P)$ equations (3) at each point P on $\mathfrak{R}$, we obtain in total $\sum_{P\in\mathfrak{R}}\{rr'(2a(P)+2b(P))+m(P)\}$ equations. The number of unknowns is $rr'(\sum_{P\in\mathfrak{R}}b(P)-g+1)$. If we obtain the number of independent relations among the left hand sides of (3), we can calculate $l(\Theta, \Theta')$.

If there exists a system $\{C_{h,k}^{(P,\rho)}\}\neq\{0\}$, $C_{h,k}^{(P,\rho)}\in C$ such that

$$\sum_{h,k,\rho,P} C_{h,k}^{(P,\rho)}\,\mathrm{Res}_P\{t^\rho(\sum\theta_{hi}\varphi_{ij}\vartheta'_{jk})\}=0 \text{ for all } \varphi_{ij}\in L(\prod_{P\in\mathfrak{R}}P^{b(P)}) \text{ then}$$

$$\sum_{i,j}[\sum_{P\in\mathfrak{R}}\mathrm{Res}_P(\sum_{h,k,\rho}C_{h,k}^{(P,\rho)}\,t^\rho\theta_{hi}\varphi_{ij}\vartheta'_{jk})]=0 \text{ for all } \varphi_{ij}\in L(\prod_{P\in\mathfrak{R}}P^{b(P)}). \tag{4}$$

Put

$$R_{kh}^{(P)}(t)=\sum_{\substack{0\leqq\rho\leqq 2a(P)+b(P)-1\\ \rho=-1 \text{ if } (h,k)\in\mathfrak{M}(P)}} C_{h,k}^{(P,\rho)}\,t^\rho \text{ and } R=\{(R_{kh}^{(P)})\} \tag{5}$$

(4) is equivalent to the following condition (6)

$$\sum_{P\in\mathfrak{R}}\mathrm{Res}_P\{(\sum_{k,h}\vartheta'_{jk}R_{kh}^{(P)}\theta_{hi})\varphi\}=0 \text{ for all } \varphi\in L(\prod_{P\in\mathfrak{R}}P^{b(P)}) \tag{6}$$

and for all $1\leqq i\leqq r$, $1\leqq j\leqq r'$.
By a lemma which was proved by Weil [1] pp. 58–59, there exists a differential matrix $dI=(dI_{jk})$ of k such that $\sum_{k,h}\vartheta'_{jk}R_{kh}^{(P)}\theta_{hi}=\frac{dI_{ji}}{dt}$ (mod $t^{b(P)}$). Hence there exists a system $M=\{M^{(P)}\}$ such that $M^{(P)}\succ_P 0$ at each point $P\in\mathfrak{R}$ and

$$\frac{dI}{dt}=\Theta_0'^{-1}R\Theta_0+M(t)\cdot t^{b(P)}.$$

From this it follows that

$$\Theta'\frac{dI}{d\tau}\Theta^{-1}=\alpha_P\Delta'_P(R+\Theta'_0M\Theta_0^{-1}\cdot t^{b(P)})\cdot\Delta_P^{-1}\,t^{1-\frac{1}{n_P}}$$

where α_P is a constant. Put $N^{(P)}=(N_{kh}^{(P)})=\Theta'_0M^{(P)}\Theta_0^{-1}\cdot t^{b(P)}$, then by the choice of $b(P)$ we have $N^{(P)}\succ_P 0$ for all $P\in\mathfrak{R}$. Since $d'_k-d_h+1-\frac{1}{n_P}\geqq 0$ at each point $P\in\mathfrak{R}$ and $d'_k-d_h+1-\frac{1}{n_P}\geqq 1$ for the pair $(h, k)\in\mathfrak{M}(P)$, we obtain $\Theta'\frac{dI}{d\tau}\Theta^{-1}\succ_P 0$ for all $P\in\mathfrak{R}$. Hence dI belongs to $\mathfrak{D}$. Moreover

$$S_{kh}=R_{kh}^{(P)}+N_{kh}^{(P)}\succ_P 0$$

for (h, k) with $d_h=d'_k$ at each parabolic point $P\in\mathfrak{R}$. Hence dI belongs to $\mathfrak{D}_0$.

