On generalized Picard varieties

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Introduction. J.P. Murre has constructed generalized Picard varieties of a complete normal projective variety [4]. They are the analogy of generalized Jacobian varieties of a curve in the case of a higher dimensional algebraic variety. There he considered new equivalence relations (called $\mathfrak{a}$-linear equivalence) and parametrized the corresponding Picard groups by the commutative group varieties, so called the generalized Picard varieties.[1]

In the paper [4] Murre defined $\mathfrak{a}$-structure on algebraic variety, but we deal with slightly special case of $\mathfrak{a}$-structure (we call it $\mathfrak{m}$-structure) in this paper. We need this restriction in order to calculate the structure of function modules as algebraic groups and consider the structure of generalized Picard Varieties. In our paper [3] we considered the structure of $\mathrm{Ext}(J, G_m)$, where J is the Jacobian variety of a curve C and G_m is 1-dimensional torus. We proved that $\mathrm{Ext}(J, G_m)$ is generated by generalized Jacobian varieties of C with respect to 0-cycles of type P_1+P_2 $(P_1 \neq P_2)$ on C. One of the purposes of this paper is to deal with this problem in the case of a higher dimensional algebraic variety. In §1~§4 we consider the structure of generalized Picard varieties and get a result analogous to the case of generalized Jacobian varieties. In §5~§8 we consider the problem stated above.

§1. Definitions and preliminaries.

Let V^r be an r-dimensional normal projective variety. We denote the rational function field of V over Ω by $\Omega(V)$ and that over a field K by $K(V)$. For a point P of V, we denote the local ring at P in $\Omega(V)$ by $\mathfrak{O}(P)$ and its maximal ideal by $\mathfrak{M}(P)$. When P is rational over a field K, we denote the local ring and its maximal ideal in $K(V)$ by $\mathfrak{O}_K(P)$ and $\mathfrak{M}_K(P)$ respectively.

Let $P_1, P_2, \cdots, P_m$ be a set of simple points on V and let $\mathfrak{M} = \bigcap_{i=1}^{m} \mathfrak{M}(P_i)^{n_i}$, where n_i $(i=1, 2, \cdots, m)$ are positive integers. When the set of simple points $\{P_1, P_2, \cdots, P_m\}$ and the module $\mathfrak{M}$ are given, we say an "$\mathfrak{M}$-structure on V" is defined on V. If all the points $P_1, \cdots, P_m$ are rational over a field k which is a field of definition of V, we say that the $\mathfrak{M}$-structure is defined over k. Through this section we

1) In the same paper of Murre [4], it is written that Oort has also constructed them by another method, simplifying Serre's construction. In this case he worked with Cartier divisors.

fix the $\mathfrak{M}$-structure defined over k which is defined as above.

Let $\mathcal{D}=\mathcal{D}(V)$ be the group of divisors on V. We denote the support of a divisor X by Supp(X). Let $\mathcal{D}_{\mathfrak{M}}=\mathcal{D}_{\mathfrak{M}}(V)=\{X|X\in\mathcal{D}, P_i\notin \text{Supp}(X)\ i=1, 2,\cdots, m\}$. $\mathcal{D}_{\mathfrak{M}}$ is a subgroup of $\mathcal{D}$. We put $\mathcal{D}_{\mathfrak{M}_a}=\mathcal{D}_{\mathfrak{M}}\cap\mathcal{D}_a$ where $\mathcal{D}_a$ is the group of divisors on V which is algebraically equivalent to zero.

We say a divosor $X\in\mathcal{D}_{\mathfrak{M}}$ is *$\mathfrak{M}$-linearly equivalent to zero* if there exists a function f in $\Omega(V)$ such that $X=(f)$ and $f\equiv 1 \pmod{\mathfrak{M}}$.

We denote this condition by $X\overset{\mathfrak{M}}{\sim}0$. (By $X\sim 0$ we denote the usual linear equivalence).

Let $\mathcal{D}_{\mathfrak{M}_l}=\{X|X\in\mathcal{D}_{\mathfrak{M}}, X\overset{\mathfrak{M}}{\sim}0\}$. For two divisors X, Y in $\mathcal{D}_{\mathfrak{M}}$ we say X is $\mathfrak{M}$ linearly equivalent to Y if $X-Y\overset{\mathfrak{M}}{\sim}0$. This is an equivalence relation in $\mathcal{D}_{\mathfrak{M}}$ and we have $\mathcal{D}_{\mathfrak{M}_l}\subset\mathcal{D}_{\mathfrak{M}_a}$. We denote the quotient group $\mathcal{D}_{\mathfrak{M}_a}/\mathcal{D}_{\mathfrak{M}_l}$ by Pic$_{\mathfrak{M}}(V)$. We denote the $\mathfrak{M}$-linear equivalence class of a divisor D in $\mathcal{D}_{\mathfrak{M}_a}$ by Cl$_{\mathfrak{M}}(D)$. (By Cl(D) we denote the usual linear equivalence class for a divisor D in $\mathcal{D}_a$).

We define an $\mathfrak{M}$-linear system of positive divisors as follows: Let $\mathcal{L}$ be an Ω-submodule of $\Omega+\mathfrak{M}$ and D be a divisor in $\mathcal{D}_{\mathfrak{M}}$ such that $D+(f)>0$ for every function f in $\mathcal{L}$. We call the family of positive divisors $\{D+(f)|f\in\mathcal{L}\}$ an $\mathfrak{M}$-linear system defined by the module $\mathfrak{L}$. We define the complete $\mathfrak{M}$-linear system $|X|_{\mathfrak{M}}$ of a divisor $X\in D_{\mathfrak{M}}$ by

$$\mathfrak{L}_{\mathfrak{M}}(X)=\{f|f\in\Omega+\mathfrak{M}, X+(f)>0\},$$
$$|X|_{\mathfrak{M}}=\{Y|Y>0,\ Y-X=(f),\ f\in\mathfrak{L}_{\mathfrak{M}}(X)\}.$$

For the $\mathfrak{M}$-linear equivalence we have following lemmas.

LEMMA 1. *Let X_i, Y_i $(i=1, 2)$ be positive divisors in $\mathcal{D}_{\mathfrak{M}}$ and (Y_1, Y_2) be a specialization of (X_1, X_2) over a field K containing k. Then $X_1\overset{\mathfrak{M}}{\sim}X_2$ implies $Y_1\overset{\mathfrak{M}}{\sim}Y_2$.*

LEMMA 2. *Let $X\in\mathcal{D}_{\mathfrak{M}}$ be a divisor on V which is rational over a field $K\supset k$. Then $\mathcal{L}_{\mathfrak{M}}(X)$ has a set of basis defined over K.*

COROLLARY. *The assumption being as in Lemma 2 the associated variety $T_{\mathfrak{M}}(X)$ of $|X|_{\mathfrak{M}}$ is defined over K.*

For the proof of them we refer to [4].

Let $\{X\}$ be an algebraic family of positive divisors on V. We say that $\{X\}$ is defined over a field K if the Chow variety W of $\{X\}$ is defined over K.

An algebraic family of positive divisors $\{X\}$ is called *total* if for a fixed member X' of $\{X\}$ there exists for every $Y\in\mathcal{D}_a(V)$ a divisor X in $\{X\}$ such that $Y\sim X-X'$. This definition is generalized for the case of $\mathfrak{M}$-linear equivalence.

An algebraic family of positive divisor $\{X\}$ of V is called *restricted $\mathfrak{M}$-total* if there exists a field of definition K of $\{X\}$ such that for every Y in $\mathfrak{D}_{\mathfrak{M}_a}$ and for every generic member X' of $\{X\}$ over $K(y)$ there exists a divisor X in $\{X\}\cap\mathfrak{D}_{\mathfrak{M}}$ such that $Y\overset{\mathfrak{M}}{\sim}X-X'$.

An algebraic family of positive divisor $\{X\}$ of V is called *$\mathfrak{M}$-total* if for every fixed divisor $X'\in\{X\}\cap\mathfrak{D}_{\mathfrak{M}}$ and for every $Y\in\mathfrak{D}_{\mathfrak{M}_a}$ there exists a divisor $X\in\{X\}\cap\mathfrak{D}_{\mathfrak{M}}$ such that $Y\overset{\mathfrak{M}}{\sim}X-X'$.

In [4] it was proved that a restricted $\mathfrak{M}$-total family $\{X\}$ of V exists when V is normal and an $\mathfrak{M}$-total family exists when V is non-singular.

