On the second cohomology groups (Schur-multipliers) of infinite discrete reflection groups

By Takeo Yokonuma

§ 0. Introduction.

This note is a continuation of Ihara-Yokonuma [3]. It is well-known that the second cohomology group of a group G is closely related to the theory of group extensions and projective representations. More precisely, for any element in the second cohomology group for the coefficient group Ω (where Ω is an abelian group) under the trivial action of G on Ω, there exists a central group extension and vice versa (cf. § 1). And if G is finite, there exists a similar correspondence between projective representations and the elements of the second cohomology group. (When we consider the projective representations, Ω is the multiplicative group of a field.) (cf. Schur [5], Yamazaki [8])

In our previous paper [3], we have determined the second cohomology group $\mathrm{H}^2(G, \boldsymbol{C}^*)$ for the coefficient group $\boldsymbol{C}^*$ (=the multiplicative group of the complex number field $\boldsymbol{C}$) under the trivial action of G on $\boldsymbol{C}^*$, in the case where G is the finite reflection group on a Euclidean space. The purpose of this note is to do the same thing when G is an infinite discrete reflection group on a Euclidean space. Our main result is the following.

Theorem. *Let G be an infinite discrete reflection group on a Euclidean space E. Then $\mathrm{H}^2(G, \boldsymbol{C}^*)$ is given as follows:*

$$\mathrm{H}^2(G, \boldsymbol{C}^*) \cong \underbrace{\boldsymbol{Z}_2 \times \cdots \times \boldsymbol{Z}_2}_{\kappa}$$

where $\boldsymbol{Z}_2$ means the cyclic group of order 2 and κ is a non negative integer which is determined as follows:

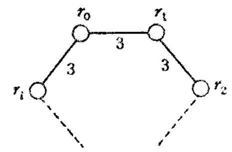
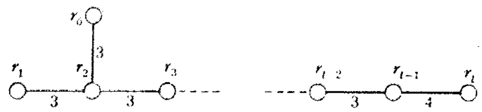
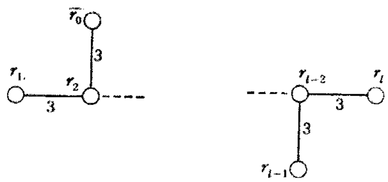
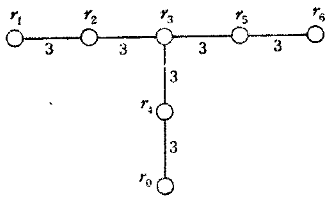
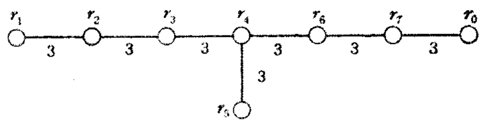
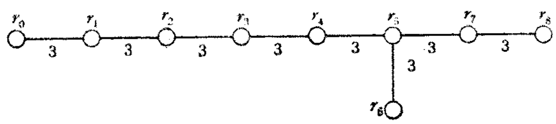
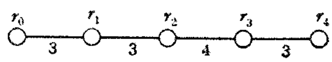
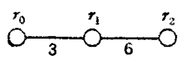
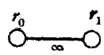
i) *If G is irreducible on E, the value of κ is given in Table I.*

ii) *Let $G \cong G_1 \times \cdots \times G_r$ be a decomposition of G into the irreducible components. Then*

$$\mathrm{H}^2(G, \boldsymbol{C}^*) \cong \prod_i \mathrm{H}^2(G_i, \boldsymbol{C}^*) \times \prod_{i<j} P(G_i, G_j)$$

where $P(G_i, G_j)$ is the group of all pairings of G_i and G_j in $\boldsymbol{C}^*$.
(See for the definition of pairing to K. Yamazaki [8] or [3, § 0].)

Table I

type of G	Diagram	κ
$\widetilde{A}_2$		1
$\widetilde{A}_3$		3
$\widetilde{A}_l (l \geqq 4)$		2
$\widetilde{B}_3$		4
$\widetilde{B}_4$		5
$\widetilde{B}_l (l \geqq 5)$		4
$\widetilde{C}_2$		3
$\widetilde{C}_3$		5
$\widetilde{C}_l (l \geqq 4)$		6
$\widetilde{D}_4$		6
$\widetilde{D}_l (l \geqq 5)$		3
$\widetilde{E}_6$		
$\widetilde{E}_7$		1
$\widetilde{E}_8$		
$\widetilde{F}_4$		3
$\widetilde{G}_2$		2
I_2		0

Our method here is similar as that of [3] except few particular considerations due to the following two facts:

a) in the present case G is not finite,

and

b) the diagram of type $\tilde{A}_l(l\geqq 2)$ is a circuit.

In §1, we shall review, for the convenience of the reader, the concepts of cohomology group and group extension, and the classification of discrete reflection groups due to Coxeter [1] and Witt [7], etc.

In §2 and §3 we shall establish the theorem by a similar procedure as [3], i.e. by constructing a bijective homomorphism $\mathrm{H}^2(G, \boldsymbol{C}^*)\to\mathfrak{N}$, where $\mathfrak{N}$ is a group consisting of "normalized" factor sets on G. However when we construct $\mathfrak{N}$, we have to pay attention to the facts a), b).