Let V be the set of all systems $R=\{R_{kh}^{(P)}\}$ $(P\in\mathfrak{R})$ which is given by (5) satisfying the relations (4). V is a vector space over C. By the fact explained above, there exists a linear mapping f from V into $\mathcal{D}_0$ defined by $f(R)\equiv\Theta_0'^{-1}R\Theta_0$ (mod $t^{b(P)}$). We shall prove that f is surjective. Put

$$R(t)=\Theta_0' \frac{dI}{dt}\Theta_0^{-1}-t^{2a(P)+b(P)}\cdot T^{(P)} \text{ where } T^{(P)}\succ_P 0, \text{ then we have}$$

$$\frac{dI}{dt}\equiv\Theta_0'^{-1}R\Theta_0 \pmod{t^{b(P)}}$$

and

$$\Theta' \frac{dI}{d\tau}\Theta^{-1}=\alpha_P(R_{kh}^{(P)}t^{d_k'-d_h+1-\frac{1}{n_P}})+\alpha_P(T_{kh}^{(P)}\cdot t^{d_k'-d_h+1-\frac{1}{n_P}})\cdot t^{2a(P)-b(P)}.$$

By the definition of $\mathcal{D}_0$ we have $R_{kh}^{(P)}(t)=p_{kh}(t)+\beta_{kh}/t$ where $p_{kh}(t)$ is a polynomial of degree $2a(P)+b(P)-1$ and β_{kh}/t appears if and only if $(h, k)\in\mathfrak{M}(P)$, and

$$\sum_{P\in\mathfrak{R}}\mathrm{Res}_P\,\{(\sum_{k,\,h}\vartheta_{jk}'R_{kh}\theta_{hi})\varphi\}=\sum_{P\in\mathfrak{R}}\mathrm{Res}_P\left(\varphi\,\frac{dI_{ji}}{dt}\right)=0 \text{ for all } \varphi\in L(\prod P^{b(P)}).$$

Therefore $R\in V$ and we see that f is surjective.

Since $\dim V=\dim\mathcal{D}_0+\dim \mathrm{Ker}(f)$, it remains to calculate $\dim \mathrm{Ker}(f)$. By the same method as in Weil [1] we have

$$\dim \mathrm{Ker}(f)=\sum_{P\in\mathfrak{R}}\{2a(P)rr'+r'i_P(\Theta_0)-ri_P(\Theta_0')\}.$$

Therefore $\dim V=d_0+(\sum_{P\in\mathfrak{R}}2a(P))\cdot rr'+r'I(\Theta_0)-rI(\Theta_0')$ which is the number of independent relations among the left hand sides of (3). Therefore the number of independent equations among (3) is

$$\sum_{P\in\mathfrak{R}}\{rr'(2a(P)+b(P))+m(P)\}-\{(\sum_{P\in\mathfrak{R}}2a(P))rr'+r'I(\Theta_0)-rI(\Theta_0')+d_0\}$$

$$=rr'\sum_{P\in\mathfrak{R}}b(P)+rI(\Theta')-r'I(\Theta)+\sum_{P\in\mathfrak{R}}\nu_P-d_0.$$

Since there are $rr'\{\sum b(P)-g+1\}$ unknowns, we obtain

$$l(\Theta,\,\Theta')=r'I(\Theta)-rI(\Theta')-rr'(g-1)-\sum_{P\in\mathfrak{R}}\nu_P+d_0 \qquad \text{(q.e.d.)}$$

§ 3. Holomorphic forms associated with a representation of Γ.

Let $\tilde{\Gamma}$ be a Fuchsian group of fractional linear transformations on the unit disk $\mathfrak{K}=\{w\,|\,|w|<1\}$ of the first kind. Let $\{\mathfrak{M}(\tilde{\sigma})\}$ be a matrix representation of degree r of $\tilde{\Gamma}$. H. Poincaré defined the zetafuchsian series

$$\tilde{\Theta}_*(w)=\sum_{\sigma\in\tilde{\Gamma}}\mathfrak{M}(\tilde{\sigma})^{-1}\tilde{F}(\tilde{\sigma}w)\left(\frac{d\tilde{\sigma}w}{dw}\right)^m$$

where $\tilde{F}(w)=(\tilde{f}_{ij}(w))$, $\tilde{f}_{ij}(w)$ is a rational function which is holomorphic on some neighbourhood of the unit circle $\{w||w|=1\}$.

Theorem (Poincaré [3] pp. 445–450).