Let $\{X\}$ be a restricted $\mathfrak{M}$-total family of V defined over k_0**), W be its associated variety and X be a generic member of $\{X\}$ over a field k_0. By $\xi=C(|X|_{\mathfrak{M}})$ we denote the Chow-point of the associated variety of $|X|_{\mathfrak{M}}$. Then by Cor. of Lemma 2, ξ is rational over $k_0(x)$, where x is the Chow-point of X. We denote by small letter $x, y, z, \cdots$ the Chow-points of divisors $X, Y, Z, \cdots$.

Let U be the locus of ξ over k_0, then U has a commutative normal law of composition given by $\xi+\eta=\zeta$ where $\xi=C(|X|_{\mathfrak{M}})$, $\eta=C(|Y|_{\mathfrak{M}})$, $\zeta=C(|Z|_{\mathfrak{M}})$, X, Y are independent generic members of $\{X\}$ over k_0 and Z is a generic member of $\{X\}$ over k_0 such that $X+Y-X_0\overset{\mathfrak{M}}{\sim}Z$. (Here X_0 is a fixed divisor in $\{X\}\cap\mathfrak{D}_{\mathfrak{M}}$). Thus there is a commutative group variety $\mathfrak{P}_{\mathfrak{M}}$ defined over k_0*) and a birational transformation $T: U\to\mathfrak{P}_{\mathfrak{M}}$ defined over k_0 which is compatible with the composition law. For the later use we define a rational mapping g from W to $\mathfrak{P}_{\mathfrak{M}}$ defined over k_0 by $g(x)=T(\xi)$ where x is a generic point of W over k_0 and $\xi=C(|X|_{\mathfrak{M}})$.

By the definition if $X\overset{\mathfrak{M}}{\sim}Y$ then we have $g(x)=g(y)$ and if $X+Y-X_0\overset{\mathfrak{M}}{\sim}Z$ then we have $g(x)+g(y)=g(z)$. $\mathfrak{P}_{\mathfrak{M}}$ is unique up to birational isomorphisms over k. In [4] the following theorem is proved.

THEOREM. *Let notations and assumption be as above. There is a surjective isomorphism* $\varphi_{\mathfrak{M}}: \mathrm{Pic}_{\mathfrak{M}}(V)\to\mathfrak{P}_{\mathfrak{M}}$ *such that*:

1) *If* $D\in\mathfrak{D}_{\mathfrak{M}_a}$ *is rational over a field* $K\supset k$, *then* $\varphi_{\mathfrak{M}}(\mathrm{Cl}_{\mathfrak{M}}(D))$ *is rational over* K. If $D'\in\mathfrak{D}_{\mathfrak{M}_a}$ *is a generic specialization of* D *over* k (*i.e.* $D\overset{k}{\leftrightarrow}D'$) *then* $\varphi_{\mathfrak{M}}(\mathrm{Cl}_{\mathfrak{M}}(D'))$ *is the unique generic specialization of* $\varphi_{\mathfrak{M}}(\mathrm{Cl}_{\mathfrak{M}}(D))$ *over* $D\overset{k}{\leftrightarrow}D'$. ($\varphi_{\mathfrak{M}}$ *is given by* $\varphi_{\mathfrak{M}}(\mathrm{Cl}_{\mathfrak{M}}(X_1-X_2))=g(x_1)-g(x_2)$ *for generic members* X_1 *and* X_2 *of* $\{X\}$ *over* k_0.)

2) *Let* G' *be a commutative group variety defined over* $k'\supset k$ *and* $\varphi': \mathrm{Pic}_{\mathfrak{M}}(V)\to G'$ *be a homomorphism such that* (i) *if* $D\in\mathfrak{D}_{\mathfrak{M}_a}$ *is rational over* $K\ni k'$ *then*

*) The group variety $\mathfrak{P}_{\mathfrak{M}}$ defined here may not be defined over k. But we can choose a group variety which is defined over k and is biregularly isomorphic to $\mathfrak{P}_{\mathfrak{M}}$ as is shown in [4].

**) We may take such a field k_0 that is separably algebraic over k.

$\varphi'(\mathrm{Cl}_{\mathfrak{M}}(D))$ is rational over K and (ii) *if D' is a generic specialization of D over k' then $\varphi'(\mathrm{Cl}_{\mathfrak{M}}(D'))$ is the unique specialization of $\varphi'(\mathrm{Cl}_{\mathfrak{M}}(D))$ over $D\overset{k}{\leftrightarrow}D'$. Then there exists a rational homomorphism $\alpha: \mathcal{P}_{\mathfrak{M}}\to G'$* defined over k' such that $\varphi'=\alpha\circ\varphi_{\mathfrak{M}}$.

§2. Covering homomorphisms.

Let V be a complete normal projective variety defined over a field k and $P_1, \cdots, P_m$ be a set of k-rational simple points of V. Put $\mathfrak{M}=\bigcap_{i=1}^{m}\mathfrak{M}(P_i)^{n_i}$ and $\mathfrak{M}'=\bigcap_{i=1}^{m}\mathfrak{M}(P_i)^{n'_i}$ $(n_i\geqslant n'_i\geqslant 0)$. Then there exist two commutative group varieties $\mathcal{P}_{\mathfrak{M}}$ and $\mathcal{P}_{\mathfrak{M}'}$ defined over k, (by Theorem in §1) such that $\mathrm{Pic}_{\mathfrak{M}}(V)\overset{\varphi_{\mathfrak{M}}}{\simeq}\mathcal{P}_{\mathfrak{M}}$ and $\mathrm{Pic}_{\mathfrak{M}'}(V)\overset{\varphi_{\mathfrak{M}'}}{\simeq}\mathcal{P}_{\mathfrak{M}'}$. By 2) of the Theorem, we have a rational homomorphism $\pi_{\mathfrak{M}\mathfrak{M}'}:\mathcal{P}_{\mathfrak{M}}\to\mathcal{P}_{\mathfrak{M}'}$ defined over k such that the following diagram is commutative

$$\begin{array}{ccc} \mathrm{Pic}_{\mathfrak{M}}(V)=\mathcal{D}_{\mathfrak{M}a}/\mathcal{D}_{\mathfrak{M}l} & \xrightarrow{\varphi_{\mathfrak{M}}} & \mathcal{P}_{\mathfrak{M}} \\ \downarrow p_{\mathfrak{M}\mathfrak{M}'} & & \downarrow \pi_{\mathfrak{M}\mathfrak{M}'} \\ \mathrm{Pic}_{\mathfrak{M}'}(V)=\mathcal{D}_{\mathfrak{M}'a}/\mathcal{D}_{\mathfrak{M}'l} & \xrightarrow{\varphi_{\mathfrak{M}'}} & \mathcal{P}_{\mathfrak{M}'} \end{array}$$

where $p_{\mathfrak{M}\mathfrak{M}'}$ is a natural homomorphism defined by $\mathcal{D}_{\mathfrak{M}a}\subset\mathcal{D}_{\mathfrak{M}'a}$, $\mathcal{D}_{\mathfrak{M}l}\subset\mathcal{D}_{\mathfrak{M}'l}$.

LEMMA 3. *There exists a field L containing k such that if u is a generic point of $\mathcal{P}_{\mathfrak{M}}$ over L then there exists a divisor $D\in\mathcal{D}_{\mathfrak{M}a}$ where D is rational over $L(u)$ with $\varphi_{\mathfrak{M}}(\mathrm{Cl}_{\mathfrak{M}}(D))=u$. We may add the condition that for any set of rational points $R_1, R_2, \cdots$ of V over k, $R_i\notin\mathrm{Supp}(D)$ $i=1, 2, \cdots$.*

Let $Q_1, \cdots, Q_t$ be a set of a sufficiently large number of independent generic points of V over k and z be a generic point of $\mathcal{P}_{\mathfrak{M}}$ over $k_0(Q_1, \cdots, Q_t)$, then the field $k_0(z, Q_1, \cdots, Q_t)$ fulfils the conditions.