Through this note, notations and terminologies accord with those in [3].

§1. Preliminaries.

Let G be a group. A function $\alpha\colon G\times G\to\Omega$ (where Ω is an abelian group) which satisfies

(1) $\alpha(a, bc)\alpha(b, c)=\alpha(a, b)\alpha(ab, c)$ for all $a, b, c\in G$ is called a factor set or a Ω-valued 2-cocycle on G (under the trivial action of G on Ω). The equivalence of two factor sets, $\mathrm{Z}^2(G, \Omega)$ and the second cohomology group $\mathrm{H}^2(G, \Omega)$ are defined as [3, §1] where $\Omega=\boldsymbol{C}^*$.

A pair (H, ρ) of a group H and a surjective homomorphism $\rho\colon H\to G$ is called a group extension of G. If the kernel A of ρ is contained in the center of H, the extension (H, ρ) is called a central extension. Let (H_i, ρ_i) $i=1, 2$ be two group extensions of the same kernel A. Then they are called to be equivalent (strongly equivalent in K. Yamazaki's [8] terminology) if there exists an isomorphism $\tau\colon H_1\to H_2$ such that $\rho_2\circ\tau=\rho_1$ and the restriction of τ to A is the identity mapping. A mapping $S\colon G\to H$ is called a section of (H, ρ) if $\rho\circ S$ is the identity mapping of G.

Let (H, ρ) be a central extension of G with kernel A and S a section of (H, ρ). The mapping $\alpha\colon G\times G\to A$ defined by $(a, b)\to\alpha(a, b)=S(a)S(b)S(ab)^{-1}$ is an A-valued 2-cocycle of G (under the trivial action of G on A). α is called the factor set associated to the section S of the extension (H, ρ). Let T be another section of (H, ρ) and β the factor set associated to T. Then α and β are equivalent. Also, equivalent central extensions of the same kernel have equivalent factor sets. Thus we have a mapping

$$\mathfrak{E}(G)\to\mathrm{H}^2(G, A)$$

where $\mathfrak{E}(G)$ is the set of all equivalence classes of central extension of G of kernel A. It is well known that this mapping is surjective for any G and for any abelian group A. (We refer to e.g. Yamazaki [8] or M. Hall [2] for the details.) In the following we consider only the case where $A=C^*$.

Moreover, let φ be a projective representation of G over C and T a section of φ. Then there exist numbers $\alpha(a, b)\in C^*$ such that

$$T(a)T(b)=\alpha(a, b)T(ab) \quad \text{for } a, b\in G.$$

The function $\alpha: G\times G\to C^*$ is a factor set of G and we have a mapping

$$\mathfrak{P}(G)\to \mathrm{H}^2(G, C^*)$$

where $\mathfrak{P}(G)$ is the set of all equivalence classes of projective representations of G over C, as stated in [3. § 1]. When G is finite, this mapping is surjective (Schur [5]).

A group G of linear transformations in an l-dimensional Euclidean space $E^{(l)}$ will be called a discrete reflection group if it is generated by reflections and a discrete subgroup of the group of orthogonal transformations in $E^{(l)}$. G being discrete, we remark that the angle between P_i and $P_j (i\neq j)$ is equal to π/m_{ij} where $m_{ij}\in\{2, 3, \cdots, \infty\}$ and $P_1, \cdots, P_k$ are the hyperplanes in $E^{(l)}$ fixed pointwise by the generating reflections $R_1, \cdots, R_k$ of G respectively. These groups were classified by Coxeter [1] and Witt [7].

Let us recall the result in [1], [7] (cf. [3, § 2]). Let G be an abstract group generated by $r_0, r_1, \cdots, r_l$ with the defining relations

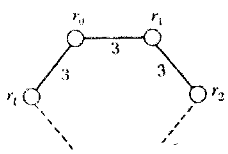
(2) $(r_ir_j)^{m_{ij}}=e \quad m_{ii}=1, m_{ij}=m_{ji}=\text{integer}\geqq 2$ or ∞. $(i\neq j)$

To such a group there is associated a diagram $\Pi(G)$ as [3, § 2].

LEMMA 1. *The discrete reflection group is isomorphic to one of those groups or direct product of several number of those groups which are represented by the following diagrams or the diagrams in Lemma* 2 *of* [3].

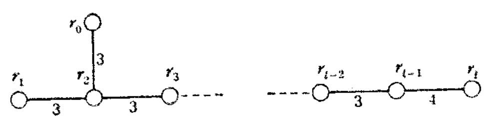
The group which is represented by the following diagram is infinite.