Let $\{w_1, \cdots, w_l\}$ be the set of all the poles of $\tilde{f}_{ij}(w)$, $(i, j=1, \cdots, r)$. If all the proper values of $\mathfrak{M}(\tilde{\gamma})$ for any parabolic transformation $\tilde{\gamma}$ have absolute value 1, then for sufficiently large positive integer m, $\tilde{\Theta}_*(w)$ converges absolutely and uniformly on any compact set contained in $\mathfrak{K}-(\bigcup_{i=1}^{l}\tilde{\Gamma}(w_i))$.

Now we come back to the upper half plane $\mathfrak{H}$ and a matrix representation $\mathfrak{M}(\sigma)$ of Γ.

From now on we consider the case where $\mathfrak{M}(\sigma)$ $(\sigma\in\Gamma)$ takes only a finite number of different matrices. Let ρ be the isomorphism from $\mathfrak{K}$ to $\mathfrak{H}$ defined by $z=\rho(w)=i\,\dfrac{1+w}{1-w}$. Put $\tilde{\sigma}=\rho^{-1}\sigma\rho$, $\tilde{\Gamma}=\rho^{-1}\Gamma\rho$, $\mathfrak{M}(\tilde{\sigma})=\mathfrak{M}(\sigma)$, then we can apply the above theorem and

$$\Theta_*(w)=\sum_{\tilde{\sigma}\in\tilde{\Gamma}}\mathfrak{M}(\tilde{\sigma})^{-1}\tilde{F}(\tilde{\sigma}w)\left(\frac{d\tilde{\sigma}w}{dw}\right)^m=\sum_{\sigma\in\Gamma}\mathfrak{M}(\sigma)^{-1}\tilde{F}(\rho^{-1}\sigma z)\left(\frac{d\rho^{-1}\sigma z}{d\rho^{-1}z}\right)^m$$

$$=\left(\sum_{\sigma\in\Gamma}\mathfrak{M}(\sigma)^{-1}F(\sigma z)\left(\frac{d\sigma z}{dz}\right)^m\right)\cdot\left(\frac{dz}{d\rho^{-1}(z)}\right)^m$$

where $F(z)=(f_{ij}(z))=\tilde{F}(\rho^{-1}z)\left(\dfrac{d\rho^{-1}(z)}{dz}\right)^m$.

Put $\Theta_*(z)=\sum_{\sigma\in\Gamma}\mathfrak{M}(\sigma)^{-1}F(\sigma z)\left(\dfrac{d\sigma z}{dz}\right)^m$, then $\Theta_*(z)$ converges absolutely and uniformly on compact set contained in $\mathfrak{H}-\bigcup_{j=1}^{l}\Gamma(z_j)$, where $\{z_1, \cdots, z_l\}$ is the set of all poles of $f_{ij}(z)$ $(i, j=1, 2, \cdots, r)$. Here we can take $F(z)=E_r\cdot\dfrac{1}{(z-\beta)^{2m}}$ where β is an arbitrary point in $\mathfrak{H}$ and we can easily find β such that $\det(\Theta_*(z))\neq 0$.

We shall consider $\Theta_*(z)$ near each parabolic point z_0. Here we may assume $z_0=i\infty$. Take $\tau=\dfrac{z}{h}$, $t=e^{2\pi i\tau}$ such that $\Gamma_\infty=\{\gamma^n|\gamma(\tau)=\tau+1\}$, then $\Gamma=\bigcup_{j=1}^{\infty}\sigma_j\Gamma_\infty$ where $\sigma_j=\dfrac{\alpha_j z+\beta_j}{\gamma_j z+\delta_j}$ $(\gamma_j\neq 0)$.

Put $F_j(z)=\mathfrak{M}(\sigma_j)^{-1}F(\sigma_j z)\left(\dfrac{d\sigma_j(z)}{dz}\right)=\mathfrak{M}(\sigma_j)^{-1}\dfrac{(\alpha_j\delta_j-\beta_j\gamma_j)^m}{(\alpha_j-\beta\gamma_j)^{2m}}\;\dfrac{1}{(z-\sigma_j^{-1}(\beta))^{2m}}$

then

$$\Theta_*(z)=\sum_{j=1}^{\infty}\sum_{n=-\infty}^{\infty}\mathfrak{M}(\gamma^n)^{-1}F_j(\gamma^n z)=A_P^{-1}\sum_{j=1}^{\infty}\sum_{n=-\infty}^{\infty}D_P^{-n}A_PF_j(\gamma^n z)$$

where A_P is a constant matrix such that $D_P=A_P\mathfrak{M}(\gamma)A_P^{-1}$

$$=\begin{pmatrix} e^{2\pi i a_1} & & 0 \\ & \ddots & \\ & & e^{2\pi i a_r} \end{pmatrix} \quad (0 \leqq a_1 \leqq a_2 \leqq \cdots \leqq a_r < 1).$$