By this Lemma, $\pi_{\mathfrak{M}\mathfrak{M}'}$ is a surjective rational homomorphism defined over k and moreover we have a rational cross section from $\mathcal{P}_{\mathfrak{M}}$ to $\mathcal{P}_{\mathfrak{M}'}$ defined over some field $L\supset k$. In fact let v be a generic point of $\mathcal{P}_{\mathfrak{M}'}$ over the field L which is chosen in Lemma 3 for $\mathcal{P}_{\mathfrak{M}'}$. Then there exists a divisor $D\in\mathcal{D}_{\mathfrak{M}'a}$ such that $\varphi_{\mathfrak{M}'}(\mathrm{Cl}_{\mathfrak{M}'}(D))=v$ and D is rational over $L(v)$. Moreover we may assume that $P_i\notin\mathrm{Supp}(D)$ $i=1, 2, \cdots, m$, i.e. $D\in\mathcal{D}_{\mathfrak{M}a}$. Thus $\varphi_{\mathfrak{M}}(\mathrm{Cl}_{\mathfrak{M}}(D))$ is a point on $\mathcal{P}_{\mathfrak{M}}$ rational over $L(v)$. If we put $s(v)=u$, s gives a rational cross section from $\mathcal{P}_{\mathfrak{M}'}$ to $\mathcal{P}_{\mathfrak{M}}$ with respect to $\pi_{\mathfrak{M}\mathfrak{M}'}$. From this the surjectiveness of $\pi_{\mathfrak{M}\mathfrak{M}'}$ follows. Therefore the kernel of $\pi_{\mathfrak{M}\mathfrak{M}'}$ is a connected subgroup of $\mathcal{P}_{\mathfrak{M}}$. If we replace $\mathcal{P}_{\mathfrak{M}'}$ by the usual Picard variety P of V defined over k, we have the following Theorem.

THEOREM 1. *Let V be a complete normal projective variety with an $\mathfrak{M}$-structure defined over k. Let $\mathcal{P}_{\mathfrak{M}}$ be the generalized Picard variety of V for the $\mathfrak{M}$-structure and P be the Picard variety both defined over k. Then there* *exists a rational homomorphism $\pi_{\mathfrak{M}}$ from $\mathcal{P}_{\mathfrak{M}}$ to P defined over k and the kernel $K_{\mathfrak{M}}$ of $\pi_{\mathfrak{M}}$ is a connected subgroup of $\mathcal{P}_{\mathfrak{M}}$.*

REMARK. The kernel $K_{\mathfrak{M}}$ of $\pi_{\mathfrak{M}}$ would be defined over a purely inseparable extension of k_0.

§ 3. Algebraic structure of function modules.

Let d be a point of $\mathcal{P}_{\mathfrak{M}}$ and D be a divisor in $\mathcal{D}_{\mathfrak{M}a}$ such that $\varphi_{\mathfrak{M}}(\mathrm{Cl}_{\mathfrak{M}}(D))=d$. For the neutral element e of P, $\pi_{\mathfrak{M}}(d)=e$ holds if and only if $D\sim 0$ (i. e. $D=(g)$ holds for a function $g\in\Omega(V)$ such that g is a unit at all points P_i $(i=1, 2, \cdots, m)$).

Let $U(P_i)$ $(i=1, 2, \cdots, m)$ be the multiplicative group of all units of $\mathfrak{O}(P_i)$ $(i=1, \cdots, m)$, and let $U(P_i)^{(n_i)}=1+\mathfrak{M}(P_i)^{n_i}$. Then $U(P_i)^{(n_i)}$ is a subgroup of $U(P_i)$. We put $R_{\mathfrak{M}}=\prod_{i=1}^{m} U(P_i)/U(P_i)^{(n_i)}$. Then a function g in $U_{\mathfrak{M}}=\bigcap_{i=1}^{m} U(P_i)$ determines an element of $R_{\mathfrak{M}}$. By this mapping we get a natural homomorphism θ from $U_{\mathfrak{M}}$ to $R_{\mathfrak{M}}$.

LEMMA 4. *Let V^r be an r-dimensional projective variety defined over k and $P_1, \cdots, P_m$ be a set of simple points of V which are rational over k. Then we can choose a set of generators $t_{i1}, \cdots, t_{ir}$ of $\mathfrak{M}_k(P_i)$ $(i=1, \cdots, m)$ such that all t_{ij} $(1\leqslant i\leqslant m,\ 1\leqslant j\leqslant r)$ are integral at every P_i $(i=1, \cdots, m)$ and t_{i1} is unit at all P_j $(i\neq j)$.*

We omitt the proof.

PROPOSITION 1. *$\bar{\theta}$ is surjective.*

PROOF. Let $(t_{i1}, t_{i2}, \cdots, t_{ir})=(t_i)$ be the generators of $\mathfrak{M}_k(P_i)$ $(i=1, \cdots, m)$ chosen as in Lemma 4. Let $\bar{g}=(\bar{g}_i)$ $(i=1, \cdots, m)$ be an element of $R_{\mathfrak{M}}$ such that $\bar{g}_i=\sum_{0\leqslant k\leqslant n_i} f_{ik}(t_i)$ where $f_{ik}(t_i)$ is a homogeneous polynomial of degree k of the form $f_{ik}=\sum a_{k_{i1}\cdots k_{ir}} t_{i1}^{k_{i1}}\cdots t_{ir}^{k_{ir}}$ and $\sum$ is taken all over the monomials with $\sum_{j=1}^{r} k_{ij}=k$.

Let $\bar{g}_1=\sum_{0\leqslant k} h_k(t_2)$ and $t_{11}^{n_1}=\alpha_2+f_2$, be the power series expansion of g_1 and $t_{11}^{n_1}$ at P_2. h_k is the homogeneous part of degree k and $\alpha_2\in\Omega$, $\alpha_2\neq 0$, $f_2\in\mathfrak{M}(P_2)$. If we put $g_1'=\bar{g}_1-\dfrac{h_0(t_2)-f_{20}(t_2)}{\alpha_2}\cdot t_{11}^{n_1}$ then $g_1'=f_{20}(t_2)+\sum_{1\leqslant k} h_k'(t_2)$. If we put

$g_1''=g_1'-\dfrac{h_1'(t_2)-f_{21}(t_2)}{\alpha_2}\cdot t_{11}^{n_1}$ then $g_1''=f_{20}(t_2)+f_{21}(t_2)+\sum_{2\leqslant k} h_k''(t_2)$. Thus we get

$g_1^{(n_2)}\equiv\bar{g}_2\ (\mathrm{mod}\ \mathfrak{M}(P_2)^{n_2})$. Also we have $g_1^{(n_2)}\equiv\bar{g}_1\ (\mathrm{mod}\ \mathfrak{M}(P_1)^{n_1})$. Let $g_2=g_1^{(n_2)}=\sum_{0\leqslant k} p_k(t_3)$, $t_{11}^{n_1}\cdot t_{21}^{n_2}=\alpha_3+f_3$ be the power series expansion of g_2, $t_{11}^{n_1}\cdot t_{12}^{n_2}$ at P_2 respectively, where $p_k(t_3)$ is the homogeneous part of degree k and $\alpha_2\neq 0$, $f_3\in\mathfrak{M}(P_3)$. If we put

$g_2'=g_2-\dfrac{p_0(t_3)-f_{30}(t_2)}{\alpha_3}\, t_{11}^{n_1}\cdot t_{21}^{n_2}$ then $g_2'=f_{30}(t_3)+\sum p_k'(t_2)$. If we put

$g_2''=g_2'-\dfrac{p_1'(t_2)-f_{31}(t_3)}{\alpha_2}\, t_{11}^{n_1}\cdot t_{21}^{n_2}$ then $g_2''=f_{30}(t_3)+f_{31}(t_3)+\sum_{2\leqslant k} p_k''(t_3)$. Thus we get

$g_3=g_2^{(n_3)}\equiv g_3 \pmod{\mathfrak{M}(P_3)^{n_3}}$. Also we have $g_2^{(n_3)}\equiv g_1 \pmod{\mathfrak{M}(P_1)^{n_1}}$ and $g_2^{(n_3)}\equiv g_2 \pmod{\mathfrak{M}(P_2)^{n_2}}$. Repeating this process we get a function $g_m=g$ such that $\bar{\theta}(g_m)=\bar{g}$.

Q. E. D.

REMARK. By the proof of the above Proposition we can choose a function which is defined over $k(\cdots, a_{k_1\cdots k_r}, \cdots)$, as a function g such that $\bar{\theta}(g)=\bar{g}$.

Let $\Delta=\{\overbrace{a, \cdots, a}^{m}|a\in\Omega^*\}$, then Δ is a subgroup of $R_{\mathfrak{M}}$. We put $H_{\mathfrak{M}}=R_{\mathfrak{M}}/\Delta$. The mapping $\bar{\theta}$ induces a bijective isomorphism θ from $K_{\mathfrak{M}}$ to $H_{\mathfrak{M}}$ as abstract groups. This bijective isomorphism is given as follows. Let $d=\varphi_{\mathfrak{M}}(\mathrm{Cl}_{\mathfrak{M}}(D))\in K_{\mathfrak{M}}$ then we have $D=(f)$ for a function $f\in U_{\mathfrak{M}}$. Let $\bar{\theta}(f_i)\in R_{\mathfrak{M}}$ and (f_i) be the image of $(\bar{f}_i)$ by the natural homomorphism $R_{\mathfrak{M}}\to R_{\mathfrak{M}}/\Delta=H_{\mathfrak{M}}$. Then we have $\theta(d)=(f_i)$. If $(f_i)=1$ in $H_{\mathfrak{M}}$ then we have $f_i=a+h_i$ $(a\in\Omega,\ h_i\in\mathfrak{M}(P_i)^{n_i})$ $(i=1, 2, \cdots, m)$ and $\frac{1}{a}f\equiv 1 \pmod{\mathfrak{M}}$. Therefore $d=e$ in $K_{\mathfrak{M}}$. This proves the injectivity of θ. Surjectivity is clear by Prop. 1.