$\tilde{A}_l (l\geqq 2)$



(*the extended Weyl group of* $SL(l+1, \boldsymbol{C})$)

$\tilde{B}_l (l\geqq 3)$

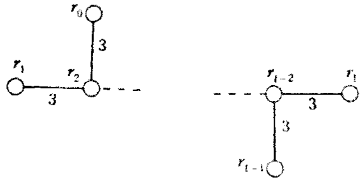


(*the extended Weyl group of* $SO(2l+1, \boldsymbol{C})$)

$\tilde{C}_l (l \geqq 2)$

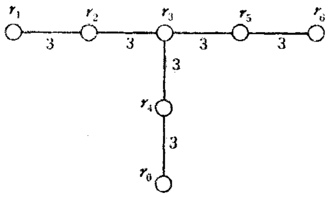
(*the extended Weyl group of* $Sp(l, \boldsymbol{C})$)

$\tilde{D}_l (l \geqq 4)$



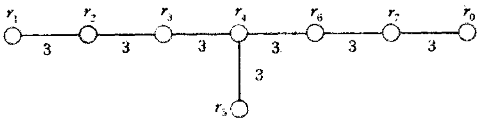
(*the extended Weyl group of* $SO(2l, \boldsymbol{C})$)

$\tilde{E}_6$



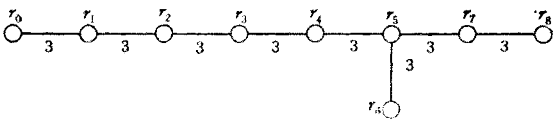
(*the extended Weyl group of complex Lie group of type* (E_6))

$\tilde{E}_7$



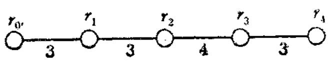
(*the extended Weyl group of complex Lie group of type* (E_7))

$\tilde{E}_8$



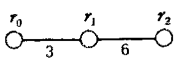
(*the extended Weyl group of complex Lie group of type* (E_8))

$\tilde{F}_4$



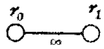
(*the extended Weyl group of complex Lie group of type* (F_4))

$\tilde{G}_2$



(*the extended Weyl group of complex Lie group of type* (G_2))

I_2



(We refer as for the details about the properties of those groups to N. Iwahori–H. Matsumoto [4, §1] or E. Stiefel [6].)

We shall now give the structure of the group $P(G_1, G_2)$ of pairings of two discrete infinite reflection groups G_1, G_2. As in the finite case (cf. [3, §2]) we have the following lemma.

LEMMA 2. i) *Let G be discrete reflection group on a Euclidean space E. Then*

(3) $\mathrm{Hom}(G, \boldsymbol{C}^*)\cong \boldsymbol{Z}_2\times\cdots\times \boldsymbol{Z}_2$ (λ-*times*)

If G is irreducible, λ is equal to the number of connected components of diagram after the segments of diagram corresponding to m_{ij}=even are erased. If $G\cong G_1\times\cdots\times G_r$ is the decomposition of G into the irreducible components, then

$$\mathrm{Hom}(G, \boldsymbol{C}^*)\cong \prod_i \mathrm{Hom}(G_i, \boldsymbol{C}^*).$$

ii) *Let G_1, G_2 be discrete reflection groups. Then*

$$P(G_1, G_2)\cong \boldsymbol{Z}_2\times\cdots\times \boldsymbol{Z}_2 \qquad (\lambda_1\cdot\lambda_2\text{-}times)$$

where λ_1, λ_2 are the numbers of $\boldsymbol{Z}_2$ in (3) for G_1, G_2 respectively.

§ 2. Normalization of a factor set.

Now, let us proceed in our subject. The process of proof is entirely same as that in [3]. Namely, first of all we may assume that G is irreducible on the Euclidean space E, and G can be identified with the group generated by $r_0, \cdots, r_l$ with fundamental relations (2), because theorem 2.1 in Yamazaki's [8] is available in this case. Then we shall construct an injective homomorphism θ from $\mathrm{H}^2(G, \boldsymbol{C}^*)$ into $\mathfrak{N}\cong(\boldsymbol{Z}_2)^{\kappa_0}$, where $\mathfrak{N}$ is a group consisting of "normalized" factor sets on G (in the sense specified below), and $(\boldsymbol{Z}_2)^{\kappa_0}$ is a direct product of κ_0 copies of cyclic group $\boldsymbol{Z}_2$. And, in next section we shall prove that θ is surjective.

Let $\{\alpha\}$ be an element of $\mathrm{H}^2(G, \boldsymbol{C}^*)$. There exist central group extensions with kernel $\boldsymbol{C}^*$ whose factor sets belong to $\{\alpha\}$. Denote one of them by (H, ρ) and let S be a section of ρ. Denote $S(r_i)$ by S_i. Then the following relations are valid according to (2):

(4) $S_i^2=\varepsilon_i e \qquad i=0, 1, \cdots, l$

(5) $(S_iS_j)^{n_{ij}}=\alpha_{ij}(S_jS_i)^{n_{ij}}$ if m_{ij} is even: $m_{ij}=2n_{ij}$

(6) $(S_iS_j)^{n_{ij}}S_i=\beta_{ij}(S_jS_i)^{n_{ij}}S_j$ if m_{ij} is odd: $m_{ij}=2n_{ij}+1$,

where e denote the identity element in H and ε_i, α_{ij}, β_{ij} belong to $\boldsymbol{C}^*$.

These numbers determine the cohomology class of the extension. Namely,

LEMMA 3. *Let G be the discrete reflection group which is generated by r_i $i=0, \cdots, l$ with fundamental relations* (2).