Put

$$\Delta_P(\tau)=\begin{pmatrix} e^{2\pi i a_1 \tau} & & 0 \\ & \ddots & \\ & & e^{2\pi i a_r \tau} \end{pmatrix}.$$

Since $\Delta_P^{-1} \sum_{n=-\infty}^{\infty} D_P^{-n} A_P F_j(\gamma^n z)$ is invariant under the transformation γ, we can express $\Delta_P^{-1} \sum_{n=-\infty}^{\infty} D_P^{-1} A_P F_j(\gamma^n z) = (\Theta_j^{(h,k)}(t))$ and $\Delta_P^{-1} A_P \Theta_*(z) = \sum_{j=1}^{\infty} \Theta_j(t)$. Since $t\Theta_j^{(h,k)}(t) = \alpha_{h,k} \sum_{n=-\infty}^{\infty} \frac{e^{-2\pi i a_h} \cdot e^{-2\pi i (a_h - 1)\tau}}{(\tau - \sigma_j^{-1}(\beta) + n)^{2m}}$ then we have $|t \cdot \Theta_j^{(h,k)}(t)| \leqq |\alpha_{h,k}| \sum_{n=-\infty}^{\infty} \frac{1}{|\tau - \sigma_j^{-1}(\beta) + n|^{2m}}$, and $t \cdot \Theta_j^{(h,k)}(t) \to 0$ ($\mathrm{Im}\, \tau \to \infty$). Hence $t \cdot \Theta_j^{(h,k)}(t)$ is zero at $t=0$. It follows that $\Theta_j^{(h,k)}(t)$ is holomorphic at $t=0$ and $\Delta_P^{-1} A_P \Theta_*(z)$ is also holomorphic at $t=0$. Therefore we have $\Theta_*(z) = A_P^{-1} \cdot \Delta_P(\tau) \Theta_0(t)$ where $\Theta_0(t) \in GL(r, k_P)$. At each finite parabolic point $z = x_0$ (x_0: real) we have $\Theta_*(z) = A_P^{-1} \cdot \Delta_P(\tau) \Theta_0(t) \left(\frac{d\tau}{dz} \right)^m$ where $\tau = -\frac{1}{h} \frac{1}{(z - x_0)}$, $t = e^{2\pi i \tau}$, $\Gamma_{x_0} = \{\gamma^n ; \gamma(\tau) = \tau + 1\}$, $\Theta_0(t) \in GL(r, k_P)$.

We denote by $\theta_*(z)$ in case $r=1$ and $\{\mathfrak{M}(\sigma)\}$ is trivial. Put

(7) $\quad \Theta(z) = \Theta_*(z)/\theta_*(z)$, then we have $\Theta(\sigma z) = \mathfrak{M}(\sigma)\Theta(z)$

for all $\sigma \in \Gamma$. Hence we can regard $\Theta(z)$ as a divisor of degree r on $\mathfrak{R}$.

PROPOSITION 2. *The total index $I(\Theta)$ of $\Theta(z)$ given by* (7), *is equal to* 0.

PROOF. Since $\frac{1}{\det(\Theta)} d(\det(\Theta))$ is Γ-invariant, then it is a differential of k and

$$i_P(\Theta) = i_P(\det(\Theta)) = \frac{1}{2\pi i} \int_P \frac{1}{\det(\Theta)} \frac{d(\det(\Theta))}{dt} = \mathrm{Res}_P \left(\frac{1}{\det(\Theta)} d(\det(\Theta)) \right)$$

By the Residue theorem, we have $I(\Theta) = \sum_{P \in \mathfrak{R}} i_P(\Theta) = \sum_{P \in \mathfrak{R}} \mathrm{Res}_P \left(\frac{1}{\det(\Theta)} d(\det(\Theta)) \right) = 0$

(q.e.d.)