The proof of the following Propositions 2–4 is essentially the same as that in Serre's Book [7] Chap. V.

Let P be a simple point of V^r and $t_1, t_2, \cdots, t_r$ be a set of local parameters of V at P in $\mathfrak{M}(P)$. Then $U(P)/U(P)^{(n)}$, which is defined as above, has, as its system of representatives, the polynomials of the form,

$$f=\sum_{0\leq K<n} a_{k_1\cdots k_r} t_1^{k_1}\cdot t_2^{k_2}\cdot \cdots \cdot t_r^{k_r} \quad \left(K=\sum_{j=1}^{r} k_j\right.$$

Therefore we may consider $U(P)/U(P)^{(n)}$ as an open subspace of N-dimensional affine space, where $N=\sum_{j=0}^{n-1} {}_rH_j$ and the group structure is compatible with this algebraic structure.

PROPOSITION 2. *$U(P)/U(P)^{(1)}$ is birationally isomorphic to G_m as algebraic groups.*

This is clear by the definition of algebraic strueture of $U(P)/U(P)^{(1)}$.

PROPOSITION 3. *Let the characteristic p of Ω be equal to zero. Then $U(P)^{(1)}/U(P)^{(n)}$ is birationally isomorphic to G_a^{N-1} as algebraic groups. The isomorphism is given in the following way. We give the lexicographic order in the set of monomials $\left\{t_1^{k_1}\cdot t_2^{k_2}\cdot \cdots \cdot t_r^{k_r} \mid 0<\sum_{j=1}^{r} k_j<n\right\}$. Let $g_{k_1\cdots k_r}$ be the formal power series in $\{t_1, \cdots, t_r\}$ such that the first term is $t_1^{k_1}\cdot t_2^{k_2}\cdots t_r^{k_r}$. Then every element g in $U(P)^{(1)}/U(P)^{(n)}$ is uniquely written as $g\equiv \Pi \exp(a_{k_1\cdots k_r}g_{k_1\cdots k_r})$ modulo formal power series of degree n, where $K=\sum_{j=1}^{r} k_j$, $a_{k_1\cdots k_r}\in\Omega$ $0<K<n$. The mapping $g\to(\cdots, a_{k_1\cdots k_r}, \cdots)$ gives the isomorphism.*

PROOF. Let $\exp(f)=1+f+\frac{f^2}{2!}+\frac{f^3}{3!}+\cdots$ for all power series f in $\{t_1, \cdots,$

$t_r\}$. Then we have $\exp(f_1)\cdot\exp(f_2)=\exp(f_1+f_2)$. Let g be an element of $U(P)^{(1)}/U(P)^{(n)}$. Then

$$g=1+b_{10\cdots 0}t_1+b_{010\cdots 0}t_2+\cdots+b_{k_1\cdots k_r}t_1^{k_1}\cdots t_r^{k_r}+\cdots$$
$$g/(1+b_{10\cdots 0}g_{10\cdots 0})=1+b'_{011\cdots 0}t_2+\cdots$$
$$g/(1+b_{10\cdots 0}g_{10\cdots 0})(1+b'_{010\cdots 0}g_{010\cdots 0})=1+b''_{0010\cdots 0}t_3+\cdots.$$

Repeating this step we get

$$g=(1+\alpha_{10\cdots 0}g_{10\cdots 0})\cdots(1+\alpha_{k_1\cdots k_r}g_{k_1\cdots k_r})\cdots(1+\alpha_{0\cdots 0n}g_{0\cdots 0n}),$$
(modulo formal power series of degree n.)

If we put $\exp(\alpha_{k_1\cdots k_r}g_{k_1\cdots k_r})=1+f_{k_1\cdots k_r}$ then $f_{k_1\cdots k_r}$ has $\alpha_{k_1\cdots k_r}t_1^{k_1\cdots k_r}\cdots t_r^{k_r}$ as its first term. Choosing suitable $(\cdots\alpha_{k_1\cdots k_r}\cdots)$ in Ω we get

$$g=\prod_{0<K<n}\exp(\alpha_{k_1\cdots k_r}g_{k_1\cdots k_r}) \qquad \text{(modulo formal power series of degree } n).$$

The uniqueness of this expression is clear. By the above argument $\alpha_{k_1\cdots k_r}$ must be a polynomial of the coefficients of g and the coefficients of g are polynomials of $(\cdots,\alpha_{k_1\cdots k_r},\cdots)$. This completes the proof of our Proposition. Q. E. D.

COROLLARY. *When $p=0$, $U(P)/U(P)^{(n)}$ is birationally isomorphic to $G_m\times G_a^{N-1}$.*

When the sharacteristic of Ω is not equal to zero we have the following

PROPOSITION 4. *Let the characteristic p of Ω be $\neq 0$. Let $0<\sum_{j=1}^{r}k_j<n$, l be a positive integer such that $l\sum_{j=1}^{r}k_j\leqslant n-1$, $(l,p)=1$ and $r_{k_1\cdots k_rl}=\mathrm{Min}\{r\mid p^r\geqslant n/l\sum k_j\}$. Let $g_{k_1\cdots k_r}$ be a formal power series of $t_1^{k_1}\cdot t_2^{k_2}\cdots t_r^{k_r}$ such that its first term is $(t_1^{k_1}\cdot t_2^{k_2}\cdots t_r^{k_r})^l$. Then every g in $U(P)^{(1)}/U(P)^{(n)}$ can be written uniquely as*

$$g=\prod_{(k_1,\cdots,k_r,l)}E(\overrightarrow{a_{k_1\cdots k_rl}}g_{k_1\cdots k_rl}) \qquad \text{(modulo formal power series of degree } n)$$

using Witt vector $\overrightarrow{a_{k_1\cdots k_rl}}$ of length $r_{k_1\cdots k_rl}$. $U(P)^{(1)}/U(P)^{(n)}$ is birationally isomorphic to $\prod_{(k_1,\cdots,k_r,l)}W_{r_{k_1\cdots k_rl}}$ as algebraic groups, where $E(x)=\exp(-x^{(0)}-x^{(1)}-x^{(2)}-\cdots)$ for a Witt vector $x=(x^{(0)},x^{(1)},x^{(2)},\cdots)$.

The proof is quite similar to that of Proposition 3.

Thus we get

PROPOSITION 5. *When $p=0$, $R_{\mathfrak{M}}$ (resp. $H_{\mathfrak{M}}$) is birationally isomorphic to $G_m^m\times G_a^M$ (resp. $G_m^{m-1}\times G_a^M$) over k, where $M=\sum_{i=1}^{m}(N_i-1)$, $N_i=\sum_{j=1}^{n-1}{}_rH_j$.*

When $p\neq 0$, $R_{\mathfrak{M}}$ (resp. $H_{\mathfrak{M}}$) is birationally isomorphic to

$$G_m^m\times\prod_{i=1}^{m}\prod_{(k_1^{(i)},\cdots k_r^{(i)},l)}W_{r_{k_1^{(i)},\cdots,k_r^{(i)}l}} \qquad \left(\text{resp. } G_m^{m-1}\times\prod_{i=1}^{m}\prod_{(k_1^{(i)},\cdots,k_r^{(i)},l)}W_{r_{k_1^{(i)},\cdots,k_r^{(i)},l}}\right)$$

over k.

REMARK. By the isomorphisms of Proposition 4 we identify these algebraic groups respectively. Then the mapping θ is given as follows:

1) $p=0$: Let $\tilde{A}=(\alpha_1, \alpha_2, \cdots, \alpha_m, \cdots, a^{(i)}_{k_1\cdots k_r l}, \cdots)\in R_{\mathfrak{M}}$ and A be the image of $\tilde{A}$ by the natural mapping $R_{\mathfrak{M}}\xrightarrow{q} R_{\mathfrak{M}}/\Delta=H_{\mathfrak{M}}$. Let g be a function in $U_{\mathfrak{M}}$ such that $g\equiv\alpha_i \prod_{0<K<n_i} \exp(a^{(i)}_{k_1\cdots k_r} g^{(i)}_{k_1\cdots k_r})$ $(i=1, 2, \cdots, m)$. Then $\theta(A)=\varphi_{\mathfrak{M}}(\mathrm{Cl}_{\mathfrak{M}}(g))$.

