Extension of group varieties and generalized Jacobian varieties

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Let A and B be commutative algebraic groups and let $\mathrm{Ext}(A, B)$ be the additive group of all extensions of A by B. The purpose of this note is to consider the structure of $\mathrm{Ext}(A, B)$ in case B is multiplicative group of the universal domain Ω, which we denote by G_m, and A is the Jacobian variety J of an algebraic curve C. Our main result is that $\mathrm{Ext}(J, G_m)$ is generated by generalized Jacobian varieties of C with respect to 0-cycles of type P_1+P_2 on C (Theorem in § 4). In § 1 we recall the composition law of $\mathrm{Ext}(A, B)$ and some of its properties. In § 2 we state about generalized Jacobian varieties. In § 3 we consider subgroups of the group of type $G_m\times\cdots\times G_m$. In the last § 4 we prove our Theorem.

§ 1. Let Ω be the universal domain. We fix an algebraically closed field k in Ω. We deal with algebraic objects (algebraic groups, rational mappings, $\cdots$ etc.) defined over k or rational over k. We recall some necessary properties of $\mathrm{Ext}(A, B)$ (cf. J. P. Serre (1)).

Let A, B, C be three commutative group varieties. In case the sequence

$$0 \longrightarrow B \xrightarrow{i} C \xrightarrow{p} A \longrightarrow 0$$

is exact in the usual sense, where i and p are separable homomorphisms, we call C an extension of A by B. We denote by $\mathrm{Ext}(A, B)$ the set of all extensions of A by B. In $\mathrm{Ext}(A, B)$ we define the composition law as follows.

1° Let C be an element of $\mathrm{Ext}(A, B)$ and $f\colon B \longrightarrow B'$, be a homomorphism from B to some commutative group variety B'. Then there exists a unique element C' of $\mathrm{Ext}(A, B')$ and a homomorphism F from C to C' such that the diagram

$$\begin{tikzcd} 0 \arrow[r] & B \arrow[r, "i"] \arrow[d, "f"] & C \arrow[r, "p"] \arrow[d, "F"] & A \arrow[r] \arrow[d, "id"] & 0 \\ 0 \arrow[r] & B' \arrow[r, "i"] & C' \arrow[r, "p"] & A \arrow[r] & 0 \end{tikzcd}$$

is commutative. For C' we may adopt the quotient group of $B'\times C$ by the subgroup $\{(f(b), -i(b));\, b\in B\}$. We denote the group C' by $f_*(C)$.

2° Let C be an element of $\mathrm{Ext}(A, B)$ and $g\colon A' \longrightarrow A$ be a homomorphism from some commutative group variety A' to A. Then there exists a unique element C' of $\mathrm{Ext}(A', B)$ and a homomorphism F from C' to C such that the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & B & \xrightarrow{i} & C' & \xrightarrow{p} & A' & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle id} & & \downarrow{\scriptstyle F} & & \downarrow{\scriptstyle g} & & \\ 0 & \longrightarrow & B & \xrightarrow{i} & C & \xrightarrow{p} & A & \longrightarrow & 0 \end{array}$$

is commutative. For C' we may adopt the sub-group $\{(c, a') \in C \times A';\ p(c) = g(a)\}$ of $C \times A'$. We denote this group C' by $g^*(C)$.

Let d; $A \longrightarrow A \times A$ be the canonical injection from A onto the diagonal of $A \times A$, and s; $B \times B \longrightarrow B$ be the mapping of the composition law of B. For two elements C and C' of $\mathrm{Ext}(A, B)$ we define the sum $C + C'$ by

$$C + C' = d^* s_*(C \times C') = s_* d^*(C \times C')\,.$$

By this composition law $\mathrm{Ext}(A, B)$ turns out to be an abelian group, and

1) $\mathrm{Ext}(A \times A, B) = \mathrm{Ext}(A, B) \times \mathrm{Ext}(A, B)$,
2) $\mathrm{Ext}(A, B \times B) = \mathrm{Ext}(A, B) \times \mathrm{Ext}(A, B)$.