Let (H, ρ) and (K, σ) be two central extensions with kernel $\boldsymbol{C}^*$ of G and

S(resp. T) a section of (H, ρ)(resp. (K, σ)) and α(resp. β) the factor set of (H, ρ)(resp. (K, σ)) associated with S(resp. T). We denote $S(r_i)$ and $T(r_i)$ by S_i and T_i respectively. Then the following relations are valid by (2).

$$\begin{array}{lll} S_i^2=\varepsilon_i e & ;\ T_i^2=\varepsilon'_i e' & i=0, 1, \cdots, l, \\ (S_iS_j)^{n_{ij}}=\alpha_{ij}(S_jS_i)^{n_{ij}} & ;\ (T_iT_j)^{n_{ij}}=\alpha'_{ij}(T_jT_i)^{n_{ij}} & \text{if } m_{ij}=2n_{ij} \\ (S_iS_j)^{n_{ij}}S_i=\beta_{ij}(S_jS_i)^{n_{ij}}S_j; & (T_iT_j)^{n_{ij}}T_i=\beta'_{ij}(T_jT_i)^{n_{ij}}T_j & \text{if } m_{ij}=2n_{ij}+1 \end{array}$$

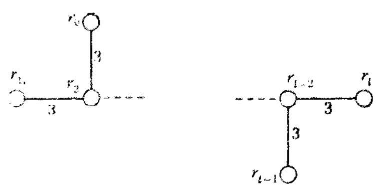
If $\varepsilon_i=\varepsilon'_i$, $\alpha_{ij}=\alpha'_{ij}$, $\beta_{ij}=\beta'_{ij}$ for all i and j in above relations, α is equivalent to β.

PROOF. To each element of G we fix once for all an expression with the generators (cf. [4, §1]). Let us construct sections $S_1: G\to H$ and $T_1: G\to K$. Suppose that $G\ni a=r_{i_1}\cdots r_{i_p}$ is the expression thus fixed. We define $S_1(a)=S_{i_1}\cdots S_{i_p}$ and $T_1(a)=T_{i_1}\cdots T_{i_p}$. We denote by α_1(resp. β_1) the factor set of (H, ρ)(resp. (K, σ)) associated with S_1(resp. T_1). It is clear that $\alpha_1(a, b)=\beta_1(a, b)$ for all $a, b\in G$. α_1 is equivalent to β_1 by the way of construction. α is equivalent to α_1, and β is equivalent to β_1. Therefore α is equivalent to β, Q.E.D.

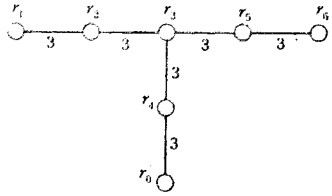
Now, let us normalize the numbers ε_i, α_{ij}, β_{ij}. We shall, however, distinguish two types of diagrams. In Case I, the diagram has no circuits. In Case II, there exists a circuit in the diagram, that is to say, G is of type $\tilde{A}_l$.

CASE I

As we showed in [3, §3], we may assume $\varepsilon_i=1$, $\beta_{ij}=1$ and $\alpha_{ij}^2=1$, by changing the section of ρ. Even if G is of type $\tilde{D}_l(l\geqq 4)$ or $\tilde{E}_6$, i.e. in the cases which are not described in [3], we may assume that all $\beta_{ij}=1$. Let G be of type $\tilde{D}_l$, i.e. diagram of G be



We can assume $\beta_{1,2}=\beta_{2,3}=\cdots=\beta_{l-2,l}=\beta_{l-2,l-1}=1$, and replacing S_0 by $\beta_{0,2}S_0$ we also may assume $\beta_{0,2}=1$. Let G be of type $\tilde{E}_6$, i.e. diagram of G be



We can assume $\beta_{1,2}=\beta_{2,3}=\beta_{3,5}=\beta_{5,6}=\beta_{3,4}=1$ and replacing S_0 by $\beta_{4,0}S_0$, we also may assume $\beta_{4,0}=1$.

And we can define a homomorphism

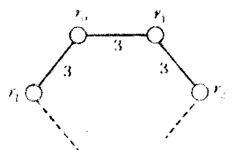
$$\theta:\ \mathrm{H}^2(G, \boldsymbol{C}^*)\rightarrow\boldsymbol{C}^*\times\cdots\times\boldsymbol{C}^*$$

by $\{\alpha\}\rightarrow\{\alpha_{ij}\}$. By lemma 3, θ is injective. Using Lemma 4 in [3], α_{ij}'s are divided into several classes: if α_{ij} and α_{kl} are necessarily equal by Lemma 4 in [3], α_{ij} and α_{kl} belong to the same class. The number of classes is denoted by κ_0, which is given in Table II, and the values of α_{ij} of κ_0 classes are denoted by $\alpha_1, \cdots, \alpha_{\kappa_0}$. Then θ is considered as a homomorphism: $\mathrm{H}^2(G, \boldsymbol{C}^*)\rightarrow\boldsymbol{Z}_2\times\cdots\times\boldsymbol{Z}_2$ (κ_0-times)$=\mathfrak{N}$.

CASE II

As before we may assume all $\varepsilon_i=1$, $\alpha_{ij}^2=1$, $\beta_{ij}^2=1$, and all α_{ij} are equal by Lemma 4 in [3] except the case $l=3$.