2) $p\neq 0$: Let $\tilde{A}=(\alpha_1, \alpha_2, \cdots, \alpha_m, \cdots, \overrightarrow{a^{(i)}_{k_1\cdots k_r l}}, \cdots)\in R_{\mathfrak{M}}$ and A be the image of $\tilde{A}$ by the natural mapping $R_{\mathfrak{M}}\to R_{\mathfrak{M}}/\Delta=H_{\mathfrak{M}}$. Let g be a function in $U_{\mathfrak{M}}$ such that $g\equiv\alpha_i \prod_{(k_1\cdots k_r l)} E(\overrightarrow{a^{(i)}_{k_1\cdots k_r l}} g^{(i)}_{k_1\cdots k_r l})$ $(i=1, 2, \cdots, m)$. Then $\theta(A)=\varphi_{\mathfrak{M}}(\mathrm{Cl}_{\mathfrak{M}}(g))$.

3) By the Remark of Prop. 1, we may assume that g is defined over $k(A)$.

§4. The structure of generalized Picard Varieties.

For the proof of Theorem 2 first we remark the following Lemma.

LEMMA 4. *Let D and D' be divisors on V such that $D\in\mathcal{D}_{\mathfrak{M}}$ and $D=(f)$ for a function $f\in\Omega(V)$. If D and D' are generic specializations of each other over a field $K\supset k$, then $D'\in\mathcal{D}_{\mathfrak{M}}$ and $D'=(f^{\sigma})$ hold, where σ is a K-isomorphism of $K(d)$ to $K(d')$ such that $D'=D^{\sigma}$ and d, d' are Chow-points of D, D' respectively.*

PROOF. For a generic point N of V over $K(d, d')$, σ can be extended to a $K(N)$-isomorphism of $K(N, d)$ to $K(N, d')$. We denote it also by σ. $f(N)=z$ may be assumed to be an element of $K(N, d)$ and z^{σ} is an element of $K(N, d')$. Thus there exists a function $f^{\sigma}(N)=z^{\sigma}$ on V defined over $K(d')$. It follows that

$$\begin{aligned} D'=D^{\sigma}=(f)^{\sigma}&=pr_V[\Gamma_f\cdot(V\times(0-\infty))]^{\sigma} \\ &=pr_V[\Gamma_f^{\sigma}\cdot(V\times(0-\infty))] \\ &=pr_V[\Gamma_{f^{\sigma}}\cdot(V\times(0-\infty))]=(f^{\sigma}). \end{aligned}$$

Clearly we have $D'=D^{\sigma}\in\mathcal{D}_{\mathfrak{M}}$.

THEOREM 2. *The bijective isomorphism θ from $H_{\mathfrak{M}}$ to $K_{\mathfrak{M}}$ is a birational isomorphism defined over k', where k' is a purely inseparable extension of k_0 over which $K_{\mathfrak{M}}$ is defined* (See Remark of Theorem 1)

PROOF. Let $A\in H_{\mathfrak{M}}$ and $g\in U_{\mathfrak{M}}$ be such that $\theta(A)=\varphi_{\mathfrak{M}}(\mathrm{Cl}_{\mathfrak{M}}(g))$. Since we may choose such a function g which is defined over $k'(A)$, $\theta(A)$ is rational over $k'(A)$. Conversely let $L=k'(z, Q_1, \cdots, Q_t)$ be the field chosen in Lemma 3, where $Q_1, \cdots, Q_t$ are independent generic points of V over k' and z is a generic point of $\mathcal{P}_{\mathfrak{M}}$ over $k'(Q_1, \cdots, Q_t)$. Let $\varphi_{\mathfrak{M}}(\mathrm{Cl}_{\mathfrak{M}}(g))=w$ be a point of $K_{\mathfrak{M}}$ and u be a generic point of $\mathcal{P}_{\mathfrak{M}}$ over $L(w)$. If we put $v=w-u$, v is a generic point of $\mathcal{P}_{\mathfrak{M}}$ over L and $L(u, v)=L(u, w)$. Let D_u and D_v be divisors in $\mathcal{D}_{\mathfrak{M}a}$, whose existence is assured in Lemma 3, such that $\varphi_{\mathfrak{M}}(\mathrm{Cl}_{\mathfrak{M}}(D_u))=u$, $\varphi_{\mathfrak{M}}(\mathrm{Cl}_{\mathfrak{M}}(D_v))=v$ and D_u, D_v are rational over $L(u)$, $L(v)$ respectively. We have $\varphi_{\mathfrak{M}}(\mathrm{Cl}_{\mathfrak{M}}(D_u+D_v))=u+v=w$. Therefore there exists a function f in $U_{\mathfrak{M}}$ which is defined over $L(u, v)$ such that $D_u+D_v=(f)$ (By [12], Theorem 10, VIII). We have also $\theta(A)=\varphi_{\mathfrak{M}}(\mathrm{Cl}_{\mathfrak{M}}(f))$. By these arguments we see that A is rational over $L(u, v)=L(u, w)$. Replacing u by another generic point of $\mathcal{P}_{\mathfrak{M}}$ over $L(w, u)$ we see that A is rational over $L(w)$. We may also replace $z, Q_1, \cdots, Q_t$ by another set of independent generic points of $\mathcal{P}_{\mathfrak{M}}$ and V over $k'(z, Q_1, \cdots, Q_t)$ respectively then we see that A is rational over $k'(w)$. Thus the mapping θ is birational at every point in point wise. Let A_1 and A_2 be two points of $H_{\mathfrak{M}}$ such that $\theta(A_1)=\varphi_{\mathfrak{M}}(\mathrm{Cl}_{\mathfrak{M}}(g_1))$, $\theta(A_2)=\varphi_{\mathfrak{M}}(\mathrm{Cl}_{\mathfrak{M}}(g_2))$. First we assume that A_1 and A_2 are generic specializations of each other over k'. Then we may assume that (g_1) and (g_2) are generic specializations of each other over k'. Therefore $\theta(A_1)$ and $\theta(A_2)$ are generic specializations of each other over k. (By Theorem in § 1)

Conversely we can show that if $\theta(A_1)$ and $\theta(A_2)$ are generic specializations of each other over k' then A_1 and A_2 are generic specializations of each other over k' then A_1 and A_2 are generic specializations of each other over k. In fact, there exists a k-isomorphism σ from $k'(\theta(A_1))$ to $k'(\theta(A_2))$ transporting $\theta(A_1)$ to $\theta(A_2)$. Let X_1 and Y_1 be generic members of $\{X\}$ over k' such that $X_1-Y_1\overset{\mathfrak{M}}{\sim}(g_1)$, where $\{X\}$ is the algebraic family defined in § 1. We have a function f in $U_{\mathfrak{M}}$ defined over $k'(x_1, y_1)$ such that $X_1-Y_1=(f_1)$. By Theorem in § 1 we have $\varphi_{\mathfrak{M}}(\mathrm{Cl}_{\mathfrak{M}}(X_1-Y_1))=g(x_1)-g(y_1)=\theta(A_1)$. Let (X_2, Y_2) be a generic specialization of (X_1, Y_1) over $\theta(A_1)\overset{k'}{\leftrightarrow}\theta(A_2)$. Then $\varphi_{\mathfrak{M}}(\mathrm{Cl}_{\mathfrak{M}}(X_2-Y_2))=g(x_2)-g(y_2)=\theta(A_2)$. σ can be extended to a k-isomorphism of $k'(\theta(A_1), x_1, y_1)$ to $k(\theta(A_2), x_2, y_2)$ transporting $(\theta(A_1), x_1, y_1)$ to $(\theta(A_2), x_2, y_2)$. We also denote this isomorphism by σ. We have $(X_1-Y_1)^{\sigma}=X_2-Y_2=(f_1{}^{\sigma})$ where $f_1{}^{\sigma}$ shall be defined as in Lemma 4. A_1 and A_2 are defined by f_1 and $f_1{}^{\sigma}$ respectively. Therefore A_1 is transformed to A_2 by k-isomorphism σ. That is to say, A_1 and A_2 are generic specializations of each other over k'. Thus we proved the above assertion.

Taking into account that θ is an isomorphism as abstract groups, we can easily see that θ gives a surjective birational isomorphism from $H_{\mathfrak{M}}$ to $K_{\mathfrak{M}}$ defined over k'. This completes the proof of Theorem. Q. E. D.