§ **2.** Here we consider generalized Jacobian varieties. Let C be a complete non-singular curve and suppose that the set of rational points over k is dense in C. Let S be a finite set of points of C and $\sum_{P \in S} n_P P$ be a k-rational cycle of C. In the rational function field $\Omega(C)$ of C over Ω the set of all functions f with $v_P(f) \geqq n_P$ makes an Ω-module $\mathfrak{m}$. For 0-cycle $\mathfrak{a}$ and $\mathfrak{b}$ are $\mathfrak{m}$-linearly equivalent if there exists a function f in $\Omega(C)$ such that $f \equiv 1 \pmod{\mathfrak{m}}$ and $\mathfrak{a} - \mathfrak{b} = (f)$. Let P_0 be a k-rational point on C outside S and $\pi = g + \sum_{P \in S} n_P - 1$, where g is the genus of curve C. For 2π independent generic points $(M_1, \cdots, M_\pi, N_1, \cdots, N_\pi)$ of C over k there exist π independent generic points $(R_1, \cdots, R_\pi)$ of C over k such that

$$\sum_{i=1}^{\pi} M_i + \sum_{i=1}^{\pi} N_i \overset{\mathfrak{m}}{\sim} \sum_{i=1}^{\pi} R_i + \pi P_0\,. \tag{1}$$

By $C^{(\pi)}$ we denote the symmetric product of π copies of C. Let M, N, R be the points on $C^{(\pi)}$ which correspond to cycles $\sum_{i=1}^{\pi} M_i$, $\sum_{i=1}^{\pi} N_i$, $\sum_{i=1}^{\pi} R_i$ respectively. Then we know

$$k(M, N) = k(M, R) = k(N, R)\,.$$

Under these situation, there exists a group variety $J_{\mathfrak{m}}$ and a birational mapping $\bar{\varphi}_{\mathfrak{m}}$ from $C^{(\pi)}$ to $J_{\mathfrak{m}}$, where $J_{\mathfrak{m}}$ and $\bar{\varphi}_{\mathfrak{m}}$ are defined over k. From (1) follows $\bar{\varphi}_{\mathfrak{m}}(M) + \bar{\varphi}_{\mathfrak{m}}(N) = \bar{\varphi}_{\mathfrak{m}}(R)$ which means the composition law in $J_{\mathfrak{m}}$. $J_{\mathfrak{m}}$ and $\bar{\varphi}_{\mathfrak{m}}$ are uniquely determined up to isomorphism over k. Let $\varphi_{\mathfrak{m}}$ be the rational mapping from C to $J_{\mathfrak{m}}$ canonically defined by $\bar{\varphi}_{\mathfrak{m}}$. $J_{\mathfrak{m}}$ is the generalized Jacobian variety of C with respect to the module $\mathfrak{m}$ or with respect to the cycle $\sum_{P \in S} n_P P$, and $\varphi_{\mathfrak{m}}$ its canonical mapping. $(J_{\mathfrak{m}}, \varphi_{\mathfrak{m}})$ has the following properties;

1. $\varphi_{\mathfrak{m}}$ is biregularly defined on C outside S and the image of $\varphi_{\mathfrak{m}}$ generates $J_{\mathfrak{m}}$.

2. If we denote by J the Jacobian variety of C, $J_{\mathfrak{m}}$ is an extension of J by the linear group $G_m^s \times K$ where $s=\#(S)-1$, K is $G_a^r(r=\sum_{P\in S}(n_P-1))$ when characteristic $p=0$ and is a product of Witt groups when $p>0$.

3. Let $\mathfrak{m}$ and $\mathfrak{m}'$ be Ω-modules in $\Omega(C)$ corresponding to the cycles $\sum_{P\in S} n_P P$ and $\sum_{P\in S} n'_P P$ respectively. If $n_P \geqq n'_P$ for all P in S, then $J_{\mathfrak{m}}$ is an extention of $J'_{\mathfrak{m}}$ by $G_m^{s'} \times K$ where $s'=\#(S)-\#(P\in S;\, n_P\neq 0)$, K is a product of G_a or Witt groups. We denote by $f_{\mathfrak{m},\mathfrak{m}'}$ the homomorphism of $J_{\mathfrak{m}}$ to $J_{\mathfrak{m}'}$.