We consider the coefficients β_{ij}. Let



be the diagram, and

$$(7)\qquad \begin{cases} S_iS_{i+1}S_i=\beta_{i,i+1}S_{i+1}S_iS_{i+1} & i=0, 1, \cdots, l-1 \\ S_lS_0S_l=\beta_{l,0}S_0S_lS_0 \end{cases}$$

are the relations (6). We can replace S_1 by $\beta_{0,1}S_1$, S_2 by $\beta_{0,1}\beta_{1,2}S_2, \cdots$, and S_l by $\beta_{0,1}\beta_{1,2}\cdots\beta_{l-1,l}S_l$ keeping the other equations invariant. Then all $\beta_{i,i+1}$ $i=0, \cdots, l-1$ may be equal to 1. If we multiply side by side of (7) we have

$$(8)\qquad S_0S_1S_0S_1S_2S_1\cdots S_lS_0S_l=\beta_{l,0}S_1S_0S_1\cdots S_0S_lS_0$$

If we put $r=r_0$, $s=r_1r_0r_1r_2\cdots r_lr_0r_l$, we have $rs=sr$ and $\beta_{l,0}=\alpha(r, s)\cdot\alpha(s, r)^{-1}$ from (8), where α is the factor set of (H, ρ) associated to S. By Lemma 1 in [3], $\beta_{l,0}$ is independent of the choice of the section. We denote $\beta_{l,0}$ by β. In this case θ is defined by $\{\alpha\}\rightarrow\{\alpha_{ij}, \beta\}$ and $\kappa_0=1(l=2), 3(l=3), 2(l\geqq4)$. The value of α_{ij} of (κ_0-1) classes are denoted by $\alpha_1, \cdots, \alpha_{\kappa_0-1}$.

Table II

type of G	normalized relations (4)~(6)	κ_0
$\tilde{A}_2$	$S_i^2=e\quad i=0,1,2$; $S_2S_0S_2=\beta S_0S_2S_0$	1
$\tilde{A}_3$	$S_i^2=e\quad i=0,\cdots,3$; $S_iS_{i+1}S_i=S_{i+1}S_iS_{i+1}\quad i=0,1,2$; $S_3S_0S_3=\beta S_0S_3S_0$; $S_0S_2=\alpha_1S_2S_0$; $S_1S_3=\alpha_2S_3S_1$	3
$\tilde{A}_l\,(l\geqq 4)$	$S_i^2=e\quad i=0,\cdots,l$; $S_iS_{i+1}S_i=S_{i+1}S_iS_{i+1}\quad i=0,1,\cdots,l-1$; $S_lS_0S_l=\beta S_0S_lS_0$; $S_iS_j=\alpha_1S_jS_i$ if $m_{ij}=2$	2
$\tilde{B}_3$	$S_i^2=e\quad i=0,\cdots,3$; $S_1S_2S_1=S_2S_1S_2$; $S_0S_2S_0=S_2S_0S_2$; $S_0S_1=\alpha_1S_1S_0$; $S_0S_3=\alpha_2S_3S_0$; $S_1S_3=\alpha_3S_3S_1$; $(S_2S_3)^2=\alpha_4(S_3S_2)^2$	4
$\tilde{B}_4$	$S_i^2=e\quad i=0,\cdots,4$; $S_iS_{i+1}S_i=S_{i+1}S_iS_{i+1}\quad i=1,2$; $S_0S_2S_0=S_2S_0S_2$; $S_0S_1=\alpha_1S_1S_0$; $S_0S_3=\alpha_2S_3S_0$; $S_iS_4=\alpha_3S_4S_i\quad i=0,1,2$; $S_1S_3=\alpha_4S_3S_1$; $(S_3S_4)^2=\alpha_5(S_4S_3)^2$	5
$\tilde{B}_l\,(l\geqq 5)$	$S_i^2=e\quad i=0,\cdots,l$; $S_iS_{i+1}S_i=S_{i+1}S_iS_{i+1}\quad i=1,\cdots,l-2$; $S_0S_2S_0=S_2S_0S_2$; $S_0S_1=\alpha_1S_1S_0$; $S_iS_j=\alpha_2S_jS_i$ $0\leqq i<j\leqq l-1,\ m_{ij}=2$; $S_iS_l=\alpha_3S_lS_i\quad i=0,\cdots,l-2$; $(S_{l-1}S_l)^2=\alpha_4(S_lS_{l-1})^2$	4
$\tilde{C}_2$	$S_i^2=e\quad i=0,1,2$; $(S_0S_1)^2=\alpha_1(S_1S_0)^2$; $S_0S_2=\alpha_2S_2S_0$; $(S_1S_2)^2=\alpha_3(S_2S_1)^2$	3
$\tilde{C}_3$	$S_i^2=e\quad i=0,1,2,3$; $(S_0S_1)^2=\alpha_1(S_1S_0)^2$; $S_0S_2=\alpha_2S_2S_0$; $S_0S_3=\alpha_3S_3S_0$; $S_1S_2S_1=S_2S_1S_2$; $S_1S_3=\alpha_4S_3S_1$; $(S_2S_3)^2=\alpha_5(S_3S_2)^2$	5
$\tilde{C}_l\,(l\geqq 4)$	$S_i^2=e\quad i=0,1,\cdots,l$; $(S_0S_1)^2=\alpha_1(S_1S_0)^2$; $S_0S_i=\alpha_2S_iS_0\quad i=2,\cdots,l-1$; $S_0S_l=\alpha_3S_lS_0$; $S_iS_{i+1}S_i=S_{i+1}S_iS_{i+1}\quad i=1,\cdots,l-2$; $S_iS_j=\alpha_4S_jS_i\quad 1\leqq i,\ j\leqq l-1,\ m_{ij}=2$; $S_iS_l=\alpha_5S_lS_i\quad i=1,\cdots,l-2$; $(S_{l-1}S_l)^2=\alpha_6(S_lS_{l-1})^2$	6
$\tilde{D}_4$	$S_i^2=e\quad i=0,\cdots,4$; $S_iS_2S_i=S_2S_iS_2\quad i=0,1,3,4$; $S_0S_1=\alpha_1S_1S_0$; $S_0S_3=\alpha_2S_3S_0$; $S_0S_4=\alpha_3S_4S_0$; $S_1S_3=\alpha_4S_3S_1$; $S_1S_4=\alpha_5S_4S_1$; $S_3S_4=\alpha_6S_4S_3$	6
$\tilde{D}_l\,(l\geqq 5)$	$S_i^2=e\quad i=0,\cdots,l$; $S_0S_1=\alpha_1S_1S_0$; $S_iS_j=\alpha_2S_jS_i\quad m_{ij}=2$, $0\leqq i<j\leqq l\ (i,j)\neq(0,1),\ (l-1,l)$; $S_0S_2S_0=S_2S_0S_2$; $S_iS_{i+1}S_i=S_{i+1}S_iS_{i+1}\ i=1,\cdots,l-2$; $S_{l-2}S_lS_{l-2}=S_{l-2}S_lS_{l-2}$; $S_{l-1}S_l=\alpha_3S_lS_{l-1}$	3
$\tilde{E}_l\,(l=6,7,8)$	$S_i^2=e\quad i=0,\cdots,l$; $S_iS_jS_i=S_jS_iS_j$ if $m_{ij}=3$; $S_iS_j=\alpha_1S_jS_i$ if $m_{ij}=2$	1
$\tilde{F}_4$	$S_i^2=e\quad i=0,\cdots,4$; $S_iS_{i+1}S_i=S_{i+1}S_iS_{i+1}\quad i=0,1,3$; $S_0S_2=\alpha_1S_2S_0$; $S_iS_j=\alpha_2S_jS_i\quad i=0,1,\ j=3,4$ or $(i,j)=(2,4)$; $(S_2S_3)^2=\alpha_3(S_3S_2)^2$	3
$\tilde{G}_2$	$S_i^2=e\quad i=0,1,2$; $S_0S_1S_0=S_1S_0S_1$; $(S_1S_2)^3=\alpha_2(S_2S_1)^3$; $S_0S_2=\alpha_1S_2S_0$	2
I_2	$S_i^2=e\quad i=1,2$	0