We get immediately the following Theorem 3 by Theorem 1 and Theorem 2.

THEOREM 3. *Let V^r be a complete normal projective variety with $\mathfrak{M}$-structure $\{P_1, \cdots, P_m; \mathfrak{M}=\bigcap_{i=1}^{m}\mathfrak{M}(P_i)^{n_i}\}$ defined over k, $\mathcal{P}_{\mathfrak{M}}$ be the generalized Picard variety of V defined over k with respect to the $\mathfrak{M}$-structure and P be the usual Picard variety of V defined over k. Then $\mathcal{P}_{\mathfrak{M}}$ is an extension of P by the linear group $G_m^{m-1}\times G_a^M$ $(M=\sum_{i=1}^{m}(N_i-1),\ N_i=\sum_{i=1}^{n_i-1}{}_rH_j)$ when the characteristic $p=0$, and by $G_m^{M-1}\times\prod_{i=1}^{m}\prod_{(k_1^{(i)},\cdots,k_r^{(i)},l^{(i)})}W_{r_{k_1^{(i)},\cdots,k_r^{(i)},l^{(i)}}}$ when $p\neq 0$, where the product is taken for all $(k_1^{(i)},\cdots,k_r^{(i)},l^{(i)})$ such that $(k_1^{(i)},\cdots,k_r^{(i)})=1$, $0<l^{(i)}\sum_{j=1}^{r}k_1^{(i)}\leqslant n_i-1$ and $W_{r_{k_1^{(i)},\cdots,k_r^{(i)},l^{(i)}}}$ is the Witt group with length $r_{k_1^{(i)}\cdots k_r^{(i)}l^{(i)}}=\mathrm{Min}\{r\,|\,p^r\geqslant n_i/l^{(i)}\sum_{j=1}^{r}k_j^{(i)}\}$.*

§ 5. *K/k*-image and *K/k*-trace of Abelian varieties.

For the proof of the statements of this section we refer to the Book of Lang [1], Chap. VIII and Appendix.

Let V^r be an r-dimensional complete normal projective variety. We fix an algebraically closed field k in Ω over which V is defined. Let W_u be a generic hyperplane section of V over k, which is defined over $k(u)$. Then W is also normal. Let A be the Albanese variety of V defined over k and A_u be that of W_u defined over $k(u)$. We denote the canonical mappings by $f: V \to A$ and $g: W_u \to A_u$, respectively. The inclusion mapping $i: W_u \to V$ induces a rational homomorphism $i^*: A_u \to A$ defined over $k(u)$. i^* is defined by $i(\mathrm{Cl}(Y)) = \mathrm{Cl}(i^{-1}(Y))$ for a divisor Y on V. By the universal mapping property of the Albanese variety we have the commutative diagram

$$(1) \qquad \begin{array}{ccc} W_u & \xrightarrow{i} & V \\ \downarrow & & \downarrow \\ A_u & \xrightarrow{i^*} & A \end{array}$$

By Theorem 4 Chap. VIII §2 [1] (A, i^*) is $k(u)/k$-image of A_u, i. e. i^* is a surjective homomorphism such that if B is an abelian variety defined over k and $\alpha: A_u \to B$ is a homomorphism defined over $k(u)$, then there exists a homomorphism $\alpha': A \to B$ defined over k and $\alpha = \alpha' \cdot i^*$.

Let $\alpha: A \to B$ be a homomorphism of Abelian varieties defined over k. We have a homomorphism $t_\alpha: \hat{B} \to \hat{A}$ defined over k, where $\hat{A}$ and $\hat{B}$ are dual abelian varieties of A and B respectively. The homomorphism t_α is defined by $t_\alpha(\mathrm{Cl}(Y)) = \mathrm{Cl}(\alpha^{-1}(Y))$ for the divisor Y on B. By Theorem 11 Chap. VIII §5 $(\hat{A}, t_{i^*})$ is a $k(u)/k$-trace of $\hat{A}_u$, i.e. t_{i^*} has finite kernel, and if B is an abelian variety defined over k and $\beta: B \to \hat{A}_u$ is a homomorphism defined over $k(u)$, then there exists a homomorphism $\beta': B \to \hat{A}$ defined over k and $\beta = t_{i^*} \cdot \beta'$.

§ 6. Homological mappings.

In this section we consider the algebraic objects (algebraic groups, rational mappings, etc.) defined or rational over a suitable field with respect to the objects considered there, but we shall not notice the field explicity.

Let A, B be commutative algebraic groups and C, D be commutative linear groups.

a) Let $\alpha: B \to A$ a rational homomorphism. Then α induces a homomorphism $\alpha_0: \mathrm{Ext}(A, C) \to \mathrm{Ext}(B, C)$. Let $\beta: C \to D$ be a homomorphism. Then β induces a homomorphism $\beta^0: \mathrm{Ext}(A, C) \to \mathrm{Ext}(A, D)$. Moreover, we get a commutative diagram

$$\begin{array}{ccc} \mathrm{Ext}(A, C) & \xrightarrow{\alpha_C^0} & \mathrm{Ext}(B, C) \\ \downarrow \beta_A^0 & & \downarrow \beta_B^0 \\ \mathrm{Ext}(A, D) & \xrightarrow{\alpha_D^0} & \mathrm{Ext}(B, D). \end{array}$$

b) The rational factor system of A with values in C is a rational mapping $f: A\times A\to C$ such that

$$f(y, z)-f(x+y, z)+f(x, y+z)-f(x, y)=0 \qquad \text{for} \qquad x, y, z\in A.$$

When $f(x, y)=f(y, x)$, f is said symmetric. For a rational mapping $g: A\to C$, $\delta g(x, y)=g(x+y)-g(x)-g(y)$ is called trivial factor system. We denote by $H^2_{\mathrm{rat}}(A, C)_s$ the group of symmetric rational factor systems modulo trivial ones. Then α induces a homomorphism $\beta^f: H^2_{\mathrm{rat}}(A, C)\to H^2_{\mathrm{rat}}(A, D)$. Moreover we get a commutative diagram

$$\begin{array}{ccc} H^2_{\mathrm{rat}}(A, C)_s & \xrightarrow{\alpha_C^f} & H^2_{\mathrm{rat}}(B, C)_s \\ \downarrow \beta_f^A & & \downarrow \beta_f^B \\ H^2_{\mathrm{rat}}(A, D)_s & \xrightarrow{\alpha_D^f} & H^2_{\mathrm{rat}}(B, D)_s. \end{array}$$

c) Let G be an element of $\mathrm{Ext}(A, C)$. Since C is a linear group, there exists a rational cross section $s: A\to G$ and $f(x, y)=s(x+y)-s(x)-s(y)$ $(x, y\in A)$ is a symmetric rational factor system. If we take another cross section $s': A\to G$ and put $f'(x, y)=s'(x+y)-s'(x)-s'(y)$, then $f(x, y)-f'(x, y)=(s(x+y)-s'(x+y))-(s(x)-s'(y))=\delta(s-s')(x, y)$ is trivial. If we write $h(G)=f$, h defines a homomorphism $h: \mathrm{Ext}(A, C)\to H^2_{\mathrm{rat}}(A, C)_s$. By Prop. 4 Chap. VII [7] h is an isomorphism onto.

PROPOSITION 6. *Let the notation be as above. We have commutative diagram*

$$\begin{array}{ccc} \mathrm{Ext}(A, C) & \xrightarrow{h_A} & H^2_{\mathrm{rat}}(A, C)_s \\ \downarrow \alpha^0 & & \uparrow \alpha^f \\ \mathrm{Ext}(B, C) & \xrightarrow{h_B} & H^2_{\mathrm{rat}}(B, C)_s \end{array}$$

PROOF. Let G be an element of $\mathrm{Ext}(A, C)$ and $\alpha^0(G)=H\in\mathrm{Ext}(B, C)$. Let $s: A\to G$ be a rational section, and $f(x, y)=s(x+y)-s(x)-s(y)$. Then $\alpha^f(f)=f\circ\alpha$. Since $s\circ\alpha$ is a rational section from B to H, we have $h_H=f\circ\alpha$ i.e. $\alpha^f(h_A(G))=f\circ\alpha$ $h_B(\alpha^0(G))$. Q.E.D.

d) Let G be an element of $\mathrm{Ext}(A, C)$ and $s: A\to G$ be a rational section. Then there exists an open set U on A on which s is everywhere regular. There is a finite open covering $\{U_i=U+a_i\}$ $a_i\in U)$ of A. If we put $s_i(x+a_i)=s(x)+s(a_i)$ $(x\in U)$ and $c_{ij}=s_j-s_i$ then (c_{ij}) is a 1-cocycle and determines an element $c\in H^1(A, C_A)$, where C_A is the sheaf of germs of regular mapping from A to C. If we write $\pi(G)=c$, π defines an into-isomorphism $\pi: \mathrm{Ext}(A, C)\to H^1(A, C_A)$ (see Prop. 5

Chap. VII n° 6 [7]). Clearly $(b_{ij}=s_j\circ\alpha-s_i\circ\alpha)$ is a 1-cocycle for finite open covering $\{V_i=\alpha^{-1}(U_i)\}$ of B and it determines an element $b\in H^1(B, C_A)$. If we put $^*\alpha(c)=b$. $^*\alpha$ defines a homomorphism $^*\alpha: H^1(A, C_A)\to H^1(B, C_B)$.