4. (Universal mapping property). Let G be a group variety defined over k and ψ be a rational mapping of C into G defined over k. Let $S=(P_1, P_2, \cdots, P_n)$ be the k-closed sub-set of C on which ψ is not defined. Then there exist a cycle $\sum_{P\in S} n_P P$ and a corresponding module $\mathfrak{m}$ of $\Omega(C)$, a k-rational point a of G and a rational homomorphism τ from the generalized Jacobian variety $(J_{\mathfrak{m}}, \varphi_{\mathfrak{m}})$ of C, with respect to $\mathfrak{m}$, to G such that for any point P of C outside S we have

$$\psi(P)=\tau(\varphi_{\mathfrak{m}}(P))+a\,.$$

§3. Let $S=(P_1, P_2, \cdots, P_n)$ be a finite set of distinct points of C and $\mathfrak{m}$ be the Ω-module of $\Omega(C)$ corresponding to the cycle $\sum_{i=1}^{n} P_i$. Then $J_{\mathfrak{m}}$ is an extension of the Jacobian variety J of C by $\overbrace{G_m\times G_m\times\cdots\times G_m}^{(n-1)}=\overbrace{G_m\times G_m\times\cdots\times G_m}^{n}/\Delta$ where Δ is the diagonal of $\overbrace{G_m\times G_m\times\cdots\times G_m}^{n}$. Hence we have the following exact sequence

$$0 \longrightarrow \Delta \longrightarrow \overbrace{G_m\times G_m\times\cdots\times G_m}^{n} \longrightarrow J_{\mathfrak{m}} \longrightarrow J \longrightarrow 0\,. \tag{2}$$

Let $\mathfrak{m}_j$ be the module corresponding to the cycle $\sum_{i\neq j} P_i$, Then $J_{\mathfrak{m}}$ is an extension of $J_{\mathfrak{m}_j}$ by G_m. If we denote this group G_m by G_j (2) can be written as

$$0 \longrightarrow \Delta \longrightarrow G_1\times G_2\times\cdots\times G_n \longrightarrow J_{\mathfrak{m}} \longrightarrow J \longrightarrow 0\,.$$

Let $\mathfrak{m}_{ij}$ be the module corresponding to the cycle P_i+P_j $(i<j)$, Then $J_{\mathfrak{m}_{ij}}=J_{ij}$ is an extension of J by $G_m=G_i\times G_j/\Delta$ and we have following commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \Delta & \longrightarrow & G_1\times G_2\times\cdots\times G_n & \longrightarrow & J_{\mathfrak{m}} & \xrightarrow{p} & J & \longrightarrow & 0 \\
 & & \big\downarrow{\scriptstyle id} & & \big\downarrow{\scriptstyle p_{ij}} & & \big\downarrow{\scriptstyle f_{ij}} & & \big\downarrow{\scriptstyle id} & & \\
0 & \longrightarrow & \Delta & \longrightarrow & G_i\times G_j & \longrightarrow & J_{ij} & \longrightarrow & J & \longrightarrow & 0\,.
\end{array}$$

Here p_{ij}; $G_1\times G_2\times\cdots\times G_n \longrightarrow G_i\times G_j$ is the projection and f_{ij} is the covering homomorphism. $G_1\times G_2\times\cdots\times G_n$ can be considered as an open set of n-dimensional affine space. Hence we can fix a coordinate system $(X_1, X_2, \cdots, X_n)$ on $G_1\times G_2\times\cdots\times G_n$. The kernel of p_{ij} is a sub-group of $G_1\times G_2\times\cdots\times G_n$ defined by $X_i=X_j$.