§ 3. Construction of projective representations.

We shall show now that the homomorphism θ: $H^2(G, C^*) \to \mathfrak{R}$ is surjective. For this purpose, we shall do the same business as § 4 in [3], i.e. we construct projective representations whose factor sets are generators of $\mathfrak{R}$, by exhibitting the matrices corresponding to r_i. The matrix which corresponds to r_i will be denoted by T_i. T_i's should satisfy the normalized relations (where S should be rewritten by T) in table II for pre-assigned values of α_i (or α_i and β if G is of type $\tilde{A}_l$).

We begin with a trivial remark. Let G, H be groups, φ a projective representation of H. If f is a homomorphism of G into H, then $\varphi \circ f$ is a projective representation of G. In the following we shall consider the case where G is a infinite discrete reflection group and H is a subgroup of G isomorphic to a finite reflection group.

Let G be generated by $\{r_0, \cdots, r_l\}$. Then we shall denote by $G_{p \cdots q}$ the subgroup of G generated by the $\{r_i\}$, $i \in \{0, \cdots, l\} - \{p, \cdots, q\}$.

For convenience of the reader, we shall repeat the notations in [3].

$\Sigma(m)$: the system of matrices $N_1, \cdots, N_{2m+1}$ in $GL(2^m, C)$ satisfying the relations

$$\begin{cases} N_k^2 = I & k = 1, \cdots, 2m+1 \\ \{N_k, N_l\} = N_k N_l + N_l N_k = 0 & 1 \leqq k \neq l \leqq 2m+1 \\ N_{2m+1} \cdots N_1 = (i)^m I \end{cases}$$

$\Lambda(l)$: the system of matrices $A_1, \cdots, A_l$ satisfying the relations

$$\begin{cases} A_j^2 = I & j = 1, \cdots, l \\ \{A_j, A_{j+1}\} = -I & j = 1, \cdots, l-1 \\ \{A_j, A_k\} = 0 & 1 \leqq j < k \leqq l \qquad |j-k| \geqq 2 \end{cases}$$

Such a system is constructed from a $\Sigma(m)$, $2m+1 \geqq l$ for example (cf. [3, Lemma 7]).