PROPOSITION 7. *Let the notations be as above a commutatiqe diagram*

$$\begin{array}{ccc} \mathrm{Ext}(A, C) & \xrightarrow{\pi_A} & H^1(A, C_A) \\ \downarrow{\scriptstyle \alpha^0} & & \downarrow{\scriptstyle ^*\alpha} \\ \mathrm{Ext}(B, C) & \xrightarrow{\pi_B} & H^1(B, C_B). \end{array}$$

PROOF. Let G be an element of $\mathrm{Ext}(A, C)$ and $\alpha^0(G)=H$. Let $s: A\to G$ be a rational section then $s\circ\alpha: B\to H$ is also a rational section. If s is everywhere regular on an open set U of A then $s\circ\alpha$ is everywhere regular on $V=\alpha^{-1}(U)$ of H. For a finite open covering $\{V+b_i=V_i\}_{i=1,\cdots,n}$ of B, $\{U+\alpha(b_i)=U_i\}$ is a finite open covering of A. If we put

$$(s\circ\alpha_i)(\alpha+b_i)=s\circ\alpha(x)+s\circ\alpha(b_i) \qquad (x\in V),$$
$$s_i(y+a_i)=s(y)+s(a_i) \qquad (y\in U,\ a_i=\alpha(b_i))$$

and

$$b_{ij}=(s\circ\alpha)_j-(s\circ\alpha)_i\,, \qquad c_{ij}=s_j-s_i\,, \qquad [b_{ij}]=b, \qquad [c_{ij}]=c.$$

Then $^*\alpha(c)=b$. We have $\alpha^*(\pi_A(G))=\pi_B(\alpha^0(G))$. Q.E.D.

e) Let $s_A: A\times A\to A$ be the composition law of A and $p_{A_i}: A\times A\to A$ $(i=1, 2)$ be the projections to the first factor $(i=1)$ and to the second factor $(i=2)$ respectively. Then we have homomorphisms $^*s_A: H^1(A, C_A)\to H^1(A\times A, C_{A\times A})$ and $^*p_{A_i}$: $H(A, C_A)\to H^1(A\times A, C_{A\times A})$ $(i=1, 2)$. An element x of $H(A, C_A)$ is called primitive if $^*s_A(s)={}^*p_{A_1}(x)+{}^*p_{A_2}(x)$.

PROPOSITION 3. *Let the notations be as above* $^*\alpha: H^1(A, C_A)\to H^1(B, C_B)$ *maps each primitive element to a primitative element.*

This follows immediately from the following Lemma.

LEMMA 6. *For a commutative diagram of algebraic groups and their homomorphism*

$$\begin{array}{ccc} A & \xrightarrow{\tau_2} & D \\ \downarrow{\scriptstyle \tau_1} & & \downarrow{\scriptstyle \tau_4} \\ B & \xrightarrow{\tau_3} & E \end{array}$$

we have commutative diagram

$$\begin{array}{ccc} H^1(A, C_A) & \xleftarrow{^*\tau_2} & H^1(D, C_D) \\ \uparrow{\scriptstyle ^*\tau_1} & & \uparrow{\scriptstyle ^*\tau_4} \\ H^1(A, C_B) & \xleftarrow{^*\tau_3} & H^1(B, C_E). \end{array}$$

The proof is quite easy.

§ 7. Geometric mappings.

Now we return to our problem. Let V be a complete normal projective variety defined over k and C be a 1-dimensional generic hyperplane section of V defined over $k(u)=k$. Let A be the Albanese variety of V defined over k and J be the Jacobian variety of C defined over $k(u)$. Let P be the Picard variety of V. We denote by α, the homomorphism $t_{i^*}: P \to J$ defined in § 1. J and A can be considered as subgroups of $H^1(J, \mathcal{O}_J^*)$ and $H^1(P, \mathcal{O}_P^*)$ respectively and they coincide with the groups of primitive elements of theme respectively, where $\mathcal{O}_J^*$ (resp. $\mathcal{O}_P^*$) is the sheaf of non-zero elements of local rings of J (resp. P). (See Chap. VII n° 16 [7]). By Prop. 3 § 2 $^*\alpha: H^1(J, \mathcal{O}_J^*) \to H^1(P, \mathcal{O}_P^*)$ maps J to A.

PROPOSITION 9. *Let the notations be as above, We have* $t_\alpha = {}^*\alpha | J$.

PROOF. Let D be a divisor on J such that $\alpha^{-1}(D)$ is defined. Then there exists a definite open covering $\{U_i\}$ of J such that D is locally defined on U_i by a rational function $R_i(x)$ (i.e. $D \cap U_i = (R_i(x)) \cap U_i$) such that $f_{ij} = R_j(x)/R_i(x)$ is unit at every point on $U_i \cap U_j$. $\{f_{ij}\}$ is 1-cocycle and determines an element $[D] = [f_{ij}] \in H^1(J, \mathcal{O}_J^*)$. By the definition $^*\alpha([D]) = {}^*\alpha([f_{ij}]) = [f_{ij} \circ \alpha] = [R_j \circ \alpha / R_i \circ \alpha]$. Putting $V_i = \alpha^{-1}(U_i)$ we get a finite open covering $\{V_i\}$ of P. The divisor $\alpha^{-1}(D)$ on P is locally defined on V_i by the rational function $R_i \circ \alpha$ on P. (See App. Theorem 3 Cor. 2 Lang [1]). Thus we have $t_\alpha([D]) = [\alpha^{-1}(D)] = [R_j \circ \alpha / R_i \circ \alpha]$ and we get $t_\alpha[D] = {}^*\alpha[D]$. Q. E. D.

By Theorem 5 Chap. VII [7] $\pi: \mathrm{Ext}(A, C) \to H^1(A, C_A)$ maps $\mathrm{Ext}(A, C)$ isomorphically onto the subgroup of primitive elements of $H^1(A, C_A)$. By Prop. 2 we have commutative diagram

$$\begin{array}{ccc} \mathrm{Ext}(J, G_m) & \xrightarrow{\pi_J} & J \subset H^1(J, \mathcal{O}_J^*) \\ \downarrow \alpha^0 & & \downarrow t_\alpha \quad \downarrow {}^*\alpha \\ \mathrm{Ext}(P, G_m) & \xrightarrow{\pi_P} & A \subset H^1(P, \mathcal{O}_P^*) \end{array}$$

where π_J and π_P are surjective isomorphisms. As is stated in § 1 t_α is a surjective homomorphism. Therefore we have

PROPOSITION 10. *Let the notations be as above,* $\alpha^0: \mathrm{Ext}(J, G_m) \to \mathrm{Ext}(P, G_m)$ *is a surjective homomorphism.*

§ 8. Structure of function modules and the correspondence of $\mathcal{P}_{\mathfrak{M}}$ and $\mathcal{J}_{\mathfrak{M}}$.

Let $P_1, P_2, \cdots, P_m$ be a set of distinct points of C. Since V is normal, C is non-singular. Therefore, $P_1, P_2, \cdots, P_m$ are simple points of C but also they are simple points on V because C is a generic hyperplane section of V over the field of definition of V. Let $\mathfrak{O}(P_i)$ and $\mathfrak{M}(P_i)$ be the local ring and its maximal ideal at P_i respectively in the rational function field $\Omega(V)$ of V over Ω. We denote by $\mathfrak{O}(P_i)$ and $\mathfrak{m}(P_i)$ the local ring and its maximal ideal of C at P_i in $\Omega(C)$ respectively. We define the $\mathfrak{M}$-structure on V and C by $\{P_1,\cdots,P_m,\ \mathfrak{M}=\bigcap_{i=1}^{m}\mathfrak{M}(P_i)^{n_i}\}$ and $\{P_1, P_2,\cdots,P_m,\ \mathfrak{m}=\bigcap_{i=1}^{m}\mathfrak{m}(P_i)^{n_i}\}$ respectively.