The sum $J_{ip,jq}$ of $J_{i,j}$ and $J_{p,q}$ in $\mathrm{Ext}(J, G_m)$ is defined by the following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \Delta \arrow[r] \arrow[d, "s"] & (G_i \times G_j) \times (G_p \times G_q) \arrow[r] \arrow[d, "s_{ip} \times s_{jq}"] & J_{i,j} \times J_{p,q} \arrow[r] \arrow[d, dotted] & J \times J \arrow[r] \arrow[d, "id"] & 0 \\
0 \arrow[r] & \Delta \arrow[r] & G_{ip} \times G_{jq} \arrow[r] & G \arrow[r] & J \times J \arrow[r] & 0 \\
0 \arrow[r] & \Delta \arrow[r] \arrow[u, "id"] & G_{ip} \times G_{jq} \arrow[r] \arrow[u, "id"] & J_{ip,jq} \arrow[r] \arrow[u, dotted] & J \arrow[r] \arrow[u, "d"] & 0
\end{tikzcd}$$

where s_{ip} (resp. s_{jq}) is the homomorphism of $G_i \times G_p$ (resp. $G_j \times G_q$) onto G_{ip} (resp. G_{jq}) defined by the group law of G_m, and G_{ip}, G_{jq} are linear group G_m. By the definition of $J_{jp,jq}$ we can define a rational homomorphism F from $J_{\mathfrak{m}}$ to $J_{ip,jq}$ and we have the following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \Delta \arrow[r] \arrow[d] & G_1 \times G_2 \times \cdots \times G_n \arrow[r] \arrow[d, "p"] & J_{\mathfrak{m}} \arrow[r] \arrow[d, "F"] & J \arrow[r] \arrow[d, "id"] & 0 \\
0 \arrow[r] & \Delta \arrow[r] & G_{ip} \times G_{jq} \arrow[r] & J_{ip,jq} \arrow[r] & J \arrow[r] & 0
\end{tikzcd}$$

where $p = (s_{ip} \times s_{jq}) \circ (p_{ip} \times p_{jq})$ i.e. $p(X_1, X_2, \cdots, X_n) = (X_i X_p, X_j X_q)$ and the kernel of p is defined by $X_i X_p = X_j X_q$ on $G_1 \times G_2 \times \cdots \times G_n$. In general we can define a homomorphism F from $J_{\mathfrak{m}}$ onto the group $J_{i_1 j_1} + J_{i_2 j_2} + \cdots + J_{i_k j_k}$ as above which induces a rational homomorphism p of $G_1 \times G_2 \times \cdots \times G_n$ onto $G_m \times G_m$ given by $p(X_1, X_2, \cdots, X_n) = (X_{i_1} X_{i_2} \cdots X_{i_k}, X_{j_1} X_{j_2} \cdots X_{j_k})$.

Let S be a sub-group with co-dimension 1 of the group $G_1 \times G_2 \times \cdots \times G_n$ which contains the diagonal Δ. Then S is defined by a unique irreducible minimal polinomial $Q(X_1, X_2, \cdots, X_n) = 0$ up to constant factor. $Q(X_1, X_2, \cdots, X_n)$ satisfies the following conditions:

a) $Q(x, x, \cdots, x) = 0$ for all x in Ω,

b) $Q(x_1, x_2, \cdots, x_n) = 0$ and $Q(y_1, y_2, \cdots, y_n) = 0$ (x_i, y_i in Ω)

imply $Q(x_1 y_1, x_2 y_2, \cdots, x_n y_n) = 0$.

From a) and b) it follows that $Q(x_1, x_2, \cdots, x_n) = 0$ implies $Q(Tx_1, Tx_2, \cdots, Tx_n) = 0$ for an indeterminate T of Ω. If we denote by $Q_j(X_1, X_2, \cdots, X_n)$ the homogeneous part with degree of $Q(X_1, X_2, \cdots, X_n)$, then we have $\sum_j Q_j(x_1, x_2, \cdots, x_n) T^j = 0$. Therefore we have $Q_j(x_1, x_2, \cdots, x_n) = 0$, $j = 0, 1, 2, \cdots, n$. By the minimality and irreducibility of Q, $Q(X_1, X_2, \cdots, X_n)$ must be homogeneous polynomial, and by a) the sum of coefficient a_i of $Q(X_1, X_2, \cdots, X_n)$ is zero

$$\sum_i a_i = 0 . \tag{3}$$

Let $G = G_1 \times G_2 \times \cdots \times G_n$ and let $(X_1, X_2, \cdots, X_n)$ and $(Y_1, Y_2, \cdots, Y_n)$ be two sets of independent indeterminate coordinates of G, the zero points of the polynomial