$\Delta(l)$: the system of matrices $D_1, \cdots, D_l$ which are defined by

$$\begin{cases} D_j = \dfrac{1}{\sqrt{2}}(N_j - N_{j+1}) & j = 1, \cdots, l-1 \\ D_l = \dfrac{1}{\sqrt{2}}(N_{l-1} + N_l) \end{cases}$$

from $\Sigma(m)$, $2m+1 \geqq l$.

I-1 $\tilde{A}_l$ $l \geqq 4$

i) $(\alpha_1, \beta) = (-1, -1)$

Let us take $\Sigma\left(\begin{bmatrix} l+1 \\ 2 \end{bmatrix}\right)$, and put $T_i=\frac{1}{\sqrt{2}}(N_i-N_{i+1})$ $i=1,\cdots,l$

$T_0=\sum_{j=1}^{l+1} k_j N_j$, where $k_1=k_{l+1}=\frac{-(l-3)\sqrt{2}}{2(l+1)}$, and $k_2=\cdots=k_l=\frac{2\sqrt{2}}{l+1}$.

ii) $(\alpha_1,\beta)=(-1,1)$

Let us take $\Sigma\left(\begin{bmatrix} l+1 \\ 2 \end{bmatrix}\right)$, and put $T_1=-N_1$, $T_i=a_{i-1}N_{i-1}+b_iN_i$ $i=2,\cdots,l$,

$T_0=\frac{1}{2}N_1+a_lN_l+cN_{l+1}$, where $a_{j-1}^2+b_j^2=1$, $2a_jb_j=-1$, $a_1=\frac{1}{2}$, $a_l^2+\frac{1}{4}+c^2=1$ (cf. [3, Lemma 7])

I-2 $\tilde{A}_3$

Using the results of the case I-1, for any values given to α_1 and β(consisting of 1 or -1 only) we can construct projective representations, though always $\alpha_2=\alpha_1$. For example, if we take T_i as the case I-1 i) and ii), then we can construct projective representations whose factor sets correspond to $(\alpha_1,\alpha_2,\beta)=(-1,-1,-1)$, $(-1,-1,1)$ respectively.

Consider the homomorphism: $G\to G_0$ defined by $r_0\to r_2$, $r_i\to r_i$ $i=1,2,3$. G_0 is a finite reflection group of type A_3. By [3], we can see that α_2 may be -1, keeping $\alpha_1=\beta=1$. For example if we take $A(3)=\{A_1,A_2,A_3\}$ and put $T_0=T_2=A_2$, $T_1=A_1$, $T_3=A_3$, we have the projective representation with $(\alpha_1,\alpha_2,\beta)=(1,-1,1)$.

I-3 $\tilde{A}_2$ $\beta=-1$

Let us take T_i as the case I-1 i).

II-1 $\tilde{B}_l$ $l\geqq 5$

Consider the homomorphism: $G\to G_0$ defined by $r_0\to r_1$, $r_i\to r_i$ $i=1,\cdots,l$. G_0 is a finite reflection group of type B_l. By [3], we can construct projective representations for any values given to α_2,α_3 and α_4 though always $\alpha_1=1$.

Next, consider the homomorphism: $G\to G_l$ defined by $r_i\to r_i$ $i=0,\cdots,l-1$, $r_l\to e$. G_l is a finite reflection group of type D_l. By [3] we can see that α_1 and α_2 can vary independently, keeping $\alpha_3=\alpha_4=1$.

II-2 $\tilde{B}_4$

Using the results of the case II-1 and of [3] with respect to the finite reflection group D_4, the image of homomorphism $G\to G_4$ defined above, we can see that α_i can vary independently.

II-3 $\tilde{B}_3$

By II-1, $\alpha_1,\alpha_2,\alpha_4$ can vary independently, keeping $\alpha_2=\alpha_3$.

For $(\alpha_1,\alpha_2,\alpha_3,\alpha_4)=(1,-1,1,1)$, let us take T_i as follows:

$$T_0=\left(\begin{array}{ccc|ccc|cc} -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{array}\right) \qquad T_1=\left(\begin{array}{ccc|ccc|cc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{array}\right)$$

$$T_2=\left(\begin{array}{ccc|ccc|cc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{array}\right)$$

$$T_3=\left(\begin{array}{ccc|ccc|cc} 2a & a & 0 & 0 & a & -2a & b & 0 \\ 4a & -2a & 0 & 0 & 2a & -4a & 0 & 0 \\ 2a & -a & 0 & 0 & -a & -2a & 0 & d \\ \hline 0 & a & -2a & 2a & a & 0 & 0 & d \\ 0 & 2a & -4a & 4a & -2a & 0 & 0 & 0 \\ 0 & -a & -2a & 2a & -a & 0 & b & 0 \\ \hline 2c & -c & 0 & 0 & -c & 2c & 0 & 0 \\ 0 & -d & 2d & 2d & -d & 0 & 0 & 0 \end{array}\right)$$

where $a^2=\frac{1}{16}$, $bc=d^2=\frac{1}{4}$.