DEFINITION. A vector $f(z) = \begin{pmatrix} f_1(z) \\ \vdots \\ f_r(z) \end{pmatrix}$ is called a holomorphic form associated with $\mathfrak{M}$ of weight m, if f satisfies the following two conditions.

1) $f(\sigma z)=\mathfrak{M}(\sigma)f(z)\left(\frac{d\sigma z}{dz}\right)^{-m}$ (for all $\sigma\in\Gamma$). Assume that f satisfies 1). At each point z_0 on $\mathfrak{H}^*$ let $\Gamma_{z_0}=\{\gamma\}$ and $\mathfrak{M}(\gamma)=A\begin{pmatrix} e^{2\pi i a_1} & & 0 \\ & \ddots & \\ 0 & & e^{2\pi i a_r}\end{pmatrix}A^{-1}$ $(0\leqq a_1\leqq a_2\leqq \cdots\leqq a_r<1)$. Then we see

$$f(z)\left(\frac{d\tau}{dz}\right)^{-m}=A\begin{pmatrix} t^{a_1} & & 0 \\ & \ddots & \\ & & t^{a_r}\end{pmatrix}F(t)\left(\frac{dt}{d\tau}\right)^{m} \text{ where } A\in GL(r,\ C),\ F(t)=\begin{pmatrix}F_1(t)\\ \vdots \\ F_r(t)\end{pmatrix}.$$

2) $F_i(t)$ is meromorphic at $t=0$ and $f(z)\left(\frac{d\tau}{dz}\right)^{-m}\succ_P 0$ for all $P\in\mathfrak{R}$.

Let $M_m(\Gamma,\ \mathfrak{M})$ be the set of all holomorphic forms associated with $\mathfrak{M}$. It is a vector space over C. We shall calculate the dimension of $M_m(\Gamma,\ \mathfrak{M})$. Let $f\in M_m(\Gamma,\ \mathfrak{M})$ and Θ be the matrix given by (7). Put $\Phi(z)=(\Phi_i(z))=\Theta^{-1}f\left(\frac{dj}{dz}\right)^{-m}$. Since $\Phi(\sigma z)=\Phi(z)$ for all $\sigma\in\Gamma$ and Φ is meromorphic at each point $P\in\mathfrak{R}$, $\Phi_i(z)$ $(1\leqq i\leqq r)$ belongs to k. By the condition 2) we have

$$\Theta\Phi\left(\frac{dj}{d\tau}\right)^{m}=f(z)\left(\frac{d\tau}{dz}\right)^{-m}\succ_P 0 \quad \text{for all } P\in\mathfrak{R}.$$

Hence $\Phi\in L\left(\Theta\left(\frac{dj}{d\tau}\right)^{m},\ 1\right)$.

Therefore, $M_m(\Gamma,\ \mathfrak{M})$ is isomorphic to $L\left(\Theta\left(\frac{dj}{d\tau}\right)^{m},\ 1\right)$ and

$$\dim M_m(\Gamma,\ \mathfrak{M})=l\left(\Theta\left(\frac{dj}{d\tau}\right)^{m},\ 1\right).$$

By Theorem 1 and Proposition 2.

$$l\left(\Theta\left(\frac{dj}{d\tau}\right)^{m},\ 1\right)=I\left(\Theta\left(\frac{dj}{d\tau}\right)^{m}\right)-r(g-1)-\sum_{P\in\mathfrak{R}}\nu_P+d_0=I(\Theta)+$$

$$+mrI\left(\frac{dj}{d\tau}\right)-r(g-1)-\sum_{P\in\mathfrak{R}}\nu_P+d_0=r\left\{(2m-1)\ (g-1)+m\sum_{\lambda=1}^{s}\left(1-\frac{1}{n_\lambda}\right)\right\}$$

$$-\sum_{\lambda=1}^{s}\sum_{h=1}^{r}<a_h-\frac{m}{n_\lambda}>+d_0.$$

It remains to calculate d_0. Since we have assumed that $\{\mathfrak{M}(\sigma)|\sigma\in\Gamma\}$ is a finite set, $\Gamma'=\{\sigma\in\Gamma|\mathfrak{M}(\sigma)=E\}$ is also a Fuchsian group of the 1st kind and all the Fuchsian functions with respect to Γ' make an algebraic function field k' which is a finite extension over k. Since Θ is invariant under each transformation of Γ'', we have $\Theta \in GL(r, k')$. From the definition it follows that $\mathfrak{D} \cong L\left(\left(\frac{dj}{d\tau}\right)^{1-m}, \Theta\right)$ and $\mathfrak{D} \supset \mathfrak{D}_0$.