$Q(X_1Y_1, X_2Y_2, \cdots; X_nY_n)$ of $(X_1, X_2, \cdots, X_n, Y_1, Y_2, \cdots, Y_n)$ contains $S\times S$ in $G\times G$. $S\times S$ is defined by the ideal $(Q(X_1, X_2, \cdots, X_n), Q(Y_1, Y_2, \cdots, Y_n))$ in $\Omega[X_1, X_2, \cdots, X_n, Y_1, Y_2, \cdots, Y_n]$. Therefore $Q(X_1Y_1, X_2Y_2, \cdots, X_nY_n)$ can be written as

$$(4)\qquad Q(X_1Y_1, \cdots, X_nY_n) = Q(X_1, \cdots, X_n)R(X_1, \cdots, X_n, Y_1, \cdots, Y_n) + Q(Y_1, \cdots, Y_n)P(X_1, \cdots, X_n, Y_1, \cdots, Y_n)$$

where $R(X_1, \cdots, X_n, Y_1, \cdots, Y_n)$ and $P(X_1, \cdots, X_n, Y_1, \cdots, Y_n)$ are elements of $\Omega[X_1, X_2, \cdots, X_n, Y_1, Y_2, \cdots, Y_n]$. Since the left hand side of (4) is symmetric with respect to $(X_1, \cdots, X_n)$ and $(Y_1, Y_2, \cdots, Y_n)$, we get

$$(5)\quad Q(X_1, \cdots, X_n)R(X_1, \cdots, X_n, Y_1, \cdots, Y_n) + Q(Y_1, \cdots, Y_n)P(X_1, \cdots, X_n, Y_1, \cdots, Y_n) = Q(Y_1, \cdots, Y_n)R(Y_1, \cdots, Y_n, X_1, \cdots, X_n)Q(X_1, \cdots, X_n)P(Y_1, \cdots, Y_n, X_1, \cdots, X_n)$$

In the equation (5), if we put $(X_1, X_2, \cdots, X_n) = (1, 1, \cdots, 1)$, we get $P(1, \cdots, 1, Y_1, \cdots, Y_n) = 1$ and if we put $(Y_1, Y_2, \cdots, Y_n) = (1, 1, \cdots, 1)$ we get $R(X_1, X_2, \cdots, X_n, 1, 1, \cdots, 1) = 1$. Therefore if we consider $R(X_1, \cdots, X_n, Y_1, \cdots, Y_n)$ (resp. $P(X_1, \cdots, X_n, Y_1, \cdots, Y_n)$) as a polynomial of $(Y_1, Y_2, \cdots, Y_n)$ (resp. $(X_1, X_2, \cdots, X_n)$), the coefficient $R_j(X_1, \cdots, X_n)$ of the monomial $M_j(Y_1, \cdots, Y_n)$ of $(Y_1, \cdots, Y_n)$ (resp. $P_j(Y_1, \cdots, Y_n)$ of the monomial $M_j(X_1, \cdots, X_n)$ of $(X_1, \cdots, X_n)$) is a multiple of $Q(X_1, \cdots, X_n)$ (resp. $Q(Y_1, \cdots, Y_n)$). Taking into account the degree of the both side of (5) we get $R_j(X_1, \cdots, X_n) = 0$ (resp. $P_j(Y_1, \cdots, Y_n) = 0$). Thus we have

$$(6)\quad Q(X_1Y_1, \cdots, X_nY_n) = Q(X_1, \cdots, X_n)R(Y_1, \cdots, Y_n) + Q(Y_1, \cdots, Y_n)R(X_1, \cdots, X_n).$$

where $R(X_1, \cdots, X_n)$ is homogeneous of degree h and deg. Q = deg. R. Let $M_j(X)$, $(j = 1, 2, \cdots, N)$ be all the monomials of degree h in $(X_1, X_2, \cdots, X_n)$. Then $Q(X_1, \cdots, X_n)$ and $R(X_1, \cdots, X_n)$ can be written as

$$Q(X_1, \cdots, X_n) = \sum_{i=1}^{N} a_iM_i(X), \qquad R(X_1, \cdots, X_n) = \sum_{i=1}^{N} b_iM_i(X).$$