III-1 $\tilde{C}_l$ $l\geqq 4$

Consider the homomorphisms $G\to G_0$, $G\to G_l$ defined by $r_0\to e$, $r_i\to r_i$ $i=1,\cdots,l$ and $r_i\to r_i$, $i=0,\cdots,l-1$, $r_l\to e$ respectively. By the results in [3] on a group of type B_l, we see that $\alpha_4,\alpha_5,\alpha_6$ can vary independently, keeping $\alpha_1=\alpha_2=\alpha_3=1$, and $\alpha_1,\alpha_2,\alpha_4$ can vary independently, keeping $\alpha_3=\alpha_5=\alpha_6=1$.

For $\alpha_3=-1$, $\alpha_i=1$ $(i\neq 3)$, let us take $\Sigma(1)=\{N_1, N_2, N_3\}$ and put $T_0=N_1$, $T_1=\cdots=T_{l-1}=I$, $T_l=N_2$.

III-2 $\tilde{C}_3, \tilde{C}_2$

The same reasoning as III-1 is available in this case, using the results on groups of type B_3 and B_2.

IV-1 $\tilde{D}_l \quad l \geqq 5$

Consider the homomorphisms $G \to G_0$, $G \to G_l$ defined by $r_0 \to r_1$, $r_i \to r_i$ $i=1, \cdots, l$ and $r_i \to r_i$ $i=0, \cdots, l-1$, $r_l \to r_{l-1}$ respectively. By the results on a group of type $D_l (l \geqq 5)$ we see that α_i can vary independently.

IV-2 $\tilde{D}_4$

Consider the homomorphisms $G \to G_0$ defined by $r_0 \to r_i$ $i=1, 2, 3$, $r_j \to r_j$ $j \neq 0$. By the results on a group of type D_4, we see that α_i can vary independently.

V-1 $\tilde{E}_6 \quad \alpha_1 = -1$

We take a $\Delta(5)$ associated to $\Sigma(3)$. Put $T_i = D_i$ $i=1, \cdots, 5$. $T_6 = \sum_{j=1}^{6} a_j N_j$, $T_0 = \sum_{j=1}^{4} a_j N_j - (a_5 N_5 + a_6 N_6)$, where $a_1 = \cdots = a_5 = -\frac{1}{\sqrt{2}}$, $a_6 = \frac{\sqrt{3}}{2\sqrt{2}}$.

V-2 $\tilde{E}_7 \quad \alpha_1 = -1$

We take a $\Delta(6)$ associated to $\Sigma(3)$. Put $T_i = D_i$ $i=1, \cdots, 6$, $T_7 = \sum_{j=1}^{7} a_j N_j$, $T_0 = -N_7$, where $a_i = -\frac{1}{2\sqrt{2}}$ $i=1, \cdots, 6$, $a_7 = \frac{1}{2}$.

V-3 $\tilde{E}_8 \quad \alpha_1 = -1$

We take a $\Delta(7)$ associated to $\Sigma(4)$. Put $T_i = D_i$ $i=1, \cdots, 7$, $T_8 = \sum_{j=1}^{8} a_j N_j$, $T_0 = \frac{1}{\sqrt{2}}(N_8 - N_1)$, where $a_i = -\frac{1}{2\sqrt{2}}$ $i=1, \cdots, 8$.

VI $\tilde{F}_4$

i) $(\alpha_1, \alpha_2, \alpha_3) = (-1, 1, 1)$

Let us take $\Delta(3) = \{A_1, A_2, A_3\}$, and put $T_0 = A_1$, $T_1 = A_2$, $T_3 = A_3$, $T_4 = T_5 = I$.

ii) $(\alpha_1, \alpha_2, \alpha_3) = (1, -1, 1)$

Let us take $\Sigma(1) = \{N_1, N_2, N_3\}$, and put $T_i = N_1$, $i=0, 1, 2$, $T_j = N_2$, $j=3, 4$.

iii) $(\alpha_1, \alpha_2, \alpha_3) = (-1, -1, -1)$

Let us take $\Sigma(2) = \{N_1, N_2, N_3, N_4, N_5\}$ and put $T_0 = -N_1$, $T_1 = \frac{1}{2} N_1 + \frac{\sqrt{3}}{2} N_2$, $T_2 = -\frac{1}{\sqrt{3}} N_2 - \frac{\sqrt{2}}{\sqrt{3}} N_3$, $T_3 = \frac{\sqrt{3}}{2} N_3 + \frac{1}{2} N_4$, $T_4 = -N_4$.

VII $\tilde{G}_2$

i) $(\alpha_1, \alpha_2) = (-1, -1)$

Let us take $\Sigma(1) = \{N_1, N_2, N_3\}$. Put $T_0 = N_1$, $T_1 = -\frac{1}{2} N_1 - \frac{\sqrt{3}}{2} N_2$, $T_2 = N_3$.

ii) $(\alpha_1, \alpha_2) = (-1, 1)$

Let us take $\Delta(3) = \{A_1, A_2, A_3\}$. Put $T_i = A_{i+1}$.

REMARK. By above considerations we can see that the mapping

$$\mathfrak{P}(G) \to H^2(G, C^*)$$

stated in § 1 is surjective for any infinite discrete reflection groups.

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