We divide two cases.

(I) $m>1$

If $\Psi=(\Psi_i) \in L\left(\left(\frac{dj}{d\tau}\right)^{1-m}, \Theta\right)$, then $\Omega=(\Omega_i)=\left(\frac{dj}{d\tau}\right)^{1-m}\Psi\Theta^{-1} \succ_P 0$ for all $P \in \mathfrak{R}$. Put $\Phi=(\Phi_i)=\Psi\Theta^{-1}=\Omega\left(\frac{dj}{d\tau}\right)^{m-1}$. Since $I(\Phi_i)=I(\Omega_i)+(m-1)I\left(\frac{dj}{d\tau}\right)>0$ and $\Phi_i \in k'$, we have $\Phi_i=0$. This implies $\mathfrak{D} \cong L\left(\left(\frac{dj}{d\tau}\right)^{1-m}, \Theta\right)=(0)$, and hence $\mathfrak{D}_0 \cong \{0\}$, $d_0=\dim \mathfrak{D}_0=0$.

(II) $m=1$

In this case we have $\mathfrak{D} \cong L(1, \Theta)$. Let $\Psi \in L(1, \Theta)$. Since $\Psi\Theta^{-1} \succ_P 0$ for all $P \in \mathfrak{R}$, we have $\Psi\Theta^{-1}=C$ (a constant vector). And $(\Psi\Theta^{-1})^\sigma=\Psi^\sigma\Theta^{-1\sigma}=\Psi \cdot \Theta^{-1} \cdot \mathfrak{M}(\sigma)^{-1}=(\Psi\Theta^{-1})\mathfrak{M}(\sigma)^{-1}$, hence $C=C\mathfrak{M}(\sigma)$ for all $\sigma \in \Gamma$. We obtain $L(1, \Theta)=\{C\Theta | C=C\mathfrak{M}(\sigma)\}$ and hence $\mathfrak{D} \cong \{C | C=C\mathfrak{M}(\sigma)$ for all $\sigma \in \Gamma\}$. Decompose $\mathfrak{M}$ into the irreducible representations of Γ. Then $\dim \mathfrak{D}$ is the multiplicity of the trivial representation in $\mathfrak{M}$.

Now we shall calculate $\dim \mathfrak{D}_0$.

(II_1): In case where Γ contains no parabolic transformations, we have $\mathfrak{D}_0=\mathfrak{D}$ and hence $d_0=\dim \mathfrak{D}$.

(II_2): In case where Γ contains at least one parabolic transformation. Let us consider the case $\{m(\sigma)\}=1$ and hence $\Theta=1$. Then $\mathfrak{D}=\{dI | \frac{dI}{d\tau}\left(\frac{dj}{d\tau}\right)^{-1} \succ_P 0$ for all $P \in \mathfrak{R}\}$ and $\mathfrak{D}_0=\{c\, dj | c\frac{dj}{dt} \cdot \left(\frac{dj}{dt} \cdot t_P\right)^{-1} \succ_P 0$ for each parabolic point $P\}$ $=\{0\}$ and $d_0=0$. In the general case we have also $d_0=0$ by decomposing $\mathfrak{M}$ into irreducible representations. Thus we have proved the following theorem.

THEOREM 2. *If $\mathfrak{M}$ is a matrix representation of degree r of Γ such that $\{\mathfrak{M}(\sigma) | \sigma \in \Gamma\}$ is a finite set, then*

$$\dim M_m(\Gamma, \mathfrak{M})=r\left\{(2m-1)(g-1)+m\sum_{\lambda=1}^{s}\left(1-\frac{1}{n_\lambda}\right)\right\}-\sum_{\lambda=1}^{s}\sum_{h=1}^{r}\left\langle a_h-\frac{m}{n_\lambda}\right\rangle+d_0$$

where d_0 is the multiplicity of the trivial representation in $\mathfrak{M}$ when $m=1$ and Γ contains no parabolic transformations, otherwise d_0 is 0.

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