By (6) we have (i) $2a_ib_i = a_i$, (ii) $a_lb_k + a_kb_l = 0$ $(k \neq l)$. The properties (i) and (ii) do not hold if $a_i \neq 0$ for more than three indices i. In fact if a_1, a_2, a_3 were not equal to zero we would have $a_1 = -a_2 = a_3 = -a_1$ by (i) and (ii). Hence we would have $a_1 = 0$, which is a contradiction. Since $\sum_i a_i = 0$ by (3) we have

$$(7)\qquad Q(X_1, X_2, \cdots, X_n) = M_i(X) - M_j(X)$$

up to a constant factor.

Lemma. *Let τ be any rational homomorphism of $G_1 \times G_2 \times \cdots \times G_n$ onto G_m. Then the kernel of τ is defined by $M(X) - N(X) = 0$, where $M(X)$ and $N(X)$ are monomial of the same degree in $(X_1, X_2, \cdots, X_n)$, and τ is defined by $\tau(X_1, X_2, \cdots, X_n) = M(X)/N(X)$ or $\tau(X_1, X_2, \cdots, X_n) = N(X)/M(X)$.*

§ 4. Let G be an element of $\mathrm{Ext}(J, G_m)$.

$$\text{i.e.}\quad 0 \longrightarrow G_m \longrightarrow G \longrightarrow J \longrightarrow 0, \qquad \text{(exact)}.$$

Since G_m is a linear group there exists a rational cross-section s of J to G. Translating the image of the canonical mapping φ of C into J by a suitable k-rational point of J, we may assume that s is defined along the image $\varphi(C)$ of C by φ. The rational mapping of C into G induced by s we denote by ψ. Then ψ must be defined biregularly on C outside a k-closed set $S=(P_1, P_2, \cdots, P_n)$. By 4) of § 2, there exists a 0-cycle $\sum_{i=1}^{n} n_i P_i$ and a corresponding Ω-module $\mathfrak{m}$ in $\Omega(C)$. If we denote by $(J_{\mathfrak{m}}, \varphi_{\mathfrak{m}})$ the generalized Jacobian variety of C with respect to the module $\mathfrak{m}$, then there exists a homomorphism τ of $J_{\mathfrak{m}}$ to G and a k-rational point a such that

$$\varphi(P)-a=\tau(\psi(P)) \qquad \text{for all } P \text{ in } C \text{ outside } S,$$

namely we have the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & G_m^{n-1}\times K & \longrightarrow & J_{\mathfrak{m}} & \longrightarrow & J & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\bar{\tau}} & & \downarrow{\scriptstyle\tau} & & \downarrow{\scriptstyle id} & & \\ 0 & \longrightarrow & G_m & \longrightarrow & G & \longrightarrow & J & \longrightarrow & 0 \end{array}$$

where $\bar{\tau}$ is the induced homomorphism of τ, and K is Witt group or a multiple of the additive group G_a of universal domain. Hence K must be in the kernel of $\bar{\tau}$, and we may assume $n_i=1$ $(i=1, 2, \cdots, n)$. Therefore we have following commutative diagram:

$$(8)\qquad \begin{array}{ccccccccccc} 0 & \longrightarrow & \Delta & \longrightarrow & G_m^n & \longrightarrow & J_{\mathfrak{m}} & \longrightarrow & J & \longrightarrow & 0 \\ & & \downarrow & & \downarrow{\scriptstyle\bar{\tau}} & & \downarrow{\scriptstyle\tau} & & \downarrow{\scriptstyle id} & & \\ & & 0 & \longrightarrow & G_m & \longrightarrow & G & \longrightarrow & J & \longrightarrow & 0\,. \end{array}$$

In the group G_m^n we fix the coordinate system $(X_1, X_2, \cdots, X_n)$ as in § 3. Then $\bar{\tau}$ is given by $\bar{\tau}(X_1, X_2, \cdots, X_n)=M_1(X)/M_2(X)$ by Lemma, where $M_1(X)$ and $M_2(X)$ are monomial of the same degree h:

$$M_1(X)=X_{i_1}X_{i_2}\cdots X_{i_h}, \qquad M_2(X)=X_{j_1}X_{j_2}\cdots X_{j_h}.$$