REMARK. The $\mathfrak{M}$-structure is determined if we give a 0-cycle $\sum_{i=1}^{m} n_iP_i$. Therefore we use sometimes 0-cycle instead of $\mathfrak{M}$-structure.

Let $U(P_i)$ and $u(P_i)$ be the unit groups of $\mathfrak{O}(P_i)$ and $\mathfrak{o}(P_i)$ respectively. We denote by $\mathfrak{O}(V, C)$ the specialization ring of C in $\Omega(V)$. The restriction of a function on V to the function on C defines an onto-homomorphism $\rho:\mathfrak{O}(V, C)\to\Omega(C)$. The homomorphism ρ maps $\mathfrak{O}(P_i)$ onto $\mathfrak{o}(P_i)$, $\mathfrak{M}(P_i)$ onto $\mathfrak{m}(P_i)$ and $U(P_i)$ onto $u(P_i)$ respectively. By [12] Chapter VIII, Prop. 10 we can choose such local parameters $\{t_{i1}, t_{i2},\cdots, t_{ir}\}$ at P_i on V and t_i at P_i on C that $\rho(t_{i1})=t_i$ and $\rho(t_{ij})=0$ $(j\geqslant 2)$. Choosing such local parameters $\bar{\rho}_i=\rho|\mathfrak{O}(P_i):\mathfrak{O}(P_i)\to\mathfrak{o}(P_i)$ is given by

$$\bar{\rho}_i\Big(\sum_{0\leqslant\Sigma k_j} a_{k_{i1}\cdots k_{ir}}t_{i1}^{k_{i1}}\cdot t_{i2}^{k_{i2}}\cdots t_{ir}^{k_{ir}}\Big)=\sum_{0\leqslant k_{i1}}\alpha_{k_{i1}0\cdots 0}t_{i1}^{k_{i1}}.$$

$\bar{\rho}_i$ induces a homomorphism $\rho_i: U(P_i)/U(P_i)^{(n_i)}\to u(P_i)/u(P_i)^{(n_i)}$.

By Prop. 2 and Prop. 3 in §4, we have

$$p=0: U(P_i)/U(P_i)^{(n_i)}\approx G_m\times G_a^{N_i-1}, \qquad u(P_i)/u(P_i)^{(n_i)}\approx G_m\times G_a^{n_i-1},$$

$$p\neq 0: U(P_i)/U(P_i)^{n_i}\approx G_m\times\prod_{(k_1^{(i)}\cdots k_r^{(i)}l^{(i)})} Wr_{k_1^{(i)}\cdots k_r^{(i)}l^{(i)}}$$

$$u(P_i)/u(P_i)^{(n_i)}\approx G_m\times\prod_{k_1^{(i)}} Wr_{k_1^{(i)}l^{(i)}}.$$

By the proof of these Propositions and the above selection of local parameters,

$$\rho_i:\quad G_m\times G_a^{N_i-1}\longrightarrow G_m\times G_a^{n_i-1}\qquad (p=0),$$

$$\rho_i:\quad G_m\times\prod_{i=1}^{m}\prod_{(k_1^{(i)},\cdots,k_r^{(i)},l^{(i)})} Wr_{k_1^{(i)}\cdots k_r^{(i)}l^{(i)}}\longrightarrow G_m\times\prod_{i=1}^{m}\prod_{k_1^{(i)}l^{(i)}} Wr_{k_1^{(i)}l^{(i)}}\qquad (p\neq 0)$$

are projections onto suitable factors.

Let $\mathcal{P}_{\mathfrak{M}}$ be the generalized Picard variety of V with respect to the $\mathfrak{m}$-structure $\{P_1,\cdots,P_m, \mathfrak{M}\}$ of V and $\mathcal{J}_{\mathfrak{m}}$ be the generalized Jacobian variety of C with respect to the $\mathfrak{m}$-structure $\{P_1,\cdots,P_m;\ \mathfrak{m}\}$ of C. P and J be the Picard and Jacobian variety of V and of C respectively. Then by Theorem 3 in §4 we have the exact sequences

$$0\longrightarrow K_{\mathfrak{M}}\xrightarrow{\ \iota_P\ }\mathcal{P}_{\mathfrak{M}}\xrightarrow{\ p_P\ }P\longrightarrow 0$$

$$0\longrightarrow k_{\mathfrak{m}}\xrightarrow{\ \iota_J\ }\mathcal{J}_{\mathfrak{m}}\xrightarrow{\ p_J\ }J\longrightarrow 0$$

where $K_{\mathfrak{M}}=[\prod_{i=1}^{m}(U(P_i)/U(P_i)^{(n_i)}]/G_m$ and $k_{\mathfrak{m}}=[\prod_{i=1}^{m}u(P_i)/u(P_i)^{(n_i)}]/G_m$

We have a homomorphism $\tau: K_{\mathfrak{M}} \to k_{\mathfrak{m}}$ defined by $\prod_{i=1}^{m} \rho_i$, and a homomorphism $\alpha: P \to J$ induced by $i: C \to V$. Therefore we have homomorphisms

$$\tau_0: \mathrm{Ext}(P, K_{\mathfrak{M}}) \longrightarrow \mathrm{Ext}(P, k_{\mathfrak{m}}), \qquad \alpha^0: \mathrm{Ext}(J, k_{\mathfrak{m}}) \longrightarrow (P, k_{\mathfrak{m}}).$$

THEOREM 4. $\tau_0(\mathcal{P}_{\mathfrak{M}}) = \alpha^0(\mathcal{J}_{\mathfrak{m}})$.

PROOF. By the definition it is clear that if $f \equiv 1 \pmod{\mathfrak{M}}$ for a function f on V, we have $\rho(f) \equiv 1 \pmod{\mathfrak{m}}$ and $i^{-1}((f)) = (\rho(f))$ (By Theorem 3 App. [1]). Therefore i^{-1} induces a homomorphism $\mathcal{D}_{\mathfrak{M}_a}(V) \to \mathcal{D}_{\mathfrak{M}_a}(C)$ and $\mathcal{D}_{\mathfrak{M}_l}(V) \to \mathcal{D}_{\mathfrak{M}_l}(C)$, and we get, taking quotient, a homomorphism $\bar{\alpha}: \mathcal{P}_{\mathfrak{M}} \to \mathcal{J}_{\mathfrak{m}}$. By the definition of the homomorphisms $\alpha, \bar{\alpha}, \tau$, we have the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K_{\mathfrak{M}} & \longrightarrow & \mathcal{P}_{\mathfrak{M}} & \longrightarrow & P & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \tau} & & \downarrow{\scriptstyle \bar{\alpha}} & & \downarrow{\scriptstyle \alpha} & & \\ 0 & \longrightarrow & k_{\mathfrak{m}} & \longrightarrow & \mathcal{J}_{\mathfrak{m}} & \longrightarrow & J & \longrightarrow & 0\,. \end{array}$$

Thus we get $\tau_0(\mathcal{P}_{\mathfrak{M}}) = \alpha^0(\mathcal{J}_{\mathfrak{m}})$. Q. E. D.

By Theorem 4 we get immediately

COROLLARY. *When $m=2$ and $n_1=n_2=1$, we have $K_{\mathfrak{M}} = k_{\mathfrak{m}} = G_m$ and τ_0 must be identity. In this case we have $\alpha^0(\mathcal{J}_{\mathfrak{m}}) = \mathcal{P}_{\mathfrak{M}}$.*

In the commutative diagram

$$\begin{array}{ccc} \mathrm{Ext}(J, G_m) & \xrightarrow{\pi_J} & J \\ \uparrow{\scriptstyle \alpha^0} & & \uparrow{\scriptstyle t_\alpha} \\ \mathrm{Ext}(P, G_m) & \xrightarrow{\pi_P} & A \end{array}$$

t_α is surjective by the statements in § 1 and π_J, π_P are surjective isomorphisms. Therefore α^0 is also surjective. By the Theorem in our paper [3], $\mathrm{Ext}(J, G_m)$ is generated by $\mathcal{J}_{\mathfrak{m}}$ defined as in Corollary of Theorem 1. By the same Corollary we conclude that $\mathrm{Ext}(P, G_m)$ is generated by the generalized Picard varieties of V with respect to 0-cycles $P_1 + P_2$ $(P_1 \neq P_2)$ on V.

Summarizing above results we get

THEOREM 5. *Let V be a normal projective variety and P be its Picard variety. Then $\mathrm{Ext}(P, G_m)$ is generated by the generalized Picard varieties of V with respect to 0-cycles $P_1 + P_2$ $(P_1 \neq P_2)$ on V.*

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(Received October 30, 1964)