Let J_{ij} be the generalized Jacobian variety of curve C with respect to the cycle P_i+P_j $(1\leqq i<j\leqq n)$. Then J_{ij} is an element of $\mathrm{Ext}(J, G_m)$. We have the commutative diagram:

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & \Delta & \longrightarrow & G_i\times G_j & \longrightarrow & J_{ij} & \longrightarrow & J & \longrightarrow & 0 \\ & & \downarrow & & \downarrow{\scriptstyle\tau_{ij}} & & \downarrow{\scriptstyle id} & & \downarrow{\scriptstyle id} & & \\ & & 0 & \longrightarrow & G_m & \longrightarrow & J_{ij} & \longrightarrow & J & \longrightarrow & 0 \end{array}$$

where τ_{ij} is given by $\tau_{ij}(X_i, X_j)=X_i/X_j$ or $\tau_{ij}(X_i, X_j)=X_j/X_i$. If $\tau_{ij}(X_i, X_j)=X_j/X_i$, take the inverse $-J_{ij}$ of J_{ij} in Ext (J, G_m). Then we have the commutative diagram:

$$\begin{tikzcd} 0 \arrow[r] & \Delta \arrow[r] \arrow[d] & G_i\times G_j \arrow[r] \arrow[d, "\tau'_{ij}"] & J_{ij} \arrow[r] \arrow[d] & J \arrow[r] \arrow[d, "id"] & 0 \\ & 0 \arrow[r] & G_m \arrow[r] & -J_{ij} \arrow[r] & J \arrow[r] & 0 \end{tikzcd}$$

where $\tau'_{ij}(X_i, X_j)=X_i/X_j$. Therefore if necessary replacing J_{ij} by $-J_{ij}$, we may assume $\tau_{ij}(X_i, X_j)=X_i/X_j$ $(i<j)$. Let $J_{i_1j_1}$, $J_{i_2j_2}$, $\cdots$, $J_{i_kj_k}$ $(i_1<j_1, \cdots, i_k<j_k)$ be generalized Jacobian varieties of curve C with respect to 0-cycles $P_{i_1}+P_{j_1}$, $\cdots$, $P_{i_k}+P_{j_k}$ respectively. Then $\tau_{ij}(X_{i_1}, X_{j_1})=X_{i_1}/X_{j_1}$, $\cdots$, $\tau_{ij}(X_{i_k}, X_{j_k})=X_{i_k}/X_{j_k}$ by our above assumption. By the first part of §3 we have the commutative diagram:

$$\begin{tikzcd} 0 \arrow[r] & \Delta \arrow[r] \arrow[d] & G_1\times G_2\times\cdots\times G_n \arrow[r] \arrow[d, "\bar{\sigma}"] & J_m \arrow[r] \arrow[d, "\sigma"] & J \arrow[r] \arrow[d, "id"] & 0 \\ & 0 \arrow[r] & G_m \arrow[r] & J_{i_1j_1}\pm\cdots\pm J_{i_kj_k} \arrow[r] & J \arrow[r] & 0 \end{tikzcd}$$

where $\bar{\sigma}$ is given by $\bar{\sigma}(X_1, X_2, \cdots, X_n)=X_{i_1}X_{i_2}\cdots X_{i_k}/X_{j_1}X_{j_2}\cdots X_{j_k}$. $\bar{\sigma}$ is the same homomorphism as $\bar{\tau}$ in the diagram (8) and the extension G of J by G_m in the same diagram must be $J_{i_1j_1}\pm\cdots\pm J_{i_kj_k}$ up to isomorphism. Therefore we have proved following Theorem:

THEOREM. *Let k be an algebraically closed field in the universal domain Ω, and let (J, φ) be Jacobian variety and canonical mapping of a complete nonsingular curve C, all defined over the field k. Then the group of all extensions of J by the group G_m of the multiplication of universal domain Ω is generated by the generalized Jocobian varieties of C with respect to the cycles of type P_1+P_2. $(P_1\neq P_2)$ on the curve C.*

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